Blatt 0. Keine Abgabe

1. Let M be any topological space. Let K be a compact subset of M and F be a closed subset of M. Prove that if $F \subset K$ then F is compact.

   Solution. Let $\{U_\alpha\}$ be an open covering of F. We need to prove that it has a finite subcover. The set $V = F^c$ is open. Then the family $\{U_\alpha, V\}$ covers the entire space M and, in particular, K. Therefore, it has a finite subcover: $\{U_{\alpha_i}, V\}$. It follows that F is covered by the union of all U_{α_i} and V. Since F and V are disjoint, we obtain that $F \subset \bigcup U_{\alpha_i}$, which finishes the proof.

2. A topological space M is called *Hausdorff* if, for any two disjoint points $x, y \in M$, there exist two disjoint open sets $U, V \subset M$ such that $x \in U$ and $y \in V$. Prove the following properties of a compact subset K of a Hausdorff topological space M.

   (a) For any $x \in K^c$ there exists an open set W_x containing x and disjoint from K.

   (b) K is a closed subset of M.

   Solution. (a) Fix $x \in K^c$. For any $y \in K$ there are disjoint open sets U_y and V_y containing x and y, respectively. All sets V_y, $y \in K$, form an open covering of K. Choose a finite subcover $\{V_{y_i}\}_{i=1}^N$ and set

   $$W_x = \bigcap_{i=1}^{n} U_{y_i}.$$

   Then W_x is an open set containing x and disjoint from all V_{y_i}. It follows that W_x is disjoint from K, which was to be proved.

   (b) Since $K^c = \bigcup_{x \in K^c} W_x$, it follows that K^c is open. Hence, K is closed.

3. Let X, Y be two topological spaces and $f : X \to Y$ be a continuous mapping. Prove that if K is a compact subset of X then $f(K)$ is a compact subset of Y.

   Solution. Let $\{U_\alpha\}$ be an open covering of $f(K)$. Then the preimages $\{f^{-1}(U_a)\}$ form an open covering of K (the sets $f^{-1}(U_\alpha)$ are open by definition of a continuous mapping). Therefore, there is a finite subcover $\{f^{-1}(U_{\alpha_i})\}_{i=1}^N$ of K, which implies that $\{U_{\alpha_i}\}_{i=1}^N$ is a finite subcover of K, thus proving the compactness of K.

4. Prove that, on any C-manifold M, there exists a countable sequence $\{\Omega_k\}$ of relatively compact open sets such that $\Omega_k \Subset \Omega_{k+1}$ (that is, Ω_k is relatively compact and $\overline{\Omega}_k \subset \Omega_{k+1}$) and the union of all Ω_k is M. Prove also that if M is connected then the sets Ω_k can also be taken connected.

   Remark. An increasing sequence $\{\Omega_k\}$ of open subsets of M whose union is M, is called an *exhaustion sequence.* If in addition $\Omega_k \Subset \Omega_{k+1}$ then the sequence $\{\Omega_k\}$ is called a *compact exhaustion sequence.*

Solution. By a lemma from lectures, there exists a countable family $\{U_i\}_{i=1}^{\infty}$ of relatively compact charts covering all M. Set

$$\Omega_k = \bigcup_{j=1}^{k} U_j \tag{1}$$

so that $\{\Omega_k\}_{k=1}^{\infty}$ is an increasing sequence of relatively compact open sets covering M. However, we may not have yet the inclusion $\overline{\Omega}_k \subset \Omega_{k+1}$. To achieve that, we will select a subsequence of $\{\Omega_k\}$. The first term to be selected is Ω_1. If we have already selected Ω_i then observe that $\overline{\Omega}_i$ is a compact set and, hence is covered by a finitely many of sets $\{\Omega_k\}$. Since this family is increasing, $\overline{\Omega}_i$ is covered by one of Ω_k. Hence, select this Ω_k as the next term in the subsequence.

Let M be connected. The sets U_j considered above are always connected as they are constructed as small balls in charts. All we need is to renumber the sequence $\{U_j\}$ in an appropriate order so that each set Ω_k defined by (1) is connected. We will do this by means of an inductive construction. At each step, some of the sets $\{U_j\}$ will be declared *selected* and denoted by $V_1, V_2, ...$ Set $V_1 = U_1$ and declare U_1 selected. Choose a non-selected set U_j with the minimal j that intersects V_1, denote it by V_2 and declare selected, etc. If $V_1, ..., V_i$ are already defined then choose a non-selected set U_j with minimal j that intersects $V_1 \cup V_2 ... \cup V_i$, denote it by V_{i+1} and declare selected. The process stops if we cannot choose V_{i+1}, and continues countably many times otherwise. By construction, all the unions $V_1 \cup V_2 ... \cup V_i$ are connected, so we need only to verify that the sequence $\{V_i\}$ covers all M.

Assume first that the sequence $\{V_i\}$ is finite. Then, at some step i, any non-selected U_j is disjoint with $V := V_1 \cup V_2 ... \cup V_i$. Let U be the union of all non-selected U_j. All selected U_j are contained in $V_1, ..., V_i$ and, hence, their union is V. Since U and V are two disjoint open sets covering M, one of them must be empty, which can be only U, whence $V = M$.

Assume now that the sequence $\{V_i\}$ is infinite, and show that it covers M. If this is not the case then there exists U_j which is not covered by $V = \bigcup_i V_i$. If U_j intersects V then it should have been selected at some step because there are selected sets $U_{j'}$ with $j' > j$. Hence, any U_j that is not covered by V is actually disjoint with V. Let U be the union of all such sets U_j. Clearly, U and V cover M and are disjoint, which implies by the connectedness of M that $U = \emptyset$ and, hence, $V = M$.

Blatt 1. Abgabe bis 24.10.2025

5. Let M be a C-manifold of dimension n. Let V be a chart on M and E be a subset of V. The compact inclusion $E \Subset V$ can be understood in two ways: in the sense of the topology of M as well as in the sense of the topology of $\mathbb{R}^n$, when identifying V with a subset of $\mathbb{R}^n$. Prove that these two meanings of $E \Subset V$ are equivalent.

   Solution. Let $\overline{E}$ be the closure of E in the topology of M and $\widetilde{E}$ be the closure of E in the topology of $\mathbb{R}^n$. We need to prove that the following two conditions are equivalent:

   (a) $\widetilde{E}$ is compact in $\mathbb{R}^n$ and $\widetilde{E} \subset V$

   (b) $\overline{E}$ is compact in M and $\overline{E} \subset V$,

   Let us prove that $(a) \Rightarrow (b)$. By Exercise 3, the set $\widetilde{E}$ is also compact in M (as a continuous image of a compact subset of $\mathbb{R}^n$). Since M is Hausdorff, $\widetilde{E}$ is also closed in M (Exercise 2). Since $E \subset \widetilde{E} \subset V$, it follows that $\overline{E} \subset \widetilde{E}$ and, hence, $\overline{E}$ is compact in M as a closed subset of a compact set (Exercise 1). Since also $\overline{E} \subset V$, we obtain (b). The converse implication $(b) \Rightarrow (a)$ is proved in the same way.

6. Prove that, on any C-manifold M, there is a countable *locally finite* family of relatively compact charts covering M.

   Remark. A family $\mathcal{F}$ of subsets of M is called locally finite if any compact subset of M intersects only finitely many sets from $\mathcal{F}$.

   Solution. By a lemma from lectures, there exists countable family $\{U_i\}$ of locally compact charts covering M. Let $\{\Omega_k\}$ be a sequence from Exercise 4. Let us construct inductively a locally finite family $\mathcal{F}$ of relatively compact charts which will also cover M. At step 0, set $\mathcal{F} = \emptyset$. At step $k \geq 1$, consider the compact set $\overline{\Omega}_k \setminus \Omega_{k-1}$ (where $\Omega_0 := \emptyset$). This set is covered by a finite number of charts from the family $\{U_i\}$; say $U_1, ..., U_m$. Then add to $\mathcal{F}$ the charts $U_i \setminus \overline{\Omega}_{k-1}$, $i = 1, ..., m$. Clearly, the newly added charts cover $\overline{\Omega}_k \setminus \overline{\Omega}_{k-1}$ and do not intersect $\overline{\Omega}_{k-1}$.

   The family of charts $\mathcal{F}$ obtained in this way covers all sets $\overline{\Omega}_k \setminus \overline{\Omega}_{k-1}$ and hence M. Let us verify that it is locally finite. Indeed, any compact set K is contained in one of the sets Ω_k. Up to the step k of the above construction, family $\mathcal{F}$ contains a finite number of chart. From step $k+1$ onwards, each added chart does not intersect Ω_k. Hence, there is only a finite number of charts in $\mathcal{F}$ intersecting Ω_k and hence K, which finishes the proof.

7. Fix some positive integers n, m, let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a C^1-function. Consider the null set of F, that is, the set

$$M = \left\{ x \in \mathbb{R}^{n+m} : F(x) = 0 \right\},$$

and assume that, for any point $x \in M$, the Jacobi matrix $F'(x)$ has the rank m. Prove that M is a C-manifold of dimension n.

Hint. Use the implicit function theorem.

Solution. The topology of M is induced from that of $\mathbb{R}^{n+m}$, that is, open sets in M are intersections of open sets in $\mathbb{R}^{n+m}$ with M. Since $\mathbb{R}^{n+m}$ has a countable base, it follows that M also has countable base. Since $\mathbb{R}^{n+m}$ is Hausdorff, the same is true also for M.

Fix a point $z \in M$ and show that there is a chart in M that covers z. The Jacobi matrix $F'(z)$ is as follows:

$$F'(z) = \begin{pmatrix} \partial_{x_1} F_1 & ... & \partial_{x_n} F_1 & \partial_{x_{n+1}} F_1 & ... & \partial_{x_{n+m}} F_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \partial_{x_1} F_m & ... & \partial_{x_n} F_m & \partial_{x_{n+11}} F_m & ... & \partial_{x_{n+m}} F_m \end{pmatrix}.$$

It has $n + m$ columns and m rows. Since the rank of $F'(z)$ is equal to m, there are m linearly independent columns. Without loss of generality, assume that the last m columns are linearly independent. Then, by the implicit function theorem, there exist open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ such that $z \in U \times V$ and that the equation $F(x) = 0$ in $U \times V$ can be resolved with respect to the last m coordinates $x^{n+1}, ..., x^{n+m}$; that is, in $U \times V$ the equation $F(x) = 0$ is equivalent

$$\left(x^{n+1}, ..., x^{n+m}\right) = f(x^1, ...x^n)$$

where $f : U \to V$ is of the class C^1. In other words, $M \cap (U \times V)$ is a graph of a continuous function $f : U \to \mathbb{R}^{n+m}$, which implies that $M \cap (U \times V)$ is a chart. Since any point $z \in M$ is covered by such a chart, we conclude that M is a C-manifold.

8. Let K be a compact subset of a smooth manifold M and $\{U_j\}_{j=1}^k$ be a finite family of open sets covering K. Prove that there exist non-negative functions $\varphi_j \in C_0^\infty(U_j)$ such that $\sum_{j=1}^k \varphi_j \equiv 1$ in an open neighbourhood of K and $\sum_{j=1}^k \varphi_j \leq 1$ in M.

   Remark. The family $\{\varphi_j\}$ is called a partition of unity at K subordinate to $\{U_j\}$. If all U_j are charts then the existence of the partition of unity was proved in lectures.

   Hint. Choose first a finite family $\{W_i\}$ of charts covering K and such that each W_i is contained in one of the sets U_j. By a theorem from lectures, there exists a partition of unity $\{\psi_i\}$ of K subordinate to $\{W_i\}$. Use functions ψ_i to construct functions φ_j.

   Solution. For any point $x \in K$, there is a chart W_x containing x. Since x is also covered by one of the sets U_j, by reducing W_x we can assume that $W_x \subset U_j$ for some j. Since the family $\{W_x\}_{x \in K}$ covers K, there exists a finite subfamily $\{W_i\}_{i=1}^m$ also covering K. Since each W_i is a chart, by a theorem from lectures there exists a partition of unity $\{\psi_i\}_{i=1}^m$ at K subordinate to $\{W_i\}$. Now define the sequence $\{\varphi_j\}_{j=1}^k$ as follows:

$$\varphi_1 = \sum_{\{i: \operatorname{supp} \psi_i \subset U_1\}} \psi_i,$$

$$\varphi_2 = \sum_{\{i: \operatorname{supp} \psi_i \subset U_2,\ \operatorname{supp} \psi_i \not\subset U_1\}} \psi_i,$$

$$\dots$$

$$\varphi_k = \sum_{\{i:\operatorname{supp}\psi_i \subset U_k,\ \operatorname{supp}\psi_i \not\subset U_l\ \forall l<k.\}} \psi_i.$$

Clearly, each φ_j is non-negative and belongs to $C_0^\infty(U_j)$. Since W_i is covered by some U_j, each ψ_i is supported in some U_j and, hence, each ψ_i has been used in the above construction exactly once. It follows that

$$\sum_j \varphi_j \equiv \sum_i \psi_i,$$

which implies that $\{\varphi_j\}$ is a partition of unity at K subordinate to $\{U_j\}$.

Blatt 2. Abgabe bis 31.10.2025

In all exercises, M is a smooth manifold of dimension n.

9. A *path* on M is any smooth mapping $\gamma : [0, a] \to M$, where $a > 0$. Set $x = \gamma(0)$. For any function $f \in C^\infty(M)$, define the derivative of f along the path γ at the point x by
$$\frac{\partial f}{\partial \gamma} := \frac{d}{dt} f(\gamma(t))\Big|_{t=0}.$$
   (a) Prove that $\frac{\partial}{\partial \gamma}$ is an $\mathbb{R}$-differentiation at x, that is, $\frac{\partial}{\partial \gamma} \in T_x M$.
   (b) Prove that any tangent vector $\xi \in T_x M$ can be represented in the form $\xi = \frac{\partial}{\partial \gamma}$ for some path γ.

Solution. (a) The operation $\frac{d}{d\gamma}$ is linear and satisfies the product rule because
$$\begin{aligned}\frac{\partial}{\partial \gamma}(fg) &= \frac{d}{dt}\left(f(\gamma(t))\, g(\gamma(t))\right)\Big|_{t=0} \\ &= \frac{d}{dt} f(\gamma(t))\, g(\gamma(t))\Big|_{t=0} + f(\gamma(t))\, \frac{d}{dt} g(\gamma(t))\Big|_{t=0} \\ &= \frac{\partial f}{\partial \gamma} g(x) + \frac{\partial g}{\partial \gamma} f(x).\end{aligned}$$

Hence, $\frac{\partial}{\partial \gamma} \in T_x M$.

(b) Let $x^1, ..., x^n$ be a local coordinate system near the point x. Fix some $\xi \in T_x M$ and find a path γ such that $\xi = \frac{\partial}{\partial \gamma}$. If $\xi = 0$ then define γ just by $\gamma(t) \equiv x$. Let $\xi \neq 0$. Assuming that the Euclidean ball $B_r(x)$ is contained in this chart, define in this coordinate system the path γ by
$$\gamma(t) = x + t\xi$$
where $t \in [0, a]$ and $a = r/|\xi|$, where $|\xi|$ is the Euclidean length of $\xi = \left(\xi^1, ..., \xi^n\right)$. Indeed, for any $t \in [0, a]$, we have
$$t|\xi| \leq a|\xi| \leq r$$
so that $\gamma(t) \in B_r(x)$. Hence, $\gamma : [0, a] \to M$ is well-defined and is obviously smooth in t. We have
$$\frac{d}{dt} f(\gamma(t)) = \frac{d}{dt} f(x + t\xi) = \frac{\partial f}{\partial x^i} \xi^i.$$
Since also
$$\xi(f) = \frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x^i} \xi^i,$$
we conclude that $\frac{d}{dt} f(\gamma(t)) = \xi(f)$ and, hence, $\frac{\partial}{\partial \gamma} = \xi$.

10. A smooth *vector field* on M is a mapping $X : C^\infty(M) \to C^\infty(M)$ such that, for any $x \in M$, the mapping

$$\begin{aligned} C^\infty(M) &\to \mathbb{R} \\ f &\mapsto X(f)(x) \end{aligned}$$

is a $\mathbb{R}$-differentiation at x. Prove that, in any chart U with the local coordinates $x^1, ..., x^n$, there are functions $a^1, ..., a^n \in C^\infty(U)$ such that

$$X(f) = \sum_{i=1}^n a^i \frac{\partial f}{\partial x^i} \text{ for any } f \in C^\infty(M).$$

Hint. Use the fact that any $\mathbb{R}$-differentiation ξ can be represented in the form

$$\xi = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i}$$

for some $\xi^i \in \mathbb{R}$.

Solution. Since any $\mathbb{R}$-differentiation ξ at x is given by

$$\sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i}$$

for some reals ξ^i, it follows that, for any $x \in U$ there are reals $a^1(x), ..., a^n(x)$ such that

$$X(f)(x) = \sum_{i=1}^n a^i(x) \frac{\partial f}{\partial x^i},$$

that is,

$$X(f) = \sum_{i=1}^n a^i \frac{\partial f}{\partial x^i}.$$

Since $X(f)$ is smooth for any smooth f, it follows that also $a^i(x)$ must be smooth functions. Indeed, there exists a function $f \in C^\infty(M)$ such that $f(x) = x^i$ in a neighborhood of some point $x_0 \in U$. Then, in this neighborhood, we have $X(f) = a^i$, which implies that a^i is smooth in this neighborhood and, hence, in U.

11. Let X and Y be two smooth vector fields on M (as in Exercise 5). Define the *Lie bracket* $[X, Y]$ of X, Y as a mapping of $C^\infty(M)$ into itself by

$$[X, Y] := XY - YX,$$

that is, $[X, Y](f) = X(Y(f)) - Y(X(f))$ for any $f \in C^\infty(M)$.

Prove that $[X, Y]$ is a smooth vector field on M.

Hint. In the local coordinates, $X(f)$ is a combination of the first partial derivatives $\frac{\partial f}{\partial x^i}$ (by Exercise 5). Hence, $XY(f)$ and $YX(f)$ contain the second derivatives of f. The point of the present claim is that the difference $XY(f) - YX(f)$ depends on the first derivatives of f only, that is, the second derivatives cancel out.

Solution. We need to prove that, for any $x \in M$, the mapping $[X, Y]$ is $\mathbb{R}$-differentiation at x. Fix a chart U around x. Then, by Exercise 5, there are smooth functions $a^1, ..., a^n$ and $b^1, ..., b^n$ in U such that

$$X(f) = \sum_{i=1}^n a^i \frac{\partial f}{\partial x^i} \quad \text{and} \quad Y(f) = \sum_{i=1}^n b^i \frac{\partial f}{\partial x^i}.$$

It follows that

$$\begin{aligned} XY(f) = X(Y(f)) &= X\left(\sum_{i=1}^n b^i \frac{\partial f}{\partial x^i}\right) \\ &= \sum_{j=1}^n a^j \frac{\partial}{\partial x^j}\left(\sum_{i=1}^n b^i \frac{\partial f}{\partial x^i}\right) \\ &= \sum_{j=1}^n \sum_{i=1}^n a^j b^i \frac{\partial^2 f}{\partial x^j \partial x^i} + \sum_{j=1}^n \sum_{i=1}^n a^j \frac{\partial b^i}{\partial x^j} \frac{\partial f}{\partial x^i}. \end{aligned}$$

Similarly, we have

$$YX(f) = \sum_{j=1}^n \sum_{i=1}^n b^j a^i \frac{\partial^2 f}{\partial x^j \partial x^i} + \sum_{j=1}^n \sum_{i=1}^n b^j \frac{\partial a^i}{\partial x^j} \frac{\partial f}{\partial x^i}.$$

By interchanging of i and j, we see that

$$\sum_{j=1}^n \sum_{i=1}^n a^j b^i \frac{\partial^2 f}{\partial x^j \partial x^i} = \sum_{j=1}^n \sum_{i=1}^n b^j a^i \frac{\partial^2 f}{\partial x^j \partial x^i}.$$

Hence,

$$\begin{aligned} XY(f) - YX(f) &= \sum_{j=1}^n \sum_{i=1}^n a^j \frac{\partial b^i}{\partial x^j} \frac{\partial f}{\partial x^i} - \sum_{j=1}^n \sum_{i=1}^n b^j \frac{\partial a^i}{\partial x^j} \frac{\partial f}{\partial x^i} \\ &= \sum_{i=1}^n C^i \frac{\partial f}{\partial x^i}, \end{aligned}$$

where

$$C^i = \sum_{j=1}^n \left(a^j \frac{\partial b^i}{\partial x^j} - b^j \frac{\partial a^i}{\partial x^j}\right). \tag{2}$$

Therefore, $[X, Y] = XY - YX$ is $\mathbb{R}$-differentiation at x, which was to be proved.

12. *(The Jacobi identity)* Prove the following identity for three smooth vector fields X, Y, Z on a smooth manifold M:

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0, \tag{3}$$

where $[\cdot, \cdot]$ is the Lie bracket defined in Exercise 5, *Hint*. By linearity, it suffices to consider the case when X, Y, Z are given in the local coordinates $x^1, ..., x^n$ by

$$X = a\frac{\partial}{\partial x^i}, \quad Y = b\frac{\partial}{\partial x^j}, \quad Z = c\frac{\partial}{\partial x^k},$$

where a, b, c are smooth functions of $x^1, ..., x^n$ and i, j, k are some indices from $1, ..., n$.

Solution. If $i \neq j$ then we have by (**??**)

$$[X, Y] = \left[a\frac{\partial}{\partial x^i}, b\frac{\partial}{\partial x^j}\right] = C^i\frac{\partial}{\partial x^i} + C^j\frac{\partial}{\partial x^j}$$

where

$$C^i = \sum_{l=1}^{n}\left(a^l\frac{\partial b^i}{\partial x^l} - b^l\frac{\partial a^i}{\partial x^l}\right) = -b\frac{\partial a}{\partial x^j}$$

and

$$C^j = \sum_{l=1}^{n}\left(a^l\frac{\partial b^j}{\partial x^l} - b^l\frac{\partial a^j}{\partial x^l}\right) = a\frac{\partial b}{\partial x^i}.$$

If $i = j$ then by (**??**)

$$[X, Y] = \left[a\frac{\partial}{\partial x^i}, b\frac{\partial}{\partial x^i}\right] = C^i\frac{\partial}{\partial x^i},$$

where

$$C^i = \sum_{l=1}^{n}\left(a^l\frac{\partial b^i}{\partial x^l} - b^l\frac{\partial a^i}{\partial x^l}\right) = a\frac{\partial b}{\partial x^i} - b\frac{\partial a}{\partial x^i}.$$

Hence, in the both cases we obtain that

$$\left[a\frac{\partial}{\partial x^i}, b\frac{\partial}{\partial x^j}\right] = -b\frac{\partial a}{\partial x^j}\frac{\partial}{\partial x^i} + a\frac{\partial b}{\partial x^i}\frac{\partial}{\partial x^j} = -Y(a)\frac{\partial}{\partial x^i} + X(b)\frac{\partial}{\partial x^j}.$$

Similarly, we have

$$\left[Z, C^i\frac{\partial}{\partial x^i}\right] = \left[c\frac{\partial}{\partial x^k}, C^i\frac{\partial}{\partial x^i}\right] = -C^i\frac{\partial c}{\partial x^i}\frac{\partial}{\partial x^k} + c\frac{\partial C^i}{\partial x^k}\frac{\partial}{\partial x^i},$$

$$\left[Z, C^j\frac{\partial}{\partial x^j}\right] = -C^j\frac{\partial c}{\partial x^j}\frac{\partial}{\partial x^k} + c\frac{\partial C^j}{\partial x^k}\frac{\partial}{\partial x^j},$$

and, hence,

$$\begin{aligned}
[Z, [X, Y]] &= -C^i\frac{\partial c}{\partial x^i}\frac{\partial}{\partial x^k} + c\frac{\partial C^i}{\partial x^k}\frac{\partial}{\partial x^i} + -C^j\frac{\partial c}{\partial x^j}\frac{\partial}{\partial x^k} + c\frac{\partial C^j}{\partial x^k}\frac{\partial}{\partial x^j} \\
&= b\frac{\partial a}{\partial x^j}\frac{\partial c}{\partial x^i}\frac{\partial}{\partial x^k} - c\frac{\partial}{\partial x^k}\left(b\frac{\partial a}{\partial x^j}\right)\frac{\partial}{\partial x^i} - a\frac{\partial b}{\partial x^i}\frac{\partial c}{\partial x^j}\frac{\partial}{\partial x^k} + c\frac{\partial}{\partial x^k}\left(a\frac{\partial b}{\partial x^i}\right)\frac{\partial}{\partial x^j} \\
&= -c\frac{\partial}{\partial x^k}\left(b\frac{\partial a}{\partial x^j}\right)\frac{\partial}{\partial x^i} + c\frac{\partial}{\partial x^k}\left(a\frac{\partial b}{\partial x^i}\right)\frac{\partial}{\partial x^j} + \left(b\frac{\partial a}{\partial x^j}\frac{\partial c}{\partial x^i} - a\frac{\partial b}{\partial x^i}\frac{\partial c}{\partial x^j}\right)\frac{\partial}{\partial x^k}.
\end{aligned}$$

By cycling permutation of a, b, c we obtain that

$$[X, [Y, Z]] = -a\frac{\partial}{\partial x^i}\left(c\frac{\partial b}{\partial x^k}\right)\frac{\partial}{\partial x^j} + a\frac{\partial}{\partial x^i}\left(b\frac{\partial c}{\partial x^j}\right)\frac{\partial}{\partial x^k} + \left(c\frac{\partial b}{\partial x^k}\frac{\partial a}{\partial x^j} - b\frac{\partial c}{\partial x^j}\frac{\partial a}{\partial x^k}\right)\frac{\partial}{\partial x^i}$$

and

$$[Y,[Z,X]] = -b\frac{\partial}{\partial x^j}\left(a\frac{\partial c}{\partial x^i}\right)\frac{\partial}{\partial x^k} + b\frac{\partial}{\partial x^j}\left(c\frac{\partial a}{\partial x^k}\right)\frac{\partial}{\partial x^i} + \left(a\frac{\partial c}{\partial x^i}\frac{\partial b}{\partial x^k} - c\frac{\partial a}{\partial x^k}\frac{\partial b}{\partial x^i}\right)\frac{\partial}{\partial x^j}.$$

It follows that, in the sum $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]]$, the coefficient in front of $\frac{\partial}{\partial x^i}$ is

$$\begin{aligned}
&- c\frac{\partial}{\partial x^k}\left(b\frac{\partial a}{\partial x^j}\right) + \left(c\frac{\partial b}{\partial x^k}\frac{\partial a}{\partial x^j} - b\frac{\partial c}{\partial x^j}\frac{\partial a}{\partial x^k}\right) + b\frac{\partial}{\partial x^j}\left(c\frac{\partial a}{\partial x^k}\right) \\
&= -c\frac{\partial b}{\partial x^k}\frac{\partial a}{\partial x^j} - cb\frac{\partial^j a}{\partial x^k \partial x^j} + \left(c\frac{\partial b}{\partial x^k}\frac{\partial a}{\partial x^j} - b\frac{\partial c}{\partial x^j}\frac{\partial a}{\partial x^k}\right) + b\frac{\partial c}{\partial x^j}\frac{\partial a}{\partial x^k} + bc\frac{\partial^j a}{\partial x^j \partial x^k} \\
&= 0.
\end{aligned}$$

Similarly, the coefficients in front of $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial x^k}$ are 0, whence (2) follows.

Blatt 3. Abgabe bis 07.11.2025

13. Let $\{V_\alpha\}$ be a family of charts covering a smooth manifold M. Prove that if a function $f : M \to \mathbb{R}$ belongs to $C^\infty(V_\alpha)$ for any α then $f \in C^\infty(M)$.

   Remark. By definition, $f \in C^\infty(M)$ if $f \in C^\infty(U)$ for *any* chart U in M.

   Solution. Fix a chart $U \subset M$ with coordinates $x^1, .., x^n$ and prove that $f \in C^\infty(U)$. It suffices to prove that f is C^∞ in a neighborhood of any point $p \in U$. Let V be a chart from the family $\{V_\alpha\}$ that contains p, let $y^1, ..., y^n$ be coordinates in V. Since $f \in C^\infty(V)$, the function f is smooth in the coordinates $y^1, ..., y^n$. Since the change of coordinates $y^i = y^i(x^1, ..., x^n)$ is given by smooth functions, we conclude that f is also smooth in the coordinates $x^1, ..., x^n$ that are defined in $U \cap V$. Hence, f is C^∞ in the chart U in a neighborhood of p, which was to be proved.

14. Prove that a smooth hypersurface in $\mathbb{R}^{n+1}$ is a smooth n-dimensional manifold.

   Remark. Recall that a smooth hypersurface is a subset M of $\mathbb{R}^{n+1}$ that is locally a graph of a smooth function. Each graph gives rise to a chart on M. You need to prove that the change of coordinates between any two of such charts is given by smooth functions.

   Solution. Assume that a point p on a smooth hypersurface M in $\mathbb{R}^{n+1}$ belongs to two charts: the first chart where M is the graph of the function

$$x^1 = f(x^2, ..., x^n)$$

   and the second chart where M is the graph of a function

$$x^n = g\left(x^1, ..., x^{n-1}\right).$$

   We need to show that the local coordinates $x^1, ..., x^{n-1}$ in the second chart are expressed as smooth functions of the local coordinates of the first chart, that is, of $x^2, ..., x^n$. Indeed, this change is given by

$$\begin{aligned} x^1 &= f(x^2, ..., x^n) \\ x^2 &= x^2 \\ &\ldots \\ x^{n-1} &= x^{n-1} \end{aligned}$$

   which is clearly a smooth change of coordinates.

15. (a) Let U be an open set in $\mathbb{R}^n$ and $\Psi : U \to \mathbb{R}^m$ be a smooth mapping. Let Γ be the graph of Ψ, that is,

$$\Gamma = \left\{(x, \Psi(x)) \in \mathbb{R}^{n+m} : x \in \mathbb{R}^n\right\}.$$

   Prove that Γ is a submanifold of $\mathbb{R}^{n+m}$ of dimension n.

   (b) Prove that any smooth hypersurface in $\mathbb{R}^{n+1}$ in a submanifold of $\mathbb{R}^{n+1}$ of dimension n.

Hint. Use the definition of a submanifold.

Solution. (a) Let $x^1, .., x^n$ be the coordinates in $\mathbb{R}^n$, $y^1, .., y^m$ be the coordinates in $\mathbb{R}^m$. Set $N = n + m$. We introduce new coordinates $z^1, ..z^N$ in $U \times \mathbb{R}^m$ such that Γ is given by the equations $z^{n+1} = ... = z^N = 0$, which will imply by definition that Γ is a submanifold of $\mathbb{R}^N$ of dimension n (with a single chart). Set

$$\begin{aligned} z^1 &= x^1 \\ &\ldots \\ z^n &= x^n \\ z^{n+1} &= y^1 - \Psi^1\left(x^1, ..., x^n\right) \\ &\ldots \\ z^{n+m} &= y^m - \Psi^m\left(x^1, ..., x^n\right). \end{aligned}$$

First of all, the mapping $(x, y) \mapsto z$ is a diffeomorphism because this mapping is smooth and there is the smooth inverse mapping

$$\begin{aligned} x^1 &= z^1 \\ &\ldots \\ x^n &= z^n \\ y^1 &= z^{n+1} + \Psi^1\left(z^1, ..., z^n\right) \\ &\ldots \\ y^m &= z^{n+m} + \Psi^m\left(z^1, ..., z^n\right). \end{aligned}$$

The equation $y = \Psi(x)$ in the coordinates z is equivalent to $z^{n+1} = ... = z^N = 0$, which finishes the proof.

(b) If M is a hypersurface in $\mathbb{R}^{n+1}$ then locally it is a graph of a function $f : U \to \mathbb{R}$ where U is an open subset of U. By (a) there is a local coordinate system $z^1, ..., z^{n+1}$ in $U \times \mathbb{R}$ such that $M \cap (U \times \mathbb{R})$ is given by the equation $z^{n+1} = 0$. Since the entire M is covered by the charts like $U \times \mathbb{R}^n$, we conclude that M is a submanifold of $\mathbb{R}^{n+1}$ of dimension n.

16. Let M be a smooth manifold of dimension n and S be its submanifold of dimension m. Let $x^1, ..., x^n$ be local coordinates in a chart U in M and $y^1, ..., y^m$ be local coordinates in a chart V on S. Assume that $V \subset U$. Then, for any point in V, its x-coordinates can be expressed as functions of its y-coordinates:

$$x^i = f^i(y^1, ..., y^m), \quad i = 1, ..., n,$$

where f^i are some real-valued functions on V. Prove that $f^i \in C^\infty(V)$.

Hint. Use the definition of a submanifold.

Solution. It suffices to prove that f^i are C^∞ in a neighborhood of any point $p \in V$. By definition of a submanifold, for any point $p \in S$ there is a chart W in M containing p such that in its local coordinates $z^1, ..., z^n$,

$$z \in S \cap W \Leftrightarrow z^{m+1} = ... = z^n = 0.$$

In this case $S \cap W$ is a chart on S with the local coordinates $z^1, ..., z^m$.

It follows that in the intersection of the domains of the local coordinates $y^1, .., y^m$ and $z^1, ..., z^m$, the change of coordinates is given by C^∞ functions:

$$z^i = z^i(y^1, ..., y^m), \ \ i = 1, ..., m. \tag{4}$$

Similarly, in the intersection of the domains of the local coordinates $x^1, ..., x^n$ and $z^1, ..., z^n$, the change of coordinates is also given by smooth functions:

$$x^i = x^i(z^1, ..., z^n).$$

In particular, on S we have

$$x^i = x^i(z^1, ..., z^m, 0, ..., 0).$$

Substituting here the smooth functions (3), we express x^i as a smooth function of $y^1, ..., y^m$, which was to be proved.

17. * Let X and Y be smooth manifolds of dimensions n and m, respectively, with $n \geq m$. A mapping $\Phi : Y \to X$ is called smooth if in local coordinates $x^1, ..., x^n$ in X and $y^1, ..., y^m$ in Y it is given by equations

$$x^i = \Phi^i(y^1, ..., y^m), \ i = 1, ..., n,$$

where Φ^i are smooth functions. Let Φ be a smooth mapping as above satisfying the following three properties:

(1) the mapping $\Phi : Y \to X$ is injective;

(2) the rank of the Jacobi matrix $J = \left(\frac{\partial \Phi^i}{\partial y^j}\right)$ of Φ is maximal at all points, that is, it is equal to m;

(3) Φ is a homeomorphism of Y onto its image $S := \Phi(Y) \subset X$.

Prove that S is a submanifold of X of dimension m.

Solution. Fix a point $p \in S$. We need to show that there is a local coordinate system $z^1, ..., z^n$ on X around point p such that in a neighborhood of p

$$S = \left\{z : z^{m+1} = ... = z^n = 0\right\}.$$

Set $q = \Phi^{-1}(p) \in Y$ and let $y^1, ..., y^m$ be local coordinates in some chart V' on Y containing q. By (1) and (3), the mapping $\Phi^{-1} : S \to Y$ is well-defined and continuous. Hence, there is an open set U' in S containing p such that $\Phi^{-1}(U') \subset V'$. The set U' is an intersection of an open set $U \subset X$ with S, which implies that the preimage $\Phi^{-1}(U)$ is contained in V'. By shrinking U, we can assume that U is a chart around p with coordinates $x^1, ..., x^n$. Now setting $V := \Phi^{-1}(U) \subset V'$.

In the local coordinates in U and V, the mapping Φ is given by the system of equations

$$\begin{aligned} x^1 &= \Phi^1\left(y^1, ..., y^m\right) \\ &... \\ x^m &= \Phi^m\left(y^1, ..., y^m\right) \end{aligned}$$

$$\begin{aligned} &.... \\ x^n &= \Phi^n\left(y^1, ..., y^m\right) \end{aligned}$$

that is, $(x^1, ..., x^n) \in S$ if and only of there is $(y^1, ..., y^m) \in V$ such that these equations are satisfied.

Since by (2) the Jacobi matrix $J = \left(\frac{\partial \Phi^i}{\partial y^j}\right)$ at q has the rank m, this matrix has m linearly independent rows, suppose, these are the rows $i = 1, ..., m$. Then the same is true in a neighborhood of q. By the inverse function theorem, the the first m equations of the above system can be solved with respect to $y^1, ..., y^m$ in a neighborhood of p, as follows:

$$y^i = y^i\left(x^1, ..., x^m\right), \ i = 1, ..., m.$$

Consider the following new coordinates in a neighborhood of q:

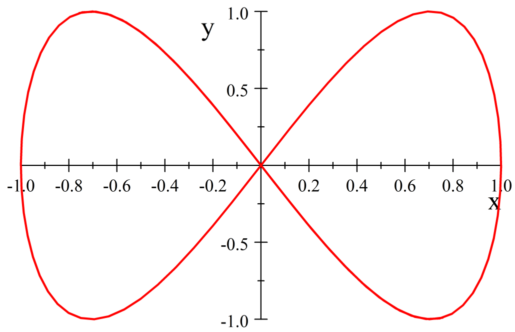
$$\begin{aligned} z^i &= y^i\left(x^1, ..., x^m\right), \ i = 1, ..., m \\ z^i &= x^i - \Phi^i\left(y^1, ..., y^m\right), \ i = m+1, ..., m \end{aligned}$$

Then in a neighborhood of p the condition that $(z^1, ..., z^n) \in S$ is equivalent to $z^{m+1} = ... = z^n = 0$, which finishes the proof.

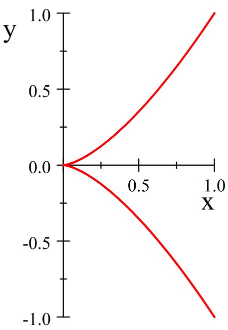
18. * Give examples to show that any of the above conditions $(1), (2), (3)$ is essential for the statement of Exercise 5.

Solution. For counterexamples, consider the following mappings $\Phi : I \to \mathbb{R}^2$ where I is an interval. In all examples S is not a a submanifold near $(0,0)$.

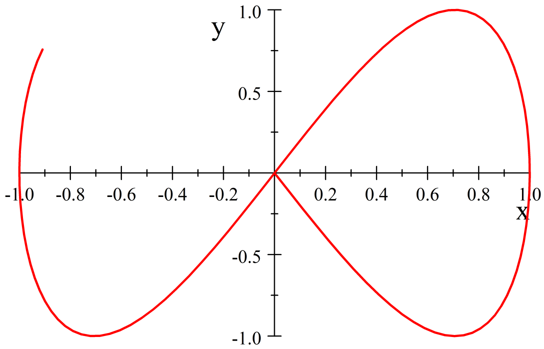
(1) $I = (-4, 4), \Phi(t) = (\sin t, \cos 2t)$ is not injective, self-intersection point $(0,0)$.



(2) $I = (1,1)$, $\Phi(t) = (t^2, t^3)$. The Jacobi matrix vanishes at $t = 0$.



(3) $I = (-2, \pi)$, $\Phi(t) = (\sin t, \cos 2t)$ is injective but the inverse mapping Φ^{-1} is not continuous at $(0, 0)$.



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19. Let M be a smooth manifold of dimension n, $F \in C^\infty(M)$ and S be a non-singular null set of F, that is,

$$S = \{x \in M : F(x) = 0\} \text{ and } \nabla F \neq 0 \text{ on } S.$$

Consequently, S is a submanifold of M of dimension $n-1$. Fix $x_0 \in S$. Every tangent vector $\xi \in T_{x_0}S$ can be regarded as an element of $T_{x_0}M$ by using the identity

$$\xi(f) := \xi(f|_S) \quad \text{for any } f \in C^\infty(M),$$

as the restriction $f|_S$ on S is a smooth function on S. Hence, the tangent space $T_{x_0}S$ is a subspace of $T_{x_0}M$. Prove that $T_{x_0}S$ as a subspace of $T_{x_0}M$ is given by the equation

$$T_{x_0}S = \{\xi \in T_{x_0}M : \langle dF, \xi\rangle = 0\}. \tag{5}$$

Hint. Verify first that every $\xi \in T_{x_0}S$ satisfies as an element of $T_{x_0}M$ the equation $\langle dF, \xi\rangle = 0$.

Solution. Note that dF is a non-zero covector, that is, a linear functional in $T_{x_0}M$, and the equation $\langle dF, \xi\rangle = 0$, indeed, determines an $(n-1)$-dimensional subspace of $T_{x_0}M$. Since $\dim T_{x_0}S = n-1$, it suffices to verify that every vector from T_xS satisfies equation (4). Indeed, if $\xi \in T_{x_0}S$ then we have by definition of dF

$$\langle dF, \xi\rangle = \xi(F) = \xi(F|_S) = \xi(0) = 0,$$

where we have used that $F|_S \equiv 0$.

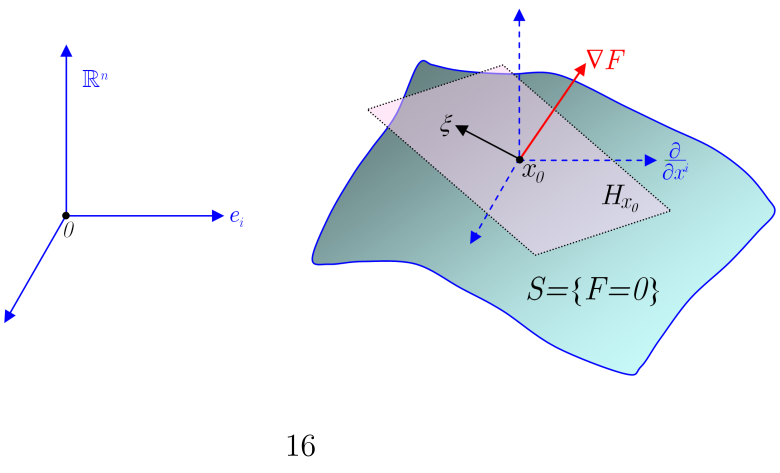
20. * In the setting of Exercise 5, let $M = \mathbb{R}^n$. Let us identify the tangent space $T_{x_0}M$ with $\mathbb{R}^n$ by using the isomorphism $I : T_{x_0}M \to \mathbb{R}^n$ defined by

$$I\left(\frac{\partial}{\partial x^i}\right) = e_i,$$

where $\{e_i\}_{i=1}^n$ is the canonical basis in $\mathbb{R}^n$. Prove that the set

$$x_0 + I(T_{x_0}S)$$

is the hyperplane H_{x_0} in $\mathbb{R}^n$ that goes through x_0 and has the normal $\nabla F(x_0)$, where $\nabla F = \left(\frac{\partial F}{\partial x^1}, \dots, \frac{\partial F}{\partial x^n}\right)$.



Remark. This result means that the tangent space $T_{x_0}S$ can be naturally identified with the tangent hyperplane H_{x_0} in $\mathbb{R}^n$ to the hypersurface S at the point x_0.

Solution. The image $I(T_{x_0}S)$ is an $n-1$-dimensional subspace of $\mathbb{R}^n$, and $x_0+I(T_{x_0}S)$ is a hyperplane that goes through x_0. It remains to verify that

$$I(T_{x_0}S) \perp \nabla F(x_0)$$

Since

$$dF = \frac{\partial F}{\partial x^i} dx^i,$$

the equation $\langle dF, \xi \rangle = 0$ of (a) for the tangent vector $\xi = \xi^i \frac{\partial}{\partial x^i}$ becomes

$$\frac{\partial F}{\partial x^i} \xi^i = 0,$$

that is,

$$\frac{\partial F}{\partial x^1} \xi^1 + ... + \frac{\partial F}{\partial x^n} \xi^n = 0.$$

For any tangent vector $\xi = \xi^i \frac{\partial}{\partial x^i} \in T_{x_0}M$, we have $I(\xi) = \xi^i e_i$. Since $\frac{\partial F}{\partial x^i}$ are the components of the gradient ∇F as a vector in $\mathbb{R}^n$, we obtain that $I(\xi) \perp \nabla F$, which was to be proved.

21. Let M be a Riemannian manifold.

(a) Prove the product rule for the operators d and ∇ on M:

$$d(uv) = udv + vdu \tag{6}$$

and

$$\nabla(uv) = u\nabla v + v\nabla u, \tag{7}$$

where u and v are smooth function on M.

(b) Prove the chain rule for the operators d and ∇ on M:

$$df(u) = f'(u)\, du$$

and

$$\nabla f(u) = f'(u)\, \nabla u,$$

where u and f are smooth functions on M and $\mathbb{R}$, respectively.

Solution. (a) In local coordinates $x^1, ..., x^n$, we have

$$du = (\partial_{x_i} u)\, dx^i \tag{8}$$

which implies

$$\begin{aligned} d(uv) &= \partial_{x_i}(uv)\, dx^i = (\partial_{x_i} u)\, v dx^i + u\, (\partial_{x_i} v)\, dx^i \\ &= vdu + vdu. \end{aligned}$$

Since

$$\nabla u = \mathbf{g}^{-1} du, \tag{9}$$

we obtain from (5)

$$\begin{aligned}\nabla (uv) &= \mathbf{g}^{-1} d (uv) = \mathbf{g}^{-1} (udv + vdu) = u\mathbf{g}^{-1} dv + v\mathbf{g}^{-1} du \\ &= u\nabla v + v\nabla u.\end{aligned}$$

(b)Using (7) and the chain rule for ∂_{x_i}, we obtain

$$df (u) = \partial_{x_i} (f (u)) dx^i = f' (u) (\partial_{x_i} u) dx^i = f' (u) du.$$

Using also (8), we obtain

$$\nabla f (u) = \mathbf{g}^{-1} df (u) = \mathbf{g}^{-1} f' (u) du = f' (u) \mathbf{g}^{-1} du = f' (u) \nabla u.$$

22. Let $(M, \mathbf{g})$ be a Riemannian manifold. Let U and V be charts on M with the local coordinates $x^1, ..., x^n$ and $y^1, ..., y^n$, respectively. Denote by g^x and g^y the matrices of the metric $\mathbf{g}$ in U and V, respectively. Let $J = \left(J_i^k\right)_{k,i=1}^n$ be the Jacobian matrix of the change $y = y (x)$ defined in $U \cap V$ by

$$J_i^k = \frac{\partial y^k}{\partial x^i}, \tag{10}$$

where k is the row index and i is the column index. Prove the following identity in $U \cap V$:

$$g^x = J^T g^y J, \tag{11}$$

where J^T denotes the transposed matrix.

Solution. By the chain rule, we have for any smooth function f in $U \cap V$

$$\frac{\partial f}{\partial x^i} = \frac{\partial y^k}{\partial x^i} \frac{\partial f}{\partial y^k} = J_i^k \frac{\partial f}{\partial y^k},$$

whence

$$\frac{\partial}{\partial x^i} = J_i^k \frac{\partial}{\partial y^k}$$

and, hence,

$$\begin{aligned} g_{ij}^x &= \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle_{\mathbf{g}} = \langle J_i^k \frac{\partial}{\partial y^k}, J_j^l \frac{\partial}{\partial y^l} \rangle_{\mathbf{g}} \\ &= J_i^k J_j^l \langle \frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^l} \rangle_{\mathbf{g}} \\ &= J_i^k g_{kl}^y J_j^l. \end{aligned}$$

Noticing that

$$J_i^k g_{kl}^y J_j^l = \left(J^T g^y J\right)_{ij},$$

we obtain

$$g_{ij}^x = \left(J^T g^y J\right)_{ij}$$

whence (10) follows.

23. Let $\mathbf{g}$, $\widetilde{\mathbf{g}}$ be two Riemannian metrics on a smooth manifold M and let g^x and $\widetilde{g}^x$ be the matrices of $\mathbf{g}$ and $\widetilde{\mathbf{g}}$, respectively, in some local coordinate system $x^1, ..., x^n$. Prove that the ratio

$$\frac{\det \widetilde{g}^x}{\det g^x}$$

does not depend on the choice of the coordinate system (although separately $\det g^x$ and $\det \widetilde{g}^x$ do depend on the coordinate system).

Hint. Use the formula (10) from Exercise 5.

Solution. Let $x^1, ..., x^n$ and $y^1, ..., y^n$ be two coordinate systems and let g^x and g^y be the matrices of $\mathbf{g}$ in these systems, respectively. By Exercise 5, we have

$$g^y = J^T g^x J$$

where J is the Jacobian matrix of the change $y = y(x)$. It follows that

$$\det g^y = (\det J)^2 \det g^x. \tag{12}$$

The same identity holds for the metric $\widetilde{\mathbf{g}}$:

$$\det \widetilde{g}^y = (\det J)^2 \det \widetilde{g}^x.$$

Dividing it by (**??**) and noticing that $(\det J)^2$ cancels out, we obtain

$$\frac{\det \widetilde{g}^y}{\det g^y} = \frac{\det \widetilde{g}^x}{\det g^x},$$

which was to be proved.

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24. Let M be a smooth manifold, S be a submanifold, and $x \in S$. Prove that, for any $f \in C^\infty(M)$,
$$d\left(f|_S\right) = (df)\,|_{T_xS}, \tag{13}$$
where d in the left hand side is differential on S, while d in the right hand side is differential on M, and $(df)\,|_{T_sS}$ means the restriction of df to the tangent space T_xS.

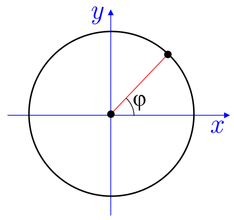
Solution. Fix $x_0 \in S$. By definition, df is an element of $T^*_{x_0}M$ such that, for any $\mathbb{R}$-differentiation $\xi \in T_{x_0}M$,
$$\langle df, \xi\rangle = \xi(f).$$
For any $\xi \in T_{x_0}S$, we have
$$\langle d\left(f|_S\right), \xi\rangle = \xi(f|_S) = \xi(f),$$
where in the second identity we consider ξ as an element of $T_{x_0}M$ as $T_{x_0}S \subset T_{x_0}M$. On the other hand, by the definition of the restriction $(df)\,|_S$ we have
$$\langle (df)\,|_{T_xS}, \xi\rangle = \langle df, \xi\rangle = \xi(f).$$
Comparing the two above equation, we obtain
$$\langle d\left(f|_S\right), \xi\rangle = \langle (df)\,|_{T_xS}, \xi\rangle \quad \forall \xi \in T_{x_0}S,$$
whence (11) follows.

Second solution. Let $x^1, ..., x^n$ be local coordinates in a neighborhood of $x_0 \in S$ such that S is given by equations
$$x^{m+1} = x^{m+2} = ... = x^n = 0.$$
Then we have for differential in M
$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i.$$
Since the basis in $T_{x_0}S$ is given by $\left\{\frac{\partial}{\partial x^j}\right\}_{j=1}^m$ and
$$\langle df, \frac{\partial}{\partial x^j}\rangle = \frac{\partial f}{\partial x^j},$$
we obtain that
$$(df)\,|_S = \sum_{i=1}^m \frac{\partial f}{\partial x^i} dx^i.$$
In the other hand,
$$f|_S(x^1, ..., x^m) = f(x^1, ..., x^m, 0, ..., 0),$$
whence
$$d\left(f|_S\right) = \sum_{i=1}^m \frac{\partial f}{\partial x^i} dx^i,$$
and (11) follows.

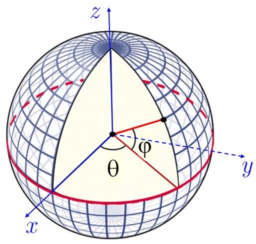
25. For any submanifold S of $\mathbb{R}^n$, denote by $\mathbf{g}_S$ the Riemannian metric on S that is induced by the canonical Euclidean metric

$$\mathbf{g}_{\mathbb{R}^n} = \left(dx^1\right)^2 + \ldots + \left(dx^n\right)^2 . \tag{14}$$

(a) Let $\mathbb{S}^1$ be the unit circle in $\mathbb{R}^2$. Express the induced metric $\mathbf{g}_{\mathbb{S}^1}$ using the polar angle φ on $\mathbb{S}^1$ as a local coordinate.



(b) Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. Express the induced metric $\mathbf{g}_{\mathbb{S}^2}$ on $\mathbb{S}^2$ in terms of the local coordinates θ, φ where θ is the longitude on $\mathbb{S}^2$ and φ is the latitude.



Hint. Express the Cartesian coordinates in terms of the polar coordinates and use the representation (12) of the metric in the Cartesian coordinates.

Solution. (a) The Cartesian coordinates x, y on $\mathbb{S}^1$ can be expressed via the polar coordinate Θ on $\mathbb{S}^1$ as follows:

$$x = \cos\varphi \quad \text{and } y = \sin\varphi.$$

Hence,

$$\mathbf{g}_{\mathbb{S}^1} = dx^2 + dy^2 = \sin^2\varphi d\varphi^2 + \sin^2\varphi d\varphi^2 = d\varphi^2.$$

(b) The Cartesian coordinates x, y, z on $\mathbb{S}^2$ are expressed via φ and θ as follows:

$$x = \cos\varphi\cos\theta$$
$$y = \cos\varphi\sin\theta$$
$$z = \sin\varphi$$

Hence, we have

$$dx = -\sin\varphi\cos\theta\, d\varphi - \cos\varphi\sin\theta\, d\theta$$
$$dy = -\sin\varphi\sin\theta\, d\varphi + \cos\varphi\cos\theta\, d\theta$$
$$dz = \cos\varphi\, d\varphi$$

whence

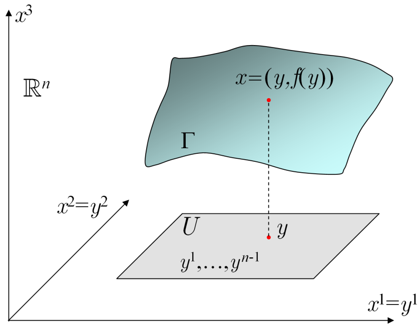
$$\begin{aligned} dx^2 + dy^2 + dz^2 &= \left(-\sin\varphi\cos\theta\, d\varphi - \cos\varphi\sin\theta\, d\theta\right)^2 \\ &+ \left(-\sin\varphi\sin\theta\, d\varphi + \cos\varphi\cos\theta\, d\theta\right)^2 \\ &+ \cos^2\varphi\, d\varphi^2 \\ &= \sin^2\varphi\cos^2\theta\, d\varphi^2 + \cos^2\varphi\sin^2\theta\, d\theta^2 + 2\sin\varphi\cos\theta\cos\varphi\sin\theta\, d\varphi d\theta \\ &+ \sin^2\varphi\sin^2\theta\, d\varphi^2 + \cos^2\varphi\cos^2\theta\, d\theta^2 - 2\sin\varphi\sin\theta\cos\varphi\cos\theta\, d\varphi d\theta \end{aligned}$$

$$
\begin{aligned}
&+\cos^2\varphi\, d\varphi^2 \\
&= \sin^2\varphi d\varphi^2 + \cos^2\varphi\, d\theta^2 + \cos^2\varphi\, d\varphi^2 \\
&= d\varphi^2 + \cos^2\varphi d\theta^2.
\end{aligned}
$$

Hence,

$$\mathbf{g}_{\mathbb{S}^2} = d\varphi^2 + \cos^2\varphi d\theta^2.$$

26. Let Γ be the graph of a smooth function $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^{n-1}$ is an open set. Let $\mathbf{g}$ be the canonical metric in $\mathbb{R}^n$. Denote by $\mathbf{g}_\Gamma$ the induced Riemannian metric on Γ considering Γ as a submanifold of $\mathbb{R}^n$.
Let $y^1, ..., y^{n-1}$ be the Cartesian coordinates in U; consider them as local coordinates in Γ. Prove that the components of the metric $\mathbf{g}_\Gamma$ in the coordinates $y^1, ..., y^{n-1}$ are as follows:



$$(g_\Gamma)_{ij} = \delta_{ij} + \frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}, \tag{15}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Hint. Use the following result from lectures: if S is a submanifold of a Riemannian manifold $(M, \mathbf{g})$ then the induced metric $\mathbf{g}_S$ is given in the local coordinates $x^1, ..., x^n$ on M and $y^1, ..., y^m$ on S by the formula

$$(g_S)_{ij} = g_{kl}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}. \tag{16}$$

Solution. Denote the Cartesian coordinates in $\mathbb{R}^n$ by $x^1, ..., x^n$. The Euclidean metric is given by

$$\mathbf{g} = \left(dx^1\right)^2 + ... + (dx^n)^2.$$

By (14) we have

$$(g_\Gamma)_{ij} = g_{kl}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j},$$

where $x = x(y)$ is the change of the coordinates that is given by

$$
\begin{aligned}
x^1 &= y^1 \\
x^2 &= y^2 \\
&\cdots \\
x^{n-1} &= y^{n-1} \\
x^n &= f\left(y^1, ..., y^{n-1}\right).
\end{aligned}
$$

Hence, we have

$$\frac{\partial x^k}{\partial y^i} = \begin{cases} \delta_i^k, & k \le n-1 \\ \frac{\partial f}{\partial y^i}, & k = n \end{cases}.$$

Since $g_{kl} = \delta_{kl}$, we obtain

$$(g_\Gamma)_{ij} = \sum_{k=1}^{n} \frac{\partial x^k}{\partial y^i} \frac{\partial x^k}{\partial y^j} = \sum_{k=1}^{n-1} \delta_i^k \delta_j^k + \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} = \delta_{ij} + \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j}.$$

Alternatively, we can use the relations

$$\begin{aligned} dx^1 &= dy^1 \\ dx^2 &= dy^2 \\ &\dots \\ dx^{n-1} &= dy^{n-1} \\ dx^n &= \frac{\partial f}{\partial y^i} dy^i \end{aligned}$$

that imply

$$\begin{aligned} \mathbf{g}_\Gamma &= \left(dx^1\right)^2 + \dots + \left(dx^n\right)^2 \\ &= \left(dy^1\right)^2 + \dots + \left(dy^{n-1}\right)^2 + \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} dy^i dy^j \end{aligned}$$

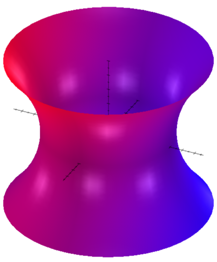
that is,

$$\begin{aligned} (g_\Gamma)_{ii} &= 1 + \left(\frac{\partial f}{\partial y^i}\right)^2 \\ (g_\Gamma)_{ij} &= \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j}, \quad i \neq j. \end{aligned}$$

27. A catenoid Cat is a surface in $\mathbb{R}^3$ that is given by the parametric equations

$$x^1 = \cosh \rho \cos \theta, \quad x^2 = \cosh \rho \sin \theta, \quad x^3 = \rho,$$

where $\rho \in \mathbb{R}$ and $\theta \in (-\pi, \pi)$.
Express the induced Riemannian metric on Cat in terms of the coordinates ρ, θ.



Catenoid

Remark. The catenoid Cat is the image of the mapping $\mathbb{R} \times (-\pi, \pi) \to \mathbb{R}^3$ given by the above equations. By using Exercise 5, it is possible to show that Cat is a submanifold of $\mathbb{R}^3$ of dimension 2.

Solution. We have

$$\begin{aligned} dx^1 &= \sinh \rho \cos \theta d\rho - \cosh \rho \sin \theta d\theta \\ dx^2 &= \sinh \rho \sin \theta d\rho + \cosh \rho \cos \theta d\theta \\ dx^3 &= d\rho \end{aligned}$$

whence

$$
\begin{aligned}
\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 &= (\sinh\rho\cos\theta d\rho - \cosh\rho\sin\theta d\theta)^2 \\
&+ (\sinh\rho\sin\theta d\rho + \cosh\rho\cos\theta d\theta)^2 \\
&+ (d\rho)^2 \\
&= \sinh^2\rho\cos^2\theta\,(d\rho)^2 + \cosh^2\rho\sin^2\theta\,(d\theta)^2 - 2\sinh\rho\cos\theta\cosh\rho\sin\theta d\rho d\theta \\
&+ \sinh^2\rho\sin^2\theta\,(d\rho)^2 + \cosh^2\rho\cos^2\theta\,(d\theta)^2 + 2\sinh\rho\sin\theta\cosh\rho\cos\theta d\rho d\theta \\
&+ (d\rho)^2 \\
&= \sinh^2\rho\,(d\rho)^2 + \cosh^2\rho\,(d\theta)^2 + (d\rho)^2 \\
&= \left(1 + \sinh^2\rho\right)(d\rho)^2 + \cosh^2\rho\,(d\theta)^2 \\
&= \cosh^2\rho\left(d\rho^2 + d\theta^2\right).
\end{aligned}
$$

Hence,the induced Riemannian metric is

$$\mathbf{g}_{Cat} = \cosh^2\rho\left(d\rho^2 + d\theta^2\right).$$

Blatt 6. Abgabe bis 28.11.2025

28. *(Product rule for divergence)* Let $(M,\mathbf{g})$ be a Riemannian manifold. Let $\nabla=\nabla_{\mathbf{g}}$ and $\operatorname{div}=\operatorname{div}_{\mathbf{g}}$ be the gradient and divergence associated with $\mathbf{g}$, respectively. Let u be any smooth function on M and v be any smooth vector field on M.

   (a) Prove the identity $\operatorname{div}(uv)=\langle\nabla u,v\rangle+u\operatorname{div}v$.

   Hint. Use the divergence theorem and the gradient product rule of Exercise 5a.

   (b) Let $(M,\mathbf{g},\mu)$ be a weighted manifold. Prove that the weighted divergence $\operatorname{div}_{\mathbf{g},\mu}$ satisfies the identity $\operatorname{div}_{\mathbf{g},\mu}(uv)=\langle\nabla u,v\rangle+u\operatorname{div}_{\mathbf{g},\mu}v$.

Solution. (a) For any $\varphi\in C_0^\infty(M)$, we obtain using the divergence theorem and the product rule (6) of gradient of Exercise 5a:

$$
\begin{aligned}
\int_M \operatorname{div}(uv)\,\varphi d\mu &= -\int_M \langle uv,\nabla\varphi\rangle d\mu = -\int_M\langle v,u\nabla\varphi\rangle d\mu\\
&= -\int_M\langle v,\nabla(u\varphi)-\varphi\nabla u\rangle d\mu\\
&= -\int_M\langle v,\nabla(u\varphi)\rangle d\mu+\int_M\langle v,\varphi\nabla u\rangle d\mu\\
&= \int_M(\operatorname{div}v)\,u\varphi\,d\mu+\int_M\langle v,\nabla u\rangle\varphi d\mu\\
&= \int_M\left((\operatorname{div}v)\,u+\langle v,\nabla u\rangle\right)\varphi d\mu
\end{aligned}
$$

whence (**??**) follows.

(b) If D is the density of μ then we have

$$\operatorname{div}_{\mathbf{g},\mu}u=\frac{1}{D}\operatorname{div}_{\mathbf{g}}(Du).$$

Using the product rule (**??**) for $\operatorname{div}_{\mathbf{g}}=\operatorname{div}$ we obtain

$$
\begin{aligned}
\operatorname{div}_{\mathbf{g},\mu}(uv) &= \frac{1}{D}\operatorname{div}(Duv)\\
&= \frac{1}{D}\left(\langle\nabla u,Dv\rangle+u\operatorname{div}(Dv)\right)\\
&= \langle\nabla u,\frac{D}{D}v\rangle+u\frac{1}{D}\operatorname{div}(Dv)\\
&= \langle\nabla u,v\rangle+u\operatorname{div}_{\mathbf{g},\mu}v,
\end{aligned}
$$

which finishes the proof.

29. Recall that the Laplace-Beltrami operator $\Delta=\Delta_{\mathbf{g}}$ on a Riemannian manifold $(M,\mathbf{g})$ is defined for any function $u\in C^\infty(M)$ by $\Delta u=\operatorname{div}(\nabla u)$.

(a) *(Product rule for the Laplacian)* Prove that, for smooth functions u and v on M,

$$\Delta(uv) = u\Delta v + 2\langle \nabla u, \nabla v\rangle + (\Delta u)\, v.$$

(b) *(Chain rule for the Laplacian)* Prove that, for functions $u \in C^\infty(M)$ and $f \in C^\infty(\mathbb{R})$,

$$\Delta f(u) = f''(u)\,|\nabla u|^2 + f'(u)\,\Delta u.$$

Solution. (a) Using the identity $\Delta = \operatorname{div} \nabla$ and the product rules for ∇ and div (cf. Exercises 5a and 5), we obtain

$$\begin{aligned}\Delta(uv) &= \operatorname{div}(\nabla(uv)) = \operatorname{div}(u\nabla v + v\nabla u)\\ &= \langle \nabla u, \nabla v\rangle + u\Delta v + \langle \nabla v, \nabla u\rangle_{\mathbf{g}} + v\Delta u\\ &= u\Delta v + 2\langle \nabla u, \nabla v\rangle + (\Delta u)\, v.\end{aligned}$$

(b) Using Exercises 5b and 5, we obtain

$$\begin{aligned}\Delta f(u) &= \operatorname{div}(\nabla f(u)) = \operatorname{div}(f'(u)\nabla u) = \langle \nabla f'(u), \nabla u\rangle + f'(u)\operatorname{div}(\nabla u)\\ &= f''(u)\,\langle \nabla u, \nabla u\rangle + f'(u)\,\Delta u.\end{aligned}$$

30. Let $(M, \mathbf{g}, \mu)$ be a weighted manifold. Prove the following identities.

(a) (*The divergence theorem*) If u is a smooth function on M and v is a smooth vector field, such that either u or v has a compact support then

$$\int_M (\operatorname{div}_{\mathbf{g},\mu} v)\; u\, d\mu = -\int_M \langle v, \nabla u\rangle\, d\mu. \tag{17}$$

(b) (*The Green formula*) If u, v are smooth functions on M and one of them has a compact support then

$$\int_M u\,\Delta_{\mathbf{g},\mu} v\, d\mu = -\int_M \langle \nabla u, \nabla v\rangle\, d\mu = \int_M v\,\Delta_{\mathbf{g},\mu} u\, d\mu. \tag{18}$$

Solution. (a) Let D be the density function, that is, $d\mu = D d\nu$ where ν is the Riemannian metric. Then we have

$$\operatorname{div}_{\mathbf{g},\mu} v = \frac{1}{D}\operatorname{div}_g(Dv)$$

whence

$$\int_M \operatorname{div}_{\mathbf{g},\mu} v\, u\, d\mu = \int_M \frac{1}{D}\operatorname{div}_{\mathbf{g}}(Dv)\; u\, D d\nu = \int_M \operatorname{div}_{\mathbf{g}}(Dv)\; u\, d\nu.$$

Using the divergence theorem on the Riemannian manifold $(M, \mathbf{g})$, we obtain

$$\int_M \operatorname{div}_{\mathbf{g}}(Dv)\; u\, d\nu = -\int_M \langle Dv, \nabla u\rangle\, d\mu = -\int_M \langle v, \nabla u\rangle\, D d\mu = -\int_M \langle v, \nabla u\rangle\, d\nu,$$

which finishes the proof of (15).

(*b*) Since $\Delta_{\mathbf{g},\mu} = \operatorname{div}_{\mathbf{g},\mu} \circ \nabla$, we obtain from (15)

$$\int_M u\,\Delta_{\mathbf{g},\mu} v\, d\mu = \int_M \operatorname{div}_{\mathbf{g},\mu} \nabla v\, u\, d\mu = -\int_M \langle \nabla v, \nabla u\rangle\, d\mu,$$

whence (16) follows.

31. (*Change of metric and measure*) Let $(M, \mathbf{g}, \mu)$ be a weighted manifold.

   (a) Let $a(x)$, $b(x)$ be smooth positive functions on M. Define new metric $\widetilde{\mathbf{g}}$ and measure $\widetilde{\mu}$ by

   $$\widetilde{\mathbf{g}} = a\,\mathbf{g} \quad \text{and} \quad d\widetilde{\mu} = b\, d\mu,$$

   where the first identity means that $\langle \xi, \eta\rangle_{\widetilde{\mathbf{g}}} = a(x)\langle \xi, \eta\rangle_{\mathbf{g}}$ for all $\xi, \eta \in T_x M$. Prove that the Laplace operator $\Delta_{\widetilde{\mathbf{g}},\widetilde{\mu}}$ of the weighted manifold $(M, \widetilde{\mathbf{g}}, \widetilde{\mu})$ is given by the formula

   $$\Delta_{\widetilde{\mathbf{g}},\widetilde{\mu}} u = \frac{1}{b}\operatorname{div}_{\mathbf{g},\mu}\left(\frac{b}{a}\nabla_{\mathbf{g}} u\right) \quad \text{for any } u \in C^\infty(M).$$

   Hint. Use the Green formula (16).

   (b) Consider the following operator L

   $$Lu = \Delta_{\mathbf{g},\mu} u + \langle \nabla v, \nabla u\rangle_{\mathbf{g}},$$

   acting on functions $u \in C^\infty(M)$, where $v \in C^\infty(M)$ is a given fixed function. Prove that $L = \Delta_{\mathbf{g},\widetilde{\mu}}$ for some measure $\widetilde{\mu}$, and determine this measure.

Solution. (*a*) By definition, we have

$$\nabla_{\mathbf{g}} f = \mathbf{g}^{-1} df,$$

which implies that

$$\nabla_{\widetilde{\mathbf{g}}} f = \frac{1}{a}\nabla_{\mathbf{g}} f.$$

Using the Green formula (16) and the identity

$$\langle \nabla_{\mathbf{g}} f, \xi\rangle_{\mathbf{g}} = \langle df, \xi\rangle \tag{19}$$

for all tangent vectors ξ, we obtain, for all $u, v \in C_0^\infty(M)$,

$$\begin{aligned}
\int_M v\Delta_{\widetilde{\mathbf{g}},\widetilde{\mu}} u\, d\widetilde{\mu} &= -\int_M \langle \nabla_{\widetilde{\mathbf{g}}} v, \nabla_{\widetilde{\mathbf{g}}} u\rangle_{\widetilde{\mathbf{g}}} d\widetilde{\mu} = -\int_M \langle dv, \nabla_{\widetilde{\mathbf{g}}} u\rangle\, b d\mu \\
&= -\int_M \langle dv, \frac{b}{a}\nabla_{\mathbf{g}} u\rangle\, d\mu \\
&= -\int_M \langle \nabla_{\mathbf{g}} v, \frac{b}{a}\nabla_{\mathbf{g}} u\rangle_{\mathbf{g}}\, d\mu = \int_M v\operatorname{div}_\mu\left(\frac{b}{a}\nabla u\right)\, d\mu \\
&= \int_M v\left(\frac{1}{b}\operatorname{div}_\mu\left(\frac{b}{a}\nabla u\right)\right) d\widetilde{\mu},
\end{aligned}$$

whence the claim follows.

(b) If $d\widetilde{\mu} = bd\mu$ then, by (a) and the product rule for weighted divergence of Exercise 5, we have

$$\begin{aligned}\Delta_{\mathbf{g},\widetilde{\mu}} u &= \frac{1}{b}\operatorname{div}_{\mathbf{g},\mu}(b\nabla u) = \operatorname{div}_{\mathbf{g},\mu}(\nabla u) + \frac{1}{b}\langle \nabla b, \nabla u\rangle_{\mathbf{g}} \\ &= \Delta_{\mathbf{g},\mu} u + \langle \nabla \log b, \nabla u\rangle_{\mathbf{g}}.\end{aligned}$$

Hence, $L = \Delta_{\mathbf{g},\widetilde{\mu}}$ provided $\log b = v$ that is, $b = e^v$.

32. * Consider in $\mathbb{R}^n$ the following differential operator

$$L = \frac{1}{b(x)}\frac{\partial}{\partial x^i}\left(a^{ij}(x)\frac{\partial}{\partial x^j}\right),$$

where $(a^{ij}(x))$ is a symmetric positive definite matrix smoothly depending on $x \in \mathbb{R}^n$, and $b(x)$ is a smooth positive function. Find in $\mathbb{R}^n$ a Riemannian metric $\mathbf{g}$ and a measure μ such that the weighted Laplace operator $\Delta_{\mathbf{g},\mu}$ coincides with L.

Solution. We have

$$\Delta_{\mathbf{g},\mu} = \frac{1}{\rho}\frac{\partial}{\partial x^i}\left(\rho g^{ij}\frac{\partial}{\partial x^j}\right)$$

where $\rho = D\sqrt{\det g}$ and D is the density of μ with respect to the Riemannian measure ν, that is, $d\mu = Dd\nu$. Since $d\nu = \sqrt{\det g}d\lambda$ where λ is the Lebesgue measure, we see that

$$d\mu = \rho d\lambda.$$

Clearly, the identity $L = \Delta_{\mathbf{g},\mu}$ holds if $\rho = b$ and $a^{ij} = \rho g^{ij}$, that is,

$$g^{ij} = b^{-1}a^{ij}.$$

In other words, the Riemannian metric is given by

$$(g_{ij}) = b\left(a^{ij}\right)^{-1},$$

and the measure μ is given by

$$d\mu = bd\lambda.$$

33. * Fix n reals $a_1, ..., a_n$ and consider the matrix

$$B = \begin{pmatrix} 1+a_1^2 & a_1a_2 & a_1a_3 & ... & a_1a_n \\ a_2a_1 & 1+a_2^2 & a_2a_3 & ... & a_2a_n \\ a_3a_1 & a_3a_2 & 1+a_3^2 & ... & a_3a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & ... & 1+a_n^2 \end{pmatrix}$$

that is, $B = (b_{ij})$ where $b_{ii} = 1 + a_i^2$ and $b_{ij} = a_ia_j$ for $i \neq j$. The purpose of this question is to prove the identity

$$\det B = 1 + a_1^2 + ... + a_n^2. \tag{20}$$

(a) Consider an auxiliary $(n+1)\times(n+1)$ matrix

$$A = \begin{pmatrix} 1 & -a_1 & -a_2 & \dots & \dots & -a_n \\ a_1 & 1 & & & & \\ a_2 & & 1 & & \mathbf{0} & \\ \vdots & & & \ddots & & \\ \vdots & & \mathbf{0} & & \ddots & \\ a_n & & & & & 1 \end{pmatrix},$$

where all the entries of the matrix outside the first column, the first row and the main diagonal are zeros. Prove that $\det A = 1 + a_1^2 + \dots + a_n^2$.

(b) Prove the identity (17).

Hint. Prove first that the matrix AA^T has the block diagonal form

$$AA^T = \begin{pmatrix} c & \mathbf{0} \\ \mathbf{0} & \boxed{B} \end{pmatrix},$$

where B is the above matrix and $c = 1 + a_1^2 + \dots + a_n^2$.

Remark. The identity (17) will be used in one of the problems in the next problem sheet in order to compute Riemannian measure on certain submanifolds.

Solution. (a) Let us expand the determinant in the first row. We obtain

$$\begin{aligned} \det A &= 1\cdot\det\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} + a_1\cdot\det\begin{pmatrix} a_1 & 0 & \dots & 0 \\ a_2 & 1 & & \\ \vdots & & \ddots & \\ a_n & & & 1 \end{pmatrix} \\ &\quad -a_2\cdot\det\begin{pmatrix} a_1 & 1 & & \\ a_2 & 0 & \dots & 0 \\ \vdots & & \ddots & \\ a_n & & & 1 \end{pmatrix} + \dots + (-1)^{n+1} a_n\cdot\begin{pmatrix} a_1 & 1 & & \\ a_2 & & 1 & \\ \vdots & & & \ddots \\ a_n & 0 & \dots & 0 \end{pmatrix} \\ &= 1 + a_1\cdot a_1\cdot\det\mathrm{id} - a_2\cdot(-a_2)\cdot\det\mathrm{id} + \dots + (-1)^{n+1} a_n\cdot(-1)^{n-1} a_n \det\mathrm{id} \\ &= 1 + a_1^2 + a_2^2 + \dots + a_n^2. \end{aligned}$$

(b) Denote by α_i the i-th row of the matrix A, where for convenience $i = 0, 1, \dots, n$. Then the elements of the product AA^T are the scalar products (α_i, α_j). Since α_0 is orthogonal to all other α_j and

$$(\alpha_0, \alpha_0) = 1 + a_1^2 + \dots + a_n^2 =: c,$$

we see that the zero row of AA^T has the form

$$c, 0, \dots., 0.$$

Since AA^T is a symmetric matrix, then the zero column has the same form. If $i, j \geq 1$ then

$$(\alpha_i, \alpha_i) = a_i^2 + 1 \quad \text{and} \quad (\alpha_i, \alpha_j) = a_i a_j.$$

Hence, we obtain that

$$AA^T = \begin{pmatrix} c & \mathbf{0} \\ \mathbf{0} & \boxed{B} \end{pmatrix}$$

with the above value of c. It follows that

$$(\det A)^2 = c \det B,$$

whence

$$\det B = \frac{1}{c} (\det A)^2 = \frac{c^2}{c} = c,$$

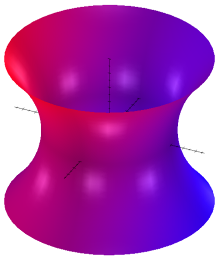
which was to be proved.

Blatt 7. Abgabe bis 05.12.2025

34. (Continuation of Exercise 5). A catenoid Cat is a surface in $\mathbb{R}^3$ that is given by the parametric equations

$$x^1 = \cosh\rho\cos\theta, \quad x^2 = \cosh\rho\sin\theta, \quad x^3 = \rho,$$

where $\rho \in (-\infty, +\infty)$ and $\theta \in (-\pi, \pi)$.


Catenoid

By Exercise 5, the Riemannian metric of Cat is given by

$$\mathbf{g}_{Cat} = \cosh^2\rho\left(d\rho^2 + d\theta^2\right).$$

Evaluate the integral

$$\int_{Cat} \frac{1}{\cosh^4\rho} d\nu,$$

where ν is the induced Riemannian measure on Cat.

Solution. Since

$$g = \begin{pmatrix} \cosh^2\rho & 0 \\ 0 & \cosh^2\rho \end{pmatrix}$$

we have

$$\det g = \cosh^4\rho.$$

Hence, the Riemannian measure is given by

$$d\nu = \sqrt{\det g}\, d\rho d\theta = \cosh^2\rho\, d\rho\, d\theta.$$

Since $\rho \in (-\infty, \infty)$ and $\theta \in (-\pi, \pi)$, we obtain

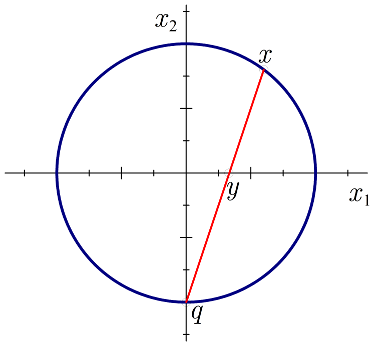
$$\begin{aligned} \int_{Cat} \frac{1}{\cosh^4\rho} d\nu &= \int_{-\infty}^{\infty}\int_{-\pi}^{\pi} \frac{1}{\cosh^4\rho}\sqrt{\det g}\, d\theta\, d\rho \\ &= \int_{-\infty}^{\infty}\int_{-\pi}^{\pi} \frac{1}{\cosh^2\rho} d\theta\, d\rho \\ &= 2\pi \int_{-\infty}^{\infty} \frac{1}{\cosh^2\rho} d\rho \\ &= 4\pi, \end{aligned}$$

where we have used that

$$\int_{-\infty}^{\infty} \frac{1}{\cosh^2\rho} d\rho = [\tanh\rho]_{-\infty}^{+\infty} = 2.$$

35. Consider the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ and set $U = \mathbb{S}^1 \setminus \{q\}$ where $q = (0, -1) \in \mathbb{S}^1$.

For any point $x \in U$ define its *stereographic projection* onto $\mathbb{R}^1$ as the point $y \in \mathbb{R}^1$ such that $(y, 0) \in \mathbb{R}^2$ lies on the straight line that goes through x and q.



(a) Prove that the stereographic projection is a homeomorphism between U and $\mathbb{R}^1$, and that it is given by

$$x_1 = \frac{2y}{1+y^2}, \quad x_2 = \frac{1-y^2}{1+y^2},$$

where $(x_1, x_2) \in U$ and $y \in \mathbb{R}^1$. Hence, U is a chart on $\mathbb{S}^1$ with the coordinate y.

(b) Prove that the canonical spherical metric $\mathbf{g}_{\mathbb{S}^1} := \mathbf{g}_{\mathbb{R}^2}|_{\mathbb{S}^1}$ has in the coordinate y the form

$$\mathbf{g}_{\mathbb{S}^1} = \frac{4}{(1+y^2)^2} dy^2.$$

(c) Evaluate $\sigma(\mathbb{S}^1)$, where σ the Riemannian measure of $(\mathbb{S}^1, \mathbf{g}_{\mathbb{S}^1})$.

Solution. (a) It follows from the definition of the stereographic projection that

$$y = \frac{x_1}{1+x_2}$$

(note that $x_2 > -1$ on U). Since also

$$x_1^2 + x_2^2 = 1,$$

it follows that

$$y^2 = \frac{x_1^2}{(1+x_2)^2} = \frac{1-x_2^2}{(1+x_2)^2} = \frac{1-x_2}{1+x_2},$$

whence

$$1 + y^2 = \frac{2}{1+x_2}$$

and

$$x_2 = \frac{2}{1+y^2} - 1 = \frac{1-y^2}{1+y^2}$$

It follows that

$$x_1 = y(1+x_2) = \frac{2y}{1+y^2}.$$

Hence, stereographic projection is a bijection between U and $\mathbb{R}^1$ that is continuous and its inverse is also continuous, which implies that it is homeomorphism.

(b) The metric $\mathbf{g}_{\mathbb{S}^1}$ in the coordinate y has the form

$$\mathbf{g}_{\mathbb{S}^1} = (g_{\mathbb{S}^1})_{11}\, dy^2$$

where

$$(g_{\mathbb{S}^1})_{11} = \sum_{k,l=1}^{2} (g_{\mathbb{R}^2})_{kl} \frac{\partial x_k}{\partial y}\frac{\partial x_l}{\partial y} = \left(\frac{\partial x_1}{\partial y}\right)^2 + \left(\frac{\partial x_2}{\partial y}\right)^2.$$

Since

$$\frac{\partial x_1}{\partial y} = \frac{d}{dy}\frac{2y}{1+y^2} = \frac{2\left(1+y^2\right)-4y^2}{\left(1+y^2\right)^2} = 2\frac{1-y^2}{\left(1+y^2\right)^2}$$

and

$$\frac{\partial x_2}{\partial y} = \frac{d}{dy}\left(\frac{2}{1+y^2}-1\right) = \frac{-4y}{\left(1+y^2\right)^2},$$

it follows that

$$\begin{aligned}(g_{\mathbb{S}^1})_{11} &= \frac{4\left(1-y^2\right)^2+16y^2}{\left(1+y^2\right)^4} = 4\frac{1-2y^2+y^4+4y^2}{\left(1+y^2\right)^4} \\ &= 4\frac{\left(1+y^2\right)^2}{\left(1+y^2\right)^4} = \frac{4}{\left(1+y^2\right)^2}.\end{aligned}$$

(c) We have $\det g_{\mathbb{S}^1} = \frac{4}{\left(1+y^2\right)^2}$, whence

$$\sigma\left(\mathbb{S}^1\right) = \sigma(U) = \int_U \sqrt{\det g_{\mathbb{S}^1}}\, dy = \int_{-\infty}^{\infty} \frac{2}{1+y^2} dy = 2\left[\arctan y\right]_{-\infty}^{\infty} = 2\pi.$$

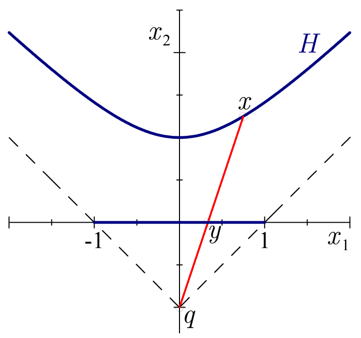
Of course, 2π is the length of the unit circle.

36. Consider in $\mathbb{R}^2$ a semi-hyperbola

$$H := \left\{(x_1,x_2)\in\mathbb{R}^2 : x_2^2 - x_1^2 = 1, \quad x_2>0\right\}$$

that is a submanifold of $\mathbb{R}^2$ of dimension 1.

For any point $x\in H$, define its *stereographic projection* onto $\mathbb{R}^1$ as the point $y\in\mathbb{R}^1$ such that $(y,0)\in\mathbb{R}^2$ lies on the straight line that goes through x and $q=(0,-1)$.



(a) Prove that the stereographic projection is a homeomorphism between H and the unit interval $I=\{y\in\mathbb{R}^1 : -1<y<1\}$, and that it is given by

$$x_1 = \frac{2y}{1-y^2}, \quad x_2 = \frac{1+y^2}{1-y^2}, \tag{21}$$

where $(x_1,x_2)\in H$ and $y\in I$. Hence, H itself is a chart with the coordinate y.

(b) Consider in $\mathbb{R}^2$ the *Minkowski metric tensor*

$$\mathbf{g}_{Mink} := dx_1^2 - dx_2^2.$$

Prove that its restriction $\mathbf{g}_H := \mathbf{g}_{Mink}|_H$ is given in the coordinate y by

$$\mathbf{g}_H = \frac{4}{(1-y^2)^2} dy^2.$$

(c) Denoting by ν the Riemannian measure of $(H, \mathbf{g}_H)$, evaluate the integral

$$\int_H \frac{1}{x_2} d\nu,$$

where x_2 is the second coordinate in $\mathbb{R}^2$ of a point $x \in H$ (as in (18)).

Solution. (a) It follows from the definition of the stereographic projection that

$$y = \frac{x_1}{1+x_2}$$

(note that $x_2 > -1$ on U). Since also

$$x_2^2 - x_1^2 = 1,$$

it follows that

$$y^2 = \frac{x_1^2}{(1+x_2)^2} = \frac{x_2^2 - 1}{(1+x_2)^2} = \frac{x_2 - 1}{1+x_2} = 1 - \frac{2}{1+x_2}$$

whence

$$x_2 = \frac{2}{1-y^2} - 1 = \frac{1+y^2}{1-y^2}$$

It follows that

$$x_1 = y(1+x_2) = \frac{2y}{1-y^2}.$$

Hence, stereographic projection is a bijection between H and I that is continuous and its inverse is also continuous, which implies that it is homeomorphism.

(b) The metric $\mathbf{g}_H$ in the coordinate y has the form

$$\mathbf{g}_H = (g_H)_{11}\, dy^2$$

where

$$(g_H)_{11} = \sum_{k,l=1}^{2} (g_{Mink})_{kl} \frac{\partial x_k}{\partial y} \frac{\partial x_l}{\partial y} = \left(\frac{\partial x_1}{\partial y}\right)^2 - \left(\frac{\partial x_2}{\partial y}\right)^2.$$

Since

$$\frac{\partial x_1}{\partial y} = \frac{d}{dy} \frac{2y}{1-y^2} = \frac{2(1-y^2) + 4y^2}{(1-y^2)^2} = 2\frac{1+y^2}{(1-y^2)^2}$$

and

$$\frac{\partial x_2}{\partial y} = \frac{d}{dy}\left(\frac{2}{1-y^2} - 1\right) = \frac{4y}{(1-y^2)^2},$$

it follows that

$$\begin{aligned}(g_H)_{11} &= \frac{4\left(1+y^2\right)^2-16y^2}{\left(1-y^2\right)^4} = 4\frac{\left(1+2y^2+y^4\right)-4y^2}{\left(1-y^2\right)^4} \\ &= 4\frac{\left(1-y^2\right)^2}{\left(1-y^2\right)^4} = \frac{4}{\left(1-y^2\right)^2}.\end{aligned}$$

(*c*) Since $\det g_H = \frac{4}{(1-y^2)^2}$, we obtain

$$\begin{aligned}\int_H \frac{1}{x_2} d\nu &= \int_H \sqrt{\det g_H} \frac{1}{x_2} dy = \int_{-1}^{1} \frac{2}{1-y^2} \frac{1-y^2}{1+y^2} dy \\ &= 2\int_{-1}^{1} \frac{dy}{1+y^2} = 2\left[\arctan y\right]_{-1}^{1} = \pi.\end{aligned}$$

37. Let Γ be the graph in $\mathbb{R}^{n+1}$ of a smooth function $f: U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^n$. Let $\mathbf{g}_\Gamma$ be the Riemannian metric on Γ that is induced by the canonical Euclidean metric in $\mathbb{R}^{n+1}$. Let $y^1, ..., y^n$ be the Cartesian coordinates in U that can be regarded as local coordinates on Γ. Denote by ν_Γ the Riemannian measure of $(\Gamma, \mathbf{g}_\Gamma)$.

(a) Prove that in the coordinates $y^1, ..., y^n$

$$d\nu_\Gamma = \sqrt{1+\left(\frac{\partial f}{\partial y^1}\right)^2 + ... + \left(\frac{\partial f}{\partial y^n}\right)^2}\, dy. \tag{22}$$

Hint. Use the result of Exercise 5 that

$$(g_\Gamma)_{ij} = \delta_{ij} + \frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}, \tag{23}$$

and then the formula (17) of Exercise 5.

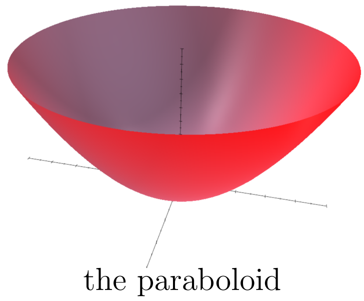
(b) Using (19), evaluate the area (=the Riemannian measure) of the paraboloid that is the graph in $\mathbb{R}^3$ of the function

$$f(x,y) = \tfrac{1}{2}\left(x^2+y^2\right)$$

in a disc

$$U = \left\{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\right\}.$$

Hint. Compute $\nu_\Gamma(\Gamma)$ using integration in the polar coordinates in $\mathbb{R}^2$.



the paraboloid

Solution. (*a*) By the definition of the Riemannian measure, we have

$$d\nu_\Gamma = \sqrt{\det g_\Gamma} dy$$

where g_Γ is the matrix of $\mathbf{g}_\Gamma$. Since g_Γ is given by (**??**), we obtain by Exercise 5 that

$$\det g_\Gamma = 1 + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial y^i}\right)^2,$$

whence (19) follows.

(*b*) In the case

$$f(x,y) = \frac{1}{2}\left(x^2+y^2\right)$$

we have

$$d\nu_\Gamma = \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}dxdy = \sqrt{1+x^2+y^2}dxdy.$$

Hence, the area of the paraboloid Γ is

$$A = \nu_\Gamma(\Gamma) = \int_U \sqrt{1+x^2+y^2}dxdy.$$

This integral can be computed in the polar coordinates (r,θ) as follows: as

$$U = \{r<1, \theta\in[0,2\pi)\} \quad \text{and} \quad dxdy = rdrd\theta$$

we obtain

$$\begin{aligned}
\nu_\Gamma(\Gamma) &= \int_0^1\left(\int_0^{2\pi}\sqrt{1+r^2}d\theta\right)rdr \\
&= 2\pi\int_0^1\sqrt{1+r^2}rdr \\
&= \pi\int_0^1\sqrt{1+r^2}d\left(r^2+1\right) \\
&= \pi\frac{2}{3}\left[\left(1+r^2\right)^{3/2}\right]_0^1 \\
&= \frac{2\pi}{3}\left(2\sqrt{2}-1\right) \approx 3.83.
\end{aligned}$$

38. * Let q be the south pole of the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, that is,

$$q = (\underbrace{0,...,0}_{n \text{ zeros}}, -1). \tag{24}$$

For any point $x \in U := \mathbb{S}^n \setminus \{q\}$, its stereographic projection is the point $y \in \mathbb{R}^n$ such that the point $(y,0)\in\mathbb{R}^{n+1}$ belongs to the straight line that goes through x and q.

(a) Prove the following relations between $x\in U$ and $y\in\mathbb{R}^n$:

$$x_i = \left(1+x_{n+1}\right)y_i, \quad i=1,...,n \tag{25}$$

and

$$|y|^2 = \frac{2}{1+x_{n+1}} - 1. \tag{26}$$

Show that the stereographic projection is a homeomorphism between U and $\mathbb{R}^n$. Hence, U is a chart on $\mathbb{S}^n$ with coordinates $y_1,...,y_n$.

(b) Prove that the canonical spherical metric $\mathbf{g}_{\mathbb{S}^n} := \mathbf{g}_{\mathbb{R}^{n+1}}|_{\mathbb{S}^n}$ has in the coordinates $y_1, ..., y_n$ the form

$$\mathbf{g}_{\mathbb{S}^n} = \frac{4}{\left(1+|y|^2\right)^2}\left(dy_1^2 + ... + dy_n^2\right).$$

Hint. Express the Euclidean metric $\mathbf{g}_{\mathbb{R}^{n+1}} = dx_1^2 + ...dx_n^2 + dx_{n+1}^2$ via dy_i using the relations (21) and (22).

Solution. (a) Let us simplify the notation by renaming x_{n+1} to t. Then the equation of the sphere $\mathbb{S}^n$ is

$$x_1^2 + ... + x_n^2 + t^2 = 1. \tag{27}$$

The point $y = (y_1, ..., y_n)$ is obtained from $(x_1, ..., x_n)$ by scaling by the factor $1+t$, which arises from comparison of the segments $[-1, t]$ and $[-1, 0]$ of the axis x_{n+1}. Hence, we obtain

$$x_i = (1+t)\, y_i, \quad i = 1, ..., n. \tag{28}$$

Substituting into (23), we obtain

$$(1+t)^2 |y|^2 + t^2 = 1$$

whence

$$|y|^2 = \frac{1-t^2}{(1+t)^2} = \frac{1-t}{1+t} = \frac{2}{1+t} - 1,$$

which proves (22). Consequently, we obtain

$$x_{n+1} = t = \frac{2}{1+|y|^2} - 1 = \frac{1-|y|^2}{1+|y|^2}.$$

From (24) we obtain

$$x_i = \frac{2y_i}{1+|y|^2}, \quad i = 1, ..., n.$$

Also, from (24) we have

$$y_i = \frac{x_i}{1+t} = \frac{x_i}{1+x_{n+1}}.$$

Hence, we see that the relation between $x \in U$ and $y \in \mathbb{R}^n$ is bijective and continuous in the both direction so that U and $\mathbb{R}^n$ are homeomorphic.

(b) The metric $\mathbf{g}_{\mathbb{S}^n}$ is obtained by restricting to $\mathbb{S}^n$ of the Euclidean metric

$$\mathbf{g}_{\mathbb{R}^{n+1}} = dx_1^2 + ... + dx_n^2 + dt^2.$$

Considering x_i, y_i and t as functions on U, we obtain from (24), for any $i = 1, ..., n$, that

$$dx_i = (1+t)\, dy_i + y_i dt$$

whence

$$dx_i^2 = (1+t)^2 dy_i^2 + (1+t)\, y_i\, (dy_i dt + dt dy_i) + y_i^2 dt^2.$$

Therefore,

$$\mathbf{g}_{\mathbb{R}^{n+1}}|_{\mathbb{S}^n} = dt^2 + \sum_{i=1}^n dx_i^2$$

$$= dt^2 + \sum_{i=1}^{n} (1+t)^2 dy_i^2 + \sum_{i=1}^{n} (1+t) y_i (dy_i dt + dt dy_i) + \sum_{i=1}^{n} y_i^2 dt^2$$

$$= dt^2 + (1+t)^2 \sum_{i=1}^{n} dy_i^2$$

$$+ (1+t) \left(\sum_{i=1}^{n} y_i dy_i \right) dt + (1+t) dt \sum_{i=1}^{n} y_i dy_i$$

$$+ |y|^2 dt^2.$$

Since by (22)

$$\sum_{i=1}^{n} y_i^2 = |y|^2 = \frac{2}{1+t} - 1,$$

it follows by differentiation of this identity that

$$\sum_{i=1}^{n} y_i dy_i = d\frac{1}{1+t} = -\frac{dt}{(1+t)^2}.$$

It follows that

$$\begin{aligned} \mathbf{g}_{\mathbb{R}^{n+1}}|_{\mathbb{S}^n} &= \left(1 + |y|^2\right) dt^2 + (1+t)^2 \sum_{i=1}^{n} dy_i^2 - 2(1+t) \frac{dt^2}{(1+t)^2} \\ &= \frac{2}{1+t} dt^2 + (1+t)^2 \sum_{i=1}^{n} dy_i^2 - \frac{2}{1+t} dt^2 \\ &= (1+t)^2 \sum_{i=1}^{n} dy_i^2 \\ &= \frac{4}{\left(1 + |y|^2\right)^2} \sum_{i=1}^{n} dy_i^2. \end{aligned}$$

39. * Define the n-dimensional hyperboloid $\mathbb{H}^n$ as the following submanifold of $\mathbb{R}^{n+1}$:

$$\mathbb{H}^n = \left\{ x \in \mathbb{R}^{n+1} : x_{n+1}^2 - x_1^2 - ... - x_n^2 = 1, \ x_{n+1} > 0 \right\}.$$

For any point $x \in \mathbb{H}^n$, its stereographic projection is the point $y \in \mathbb{R}^n$ such that the point $(y, 0) \in \mathbb{R}^{n+1}$ belongs to the straight line that goes through x and q (where q is given by (20)).

(a) Prove that the stereographic projection is a homeomorphism of $\mathbb{H}^n$ onto the unit ball $\mathbb{B}^n = \{y \in \mathbb{R}^n : |y| < 1\}$. Prove also the following relations between $x \in \mathbb{H}^n$ and $y \in \mathbb{B}^n$:

$$x_i = (1 + x_{n+1}) y_i, \ i = 1, ..., n \tag{29}$$

and

$$|y|^2 = 1 - \frac{2}{1 + x_{n+1}}. \tag{30}$$

(b) Define the Minkowski metric tensor $\mathbf{g}_{Mink}$ in $\mathbb{R}^{n+1}$ by

$$\mathbf{g}_{Mink} = dx_1^2 + ... + dx_n^2 - dx_{n+1}^2.$$

The induced metric $\mathbf{g}_{\mathbb{H}^n} = \mathbf{g}_{Mink}|_{\mathbb{H}^n}$ is called the *hyperbolic metric* on $\mathbb{H}^n$. Prove that the hyperbolic metric has in the coordinates $y_1, ..., y_n$ the form

$$\mathbf{g}_{\mathbb{H}^n} = \frac{4}{\left(1 - |y|^2\right)^2} \left(dy_1^2 + ... + dy_n^2\right). \tag{31}$$

Remark. Observe that the metric $\mathbf{g}_{\mathbb{H}^n}$ is positive definite and, hence, is Riemannian, although the Minkowski metric in $\mathbb{R}^{n+1}$ is not positive definite (it is called *pseudo-Riemannian*). The Riemannian manifold $(\mathbb{H}^n, \mathbf{g}_{\mathbb{H}^n})$ is called the *hyperbolic space*. The ball $\mathbb{B}^n$ with the metric (27) is called the *Poincaré model* of the hyperbolic space.

Solution. (a) Let us simplify the notation by renaming x_{n+1} to t. Then the equation of the hyperboloid $\mathbb{H}^n$ is

$$t^2 - x_1^2 - ... - x_n^2 = 1. \tag{32}$$

The point $y = (y_1, ..., y_n)$ is obtained from by scaling $(x_1, ..., x_n)$ by the factor $1 + t$, which arises from comparison of the segments $[-1, t]$ and $[-1, 0]$ of the axis x_{n+1}. Hence, we have

$$x_i = (1 + t)\, y_i, \quad i = 1, ..., n, \tag{33}$$

which is equivalent to (25). Substituting into (28), we obtain

$$t^2 - (1 + t)^2 |y|^2 = 1$$

whence

$$|y|^2 = \frac{t^2 - 1}{(1 + t)^2} = \frac{t - 1}{t + 1} = 1 - \frac{2}{1 + t},$$

which proves (26). In particular, we see that $|y| < 1$ so that $y \in \mathbb{B}^n$.

Consequently, for any $y \in \mathbb{B}^n$ we obtain from the above equation

$$x_{n+1} = t = \frac{2}{1 - |y|^2} - 1 = \frac{1 + |y|^2}{1 - |y|^2}.$$

From (29) we obtain

$$x_i = \frac{2y_i}{1 - |y|^2}, \quad i = 1, ..., n.$$

Hence, we see that the relation between $x \in H$ and $y \in \mathbb{B}^n$ is bijective and continuous in the both direction so that H and $\mathbb{B}^n$ are homeomorphic.

(b) The metric $\mathbf{g}_H$ is obtained by restricting to H of the Minkowski metric

$$\mathbf{g}_{Mink} = -dt^2 + dx_1^2 + ... + dx_n^2.$$

Considering x_i, y_i and t as functions on H, we obtain from (29), for any $i = 1, ..., n$, that

$$dx_i = (1 + t)\, dy_i + y_i dt$$

whence

$$dx_i^2 = (1+t)^2 \, dy_i^2 + (1+t) \, y_i \, (dy_i dt + dt dy_i) + y_i^2 dt^2.$$

Therefore,

$$\begin{aligned}
\mathbf{g}_{Mink}|_H &= -dt^2 + \sum_{i=1}^n dx_i^2 \\
&= -dt^2 + \sum_{i=1}^n (1+t)^2 \, dy_i^2 + \sum_{i=1}^n (1+t) \, y_i \, (dy_i dt + dt dy_i) + \sum_{i=1}^n y_i^2 dt^2 \\
&= -dt^2 + (1+t)^2 \sum_{i=1}^n dy_i^2 \\
&+ (1+t) \left(\sum_{i=1}^n y_i dy_i \right) dt + (1+t) \, dt \sum_{i=1}^n y_i dy_i \\
&+ |y|^2 \, dt^2.
\end{aligned}$$

Since by (26)

$$\sum_{i=1}^n y_i^2 = |y|^2 = 1 - \frac{2}{1+t},$$

it follows by differentiation of this identity that

$$\sum_{i=1}^n y_i dy_i = -d \frac{1}{1+t} = \frac{dt}{(1+t)^2}.$$

It follows that

$$\begin{aligned}
\mathbf{g}_{Mink}|_H &= -\left(1 - |y|^2\right) dt^2 + (1+t)^2 \sum_{i=1}^n dy_i^2 + 2 \, (1+t) \, \frac{dt^2}{(1+t)^2} \\
&= -\frac{2}{1+t} dt^2 + (1+t)^2 \sum_{i=1}^n dy_i^2 + \frac{2}{1+t} dt^2 \\
&= (1+t)^2 \sum_{i=1}^n dy_i^2 \\
&= \frac{4}{\left(1 - |y|^2\right)^2} \sum_{i=1}^n dy_i^2.
\end{aligned}$$

Blatt 8. Abgabe bis 12.12.2025

Die mit *markierten Aufgaben sind zusätzlich und werden korrigiert
Die mit **markierten Aufgaben sind zusätzlich und werden nicht korrigiert.

40. Prove that if a Riemannian manifold $(M, \mathbf{g})$ is connected then $d(x, y) < \infty$ for all $x, y \in M$, where d is the geodesic distance function.

    Hint: Show that, for any $x \in M$, the set $N := \{y \in M : d(x, y) < \infty\}$ is open and closed.

    Solution. Fix a point $x \in M$ and consider the set

    $$N = \{y \in M : d(x, y) < \infty\}.$$

    We need to show that $N = M$. It suffices to prove that the set N is open and closed. Then, by the connectedness of M we will conclude that either $N = \emptyset$ or $N = M$. Since N contains x, then we obtain $N = M$, which finishes the proof.

    Observe that, by definition of N,

    $$N = \bigcup_{k=1}^{\infty} B(x, k),$$

    where

    $$B(x, r) = \{y \in M : d(x, y) < r\}$$

    is geodesic ball of radius r. Since the topology of the smooth manifold M coincides with the topology of the metric space (M, d), all geodesic balls are open sets, which implies that N is also open.

    Let us show that N is closed. For that, we need to verify that the complement

    $$N^c = \{y \in M : d(x, y) = \infty\}$$

    is open. This will follows if we show that for, any $y \in N^c$ and any $\varepsilon > 0$, the ball $B(y, \varepsilon)$ is a subset of N^c. Indeed, for any $z \in B(y, \varepsilon)$ we have by the triangle inequality

    $$d(x, y) \leq d(x, z) + d(z, x)$$

    that is,

    $$\infty \leq \varepsilon + d(x, z),$$

    whence $d(x, z) = \infty$ and $z \in N^c$. Therefore, $B(y, \varepsilon) \subset N^c$.

41. Let $(M, \mathbf{g})$ be a Riemannian model, and let x', x'' be two points in $M \setminus \{o\}$ with the polar coordinates (r', θ') and (r'', θ''), respectively.

    (a) Prove that, for any piecewise C^1 path γ on M connecting the points x' and x'',

    $$\ell_{\mathbf{g}}(\gamma) \geq |r' - r''|.$$

    Deduce that $d(x', x'') \geq |r' - r''|$, where d is the geodesic distance on $(M, \mathbf{g})$.

    Hint. Use the metric $\mathbf{g}$ in the polar coordinates on M.

(b) Prove that if $\theta' = \theta''$ then $d(x', x'') = |r' - r''|$.

(c) Prove that, for any point $x = (r, \theta)$, we have $d(o, x) = r$.

(d) Conclude that in $(\mathbb{R}^n, \mathbf{g}_{\mathbb{R}^n})$ the geodesic distance $d(x, y)$ is equal to $|x - y|$ for all $x, y \in \mathbb{R}^n$.

Solution. (a) Denoting $\theta^0 = r$ and using that the metric $\mathbf{g}$ has the form

$$\mathbf{g} = \sum_{i,j=0}^{n-1} g_{ij}\theta^i\theta^j = \left(d\theta^0\right)^2 + \sum_{i,j=1}^{n-1} g_{ij}\theta^i\theta^j,$$

we obtain, for any piecewise C^1 path $\gamma : [a, b] \to M$,

$$|\dot{\gamma}|^2_{\mathbf{g}} = \sum_{i,j=0}^{n-1} g_{ij}\dot{\gamma}^i\dot{\gamma}^j = \left|\dot{\gamma}^0\right|^2 + \sum_{i,j=1}^{n-1} g_{ij}\dot{\gamma}^i\dot{\gamma}^j \geq \left|\dot{\gamma}^0\right|^2,$$

whence it follows that

$$\ell_{\mathbf{g}}(\gamma) = \int_a^b |\dot{\gamma}|_{\mathbf{g}}\, dt \geq \int_a^b \left|\dot{\gamma}^0\right| dt \geq \left|\int_a^b \dot{\gamma}^0 dt\right| = \left|\gamma^0(b) - \gamma^0(a)\right|.$$

If γ connects x' and x'' then $\gamma^0(a) = r'$ and $\gamma^0(b) = r''$, which implies

$$\ell_{\mathbf{g}}(\gamma) \geq |r' - r''|.$$

Minimizing in all γ connecting x' and x'', we obtain $d(x', x'') \geq |r' - r''|$.

(b) If $\theta' = \theta'' =: \theta$ then the path

$$\gamma(t) = (r'(1-t) + r''t, \theta), \quad t \in [0, 1],$$

connects x' and x'', because $\gamma(0) = x'$ and $\gamma(1) = x''$. Since $\dot{\gamma}(t) = (r'' - r', 0)$ and $|\dot{\gamma}|_{\mathbf{g}} = |r'' - r'|$, we obtain

$$\ell_{\mathbf{g}}(\gamma) = \int_a^b |\dot{\gamma}|_{\mathbf{g}}\, dt = r'' - r'.$$

(c) Let us show that if γ is a piecewise C^1 path connecting the points o and $x = (r, \theta)$, then

$$\ell_g(\gamma) \geq r.$$

Fix any $r' \in (0, r)$. Then γ interests the sphere $S_{r'} = \{y \in \mathbb{R}^n : |y| = r'\}$, say, at a point x'. By (b), the length of a part of γ between x' and x is $\geq r - r'$, which implies

$$\ell_{\mathbf{g}}(\gamma) \geq r - r'.$$

Since r' is arbitrary, it follows that $\ell_{\mathbf{g}}(\gamma) \geq r$ and, hence, $d(o, x) \geq r$.

On the other hand, the path $\gamma(t) = (tr, \theta)$ defined for $t \in [0, 1]$, connects o and x, and it is easy to see that $\ell_{\mathbf{g}}(\gamma) = r$. Hence, $d(0, x) = r$, which was to be proved.

(d) In $\mathbb{R}^n$, the above argument proves that $d(0, x) = |x|$. Since the origin o of the polar coordinates in $\mathbb{R}^n$ may be at any point, setting it to y we obtain that $d(x, y) = |x - y|$.

42. Let $\gamma(t) : (a, b) \to M$ be a parametric C^1 curve on a Riemannian manifold $(M, \mathbf{g})$.

(a) Consider a *time change* $\tau : (\alpha, \beta) \to (a, b)$ where the function τ is bijective and C^1 smooth. Then τ determines a new parametric curve

$$\widetilde{\gamma} : (\alpha, \beta) \to M$$
$$\widetilde{\gamma}(s) = \gamma(\tau(s)).$$

Prove that $\ell_{\mathbf{g}}(\widetilde{\gamma}) = \ell_{\mathbf{g}}(\gamma)$.

Remark. This identity means that the length of the parametric curve does not depend on a specific parametrization.

(b) Assume in addition that γ is C^∞ smooth, injective, $\dot{\gamma}(t) \neq 0$ for all $t \in (a, b)$ and that γ is a homeomorphism of (a, b) onto the image $S = \gamma(a.b)$. Then, by Exercise 5, S is a submanifold of dimension 1. Let ν_S be the induced metric on S. Prove that

$$\ell_{\mathbf{g}}(\gamma) = \nu_S(S).$$

Hint. Write down the induced metric $\mathbf{g}_S$ using the local coordinate t on S.

Solution. (a) We have

$$\frac{d}{ds}\widetilde{\gamma}(s) = \dot{\gamma}(\tau(s))\frac{d\tau}{ds}$$

and

$$\ell_{\mathbf{g}}(\widetilde{\gamma}) = \int_\alpha^\beta |\dot{\gamma}(\tau(s))| \left|\frac{d\tau}{ds}\right| ds.$$

Since τ is bijective and C^1, it must be either monotone increasing or monotone decreasing, that is, either $\tau' \geq 0$ on (α, β) or $\tau' \leq 0$ on (α, β). Indeed, assume from the contrary that $\tau'(s_1) < 0$ and $\tau'(s_2) > 0$. Suppose $s_1 < s_2$. Let $s_0 \in [s_1, s_2]$ be the point of minimum of τ on $[s_1, s_2]$. Then the function τ on $[s_1, s_0]$ takes all the values from $\tau(s_0)$ to $\tau(s_1)$, and on the interval $[s_0, s_2]$ it takes all values from $\tau(s_0)$ to $\tau(s_2)$. Hence, some value $\tau(s_0) + \varepsilon$ is taken twice, which contradicts the hypothesis that τ is bijective.

Suppose that τ is monotone increasing. Then necessarily $\tau(\alpha) = a$ and $\tau(\beta) = b$, and we obtain by change $t = \tau(s)$ that

$$\ell_{\mathbf{g}}(\widetilde{\gamma}) = \int_\alpha^\beta |\dot{\gamma}(\tau(s))| \frac{d\tau}{ds} ds = \int_a^b |\dot{\gamma}(t)| dt = \ell_{\mathbf{g}}(\gamma),$$

which was to be proved.

(b) By a formula from lectures, if $x^1, ..., x^n$ are local coordinates on M and $y^1, ..., y^m$ are local coordinates on submanifold S then the induced metric $\mathbf{g}_S$ is given by

$$(g_S)_{ij} = g_{kl}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}$$

where $i, j = 1, ..., m$ and $x^i(y^1, ..., y^m)$ is the coordinate in M of the point $y^1, ..., y^m$ on S.

In our case $m = 1$, $y^1 = t$ and $x^i(t) = \gamma^i(t)$. Hence,

$$(g_S)_{11} = g_{kl}\dot{\gamma}^k\dot{\gamma}^l = |\gamma|^2_{\mathbf{g}}.$$

It follows that

$$d\nu_S = \sqrt{\det g_S}dt = |\gamma|_{\mathbf{g}}\, dt$$

and

$$\nu_S(S) = \int_a^b |\gamma|_{\mathbf{g}}\, dt = \ell_{\mathbf{g}}(\gamma),$$

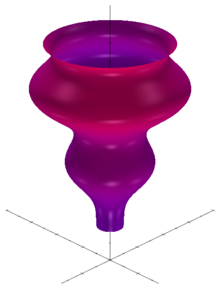
which was to be proved.

43. Let I be an open interval in $\mathbb{R}$ and S be a *surface of revolution* in $\mathbb{R}^{n+1}$ around I that is given by the equation

$$|x'| = \varphi(x^{n+1}), \quad x^{n+1} \in I,$$

where $x' = (x^1, ..., x^n)$ and $\varphi(t)$ is a smooth positive function on I.

Here is an example of a surface of revolution:



(a) Prove that S is a submanifold of $\mathbb{R}^{n+1}$ of dimension n.

(b) Let us introduce on S the *prepolar coordinates* (t, θ) as follows: for any point $(x', x^{n+1}) \in S$, set

$$t = x^{n+1} \in I \quad \text{and} \quad \theta = \frac{x'}{|x'|} \in \mathbb{S}^{n-1}.$$

Prove that in the coordinates (t, θ) the induced metric $\mathbf{g}_S := \mathbf{g}_{\mathbb{R}^{n+1}}|_S$ has the form

$$\mathbf{g}_S = \left(1 + \varphi'(t)^2\right) dt^2 + \varphi^2(t)\, \mathbf{g}_{\mathbb{S}^{n-1}}.$$

Hint. Express all x^i in terms of t and the Cartesian coordinates $f^i(\theta)$ of θ.

(c) Define the *polar coordinates* (r, θ) on S as follows: θ is as above, while $r = r(t)$ is defined by

$$r = \int_{t_0}^{t} \sqrt{1 + \varphi'(\xi)^2} d\xi, \tag{34}$$

where t_0 is any fixed point from I. Prove that the metric $\mathbf{g}_S$ has in the coordinates (r, θ) the *model* form

$$\mathbf{g}_S = dr^2 + \psi^2(r)\, \mathbf{g}_{\mathbb{S}^{n-1}}, \tag{35}$$

where the function ψ is defined by the identity $\psi(r(t)) = \varphi(t)$.

Hint. Use (30) to express dr via dt.

Remark. The manifold $(S, \mathbf{g}_S)$ is called a *cylindrical* model, which refers the fact that S is homeomorphic to a cylinder $I \times \mathbb{S}^{n-1}$ (rather than to a ball).

(d) Represent in the model form (31) the induced metric of the cone

$$Cone = \left\{x \in \mathbb{R}^{n+1} : |x'| = \alpha x^{n+1} + \beta, \quad x^{n+1} > 0\right\},$$

where $\alpha > 0$ and $\beta \geq 0$.

Solution. (a) The set S is given as a subset of $\mathbb{R}^{n+1}$ by the equation $F(x)=0$, where

$$F(x)=|x'|^2-\varphi^2\left(x^{n+1}\right)\in C^\infty.$$

Clearly, we have

$$\frac{\partial F}{\partial x^i}=2x^i,\quad i=1,...,n.$$

Since $\varphi>0$ on I, it follows that on S we have $|x'|>0$ and, hence, at least one of the partial derivatives $\frac{\partial F}{\partial x^i}$ does not vanish. It follows that $dF\neq 0$ on S and, hence, S is a submanifold.

(b) Let the Cartesian coordinates in $\mathbb{R}^n$ of a point $\theta\in\mathbb{S}^{n-1}$ be $f^1(\theta),...,f^n(\theta)$. For any point $(t,\theta)\in S$, we have

$$x'=|x'|\,\theta=\varphi(t)\,\theta,$$

which implies that the Cartesian coordinates of (t,θ) are as follows:

$$\begin{aligned}x^i&=\varphi(t)\,f^i(\theta)\,,\ i=1,...,n\\ x^{n+1}&=t.\end{aligned}$$

Therefore, the metric $\mathbf{g}_S$ is given by

$$\begin{aligned}\mathbf{g}_S&=\mathbf{g}_{\mathbb{R}^{n+1}}|_S=\left(dx^1\right)^2+...+\left(dx^n\right)^2+\left(dx^{n+1}\right)^2\\ &=\sum_{i=1}^n d\left(\varphi(t)\,f'(\theta)\right)^2+dt^2\\ &=\sum_{i=1}^n\left(f^i\varphi'dt+\varphi df^i\right)^2+dt^2\\ &=\sum_{i=1}^n\left[\left(f^i\right)^2\left(\varphi'\right)^2dt^2+\varphi\varphi'dt\left(f^idf^i\right)+\left(f^idf^i\right)\varphi\varphi'dt+\varphi^2\left(df^i\right)^2\right]+dt^2.\end{aligned}$$

Using that

$$\sum_{i=1}^n\left(f^i(\theta)\right)^2=|\theta|^2=1$$

$$\sum_{i=1}^n f^idf^i=0$$

and

$$\sum_{i=1}^n\left(df^i\right)^2=\mathbf{g}_{\mathbb{S}^{n-1}}$$

(see lectures), we obtain that

$$\mathbf{g}_S=\left(1+\left(\varphi'\right)^2\right)dt^2+\varphi^2\mathbf{g}_{\mathbb{S}^{n-1}}.$$

(c) The change $r=\int_{t_0}^t\sqrt{1+\varphi'(\xi)^2}d\xi$ obviously implies

$$dr^2=\left(1+\left(\varphi'\right)^2\right)dt^2,$$

whence

$$\mathbf{g}_S = dr^2 + \varphi^2(t)\,\mathbf{g}_{\mathbb{S}^{n-1}} = dr^2 + \psi^2(r)\,\mathbf{g}_{\mathbb{S}^{n-1}},$$

where ψ is defined by $\psi(r) = \varphi(t)$.

(d) For the cone we have $\varphi(t) = \alpha t + \beta$ on $I = (0, \infty)$ and

$$r = \int_0^t \sqrt{1 + (\varphi'(\xi))^2}d\xi = \int_0^t \sqrt{1+\alpha^2}d\xi = \sqrt{1+\alpha^2}t$$

and, hence,

$$\psi(r) = \varphi(t) = \alpha t + \beta = \frac{\alpha}{\sqrt{1+\alpha^2}}r + \beta.$$

It follows that

$$\mathbf{g}_{Cone} = dr^2 + \left(\frac{\alpha}{\sqrt{1+\alpha^2}}r + \beta\right)^2 \mathbf{g}_{\mathbb{S}^{n-1}}.$$

44. * The purpose of this question is to compute the induced metric $\mathbf{g}_S$ on surfaces of revolution given in parametric form.

(a) Assume that a surface of revolution S in $\mathbb{R}^{n+1}$ is given by the parametric equations

$$x^{n+1} = a(s) \quad \text{and} \quad |x'| = b(s),$$

where a, b are smooth functions of s on some interval and $a'(s) > 0$. Prove that the polar radius r on S (see (30)) can be computed as a function of s by

$$r = \int_{s_0}^s \sqrt{(a'(\xi))^2 + (b'(\xi))^2}d\xi,$$

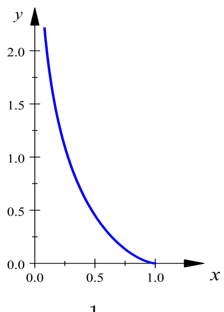
and the function ψ in (31) is determined by the equation $\psi(r(s)) = b(s)$.

(b) The *pseudo-sphere* PS in $\mathbb{R}^{n+1}$ is given by the parametric equations

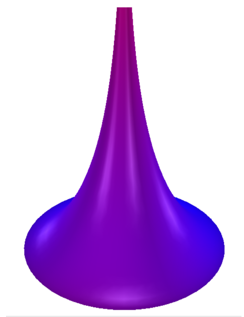
$$x^{n+1} = s - \tanh s \quad \text{and} \quad |x'| = \frac{1}{\cosh s}, \quad s > 0.$$

Prove that the induced metric on PS has in the polar coordinates the form

$$\mathbf{g}_{PS} = dr^2 + e^{-2r}\mathbf{g}_{\mathbb{S}^{n-1}}.$$



A tractrix $x = \frac{1}{\cosh s}$, $y = s - \tanh s$


A pseudosphere in $\mathbb{R}^3$

Remark. The pseudo-sphere is the surface of revolution of a *tractrix*.

Solution. (b) The surface S is represented in the form

$$|x'| = \varphi\left(x^{n+1}\right)$$

where $\varphi(t) = b\left(a^{-1}(t)\right)$ (the condition $a' > 0$ ensures that a^{-1} exists). Making change $t = a(s)$, we obtain

$$r = \int_{t_0}^{t} \sqrt{1 + \left(\frac{d\varphi}{dt}\right)^2} dt = \int_{s_0}^{s} \sqrt{1 + \left(\frac{b'(s)}{a'(s)}\right)^2} a'(s)\, ds = \int_{s_0}^{s} \sqrt{(a'(s))^2 + (b'(s))^2} ds.$$

The function $\psi(t)$ is defined by condition $\psi(r) = \varphi(t) = b\left(a^{-1}(t)\right) = b(s)$, which finishes the proof.

(c) For PS, we have

$$a(s) = s - \tanh s$$
$$b(s) = \frac{1}{\cosh s}.$$

Note that the function $a(s)$ has the derivative

$$a' = (s - \tanh s)' = 1 - \frac{1}{\cosh^2 s} = \frac{\sinh^2 s}{\cosh^2 s} = \tanh^2 s > 0.$$

Using also

$$b' = \left(\frac{1}{\cosh s}\right)' = -\frac{\sinh s}{\cosh^2 s} = -\frac{\tanh s}{\cosh s},$$

we obtain

$$r = \int_0^s \sqrt{\frac{\tanh^2 s}{\cosh^2 s} + \tanh^4 s ds} = \int_0^s \sqrt{\tan^2 s \left(\frac{1}{\cosh^2 s} + \frac{\sinh^2 s}{\cosh^2 s}\right)} ds$$
$$= \int_0^s \tanh s ds = \ln \cosh s.$$

The function ψ is determined by

$$\psi(r) = b(s) = \frac{1}{\cosh s} = e^{-r},$$

where $r \in (0, \infty)$. Hence, the metric of PS in the polar coordinates is

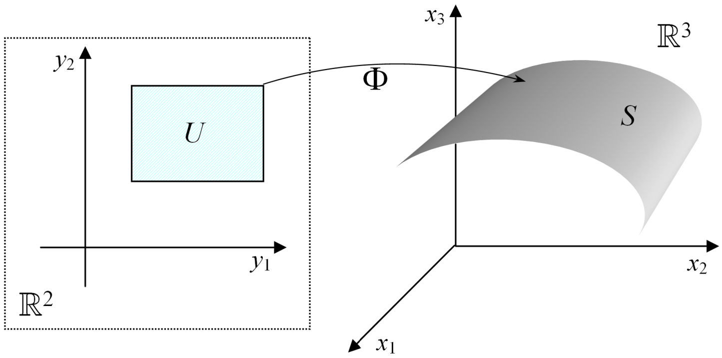
$$\mathbf{g}_{PS} = dr^2 + e^{-2r} \mathbf{g}_{\mathbb{S}^{n-1}}.$$

45. * Let a surface S in $\mathbb{R}^3$ be given in a parametric form as follows:

$$S = \left\{x \in \mathbb{R}^3 : x = \Phi(y),\ y \in U\right\},$$

where U is an open subset of $\mathbb{R}^2$ and $\Phi : U \to \mathbb{R}^3$ is a smooth injective mapping.

Assume that the Jacobi matrix J of Φ has rank 2 at all points.



Assume also that Φ is a homeomorphism of U onto S. Then by Exercise 5 S is a 2-dimensional submanifold of $\mathbb{R}^3$.

Let the components of Φ be Φ^i, $i = 1, 2, 3$. Denoting by y^1, y^2 the Cartesian coordinates in U, consider at any point of U the following two 3-dimensional vectors:

$$u := \left(\frac{\partial \Phi^1}{\partial y^1}, \frac{\partial \Phi^2}{\partial y^1}, \frac{\partial \Phi^3}{\partial y^1}\right) \quad \text{and} \quad v := \left(\frac{\partial \Phi^1}{\partial y^2}, \frac{\partial \Phi^2}{\partial y^2}, \frac{\partial \Phi^3}{\partial y^2}\right).$$

(a) Prove that the induced metric $\mathbf{g}_S = \mathbf{g}_{\mathbb{R}^n}|_S$ is given in the local coordinates y^1, y^2 by the matrix

$$g_S = \begin{pmatrix} u \cdot u & u \cdot v \\ u \cdot v & v \cdot v \end{pmatrix}$$

where "$\cdot$" denotes the scalar product of vectors in $\mathbb{R}^3$. Prove also that

$$\det g_S = |u \times v|^2, \tag{36}$$

where "$\times$" denotes the cross product of vectors in $\mathbb{R}^3$.

(b) Using (32), compute the induced measure ν_S for the surface S that is given by the parametric equations

$$x^1 = \sin\varphi\cos\theta, \quad x^2 = \sin\varphi\sin\theta, \quad x^3 = \cos\varphi,$$

where $\varphi \in (0, \pi)$ and $\theta \in (-\pi, \pi)$.

Solution. (a) Let x^1, x^2, x^3 be the Cartesian coordinates in $\mathbb{R}^3$. Then the relation between x^i and y^j are given by

$$x^i = \Phi^i(y^1, y^2).$$

The Jacobi matrix of the change of coordinates coincides with the Jacobi matrix of Φ:

$$J = \begin{pmatrix} \frac{\partial x^1}{\partial y^1} & \frac{\partial x^1}{\partial y^2} \\ \frac{\partial x^2}{\partial y^1} & \frac{\partial x^2}{\partial y^2} \\ \frac{\partial x^3}{\partial y^1} & \frac{\partial x^3}{\partial y^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi^1}{\partial y^1} & \frac{\partial \Phi^1}{\partial y^2} \\ \frac{\partial \Phi^2}{\partial y^1} & \frac{\partial \Phi^2}{\partial y^2} \\ \frac{\partial \Phi^3}{\partial y^1} & \frac{\partial \Phi^3}{\partial y^2} \end{pmatrix}.$$

Since $\mathbf{g}_{\mathbb{R}^n} = \mathrm{id}$, we have by a lemma from lectures that the matrix g of $\mathbf{g}_S$ is given by

$$g = J^T g_{\mathbb{R}^n} J = J^T J.$$

Since

$$J = \left(\begin{array}{|c|c|} u^T & v^T \end{array} \right)$$

and, hence,

$$J^T = \left(\begin{array}{|c|} \hline u \\ \hline v \\ \hline \end{array} \right),$$

we obtain

$$g = J^T J = \begin{pmatrix} u \\ v \end{pmatrix} \begin{pmatrix} u^T & v^T \end{pmatrix} = \begin{pmatrix} u \cdot u & u \cdot v \\ u \cdot v & v \cdot v \end{pmatrix}.$$

It follows that

$$\det g = (u \cdot u)(v \cdot v) - (u \cdot v)^2 .$$

Denoting $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, we obtain

$$\begin{aligned} \det g &= \left(u_1^2 + u_2^2 + u_3^2\right)\left(v_1^2 + v_2^2 + v_3^2\right) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2 \\ &= (u_1 v_2 - u_2 v_1)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_2 v_3 - u_3 v_2)^2 \\ &= |u \times v|^2 \end{aligned}$$

because

$$u \times v = \left((u_2 v_3 - u_3 v_2),\ (u_3 v_1 - u_1 v_3),\ (u_1 v_2 - u_2 v_1)\right).$$

(b) Since

$$x^1 = \sin\varphi \cos\theta, \quad x^2 = \sin\varphi \sin\theta, \quad x^3 = \cos\varphi,$$

we have

$$u = \left(\tfrac{\partial x^1}{\partial \varphi}, \tfrac{\partial x^2}{\partial \varphi}, \tfrac{\partial x^3}{\partial \varphi}\right) = (\cos\varphi \cos\theta, \cos\varphi \sin\theta, -\sin\varphi)$$

and

$$v := \left(\tfrac{\partial x^1}{\partial \theta}, \tfrac{\partial x^2}{\partial \theta}, \tfrac{\partial x^3}{\partial \theta}\right) = (-\sin\varphi \sin\theta, \sin\varphi \cos\theta, 0)$$

whence

$$\begin{aligned} u \times v &= \left(\cos\theta \sin^2\varphi, \sin\theta \sin^2\varphi, \left(\cos^2\theta + \sin^2\theta\right)\cos\varphi \sin\varphi\right) \\ &= \left(\cos\theta \sin^2\varphi, \sin\theta \sin^2\varphi, \cos\varphi \sin\varphi\right) \end{aligned}$$

and

$$\begin{aligned} \det g_S = |u \times v|^2 &= \cos^2\theta \sin^4\varphi + \sin^2\theta \sin^4\varphi + \cos^2\varphi \sin^2\varphi \\ &= \sin^4\varphi + \cos^2\varphi \sin^2\varphi \\ &= \sin^2\varphi. \end{aligned}$$

Hence,

$$d\nu_S = \sqrt{\det g_S}\, d\varphi\, d\theta = \sin\varphi\, d\varphi\, d\theta.$$

46. ** Prove that, for any $n \geq 1$,

$$\omega_n = 2\frac{\pi^{n/2}}{\Gamma(n/2)}, \tag{37}$$

where ω_n is the surface area of $\mathbb{S}^{n-1}$ and Γ is the gamma function.

Hint. Consider the integrals

$$I_n = \int_0^\pi \sin^n r dr$$

and, using integration by parts, prove that

$$I_n = \frac{n-1}{n} I_{n-2}.$$

By induction obtain that

$$I_n = \sqrt{\pi}\frac{\Gamma((n+1)/2)}{\Gamma((n+2)/2)}.$$

Then prove (33) by means of the inductive relation $\omega_{n+1} = \omega_n I_{n-1}$ from lectures.

Remark. The gamma function is defined for all $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

It is known that $\Gamma(x) = (x-1)!$ for a positive integer x. The following identities are satisfied for all $x > -1$:

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(1) = 1 \text{ and } \Gamma(1/2) = \sqrt{\pi}.$$

Solution. Let us first evaluate the integral

$$I_n = \int_0^\pi \sin^n r dr,$$

where n is a non-negative integer. Assuming $n \geq 2$ and integrating by parts as $\sin^{n-1} r d\cos r$, we obtain

$$\begin{aligned}
I_n &= -\int_0^\pi \sin^{n-1} r \, d\cos r \\
&= -\left[\sin^{n-1} r \cos r\right]_0^\pi + (n-1)\int_0^\pi \cos^2 r \sin^{n-2} r dr \\
&= (n-1)\int_0^\pi \left(1 - \sin^2 r\right) \sin^{n-2} r dr \\
&= (n-1) I_{n-2} - (n-1) I_n,
\end{aligned}$$

whence

$$I_n = \frac{n-1}{n} I_{n-2}. \tag{38}$$

Let us prove by induction that, for all $n \geq 0$,

$$I_n = \frac{\sqrt{\pi}\Gamma((n+1)/2)}{\Gamma((n+2)/2)} \tag{39}$$

For $n = 0$ we have $I_0 = \pi$, which matches the right hand side of (35) because $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1) = 1$. For $n = 1$ we have $I_1 = 2$, which again matches the right hand side of (35) because $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$. For $n \geq 2$ we obtain, using the inductive hypothesis for I_{n-2}, (34), and the identity $z\Gamma(z) = \Gamma(z+1)$, that

$$I_n = \frac{n-1}{n}\sqrt{\pi}\frac{\Gamma((n-1)/2)}{\Gamma(n/2)} = \sqrt{\pi}\frac{\Gamma((n+1)/2)}{\Gamma((n+2)/2)},$$

which proves (35).

Combining (35) with $\omega_{n+1} = \omega_n I_{n-1}$, we obtain, for all $n \geq 1$,

$$\omega_{n+1} = \omega_n \frac{\sqrt{\pi}\Gamma(n/2)}{\Gamma((n+1)/2)}, \tag{40}$$

which easily implies (33) by induction in n. Sometimes the following consequence of (**??**) is useful:

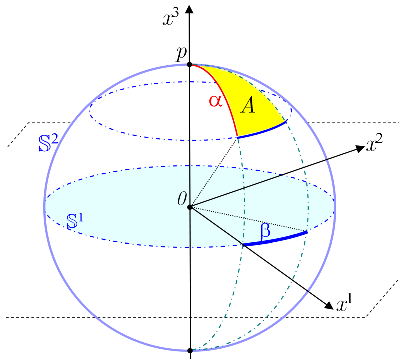
$$\omega_{n+2} = \omega_n \frac{2\pi}{n}.$$

Blatt 9. Abgabe bis 19.12.2025

Die mit *markierten Aufgaben sind zusätzlich und werden korrigiert
Die mit **markierten Aufgaben sind zusätzlich und werden nicht korrigiert.

47. Denote by σ_n the canonical Riemannian measure on the sphere $\mathbb{S}^n$ and by η_n the canonical Riemannian measure on the hyperbolic space $\mathbb{H}^n$.

   (a) Let (φ, θ) be the polar coordinates on $\mathbb{S}^2$, where $\varphi \in (0, \pi)$ is the polar radius and $\theta \in (0, 2\pi)$ is the polar angle. Compute $\sigma_2(A)$ for the following subset A of $\mathbb{S}^2$:



$$A = \{(\varphi, \theta) : 0 < \varphi < \alpha, \quad 0 < \theta < \beta\},$$

   where $\alpha \in (0, \pi)$ and $\beta \in (0, 2\pi)$ are given.

   (b) Let (r, φ, θ) the *spherical coordinates* on $\mathbb{S}^3$, where $r \in (0, \pi)$ is the polar radius and (φ, θ) are the polar coordinates on $\mathbb{S}^2$ as in (a). Compute $\sigma_3(B)$ for the following subset B of $\mathbb{S}^3$:

$$B = \{(r, \varphi, \theta) : 0 < r < R, \quad 0 < \varphi < \alpha, \quad 0 < \theta < \beta\},$$

   and $0 < R < \pi$, $\alpha \in (0, \pi)$, $\beta \in (0, 2\pi)$ are given.

   (c) Let (r, θ) be the polar coordinates in $\mathbb{H}^2$. Compute $\eta_2(C)$ for the following subset C of $\mathbb{H}^2$:

$$C = \{(r, \theta) : 0 < r < R, \; 0 < \theta < \beta\},$$

   where $R > 0$ and $\beta \in (0, 2\pi)$ are given.

   (d) Let (r, φ, θ) be the *spherical coordinates* on $\mathbb{H}^3$, where $r > 0$ is the polar radius and (φ, θ) are the polar coordinates on $\mathbb{S}^2$ as in (a). Compute $\eta_3(D)$ for the following subset D of $\mathbb{H}^3$:

$$D = \{(r, \varphi, \theta) : 0 < r < R, \; 0 < \varphi < \alpha, \; 0 < \theta < \beta\},$$

   where $R > 0$, $\alpha \in (0, \pi)$, $\beta \in (0, 2\pi)$ are given.

Solution. Let $(M, \mathbf{g})$ be an n-dimensional model manifold with a profile function $\psi(r)$, that is,

$$\mathbf{g} = dr^2 + \psi^2(r)\mathbf{g}_{\mathbb{S}^{n-1}}. \tag{41}$$

Recall that the Riemannian measure ν in $(M, \mathbf{g})$ is given by

$$d\nu = \psi^{n-1}(r) dr \, d\sigma_{n-1}, \tag{42}$$

where σ_{n-1} is the Riemannian measure on $\mathbb{S}^{n-1}$.
Since on $\mathbb{S}^n$ we have $\psi(r) = \sin r$, it follows from (19) that

$$d\sigma_n = \sin^{n-1} r \, dr \, d\sigma_{n-1}$$

In the case $n = 1$, using an angle θ on $\mathbb{S}^1$, we have

$$\mathbf{g}_{\mathbb{S}^1} = d\theta^2, \quad d\sigma_1 = d\theta, \quad \Delta_{\mathbb{S}^1} = \frac{\partial^2}{\partial \theta^2}. \tag{43}$$

In $\mathbb{H}^n$ we have $\psi(r) = \sinh r$ whence

$$d\eta_n = \sinh^{n-1} r \, dr \, d\sigma_{n-1}.$$

(a) On $\mathbb{S}^2$ we obtain in the polar coordinates (φ, θ)

$$d\sigma_2 = \sin \varphi \, d\varphi \, d\sigma_1 = \sin \varphi \, d\varphi \, d\theta$$

We have

$$\sigma_2(A) = \int_A d\sigma_2 = \int_0^\beta \left(\int_0^\alpha \sin \varphi d\varphi \right) d\theta = (1 - \cos \alpha)\, \beta.$$

(b) On $\mathbb{S}^3$ we obtain the spherical coordinates (r, φ, θ)

$$d\sigma_3 = \sin^2 r \, dr \, d\sigma_2 = \sin^2 r \sin \varphi \, dr \, d\varphi \, d\theta.$$

We have

$$\sigma_3(B) = \int_A d\sigma_3 = \int_0^\beta \left(\int_0^\alpha \left(\int_0^R \sin^2 r dr \right) \sin \varphi d\varphi \right) d\theta$$
$$= \frac{1}{4} (2R - \sin 2R)\, (1 - \cos \alpha)\, \beta.$$

(c) In $\mathbb{H}^2$ we have

$$d\eta_2 = \sinh r \, dr \, d\sigma_1 = \sinh r \, dr \, d\theta,$$

whence

$$\eta_2(C) = \int_A d\eta_2 = \int_0^\beta \left(\int_0^R \sinh r \, dr \right) d\theta = (\cosh R - 1)\, \beta.$$

(d) In $\mathbb{H}^3$ we have

$$d\eta_3 = \sinh^2 r \, dr \, d\sigma_2 = \sinh^2 r \sin \varphi \, dr \, d\varphi \, d\theta$$

whence

$$\eta_4(D) = \int_A d\eta_3 = \int_0^\beta \int_0^\alpha \left(\int_0^R \sinh^2 r dr \right) \sin \varphi \, d\varphi \, d\theta$$
$$= \frac{1}{4} (\sinh 2R - 2R)\, (1 - \cos \alpha)\, \beta.$$

48. Let $(M, \mathbf{g})$ be a Riemannian manifold. A smooth function u in an open set $\Omega \subset M$ is called *harmonic* in Ω if $\Delta_{\mathbf{g}} u = 0$ in Ω. Suppose that M is a model manifold of radius r_0 with the area function $S(r)$. A function u in $M \setminus \{o\}$ is called *radial* if it depends only on the polar radius r (and does not depend on the polar angle θ).

(a) Prove that a smooth radial function u in $M \setminus \{o\}$is harmonic if and only if

$$u(r) = C_1 \int_{r_1}^{r} \frac{dt}{S(t)} + C_2, \tag{44}$$

where C_1, C_2 are arbitrary real constants and $r_1 \in (0, r_0)$ is arbitrary.
Hint. Use the representation of the Laplace-Beltrami operator $\Delta_{\mathbf{g}}$ in polar coordinates using the area function.

(b) With help of (38) find all radial harmonic functions in $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{H}^n$ for $n = 2, 3$.

Solution. (a) Recall that

$$\Delta_{\mathbf{g},\mu} = \frac{\partial^2}{\partial r^2} + \frac{S'}{S}\frac{\partial}{\partial r} + \frac{1}{\psi^2(r)}\Delta_{\mathbb{S}^{n-1}}. \tag{45}$$

Since u does not depend on the polar angle, we obtain $\Delta_{\mathbb{S}^{n-1}} u = 0$ and the equation $\Delta_{\mathbf{g},\mu} u = 0$ becomes

$$u'' + \frac{S'}{S} u = 0$$

This equation equivalent to

$$(Su')' = 0,$$

that is, to

$$\begin{aligned} Su' &= C_1 \\ u' &= \frac{C_1}{S}. \end{aligned}$$

Fix some $r_1 \in (0, r_0)$ and consider the function

$$v(r) = C_1 \int_{r_1}^{r} \frac{dt}{S(t)}.$$

Then $v' = \frac{C_1}{S}$, and we obtain that $u' = v'$, whence

$$u - v = C_2,$$

that is,

$$u(r) = C_1 \int_{r_1}^{r} \frac{dt}{S(t)} + C_2. \tag{46}$$

(b) In $\mathbb{R}^n$ we have $S(r) = \omega_n r^{n-1}$ and (40) yields

$$u(r) = C_2 + C_1 \begin{cases} \ln r, & \text{in } \mathbb{R}^2, \\ r^{-1}, & \text{in } \mathbb{R}^3, \end{cases}$$

where we have used that

$$\int \frac{dr}{r} = \ln r + C$$

and

$$\int \frac{dr}{r^2} = -\frac{1}{r} + C.$$

Since in $\mathbb{S}^n$ we have $S(r) = \omega_n \sin^{n-1} r$, we obtain from (40)

$$u(r) = C_2 + C_1 \begin{cases} \ln \tan \frac{r}{2}, & \text{in } \mathbb{S}^2, \\ \cot r, & \text{in } \mathbb{S}^3, \end{cases}$$

where we have used that

$$\int \frac{dr}{\sin r} = \ln \tan \frac{r}{2} + C$$

and

$$\int \frac{dr}{\sin^2 r} = -\cot r + C$$

Similarly, using that

$$\int \frac{dr}{\sinh r} = \ln \tanh \frac{r}{2} + C$$

and

$$\int \frac{dr}{\sinh^2 r} = -\coth r + C,$$

we obtain

$$u(r) = C_2 + C_1 \begin{cases} \ln \tanh \frac{r}{2}, & \text{in } \mathbb{H}^2, \\ \coth r, & \text{in } \mathbb{H}^3. \end{cases}$$

Everywhere the value r_1 is absorbed into C_2.

49. (Continuation of Exercises 5 and 5). A catenoid *Cat* is a surface in $\mathbb{R}^3$ that is given by the parametric equations

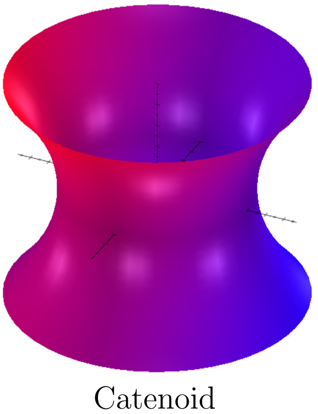
$$x^1 = \cosh \rho \cos \theta, \quad x^2 = \cosh \rho \sin \theta, \quad x^3 = \rho,$$

where $\rho \in \mathbb{R}$ and $\theta \in (-\pi, \pi)$.

(a) Write down the Laplace-Beltrami operator $\Delta_{\mathbf{g}}$ on *Cat* in the coordinates ρ, θ.

(b) Considering the Cartesian coordinates x^1, x^2, x^3 as functions on the catenoid, prove that they are harmonic, that is,

$$\Delta_{\mathbf{g}} x^1 = \Delta_{\mathbf{g}} x^2 = \Delta_{\mathbf{g}} x^3 = 0.$$



Catenoid

Hint. Use the Riemannian metric on *Cat* stated in Exercise 5.

Solution. The Riemannian metric on *Cat* has in the coordinates ρ, θ the form

$$\mathbf{g} = \cosh^2 \rho \, \left(d\rho^2 + d\theta^2\right).$$

(a) It follows from that

$$\sqrt{\det g} = \cosh^2 \rho$$

and

$$\left(g^{ij}\right) = g^{-1} = \begin{pmatrix} \frac{1}{\cosh^2 \rho} & 0 \\ 0 & \frac{1}{\cosh^2 \rho} \end{pmatrix}.$$

Using that

$$\Delta_{\mathbf{g}} = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial y^i} \left(\sqrt{\det g} g^{ij} \frac{\partial}{\partial y^j} \right)$$

with $y^1 = \rho$, $y^2 = \theta$, we obtain

$$\Delta_{\mathbf{g}} = \frac{1}{\cosh^2 \rho} \left(\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \theta^2} \right).$$

(b) Since $x^1 = \cosh \rho \cos \theta$, it follows by (a) that

$$\Delta_{\mathbf{g}} x^1 = \frac{1}{\cosh^2 \rho} \left(\cosh \rho \cos \theta - \cosh \rho \cos \theta \right) = 0.$$

In the same way, for $x^2 = \cosh \rho \sin \theta$, we obtain

$$\Delta_{\mathbf{g}} x^2 = \frac{1}{\cosh^2 \rho} \left(\cosh \rho \sin \theta - \cosh \rho \sin \theta \right) = 0$$

and for $x^3 = \rho$ we obtain $\Delta_{\mathbf{g}} x^3 = \Delta_{\mathbf{g}} \rho = 0$.

50. A non-zero smooth function v on a Riemannian manifold $(M, \mathbf{g})$ is called an *eigenfunction* of the Laplace-Beltrami operator $\Delta_{\mathbf{g}}$ if, for some constant λ,

$$\Delta_{\mathbf{g}} v + \lambda v = 0,$$

where the constant λ is called an *eigenvalue* of $\Delta_{\mathbf{g}}$. The *multiplicity* of the eigenvalue λ is defined as the dimension of the *eigenspace*

$$E_\lambda = \left\{ v \in C^\infty(M) : \Delta_{\mathbf{g}} v + \lambda v = 0 \right\}.$$

Prove that all the eigenvalues of the Laplace-Beltrami operator $\Delta_{\mathbb{S}^1}$ on the unit circle $\mathbb{S}^1$ are given by the sequence $\{m^2\}_{m=0}^{\infty}$, where the eigenvalue 0 has the multiplicity 1, and each eigenvalue m^2 with $m \geq 1$ has the multiplicity 2.

Hint. Write down the equation $\Delta_{\mathbb{S}^1} v + \lambda v = 0$ using the angle $\theta \in (-\pi, \pi)$ as a local coordinate on $\mathbb{S}^1$, and find solutions $v(\theta)$ that are 2π-periodic in θ.

Solution. (a) Using the polar angle $\theta \in (-\pi, \pi)$ on $\mathbb{S}^1$, we have $\mathbf{g}_{\mathbb{S}^1} = d\theta^2$ (see Exercise 5) and $\Delta_{\mathbb{S}^1} = \frac{d^2}{d\theta^2}$ (cf. solution of Exercise 5(f)vO). The eigenvalue problem becomes

$$\frac{d^2 v}{d\theta^2} + \lambda v = 0, \tag{47}$$

where v is any smooth function on $\mathbb{S}^1$ (we know that any eigenfunction must be smooth, while any smooth function on a compact manifold belongs to $\mathcal{D}$ and, hence, to W_0^1). Since $v(\theta)$ has to be defined and smooth not only on the interval $(-\pi,\pi)$ but on the entire $\mathbb{S}^1$, the function $v(\theta)$ must be 2π-periodic, in particular, it must satisfy

$$v(-\pi)=v(\pi) \quad \text{and} \quad v'(-\pi)=v'(\pi). \tag{48}$$

If $\lambda=0$ then we obtain a solution $v=C_1\theta+C_2$ of (**??**) that satisfies (**??**) only if $C_1=0$. Hence, $\lambda=0$ is an eigenvalue with the eigenfunction $v\equiv\text{const}$.

Assume $\lambda>0$. Then all solutions of (**??**) are given by

$$v=C_1\cos\left(\sqrt{\lambda}\theta\right)+C_2\sin\left(\sqrt{\lambda}\theta\right),$$

where C_1,C_2 are arbitrary constants. The boundary conditions (**??**) become

$$C_1\cos\left(\sqrt{\lambda}\pi\right)-C_2\sin\left(\sqrt{\lambda}\pi\right)=C_1\cos\left(\sqrt{\lambda}\pi\right)+C_2\sin\left(\sqrt{\lambda}\pi\right)$$

and

$$-C_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right)+C_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right)=C_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right)+C_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right)$$

that are equivalent to

$$\sin\left(\sqrt{\lambda}\pi\right)=0$$

(because either C_1 or C_2 must be non-zero). The latter condition is satisfied if and only if $\sqrt{\lambda}=m$ where m is a positive integer. Hence, we obtain that the sequence of the eigenvalues of $\mathbb{S}^1$ is given by $\{m^2\}_{m=0}^{\infty}$. The eigenvalue 0 is simple with the eigenfunction $v=\text{const}$, while m^2 with $m\geq 1$ is double, with independent eigenfunctions $\cos m\theta$ and $\sin m\theta$.

51. * Consider in $\mathbb{H}^3$ a function u given in the polar coordinates (r,θ) by

$$u=\frac{r}{\sinh r}.$$

   (a) Prove that, away from the pole of $\mathbb{H}^3$, the function u satisfies the equation

$$\Delta_{\mathbb{H}^3}u+u=0. \tag{49}$$

   Hint. Use the representation of $\Delta_{\mathbb{H}^3}$ in the polar coordinates.

   (b) Prove that the function u extends to a smooth function on the entire space $\mathbb{H}^3$ and, hence, satisfies (41) on $\mathbb{H}^3$.

   Hint. Show first that the function $v=r^2$ is a smooth function on $\mathbb{H}^3$ (as well as on any model manifold). Then represent u as a smooth function of r^2.

Solution. (*a*) Let (r,θ) be the polar coordinates in $\mathbb{H}^3$, where $r>0$ is the polar radius and $\theta\in\mathbb{S}^2$. Then we have

$$\Delta_{\mathbb{H}^3}=\frac{\partial^2}{\partial r^2}+2\coth r\frac{\partial}{\partial r}+\frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^2}.$$

Since the function u depends only on r, we obtain

$$\Delta_{\mathbb{H}^3} u = u'' + 2(\coth r)\, u'.$$

A direct computation yields

$$\left(\frac{r}{\sinh r}\right)' = \frac{1}{\sinh^2 r}(\sinh r - r\cosh r)$$

$$\left(\frac{r}{\sinh r}\right)'' = \frac{1}{2\sinh^3 r}(3r - 2\sinh 2r + r\cosh 2r),$$

whence

$$\begin{aligned} u'' + 2(\coth r)\, u' &= \frac{1}{2\sinh^3 r}(3r - 2\sinh 2r + r\cosh 2r) \\ &+ 2\frac{\cosh r}{\sinh r}\frac{1}{\sinh^2 r}(\sinh r - r\cosh r) \\ &= -\frac{r}{\sinh r}, \end{aligned}$$

that is, $u'' + 2(\coth r)\, u' = -u$, whence (41) follows.

(b) As a model manifold, $\mathbb{H}^3$ can be identified with $\mathbb{R}^3$ with the metric

$$\mathbf{g}_{\mathbb{H}^n} = dr^2 + \sinh^2 r \mathbf{g}_{\mathbb{S}^{n-1}}$$

where (r, θ) are the polar coordinates not only in $\mathbb{H}^3$ but also in $\mathbb{R}^3$. In $\mathbb{R}^3$ the function

$$r^2 = \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2$$

is clearly C^∞, whence the same follows for $\mathbb{H}^3$ (and in the same way for any other model manifold).

We have

$$\sinh r = \frac{e^r - r^{-r}}{2} = r + \frac{r^3}{3!} + \frac{r^5}{5!} + ...$$

whence

$$u = \frac{r}{\sinh r} = \frac{1}{1 + \frac{r^2}{3!} + \frac{r^4}{5!} + ...}.$$

Obviously, the right hand side is a smooth function of r^2, which implies that u is a smooth function on the entire $\mathbb{H}^3$.

52. ** Consider the Riemannian manifold $(M, \mathbf{g})$, where

$$M = \mathbb{R}^n_+ := \left\{\left(x^1, ..., x^n\right) \in \mathbb{R}^n : x^n > 0\right\}$$

and

$$\mathbf{g} = \frac{\left(dx^1\right)^2 + ... + \left(dx^n\right)^2}{\left(x^n\right)^2}.$$

Consider on $\mathbb{R}^n_+$ the following function

$$u(x) = (x^n)^s$$

where s is any real parameter. Prove that $\Delta_{\mathbf{g}} u = \lambda u$, where $\lambda = s(s-n+1)$.

Solution. For the Laplace-Beltrami operator we have

$$\Delta_{\mathbf{g}} = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \frac{\partial}{\partial x^j} \right).$$

Here

$$(g_{ij}) = \frac{1}{(x^n)^2} \operatorname{id},$$

which implies

$$\left(g^{ij}\right) = (x^n)^2 \operatorname{id}$$

and

$$\det g = \frac{1}{(x^n)^{2n}}.$$

Hence,

$$\begin{aligned} \Delta_{\mathbf{g}} &= (x^n)^n \sum_{i=1}^n \frac{\partial}{\partial x^i} \left((x^n)^{2-n} \frac{\partial}{\partial x^i} \right) \\ &= (x^n)^2 \sum_{i=1}^n \frac{\partial^2}{(\partial x^i)^2} + (x^n)^n \frac{\partial}{\partial x^n} \left((x^n)^{2-n} \frac{\partial}{\partial x^n} \right). \end{aligned}$$

For $u(x) = (x^n)^s$ we obtain

$$\begin{aligned} \Delta_{\mathbf{g}} u &= (x^n)^n \frac{\partial}{\partial x^n} \left((x^n)^{2-n} \frac{\partial u}{\partial x^n} \right) \\ &= (x^n)^n \frac{\partial}{\partial x^n} \left((x^n)^{2-n} s (x^n)^{s-1} \right) \\ &= (x^n)^n s \frac{\partial}{\partial x^n} (x^n)^{s-n+1} \\ &= (x^n)^n s(s-n+1) (x^n)^{s-n} \\ &= s(s-n+1)(x^n)^s \\ &= s(s-n+1) u, \end{aligned}$$

which was to be proved.

53. ** A function $P : \mathbb{R}^N \to \mathbb{R}$ is called a *polynomial* if $P(x)$ is a finite $\mathbb{R}$-linear combination of the *monomials* $x_1^{m_1} ... x_N^{m_N}$ where $m_1, ..., m_N$ are non-negative integers. The sum $m_1 + ... + m_N$ is called the *degree* of the monomial. A polynomial P is called *homogeneous* of degree m if all non-zero monomials of P have the same degree m.

   (a) Let P be a homogeneous polynomial of degree m on $\mathbb{R}^{n+1}$, where m is a non-negative integer. Assume that P is harmonic, that is, P satisfies the equation

   $$\Delta_{\mathbb{R}^{n+1}} P = 0 \text{ in } \mathbb{R}^{n+1}.$$

   Prove that the function $v = P|_{\mathbb{S}^n}$ is an eigenfunction of the Laplace-Beltrami operator $\Delta_{\mathbb{S}^n}$ with the eigenvalue $\lambda = m(m+n-1)$, that is,

   $$\Delta_{\mathbb{S}^n} v + \lambda v = 0 \text{ in } \mathbb{S}^n.$$

Hint. Use the identity $P(x) = \alpha^m P\left(\frac{x}{\alpha}\right)$ for all $\alpha \in \mathbb{R} \setminus \{0\}$ and all $x \in \mathbb{R}^{n+1}$ that follows from the homogeneity of P, and represent $\Delta_{\mathbb{R}^{n+1}}$ in the polar coordinates.

(b) Prove that a polynomial in $\mathbb{R}^3$

$$P(x) = C_1 x_1^3 x_2 x_3 + C_2 x_1 x_2^3 x_3 + C_3 x_1 x_2 x_3^3$$

is harmonic for some non-zero coefficients C_1, C_2, C_3. Hence, prove that $\lambda = 30$ is an eigenvalue of the Laplace-Beltrami operator $\Delta_{\mathbb{S}^2}$.

Remark. It is possible to prove that all the eigenvalues of $\Delta_{\mathbb{S}^n}$ are given by the sequence $\{m(m+n-1)\}_{m=0}^{\infty}$, and the eigenvalue $m(m+n-1)$ has the multiplicity

$$\frac{(n+m-2)!\,(n+2m-1)}{(n-1)!m!}$$

if $m \geq 1$, and 1 if $m = 0$.

Solution. (a) Let (r, θ) be the polar coordinates on $\mathbb{R}^{n+1}$. For any $x = (r, \theta)$ we have by the homogeneity of P that

$$P(x) = r^m P\left(\frac{x}{r}\right) = r^m P(\theta) = r^m v(\theta).$$

By the formula for the Laplace operator in $\mathbb{R}^{n+1}$ in the polar coordinates, we have

$$\Delta_{\mathbb{R}^{n+1}} P = \frac{\partial^2 P}{\partial r^2} + \frac{n}{r}\frac{\partial P}{\partial r} + \frac{1}{r^2}\Delta_{\mathbb{S}^n} P.$$

Since $\Delta_{\mathbb{R}^{n+1}} P = 0$, we obtain that in $\mathbb{R}^{n+1} \setminus \{0\}$

$$\frac{\partial^2 P}{\partial r^2} + \frac{n}{r}\frac{\partial P}{\partial r} + \frac{1}{r^2}\Delta_{\mathbb{S}^n} P = 0.$$

Substituting here $P = r^m v(\theta)$, we obtain

$$m(m-1)\, r^{m-2} v(\theta) + \frac{n}{r} m r^{m-1} v(\theta) + \frac{1}{r^2} r^m \Delta_{\mathbb{S}^n} v(\theta) = 0$$

whence after cancelling by r^{m-2}

$$-\Delta_{\mathbb{S}^n} v = (m(m-1) + nm)\, v = \lambda v.$$

Hence, λ is an eigenvalue of $\Delta_{\mathbb{S}^n}$ with the eigenfunction v, which was to be proved.

(b) We have

$$\Delta_{\mathbb{R}^3}\left(x_1^3 x_2 x_3\right) = \left(\frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2}\right)\left(x_1^3 x_2 x_3\right) = 6 x_1 x_2 x_3,$$

whence it follows that

$$\Delta_{\mathbb{R}^3} P = (6C_1 + 6C_2 + 6C_3)\, x_1 x_2 x_3.$$

Choosing $C_1 = C_2 = 1$ and $C_3 = -2$, we obtain $\Delta_{\mathbb{R}^3} P = 0$. Hence, P is a harmonic homogeneous polynomial of degree $m = 5$. Therefore, by (a), $\Delta_{\mathbb{S}^2}$ has an eigenvalue $\lambda = m(m+n-1) = 5 \cdot (5+2-1) = 30$.

Blatt 10. Abgabe bis 09.01.2026

Die mit *markierten Aufgaben sind zusätzlich und werden korrigiert
Die mit **markierten Aufgaben sind zusätzlich und werden nicht korrigiert.

54. Let $(M, \mathbf{g})$ be a Riemannian manifold of dimension n. Let $F : M \to \mathbb{R}$ be a smooth function on M such that F is non-singular[1] on the null set $S = \{x \in M : F(x) = 0\}$. In particular, S is a submanifold of dimension $n-1$.

   (a) Prove that, at any point $p \in S$, the gradient $\nabla F(p)$ is orthogonal to T_pS in the tangent space T_pM.
   Hint. Use Exercise 5.

   (b) Consider the set
   $$\Omega := \{x \in M : F(x) < 0\} \tag{50}$$
   and prove that $S = \partial\Omega$.
   Remark. An open set $\Omega \subset M$ is called a *region* if it can be represented in the form (42), where F is a smooth function on M that is non-singular on its null set.

Solution. (a) Since $T_pS \subset T_pM$ and $\nabla F(p) \in T_pM$, we need to prove that, for any $\xi \in T_pS$,
$$\langle \nabla F, \xi \rangle_{\mathbf{g}} = 0$$
that is,
$$\langle dF, \xi \rangle = 0.$$
The latter holds by Exercise 5. Or, we have
$$\langle dF, \xi \rangle = \xi(F) = \xi(F|_S) = \xi(0) = 0.$$

(b) By definition, the condition $x \in \partial\Omega$ holds if in any neighborhood of x there are points from Ω and Ω^c. If $x \in \partial\Omega$ then in any neighborhood of x there are points where $F < 0$ and $F \geq 0$, which implies by continuity that $F(x) = 0$ and $x \in S$. Conversely, let $x \in S$ that is, $F(x) = 0$. Then we need to verify that in any neighborhood of x there are points with $F < 0$. If this is not the case then in some neighborhood of x we have $F \geq 0$, which implies that x is a point of local minimum of F, which implies $dF(x) = 0$. However, this contradicts to the fact that dF is non-singular on S.

55. Let H be the semi-hyperbola
$$H = \left\{(x_1, x_2) \in \mathbb{R}^2 : \ x_2^2 - x_1^2 = 1, \ x_2 > 0\right\}.$$
For any $s > 0$, consider the following subset of H:
$$H_s = \{(x_1, x_2) \in H : \ 0 < x_1 < s\}.$$

[1] Recall that F is non-singular on a set S if $dF(x) \neq 0$ at any point $x \in S$.

Let ν be the Riemannian measure of $(H, \mathbf{g}_H)$, where $\mathbf{g}_H$ is the hyperbolic metric on H. Prove that

$$\nu(H_s) = \ln\left(s + \sqrt{s^2+1}\right).$$

Remark. Note that the function $\ln\left(s + \sqrt{s^2+1}\right)$ is the inverse to sinh.

Hint. Use the chart on H with the coordinate y from Exercise 5.

Solution. By Exercise 5 and (18), there is a chart on H with the coordinate $y \in (-1, 1)$ such that, for the point $(x_1, x_2) \in H$,

$$x_1 = \frac{2y}{1-y^2}, \quad x_2 = \frac{1+y^2}{1-y^2}.$$

Besides the hyperbolic metric on H is given by

$$\mathbf{g}_H = \frac{4}{(1-y^2)^2} dy^2.$$

The condition $\{0 < x_1 < s\}$ is equivalent to

$$0 < \frac{2y}{1-y^2} < s,$$

which is equivalent to

$$0 < y < r$$

where r is determined by the equation

$$s = \frac{2r}{1-r^2}.$$

Solving this equation in r, we obtain

$$r = \frac{\sqrt{1+s^2}-1}{s}.$$

Since $\det g_H = \frac{4}{(1-y^2)^2}$, we obtain

$$\nu(H_s) = \int_0^r \sqrt{\det g_H} dy = \int_0^r \frac{2}{1-y^2} dy = \ln \frac{1+r}{1-r}.$$

Note that

$$\frac{1+r}{1-r} = \frac{s-1+\sqrt{1+s^2}}{s+1-\sqrt{1+s^2}}.$$

Set $u = s + \sqrt{1+s^2}$ and observe that

$$\frac{1}{u} = \sqrt{1+s^2} - s$$

because

$$\left(s + \sqrt{1+s^2}\right)\left(\sqrt{1+s^2} - s\right) = \left(1+s^2\right) - s^2 = 1.$$

Hence,

$$\frac{1+r}{1-r}=\frac{u-1}{1-\frac{1}{u}}=u,$$

whence it follows that

$$\nu\left(H_s\right)=\ln u=\ln\left(s+\sqrt{1+s^2}\right).$$

56. For any two-dimensional Riemannian manifold $(M, \mathbf{g})$, the Gauss curvature $K_{\mathbf{g}}(x)$ is defined in a certain way as a function on M. It is known that if the metric $\mathbf{g}$ has in coordinates x^1, x^2 the form

$$\mathbf{g}=\frac{\left(dx^1\right)^2+\left(dx^2\right)^2}{f^2\left(x\right)}, \tag{51}$$

where f is a smooth positive function, then the Gauss curvature can be computed in this chart as follows

$$K_{\mathbf{g}}=f^2\Delta\ln f, \tag{52}$$

where $\Delta=\frac{\partial^2}{(\partial x^1)^2}+\frac{\partial^2}{(\partial x^2)^2}$ is the Euclidean two-dimensional Laplace operator in the coordinates x^1, x^2.

(a) Compute the Gauss curvature of $\mathbb{R}^2$ and the catenoid Cat (see Exercise 5).

(b) Let $(M, \mathbf{g})$ be a two-dimensional model manifold with the profile function ψ, so that in the polar coordinates (r, θ)

$$\mathbf{g}=dr^2+\psi^2\left(r\right)d\theta^2. \tag{53}$$

Prove that

$$K_{\mathbf{g}}=-\frac{\psi''\left(r\right)}{\psi\left(r\right)}. \tag{54}$$

Hint. Find other coordinates (ρ, θ) on M where the metric (45) has the form

$$\mathbf{g}=\frac{d\rho^2+d\theta^2}{f^2\left(\rho\right)},$$

and then use (44).

(c) Using (46), compute the Gauss curvature of the sphere $\mathbb{S}^2$, the hyperbolic plane $\mathbb{H}^2$, and the two-dimensional pseudosphere PS from Exercise 5(f)vJ.

Solution. (a) The metric of $\mathbb{R}^n$ is given by (43) with $f\equiv 1$ whence

$$K_{\mathbb{R}^n}=0.$$

The metric of Cat in Exercise 5 is given by

$$\mathbf{g}_{Cat}=\cosh^2\rho\left(d\rho^2+d\theta^2\right)$$

which matches (43) with

$$f\left(\rho\right)=\frac{1}{\cosh\rho}.$$

Since $\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \theta^2}$, we have

$$\Delta \ln f = (\ln f)'' = \frac{1}{\cosh^2 \rho} \sinh^2 \rho - 1 = -\frac{1}{\cosh^2 \rho}$$

whence

$$K_{Cat} = f^2 \Delta \ln f = -\frac{1}{\cosh^4 \rho}.$$

(b) Let us change the variable

$$\rho = \int \frac{dr}{\psi(r)}$$

so that $d\rho = \frac{dr}{\psi(r)}$. Since $d\rho = \frac{1}{\psi(r)} dr$ and, hence, $dr^2 = \psi^2(r)\, d\rho^2$, the metric $\mathbf{g}$ has in the coordinates ρ, θ the form

$$\mathbf{g} = \psi^2(r) \left(d\rho^2 + d\theta^2\right),$$

which matches (43) with $f(\rho) = \frac{1}{\psi(r)}$ (hence, $x^1 = \rho$, $x^2 = \theta$). Since in this case $\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \theta^2}$ and ψ does not depend on θ, we obtain by (44)

$$K_{\mathbf{g}} = -\frac{1}{\psi^2} \frac{d^2}{d\rho^2} \ln \psi.$$

We have

$$\frac{d}{d\rho} \ln \psi = \frac{1}{\psi} \frac{dr}{d\rho} \frac{d\psi}{dr} = \psi'$$

and

$$\frac{d^2}{d\rho^2} \ln \psi = \frac{d}{d\rho} \psi' = \frac{dr}{d\rho} \psi'' = \psi \psi'',$$

whence

$$K_{\mathbf{g}} = -\frac{\psi \psi''}{\psi^2} = -\frac{\psi''}{\psi}.$$

(c) Since in $\mathbb{S}^2$

$$\mathbf{g} = dr^2 + \sin^2 r\, d\theta^2$$

then

$$K_{\mathbb{S}^2} = -\frac{(\sin r)''}{\sin r} = 1.$$

Since in $\mathbb{H}^2$

$$\mathbf{g} = dr^2 + \sinh^2 r\, d\theta^2,$$

we have

$$K_{\mathbb{H}^2} = -\frac{(\sinh r)''}{\sinh r} = 1.$$

Since on PS

$$\mathbf{g} = dr^2 + e^{-2r} d\theta^2,$$

we obtain

$$K_{PS} = -\frac{\left(e^{-2r}\right)''}{e^{-2r}} = -4.$$

57. Let $\mathbf{g}$ be the metric (43) on a two-dimensional manifold M. Consider the metric $\widetilde{\mathbf{g}} = \frac{1}{h^2}\mathbf{g}$ where h is a smooth positive function on M. Prove that

$$K_{\widetilde{\mathbf{g}}} = \left(K_{\mathbf{g}} + \Delta_{\mathbf{g}} \log h\right) h^2,$$

where $\Delta_{\mathbf{g}}$ is the Laplace-Beltrami operator of the metric $\mathbf{g}$.

Solution. (a) Let us write down the Laplace operator $\Delta_{\mathbf{g}}$ in the coordinates x^1, x^2 using the fact that the matrix $g = (g_{ij})$ of the metric $\mathbf{g}$ has the form

$$g = \begin{pmatrix} f^{-2} & 0 \\ 0 & f^{-2} \end{pmatrix}.$$

Since $\det g = f^{-4}$ and

$$g^{-1} = \begin{pmatrix} f^2 & 0 \\ 0 & f^2 \end{pmatrix},$$

we obtain

$$\begin{aligned} \Delta_{\mathbf{g}} &= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^1}\left(\sqrt{\det g}\, g^{11} \frac{\partial}{\partial x^1}\right) + \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^2}\left(\sqrt{\det g}\, g^{22} \frac{\partial}{\partial x^2}\right) \\ &= f^2 \frac{\partial^2}{(\partial x^1)^2} + f^2 \frac{\partial^2}{(\partial x^2)^2} \\ &= f^2 \Delta, \end{aligned}$$

that is

$$\Delta_{\mathbf{g}} = f^2 \Delta.$$

Since

$$\widetilde{\mathbf{g}} = \frac{(dx^1)^2 + (dx^2)^2}{(fh)^2},$$

the formula (44) gives for this metric

$$K_{M,\widetilde{\mathbf{g}}} = (fh)^2 \Delta \ln (fh) = h^2 \left(f^2 \Delta \ln f + f^2 \Delta \ln h\right) = h^2 \left(K_{M,\mathbf{g}} + \Delta_{\mathbf{g}} \ln h\right),$$

which was to be proved.

(b)

58. * Let $\mathbf{g}$, $\widetilde{\mathbf{g}}$ be two Riemannian metric tensors on a smooth n-dimensional manifold M. Assume that, for some constant C,

$$\widetilde{\mathbf{g}} \le C\mathbf{g}, \tag{55}$$

that is, for all $x \in M$ and $\xi \in T_x M$,

$$\widetilde{\mathbf{g}}(x)(\xi, \xi) \le C\mathbf{g}(x)(\xi, \xi). \tag{56}$$

(a) Prove that if ν and $\widetilde{\nu}$ are the Riemannian measures of $\mathbf{g}$ and $\widetilde{\mathbf{g}}$, respectively, then

$$\frac{d\widetilde{\nu}}{d\nu} \le C^{n/2}.$$

(b) Prove that, for any smooth function f on M,

$$|\nabla f|_{\mathbf{g}}^2 \leq C\, |\nabla f|_{\widetilde{\mathbf{g}}}^2 .$$

Hint. Fix $x_0 \in M$ and consider $T_{x_0}M$ as a Euclidean space with the inner product $\mathbf{g}$. Since $\widetilde{\mathbf{g}}$ is a symmetric bilinear form in this space, there exists a $\mathbf{g}$-orthonormal basis $\{e_1, ..., e_n\}$ in $T_{x_0}M$ in which $\widetilde{\mathbf{g}}$ has a diagonal form, that is, $(\widetilde{g_{ij}}) = \operatorname{diag}\{\alpha_1, ..., \alpha_n\}$ with some reals α_i. By a linear change of coordinates in a neighborhood of x_0, you can assume that $\frac{\partial}{\partial x^i} = e_i$. For (a) note also that, by Exercise 5, the ratio $\frac{\det \widetilde{g}(x_0)}{\det g(x_0)}$ does not depend on the choice of local coordinates.

Solution. (a) Since in any chart

$$d\nu = \sqrt{\det g} d\lambda, \tag{57}$$

where λ is the Lebesgue measure in this chart, it suffices to verify that, for any $x_0 \in M$,

$$\frac{\det \widetilde{g}\,(x_0)}{\det g\,(x_0)} \leq C^n. \tag{58}$$

By Exercise 5, this ratio does not depend on the choice of the local coordinates. Fix a point $x_0 \in M$ and choose an orthonormal basis $e = \{e_1, ..., e_n\}$ in $T_{x_0}M$ with respect to the inner product $\langle , \rangle_{\mathbf{g}}$ where the quadratic form $\widetilde{\mathbf{g}}\,(x)$ is diagonal, say $\widetilde{g}_{ii} = \alpha_i$ and $\widetilde{g}_{ij} = 0$ if $i \neq j$. By linear change of coordinates in a neighborhood of x_0 we can always assume that $\frac{\partial}{\partial x^i} = e_i$.

Then we have in the basis $\left\{\frac{\partial}{\partial x^i}\right\}$

$$\det g\,(x_0) = 1 \text{ and } \quad \det \widetilde{g}\,(x_0) = \lambda_1 ... \lambda_n.$$

On the other hand, by (48)

$$\lambda_i = \widetilde{g}_{ii} = \langle e_i, e_i \rangle_{\widetilde{\mathbf{g}}} \leq C \langle e_i, e_i \rangle_{\mathbf{g}} = C g_{ii} = C,$$

whence

$$\det \widetilde{g}\,(x_0) \leq C^n = C^n \det g\,(x_0)\,,$$

which proves (50).

(b) It follows from (47) that

$$\widetilde{\mathbf{g}}^{-1} \geq C^{-1} \mathbf{g}^{-1},$$

where $\mathbf{g}^{-1}$ is the metric tensor on covectors, whose matrix in the local coordinates is (g^{ij}). Indeed, in the basis $\left\{\frac{\partial}{\partial x^i}\right\}$ as in part (a), the matrix of $\mathbf{g}^{-1}$ is the identity matrix, while that of $\widetilde{\mathbf{g}}^{-1}$ is the diagonal matrix with the diagonal entries $\alpha_i^{-1} \geq C^{-1}$, whence the claim follows. Using the identity

$$\langle \nabla f, \nabla h \rangle_{\mathbf{g}} = \langle df, dh \rangle_{\mathbf{g}^{-1}}, \tag{59}$$

we obtain

$$|\nabla f|_{\mathbf{g}}^2 = \langle df, df \rangle_{\mathbf{g}^{-1}} \leq C \langle df, df \rangle_{\widetilde{\mathbf{g}}^{-1}} = C\, |\nabla f|_{\widetilde{\mathbf{g}}}^2 ,$$

which finishes the proof.

59. * Consider two Riemannian manifolds $(X, \mathbf{g}_X)$ and $(Y, \mathbf{g}_Y)$. Let us define a Riemannian metric tensor $\mathbf{g}$ on the product manifold $M = X \times Y$ as follows

$$\mathbf{g} = \mathbf{g}_X + \psi^2(x)\,\mathbf{g}_Y, \tag{60}$$

where ψ is a smooth positive function on X. The Riemannian manifold $(M, \mathbf{g})$ with this metric is called a *warped product* of $(X, \mathbf{g}_X)$ and $(Y, \mathbf{g}_Y)$ with profile ψ.

(a) Prove that the Riemannian measure $\nu_{\mathbf{g}}$ of the metric (51) is given by

$$d\nu_{\mathbf{g}} = \psi^m(x)\,d\nu_X d\nu_Y, \tag{61}$$

where ν_X and ν_Y are the Riemannian measures of $(X, \mathbf{g}_X)$ and $(Y, \mathbf{g}_Y)$, respectively, and $m = \dim Y$.

(b) Prove that the Laplace-Beltrami operator $\Delta_{\mathbf{g}}$ of the metric (51) is given by

$$\Delta_{\mathbf{g}} f = \Delta_X f + m\langle \nabla_X \ln\psi, \nabla_X f\rangle_{\mathbf{g}_X} + \frac{1}{\psi^2(x)}\Delta_Y f, \tag{62}$$

where ∇_X is gradient on X and Δ_X, Δ_Y are the Laplace-Beltrami operators on X and Y, respectively.

Solution. Fix a chart U on X with coordinates $x^1, ..., x^n$ and a chart V on Y with coordinates $y^1, ..., y^m$. Then $U \times V$ is the chart on M with coordinates $x^1, ..., x^n, y^1, ..., y^m$. The matrix of the metric $\mathbf{g}$ in this chart has the form

$$g(x,y) = \begin{pmatrix} \boxed{g_X(x)} & 0 \\ 0 & \psi^2(x)\boxed{g_Y(y)} \end{pmatrix} \tag{63}$$

where g_X and g_Y are the matrices of $\mathbf{g}_X$ and $\mathbf{g}_Y$ in U and V, respectively.

(a) It follows from (52) that

$$\det g(x,y) = \psi^{2m}(x)\det g_X(x)\det g_Y(y).$$

Denoting by dx and dy the Lebesgue measures on U and V, respectively, we obtain

$$d\nu_{\mathbf{g}} = \sqrt{\det g}\,dxdy = \psi^m(x)\sqrt{\det g_X(x)}dx\sqrt{\det g_Y(y)}dy = \psi^m(x)\,d\nu_X d\nu_Y.$$

(b) It also follows from (52) that

$$g^{-1} = \begin{pmatrix} \boxed{g_X^{-1}} & 0 \\ 0 & \psi^{-2}(x)\boxed{g_Y^{-1}} \end{pmatrix}. \tag{64}$$

By definition, we have the following the formula for the gradient

$$(\nabla_{\mathbf{g}} u)^i = g^{ij}\frac{\partial u}{\partial x^j} \tag{65}$$

that is

$$[\nabla_g u] = g^{-1}\left[\frac{\partial u}{\partial x^j}\right],$$

where $[\cdot]$ denotes a column-vector and the right hand side is the product of the matrix g^{-1} with the column-vector.

Using (??), we see that the gradient $\nabla_{\mathbf{g}}$ for the metric (51) is given by the column-vector

$$[\nabla_{\mathbf{g}} u] = \begin{bmatrix} \nabla_X u \\ \psi^{-2}\nabla_Y u \end{bmatrix}.$$

Consider a vector field $v = \begin{bmatrix} v_X \\ v_Y \end{bmatrix}$ on M. By the definition of divergence

$$\operatorname{div} v = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(\sqrt{\det g}\, v^i\right), \tag{66}$$

we obtain the following formula for the divergence div on M

$$\begin{aligned}\operatorname{div}\ v &= \frac{1}{\psi^m(x)\sqrt{\det g_Y}}\operatorname{div}_X\left(\psi^m(x)\sqrt{\det g_Y}\, v_X\right)\\ &+ \frac{1}{\psi^m(x)\sqrt{\det g_X}}\operatorname{div}_Y\left(\psi^m(x)\sqrt{\det g_X}\, v_Y\right)\\ &= \operatorname{div}_X v_X + \frac{1}{\psi^m}\langle\nabla_X\psi^m, v_X\rangle + \operatorname{div}_Y v_Y.\end{aligned}$$

Finally, applying this to $v = \nabla_{\mathbf{g}} u$ we obtain

$$\begin{aligned}\Delta u = \operatorname{div}\nabla u &= \operatorname{div}_X(\nabla_X u) + \frac{1}{\psi^m(x)}\langle\nabla_X\psi^m, \nabla_X u\rangle + \operatorname{div}_Y(\psi^{-2}(x)\nabla_Y u)\\ &= \Delta_X u + \langle\nabla_X \ln\psi^m, \nabla_X u\rangle + \frac{1}{\psi^2(x)}\Delta_Y u.\end{aligned}$$

60. ** Let X, Y be smooth manifolds of the same dimension n and let $\Phi : Y \to X$ be a diffeomorphism. Let S be a submanifold of Y, and set $R = \Phi(S)$.

    (a) Prove that R is a submanifold of X and that $\Psi := \Phi|_S$ is a diffeomorphism of S onto R.

    (b) Prove that, for any $y \in S$,
    $$d\Phi|_{T_yS} = d\Psi$$
    (that is, for any $\xi \in T_yS$, we have $d\Phi\xi = d\Psi\xi$).

    (c) Let $\mathbf{g}(x)$ be a bilinear form on any space T_xX (for example, a Riemannian metric) and consider the induced form $\mathbf{g}_R := \mathbf{g}|_R$. Prove that
    $$(\Phi_*\mathbf{g})_S = \Psi_*(\mathbf{g}_R).$$

    (d) Prove that if X and Y are Riemannian manifolds and Φ is a Riemannian isometry of Y and X then Ψ is a Riemannian isometry of the Riemannian manifolds S and R with the induced metrics.

Solution. (a) Let $\dim S = m$. Let (U, φ) be a chart in a neighborhood of a point $p \in S$ with coordinates $y^1, ..., y^n$ where

$$S \cap U = \left\{ y \in U : y^{m+1} = ... = y^n = 0 \right\}.$$

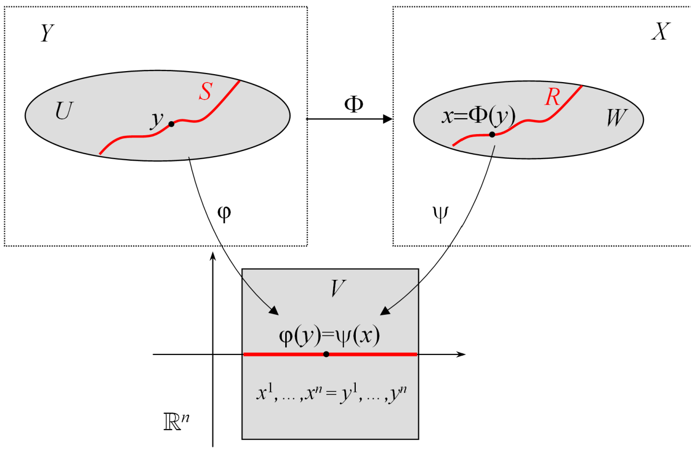
Set $V = \varphi(U) \subset \mathbb{R}^n$, $W = \Phi(U) \subset X$ and consider the mapping

$$\psi := \varphi \circ \Phi^{-1} : W \to V$$

that is a diffeomorphism. Hence, (W, ψ) is a chart in X, denote the coordinates in this chart (that come from V) by $x^1, ..., x^n$.

Any point $y \in U$ has the image $\varphi(y)$ in V, and the point $x = \Phi(y)$ has the image in V

$$\psi(x) = \psi(\Phi(y)) = \varphi(\Phi^{-1}(\Phi(y))) = \varphi(y).$$



Hence, the points $y \in U$ and $x = \Phi(y) \in W$ have the same coordinates from V, whence

$$x^i = y^i \text{ for all } i = 1, ..., n.$$

Since $R \cap W = \Phi(S \cap U)$, it follows that

$$R \cap W = \left\{ x \in W : x^{m+1} = ... = x^n = 0 \right\}$$

so that R is a submanifold of X.

Note that we have constructed the local coordinates $\{x^1, .., .x^m\}$ on R and $\{y^1, ..., y^m\}$ on S.

The mapping $\Psi : S \to R$ is clearly bijective. It is given in the above local coordinates $\{x^1, .., .x^m\}$ on S and $\{y^1, ..., y^m\}$ on R by

$$x^i = y^i, \ i = 1, ..., m,$$

which implies that it is smooth and its inverse is also smooth, so that Ψ is a diffeomorphism.

(b) Fix $y \in S$ and $\xi \in T_y S$. Set $x = \Phi(y) \in R$ and observe that $d\Phi\xi \in T_x X$ and $d\Psi\xi \in T_x R \subset T_x X$. Hence, we consider $d\Phi\xi$ and $d\Psi\xi$ as elements of the same space $T_x X$. For any $f \in C^\infty(Y)$, we have

$$d\Phi\xi(f) = \xi(f \circ \Phi) = \xi(f \circ \Phi|_S) = \xi(f \circ \Psi).$$

Since $f \circ \Psi = f|_S \circ \Psi$, it follows that

$$d\Phi\xi(f) = \xi(f|_S \circ \Psi) = \Psi\xi(f|_S) = \Psi\xi(f),$$

whence $d\Phi\xi = d\Psi\xi$ follows.

(c) By definition, for all $\xi, \eta \in T_y Y$,

$$\Phi_* \mathbf{g}(\xi, \eta) = \mathbf{g}(d\Phi\xi, d\Phi\eta),$$

where $d\Phi\xi$ and $d\Phi\eta \in T_x X$, $x = \Phi(y)$. If $\xi, \eta \in T_y S$ then by (b)

$$\begin{aligned} \mathbf{g}(d\Phi\xi, d\Phi\eta) &= \mathbf{g}(d\Psi\xi, d\Psi\eta) = \mathbf{g}_R(d\Psi\xi, d\Psi\eta) \\ &= \Psi_* \mathbf{g}_R(\xi, \eta), \end{aligned}$$

which proves that

$$\Phi_* \mathbf{g}(\xi, \eta) = \Psi_* \mathbf{g}_R(\xi, \eta) \text{ for all } \xi, \eta \in T_y S,$$

that is,

$$(\Phi_* \mathbf{g})_S = \Psi_*(\mathbf{g}_R),$$

which was to be proved.

(d) Let $\mathbf{g}$ be the Riemannian metric on X and g' – on Y. Since Φ is a Riemannian isometry then

$$\mathbf{g}' = \Phi_* \mathbf{g}.$$

It follows from (a) that

$$\mathbf{g}'_S = (\Phi_* \mathbf{g})_S = \Psi_*(\mathbf{g}_R),$$

that is, Ψ is a Riemannian isometry of $(S, \mathbf{g}'_S)$ and $(R, \mathbf{g}_R)$.

61. ** Fix a point a on a Riemannian manifold $(M, \mathbf{g})$ and consider on M the function $\rho(x) = d(x, a)$. Assume that ρ is finite and smooth in a neighborhood of a point $b \in M \setminus \{a\}$. The purpose of this Exercise is to prove that

$$|\nabla\rho(b)|_{\mathbf{g}} \le 1. \tag{67}$$

(a) Let $\gamma : [0, \varepsilon] \to M$ be a smooth path on M such that $\gamma(0) = b$ and $\dot\gamma(0) = \xi \in T_b M$. Prove that

$$\left. \frac{d}{dt}(\rho(\gamma(t))) \right|_{t=0} \le |\xi|_{\mathbf{g}}. \tag{68}$$

Hint. Use the definition of the geodesic distance d and the triangle inequality.

(b) Prove (53).

Hint. It suffices to prove that, for any $\xi \in T_b M$,

$$\langle \nabla \rho(b), \xi \rangle_{\mathbf{g}} \leq |\xi|_{\mathbf{g}}. \tag{69}$$

Use (54) to prove (55).

Solution. (a) Using the definition of the geodesic distance and the triangle inequality, we obtain, for any $t \in [0, \varepsilon]$,

$$\rho(\gamma(t)) - \rho(\gamma(0)) = d(\gamma(t), a) - d(\gamma(0), a) \leq d(\gamma(t), \gamma(0)) \leq \ell\left(\gamma|_{[0,t]}\right). \tag{70}$$

Since

$$\ell\left(\gamma|_{[0,t]}\right) = \int_0^t |\dot{\gamma}(s)|\, ds,$$

dividing (56) by t and letting $t \to 0$, we obtain (54).

(b) It suffices to prove that

$$\langle \nabla \rho(b), \xi \rangle_{\mathbf{g}} \leq |\xi|_{\mathbf{g}},$$

for any tangent vector $\xi \in T_b M$. By the definition of $\nabla \rho$, we have

$$\langle \nabla \rho(b), \xi \rangle_{\mathbf{g}} = \langle d\rho, \xi \rangle = \xi(\rho),$$

so that we need to prove that

$$\xi(\rho) \leq |\xi|_{\mathbf{g}}.$$

Consider any smooth path $\gamma : [0, \varepsilon] \to M$ for some $\varepsilon > 0$ such that $\gamma(0) = b$ and $\dot{\gamma}(0) = \xi$. Then

$$\xi(\rho) = \frac{\partial \rho}{\partial \xi} = \xi^i \frac{\partial \rho}{\partial x^i} = \frac{d}{dt}(\rho(\gamma(t)))\bigg|_{t=0}.$$

By (a) we conclude that $\xi(\rho) \leq |\xi|_{\mathbf{g}}$, which finishes the proof.

62. ** Consider the Riemannian manifold $(\mathbb{R}^n_+, \mathbf{g})$ where

$$\mathbf{g} = \frac{(dx^1)^2 + ... + (dx^n)^2}{(x^n)^2}.$$

Prove that $(\mathbb{R}^n_+, \mathbf{g})$ is isometric to the hyperbolic space $\mathbb{H}^n$.

Remark. This manifold $(\mathbb{R}^n_+, \mathbf{g})$ is called the *Poincaré half-space model* of the hyperbolic space.

Hint. By Exercise 5f, $\mathbb{H}^n$ is isometric to the *Poincaré ball*, that is, the unit ball

$$\mathbb{B}^n = \{y \in \mathbb{R}^n : |y| < 1\}$$

with the metric

$$\mathbf{g}_{\mathbb{B}^n} = 4\frac{(dy^1)^2 + ... + (dy^n)^2}{\left(1 - |y|^2\right)^2}.$$

Set $p=(0,...,0,1)\in\mathbb{R}^n$ and consider the mapping $\Phi:\mathbb{R}^n\setminus\{-p\}\to\mathbb{R}^n$ given by

$$\Phi(y)=\frac{2(y+p)}{|y+p|^2}-p$$

(in fact, Φ is the inversion in the sphere of radius 2 centered at $-p$). Prove that Φ is a diffeomorphism of $\mathbb{R}^n\setminus\{-p\}$ onto itself, and that $y\in\mathbb{B}^n\Leftrightarrow x=\Phi(y)\in\mathbb{R}^n_+$. Conclude that Φ is a diffeomorphism of $\mathbb{B}^n$ onto $\mathbb{R}^n_+$. Then prove that Φ is isometry, that is, $\Phi_*\mathbf{g}=\mathbf{g}_{\mathbb{B}^n}$.

Solution. The mapping

$$x=\Phi(y)=\frac{2(y+p)}{|y+p|^2}-p. \tag{71}$$

is invertible in $\mathbb{R}^n\setminus\{-p\}$ because it can be solved with respect to y as follows. Firstly, it follows from (57) that

$$x+p=\frac{2(y+p)}{|y+p|^2},$$

in particular, $x\neq -p$, that is, the image of Φ is in $\mathbb{R}^n\setminus\{-p\}$. Next, we have

$$|x+p|=\frac{2}{|y+p|},$$

$$|y+p|=\frac{2}{|x+p|}$$

and

$$y=\frac{1}{2}(x+p)|y+p|^2-p=\frac{2(x+p)}{|x+p|^2}-p.$$

In particular, the inverse Φ^{-1} coincides with Φ. Since $\Phi\in C^\infty$, we obtain that Φ is diffeomorphism of $\mathbb{R}^n\setminus\{-p\}$ onto itself.

Let us show that $x\in\mathbb{R}^n_+$ if and only if $y\in\mathbb{B}^n$, where x and y are related by (57). Indeed, setting $y=(y',y^n)$, where $y'=(y^1,...,y^{n-1})$, we obtain

$$\begin{aligned}x^n&=\frac{2(y^n+1)}{|y+p|^2}-1=\frac{2(y^n+1)}{|y'|^2+(y^n+1)^2}-1\\&=\frac{2(y^n+1)-(y^n+1)^2-|y'|^2}{|y+p|^2}\\&=\frac{1-(y^n)^2-|y'|^2}{|y+p|^2}\\&=\frac{1-|y|^2}{|y+p|^2}\end{aligned}\tag{72}$$

whence we see that $x^n>0$ is equivalent to $|y|<1$ that is, $x\in\mathbb{R}^n_+$ is equivalent to $y\in\mathbb{B}^n$. Hence, Φ is a diffeomorphism of $\mathbb{B}^n$ onto $\mathbb{R}^n_+$. I

In order to prove that $\Phi_*\mathbf{g}=\mathbf{g}_{\mathbb{B}^n}$, we need to compute the Jacobi matrix $J=\left(\frac{\partial\Phi^i}{\partial y^j}\right)$ and then verify that

$$J^TgJ=g_{\mathbb{B}^n}.$$

Note that

$$g_{\mathbb{B}^n} = \frac{4}{\left(1-|y|\right)^2}\,\mathrm{id} \quad \text{and} \quad g = \frac{1}{\left(x^n\right)^2}\,\mathrm{id}\,.$$

Since g is a diagonal matrix, it commutes with any other matrix, and we need to prove that

$$J^T J g = g_{\mathbb{B}^n}\,,$$

that is,

$$\frac{1}{\left(x^n\right)^2} J^T J = \frac{4}{\left(1-|y|^2\right)^2}\,\mathrm{id}$$

Substituting here (**??**), we see that we need to prove that

$$J^T J = \frac{4}{|y+p|^4}\,\mathrm{id}\,.$$

Changing variables $y+p =: z$ and rewriting Φ in the form

$$\Phi\left(z\right) = \frac{2z}{|z|^2} - p,$$

we need to prove that, for $J = \left(\frac{\partial \Phi^i}{\partial z^j}\right)$,

$$J^T J = \frac{4}{|z|^4}\,\mathrm{id}\,.$$

Using that

$$\frac{\partial}{\partial z^j}\,|z|^2 = 2z^j \quad \text{and} \quad \frac{\partial z^i}{\partial z^j} = \delta^i_j,$$

we obtain

$$\frac{\partial \Phi^i}{\partial z^j} = \frac{\partial}{\partial z^j}\frac{2z^i}{|z|^2} = \frac{2\delta^i_j\,|z|^2 - 4z^i z^j}{|z|^4} = \frac{2}{|z|^2}\delta^i_j - \frac{4}{|z|^4} z^i z^j.$$

Denote $Z = \left(z^1, ..., z^n\right)$ and consider Z as an $1\times n$ matrix. Then we have

$$Z^T Z = \begin{pmatrix} z^1 \\ \vdots \\ z^n \end{pmatrix}\left(z^1, ..., z^n\right) = \left(z^i z^j\right)^n_{i,j=1}$$

and, hence,

$$J = \frac{2}{|z|^2}\,\mathrm{id} - \frac{4}{|z|^4} Z^T Z = \frac{2}{|z|^2}\left(\mathrm{id} - \frac{2}{|z|^2} Z^T Z\right).$$

It follows that $J^T = J$ and

$$J^T J = \frac{4}{|z|^4}\left(\mathrm{id} - \frac{4}{|z|^2} Z^T Z + \frac{4}{|z|^4} Z^T Z Z^T Z\right).$$

Noticing that ZZ^T is an 1×1 matrix

$$ZZ^T = \left(z^1, ..., z^n\right)\left(z^1, ..., z^n\right)^T = |z|^2\,\mathrm{id}\,,$$

we obtain

$$J^T J = \frac{4}{|z|^4}\left(\mathrm{id} - \frac{4}{|z|^2} Z^T Z + \frac{4}{|z|^4} Z^T |z|^2 Z\right)$$
$$= \frac{4}{|z|^4}\left(\mathrm{id} - \frac{4}{|z|^2} Z^T Z + \frac{4}{|z|^2} Z^T Z\right)$$

and, hence,

$$J^T J = \frac{4}{|z|^4}\,\mathrm{id},$$

which finishes the proof.

63. ** Fix a real α and consider the mapping $x = Q(y)$ of $\mathbb{R}^{n+1}$ onto itself given by

$$\begin{aligned} x^1 &= y^1 \\ &\vdots \\ x^{n-1} &= y^{n-1} \\ x^n &= y^n \cosh\alpha + y^{n+1}\sinh\alpha \\ x^{n+1} &= y^n \sinh\alpha + y^{n+1}\cosh\alpha. \end{aligned} \tag{73}$$

The mapping Q is called a *hyperbolic rotation* or the *Lorentz transformation*[2].

(a) Prove that Q is an isometry of $\mathbb{R}^{n+1}$ with respect to the Minkowski metric

$$\mathbf{g}_{Mink} = \left(dx^1\right)^2 + ... + \left(dx^n\right)^2 - \left(dx^{n+1}\right)^2.$$

(b) Prove that Q maps $\mathbb{H}^n$ onto itself. Prove that the restriction of Q to $\mathbb{H}^n$ is a Riemannian isometry of $(\mathbb{H}^n, \mathbf{g}_{\mathbb{H}^n})$.

Hint. Recall that the hyperbolic space $\mathbb{H}^n$ is defined as the hyperboloid

$$\left(y^1\right)^2 + ... + \left(y^n\right)^2 - \left(y^{n+1}\right)^2 = -1, \quad y^{n+1} > 0,$$

with the metric tensor $\mathbf{g}_{\mathbb{H}^n} = \mathbf{g}_{Mink}|_{\mathbb{H}^n}$.

Solution. (a) It is obvious that the mapping Q is a linear bijection of $\mathbb{R}^{n+1}$ and, hence, a diffeomorphism. We need to prove that the metric tensor $\mathbf{g} \equiv \mathbf{g}_{Mink}$ is preserved by Q, that is,

$$Q_* \mathbf{g} = \mathbf{g}.$$

[2] Assuming $n = 1$ and denoting $x = x^1$, $t = x^2$, $x' = y^1, t' = y^2$, we obtain from (58)

$$x = \frac{x' + vt'}{\sqrt{1 - v^2}}, \quad t = \frac{t' + vx'}{\sqrt{1 - v^2}}$$

where $v = \tanh\alpha$. These are classical Lorentz transformations in the 2-dimensional space-time that describe in the Relativity Theory the change of coordinates in the inertial frame (x', t') moving at a speed v with respect to the frame (x, t). Note that $v < 1$ where 1 is the speed of light.

Note that the Jacobi matrix $J = \left(\frac{\partial x^j}{\partial y^i}\right)$ of Q is equal to

$$J = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & \cosh\alpha & \sinh a \\ 0 & & & & \sinh\alpha & \cosh\alpha \end{pmatrix}$$

Since

$$Q^*\left(dx^j\right) = \frac{\partial x^j}{\partial y^i} dy^i,$$

we obtain

$$\begin{aligned} Q_*\mathbf{g} = & Q_*\left(\left(dx^1\right)^2 + ... + \left(dx^{n-1}\right)^2 + \left(dx^n\right)^2 - \left(dx^{n+1}\right)^2\right) \\ & = \left(dy^1\right)^2 + ... + \left(dy^{n-1}\right)^2 \\ & + \cosh^2\alpha\left(dy^n\right)^2 + \sinh^2\alpha\left(dy^{n+1}\right)^2 + \cosh\alpha\sinh\alpha\left(dy^n dy^{n+1} + dy^{n+1}dy^n\right) \\ & - \sinh^2\alpha\left(dy^n\right)^2 - \cosh\alpha^2\alpha\left(dy^{n+1}\right)^2 - \cosh\alpha\sinh\alpha\left(dy^n dy^{n+1} + dy^{n+1}dy^n\right) \\ & = \left(dy^1\right)^2 + ... + \left(dy^{n-1}\right)^2 + \left(dy^n\right)^2 - \left(dy^{n+1}\right)^2, \end{aligned}$$

which finishes the proof.

Alternatively, one can directly verify that

$$J^T g J = g,$$

where

$$g = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}$$

is the matrix of $\mathbf{g}$.

(b) Recall that $\mathbb{H}^n$ is a hyperboloid in $\mathbb{R}^{n+1}$, given by the equation

$$\left(y^1\right)^2 + ... + \left(y^n\right)^2 - \left(y^{n+1}\right)^2 = -1, \quad y^{n+1} > 0.$$

Similarly to the computation in (a), we obtain, for x defined by (58),

$$\begin{aligned} & \left(x^1\right)^2 + ... + \left(x^n\right)^2 - \left(x^{n+1}\right)^2 \\ & = \left(y^1\right)^2 + ... + \left(y^{n-1}\right)^2 \\ & + \cosh^2\alpha\left(y^n\right)^2 + \sinh^2\alpha\left(y^{n+1}\right)^2 + \cosh\alpha\sinh\alpha\left(2y^n y^{n+1}\right) \\ & - \sinh^2\alpha\left(y^n\right)^2 - \cosh\alpha^2\alpha\left(y^{n+1}\right)^2 - \cosh\alpha\sinh\alpha\left(2y^n y^{n+1}\right) \\ & = \left(y^1\right)^2 + ... + \left(y^{n-1}\right)^2 + \left(y^n\right)^2 - \left(y^{n+1}\right)^2 \\ & = -1 \end{aligned}$$

Hence, Q maps $\mathbb{H}^n$ into itself, and the same holds for Q^{-1}. Hence, $Q(\mathbb{H}^n) = \mathbb{H}^n$. By the same argument as in Exercise 5(f)v(d), $Q|_{\mathbb{H}^n}$ preserves the induced metric of $\mathbb{H}^n$, that is $\mathbf{g}_{\mathbb{H}^n}$.

64. ** We are concerned here with Riemannian isometries of $\mathbb{H}^n$.

(a) Prove that, for any point $a \in \mathbb{H}^n$, there exists a Riemannian isometry

$$\Phi : \mathbb{H}^n \to \mathbb{H}^n$$

such that $\Phi(a) = p$ where $p = (0, .., 0, 1) \in \mathbb{R}^{n+1}$ is the pole of $\mathbb{H}^n$.

(b) Prove that, for any four points $a, b, a', b' \in \mathbb{H}^n$ such that

$$d(a', b') = d(a, b), \tag{74}$$

there exists a Riemannian isometry Φ of $\mathbb{H}^n$ such that $\Phi(a') = a$ and $\Phi(b') = b$.
Hint. Use the hyperbolic rotation of Exercise 5(f)v.

Solution. (a) By rotation in the subspace

$$\mathbb{R}^n = \{x \in \mathbb{R}^{n+1} : x^{n+1} = 0\}$$

we can assume that the projection of a onto $\mathbb{R}^n$ lies on the axis x^n, that is,

$$a = (0, ..., 0, a^n, a^{n+1}).$$

Since $a \in \mathbb{H}^n$, we have

$$(a^{n+1})^2 - (a^n)^2 = 1.$$

Then there exists real α such that

$$a^{n+1} = \cosh\alpha \text{ and } a^n = -\sinh\alpha.$$

Let Φ be the hyperbolic rotation (58) of Exercise 5(f)v with this parameter α. Then we obtain from (58)

$$\begin{aligned} \Phi(a)^n &= -\sin\alpha\cosh\alpha + \cosh\alpha\sin\alpha = 0 \\ \Phi(a)^{n+1} &= -\sinh\alpha\sin\alpha + \cosh\alpha\cosh\alpha = 1, \end{aligned}$$

and $(\Phi a)^i = 0$ for $i < n$. Hence, $\Phi(a) = p$.

(b) Consider first the case $a = a' = p$. If $b, b' \in \mathbb{H}^n$ are two points such that $d(p, b) = d(p, b')$ then, in the polar coordinates on $\mathbb{H}^n$, the points b and b' have the same polar radius (cf. Exercise 5(f)ivD). Therefore, for a suitable rotation Φ of the polar angle, we obtain $\Phi(b) = b'$, while $\Phi(p) = p$, that is, $\Phi(a) = a'$.

Consider now the general case, when points $a, b, a', b' \in \mathbb{H}^n$ satisfy (59). Let Φ be an isometry of $\mathbb{H}^n$ such that $\Phi(a) = p$ and Φ' be an isometry such that $\Phi'(a') = p$. Then

$$\begin{aligned} d(p, \Phi'(b')) &= d(\Phi'(a'), \Phi(b')) = d(a', b') \\ &= d(a, b) = d(\Phi(p), \Phi(b)) = d(p, \Phi(b)). \end{aligned}$$

Hence, there exists an isometry Ψ such that

$$\Psi\left(\Phi'\left(b'\right)\right)=\Phi\left(b\right)\quad\text{and}\quad\Psi\left(p\right)=p.$$

Since $p=\Phi\left(a\right)=\Phi'\left(a'\right)$, we obtain

$$\Psi\left(\Phi'\left(a'\right)\right)=\Phi\left(a\right).$$

It follows that

$$\Phi^{-1}\Psi\Phi'\left(a'\right)=a$$

and

$$\Phi^{-1}\Psi\Phi'\left(a'\right)=b,$$

so that $\Phi^{-1}\Psi\Phi'$ is the required isometry.

65. ** Consider the weighted manifold $(\mathbb{R},\mathbf{g},\mu)$ where $\mathbf{g}=\mathbf{g}_{\mathbb{R}^n}$ is the canonical Euclidean metric and $d\mu=e^{-x^2}dx$. Consider also the corresponding weighted Laplace operator $\Delta_{\mathbf{g},\mu}$. Prove that the *Hermite polynomial*

$$h_k\left(x\right)=e^{x^2}\frac{d^k}{dx^k}e^{-x^2}$$

of degree k (where k is a non-negative integer) satisfies the equation

$$\Delta_{\mathbf{g},\boldsymbol{\mu}}h_k+2kh_k=0.$$

That is, h_k is an eigenfunction of $\Delta_{\mathbf{g},\mu}$.

Hint. Show first that the function $g\left(x\right)=e^{-x^2}$ satisfies the equation

$$\frac{d^{k+2}}{dx^{k+2}}g+2x\frac{d^{k+1}}{dx^{k+1}}g+\left(2k+2\right)\frac{d^k}{dx^k}g=0. \tag{75}$$

Solution. If $k=0$ then (60) becomes

$$g''+2xg'+2g=0. \tag{76}$$

Indeed, we have

$$\begin{aligned} g' &= -2xe^{-x^2} \\ g'' &= 4x^2e^{-x^2}-2e^{-x^2}, \end{aligned}$$

whence (**??**) follows. If (60) is proved for some k, then differentiating of (60) in x gives

$$\frac{d^{k+3}}{dx^{k+3}}g+2x\frac{d^{k+2}}{dx^{k+2}}g+2\frac{d^{k+1}}{dx^{k+1}}g+\left(2k+2\right)\frac{d^{k+1}}{dx^{k+1}}g=0,$$

that is,

$$\frac{d^{k+3}}{dx^{k+3}}g+2x\frac{d^{k+2}}{dx^{k+2}}g+\left(2k+4\right)\frac{d^{k+1}}{dx^{k+1}}g=0,$$

which finishes the inductive step.

The weighted Laplace operator in given by

$$\Delta_{\mathbf{g},\mu} f = e^{x^2} \frac{d}{dx} \left(e^{-x^2} \frac{d}{dx} f \right).$$

For

$$f = h_k = e^{x^2} \frac{d^k}{dx^k} e^{-x^2} = e^{x^2} \frac{d^k}{dx^k} g$$

we obtain

$$\frac{d}{dx} h_k = e^{x^2} \frac{d^{k+1}}{dx^{k+1}} e^{-x^2} + 2x e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$$

and

$$\begin{aligned}
\Delta_{\mathbf{g},\mu} h_k &= e^{x^2} \frac{d}{dx} \left(e^{-x^2} \frac{d}{dx} h_k \right) \\
&= e^{x^2} \frac{d}{dx} \left(\frac{d^{k+1}}{dx^{k+1}} e^{-x^2} + 2x \frac{d^k}{dx^k} e^{-x^2} \right) \\
&= e^{x^2} \left(\frac{d^{k+2}}{dx^{k+2}} g + 2x \frac{d^{k+1}}{dx^{k+1}} g + 2 \frac{d^k}{dx^k} g \right).
\end{aligned}$$

Using (60) we see that the value of the bracket is

$$-2k \frac{d^k}{dx^k} g = -2k e^{-x^2} h_k,$$

whence it follows that

$$\Delta_{\mathbf{g},\mu} h_k = -2k h_k,$$

which was to be proved.

Blatt 11. Abgabe bis 16.01.2026

Die mit *markierten Aufgaben sind zusätzlich und werden korrigiert
Die mit **markierten Aufgaben sind zusätzlich und werden nicht korrigiert.

In all questions, $(M, \mathbf{g}, \mu)$ is a weighted manifold, $\Delta = \Delta_{\mathbf{g},\boldsymbol{\mu}}$, and Ω is an open subset of M. The quantity $\lambda_1(\Omega)$ is defined in Exercise 5(f)v.

66. Let R_α be the resolvent operator in Ω. Prove that, for any $\alpha > 0$ and $f \in L^2(\Omega)$, the function $u = \alpha R_\alpha f$ is a unique minimizer of the functional

$$E(v) := \|\nabla v\|^2_{\vec{L}^2} + \alpha \|v - f\|^2_{L^2}$$

in the domain $v \in W^1_0(\Omega)$.

Hint. Show that $E(u + \varphi) \geq E(u)$ for any $\varphi \in W^1_0(\Omega)$.

Solution. The function $u = \alpha R_\alpha f = R_\alpha(\alpha f)$ satisfies the identity

$$(\nabla u, \nabla \varphi)_{\vec{L}^2} + \alpha (u, \varphi)_{L^2} = (\alpha f, \varphi)_{L^2}$$

for all $\varphi \in W^1_0$. Hence, for any $\varphi \in W^1_0$, we have (all norms are in $L^2(\Omega)$):

$$\begin{aligned} E(u+\varphi) &= \|\nabla(u+\varphi)\|^2 + \alpha\|u+\varphi-f\|^2 \\ &= \|\nabla u\|^2 + 2(\nabla u, \nabla\varphi) + \|\nabla\varphi\|^2 + \alpha\|u-f\|^2 + 2\alpha(u-f,\varphi) + \alpha\|\varphi\|^2 \\ &= \left(\|\nabla u\|^2 + \alpha\|u-f\|^2\right) + \|\nabla\varphi\|^2 + \alpha\left\|\varphi^2\right\| + 2\left((\nabla u,\nabla\varphi) + \alpha(u,\varphi) - (\alpha f,\varphi)\right) \\ &= E(u) + \|\nabla\varphi\|^2 + \alpha\|\varphi\|^2. \end{aligned}$$

It follows that $E(u+\varphi) > E(u)$ unless $\varphi \equiv 0$, which was to be proved.

67. For any open subset $\Omega \subset M$, define $\lambda_1(\Omega)$ by

$$\lambda_1(\Omega) := \inf_{f \in \mathcal{D}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla f|^2 d\mu}{\int_\Omega f^2 d\mu}. \tag{77}$$

Prove the following properties of $\lambda_1(\Omega)$.

(a) If $\Omega_1 \subset \Omega_2$ are two open sets then

$$\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2).$$

(b) If $\{\Omega_k\}_{k=1}^\infty$ is an increasing sequence of open sets (that is, $\Omega_k \subset \Omega_{k+1}$) and $\Omega = \bigcup_k \Omega_k$ then

$$\lambda_1(\Omega) = \lim_{k\to\infty} \lambda_1(\Omega_k) = \inf_k \lambda_1(\Omega_k).$$

Remark. For any non-zero function $f \in \mathcal{D}(M)$, define its *Rayleigh quotient* by

$$\mathcal{R}(f) := \frac{\|\nabla f\|^2_{\vec{L}^2}}{\|f\|^2_{L^2}}.$$

Then (61) can be rewritten in the form

$$\lambda_1(\Omega) := \inf_{f \in \mathcal{D}(\Omega) \setminus \{0\}} \mathcal{R}(f).$$

Solution. (a) Since

$$\lambda_1(\Omega) = \inf_{f \in \mathcal{D}(\Omega) \setminus \{0\}} \mathcal{R}(f)$$

and $\mathcal{D}(\Omega_1) \subset \mathcal{D}(\Omega_2)$, we obtain $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$.

(b) By part (a), the sequence $\{\lambda_1(\Omega_k)\}$ decreases and

$$\lambda_1(\Omega_k) \geq \lambda_1(\Omega).$$

It follows that

$$\lim_{k\to\infty} \lambda_1(\Omega_k) = \inf_k \lambda_1(\Omega_k) \geq \lambda_1(\Omega). \tag{78}$$

To prove the opposite inequality, observe that, for any $f \in \mathcal{D}(\Omega)$, the support of f is covered by $\{\Omega_k\}$. Hence, by the compactness of $\operatorname{supp} f$, there is a finite sequence $\Omega_{k_1}, ..., \Omega_{k_m}$ that covers $\operatorname{supp} f$. Assuming that $k_1 < k_2 < ... < k_m$ and using the monotonicity of $\{\Omega_k\}$, we see that Ω_{k_m} covers $\operatorname{supp} f$. Hence, $f \in \mathcal{D}(\Omega_{k_m})$ and, therefore,

$$\lambda_1(\Omega_{k_m}) \leq \mathcal{R}(f).$$

It follows that

$$\inf_k \lambda_1(\Omega_k) \leq \mathcal{R}(f).$$

Taking infimum over $f \in \mathcal{D}(\Omega)$, we obtain

$$\inf_k \lambda_1(\Omega_k) \leq \lambda_1(\Omega),$$

which together with (62) proves the claim.

68. Assume that $\lambda_1(\Omega) > 0$.

(a) Prove that the weak Dirichlet problem in Ω

$$\begin{cases} \Delta u = -f & \text{weakly in } \Omega, \\ u \in W_0^1(\Omega), \end{cases} \tag{79}$$

has exactly one solution u for any $f \in L^2(\Omega)$.

Hint. Set $[u, v] := (\nabla u, \nabla v)_{\vec{L}^2}$ for all $u, v \in W_0^1(\Omega)$ and prove that $[\cdot, \cdot]$ is an inner product in $W_0^1(\Omega)$. For that, use the hypothesis $\lambda_1(\Omega) > 0$.

(b) Prove that, for the solution u of (63),

$$\|u\|_{L^2} \leq \lambda_1(\Omega)^{-1} \|f\|_{L^2} \tag{80}$$

and

$$\|\nabla u\|_{\vec{L}^2} \leq \lambda_1(\Omega)^{-1/2} \|f\|_{L^2}. \tag{81}$$

Solution. (a) By definition of $\lambda_1 = \lambda_1(\Omega)$, we have, for any $f \in W_0^1(\Omega)$

$$\|\nabla f\|_{\vec{L}^2}^2 \geq \lambda_1 \|f\|_{L^2}^2 . \tag{82}$$

It follows that

$$\|\nabla f\|_{\vec{L}^2}^2 \simeq \|\nabla f\|_{\vec{L}^2}^2 + \|f\|_{L^2}^2 = \|f\|_{W^1}^2 ,$$

which implies that

$$[u, v] := (\nabla u, \nabla v)_{\vec{L}^2}$$

is an inner product in $W_0^1(\Omega)$, and its norm is equivalent to the standard norm in $W_0^1(\Omega)$. Hence, $W_0^1(\Omega)$ with the inner product $[\cdot, \cdot]$ is a Hilbert space.

The weak Dirichlet problem has the following formulation:

$$\begin{cases} (\nabla u, \nabla \varphi)_{\vec{L}^2} = (f, \varphi)_{L^2} \quad \forall \varphi \in W_0^1(\Omega) \\ u \in W_0^1(\Omega) \end{cases} \tag{83}$$

which can be rewritten in the form

$$[u, v] = l(\varphi) \quad \forall \varphi \in W_0^1(\Omega)$$

where $l(\varphi) = (f, \varphi)_{L^2}$ is a continuous linear functional in $W_0^1(\Omega)$. Hence, the above problem has a unique solution by the Riesz representation theorem.

(b) Substituting into (65) $\varphi = u$, we obtain

$$\|\nabla u\|_{\vec{L}^2}^2 = (f, u)_{L^2} \leq \|f\|_{L^2} \|u\|_{L^2} . \tag{84}$$

Since by (64)

$$\|\nabla u\|_{\vec{L}^2}^2 \geq \lambda_1 \|u\|_{L^2}^2 ,$$

it follows that

$$\lambda_1 \|u\|_{L^2}^2 \leq \|f\|_{L^2} \|u\|_{L^2} ,$$

whence

$$\|u\|_{L^2} \leq \lambda_1^{-1} \|f\|_{L^2} .$$

Substituting into (**??**), we obtain

$$\|\nabla u\|_{\vec{L}^2}^2 \leq \lambda_1^{-1} \|f\|_{L^2}^2$$

and

$$\|\nabla u\|_{\vec{L}^2} \leq \lambda_1^{-1/2} \|f\|_{L^2}$$

69. Consider the following version of the weak Dirichlet problem in Ω: given a real constant α and functions $f \in L^2(\Omega)$, $g \in W^1(\Omega)$, find a function $u \in W^1(\Omega)$ such that

$$\begin{cases} \Delta u - \alpha u = -f \quad \text{weakly in } \Omega, \\ u - g \in W_0^1(\Omega) . \end{cases} \tag{85}$$

Prove that if $\alpha > -\lambda_1(\Omega)$ then the problem (66) has exactly one solution.

Solution. The weak equation $\Delta u - \alpha u = -f$ means that

$$(\nabla u, \nabla \varphi)_{\vec{L}^2} + \alpha (u, \varphi)_{L^2} = (f, \varphi)_{L^2} \quad \text{for any } \varphi \in W_0^1(\Omega).$$

Setting $v = u - g$ and replacing u in the above equation by $u = v + g$, we obtain the following equation for $v \in W_0^1$:

$$(\nabla v, \nabla \varphi) + \alpha (v, \varphi) = -(\nabla g, \nabla \varphi) + (f - \alpha g, \varphi) \quad \text{for any } \varphi \in W_0^1, \tag{86}$$

where the brackets mean the inner product in L^2 or $\vec{L}^2$.

Let us show that the bilinear form

$$[v, \varphi]_\alpha := (\nabla v, \nabla \varphi) + \alpha (v, \varphi)$$

defines an inner product in W_0^1, which is equivalent to the standard inner product $[v, \varphi]_1$. If $\alpha > 0$ then this is trivial and was already in lectures. We need to prove the same in under the weaker hypothesis $\alpha > -\lambda_1$.

It suffices to show that

$$[\varphi, \varphi]_\alpha \geq \varepsilon [\varphi, \varphi]_1 \tag{87}$$

for some $\varepsilon \in (0, 1)$ and all $\varphi \in W_0^1$, which is equivalent to

$$\|\nabla \varphi\|_{L^2}^2 \geq \frac{\varepsilon + \alpha}{1 - \varepsilon} \|\varphi\|_{L^2}^2. \tag{88}$$

By Exercise 5(f)v we have

$$\|\nabla \varphi\|_{L^2}^2 \geq \lambda_1 \|\varphi\|_{L^2}^2.$$

Hence, define ε from the equation

$$\frac{\varepsilon + \alpha}{1 - \varepsilon} = \lambda_1,$$

which yields

$$\varepsilon = \frac{\lambda_1 + \alpha}{1 + \lambda_1} > 0.$$

Hence, the Riesz representation theorem yields that (67) has a unique solution $v \in W_0^1$ because the right hand side of (67) is a bounded linear functional of φ in W^1.

70. * Let $f \in L^2(\Omega)$ and assume that u is a solution of the following weak Dirichlet problem:

$$\begin{cases} \Delta u = -f & \text{weakly in } \Omega, \\ u \in W_0^1(\Omega). \end{cases}$$

Prove that

$$\|u\|_{W^1}^2 \leq c \left(\|u\|_{L^2}^2 + \|f\|_{L^2}^2 \right), \tag{89}$$

where $c = \frac{1+\sqrt{2}}{2}$.

Solution. By the definition of a weak solution of $\Delta u = -f$, we have, for any $v \in W_0^1(\Omega)$,

$$(\nabla u, \nabla v)_{\vec{L}^2} = (f, v)_{L^2}$$

Setting here $v = u$, we obtain

$$\|\nabla u\|_{\vec{L}^2}^2 = (\nabla u, \nabla u)_{\vec{L}^2} = (f, u)_{L^2}.$$

Using the inequality

$$ab \leq \frac{s}{2}a^2 + \frac{1}{2s}b^2,$$

which holds for all real a, b and $s > 0$, we obtain

$$\begin{aligned}(f, u)_{L^2} &= \int_\Omega f u d\mu \leq \int_\Omega \left(\frac{s}{2}u^2 + \frac{1}{2s}f^2\right) d\mu \\ &= \frac{s}{2}\|u\|_{L^2}^2 + \frac{1}{2s}\|f\|_{L^2}^2.\end{aligned}$$

Therefore,

$$\|u\|_{W^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{\vec{L}^2}^2 \leq \left(1 + \frac{s}{2}\right)\|u\|_{L^2}^2 + \frac{1}{2s}\|f\|_{L^2}^2.$$

Therefore, (70) holds with

$$c = \max\left(1 + \frac{s}{2}, \frac{1}{2s}\right).$$

The minimum value of c is attained if

$$1 + \frac{s}{2} = \frac{1}{2}s^{-1},$$

which leads to $s = \sqrt{2} - 1$ and $c = \frac{1+\sqrt{2}}{2}$.

71. * (*Cheeger's inequality*) The *Cheeger constant* of Ω is defined by

$$h(\Omega) := \inf_{\varphi \in \mathcal{D}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla \varphi| \, d\mu}{\int_\Omega |\varphi| \, d\mu}. \tag{90}$$

Prove that

$$\lambda_1(\Omega) \geq \frac{1}{4}h^2(\Omega).$$

Hint. Substitute in the right hand side of (71) φ^2 in place of φ and use the definition of $\lambda_1(\Omega)$.

Solution. By (71) we have, for any $\varphi \in \mathcal{D}(\Omega)$, that

$$\int_\Omega |\nabla \varphi|_{\mathbf{g}} \, d\mu \geq h \int_\Omega |\varphi| \, d\mu.$$

Let us apply this inequality to φ^2 in place of φ. Since $\nabla \varphi^2 = 2\varphi \nabla \varphi$, we obtain

$$\int_\Omega 2|\varphi| |\nabla \varphi|_{\mathbf{g}} d\mu \geq h \int_\Omega \varphi^2 d\mu.$$

Since

$$\int_\Omega |\varphi| |\nabla \varphi|_{\mathbf{g}} d\mu \leq \left(\int_\Omega \varphi^2 d\mu\right)^{1/2} \left(\int_\Omega |\nabla \varphi|_{\mathbf{g}}^2 d\mu\right)^{1/2},$$

it follows that

$$\int_\Omega |\nabla\varphi|_{\mathbf{g}}^2 \, d\mu \geq \frac{h^2}{4}\int_\Omega \varphi^2 d\mu.$$

By definition of $\lambda_1(\Omega)$, we conclude that

$$\lambda_1(\Omega) = \inf_{\varphi\in\mathcal{D}(\Omega)\setminus\{0\}} \frac{\|\nabla\varphi\|_{\vec{L}^2}^2}{\|\varphi\|_{L^2}^2} = \inf_{\varphi\in\mathcal{D}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla\varphi|_{\mathbf{g}}^2 \, d\mu}{\int_\Omega \varphi^2 d\mu} \geq \frac{h^2}{4}.$$

72. ** Let d be the geodesic distance on a connected Riemannian manifold $(M, \mathbf{g})$. A function $f : M \to \mathbb{R}$ is called *Lipschitz* if there exists a constant L such that

$$|f(x) - f(y)| \leq L d(x, y) \quad \text{for all } x, y \in M.$$

The number L is called the *Lipschitz constant* of f. Prove that if f is Lipschitz with the Lipschitz constant L then the weak gradient ∇f exists and

$$\|\nabla f\|_{\vec{L}^\infty} \leq L. \tag{91}$$

Hint. In $\mathbb{R}^n$ this statement can be taken as known. In order to reduce the general case to that in $\mathbb{R}^n$, prove first the following claim: for any point $p \in M$ and for any $C > 1$, there exists a chart $U \ni p$ such that, for all $x \in U$ and $\xi \in T_x M$,

$$C^{-2}\left(\left(\xi^1\right)^2 + ... + (\xi^n)^2\right) \leq g_{ij}(x)\,\xi^i\xi^j \leq C^2\left(\left(\xi^1\right)^2 + ... + (\xi^n)^2\right).$$

This inequality was proved in lectures, however, with *some* constant C. Show that the constant C can be chosen arbitrarily close to 1.

Solution. Let us first prove the following claim.

Claim. For any point $p \in M$ and for any $C > 1$, there exists a chart $U \ni p$ such that for all $x \in U$, $\xi \in T_x M$

$$C^{-2}\left(\left(\xi^1\right)^2 + ... + (\xi^n)^2\right) \leq g_{ij}(x)\,\xi^i\xi^j \leq C^2\left(\left(\xi^1\right)^2 + ... + (\xi^n)^2\right). \tag{92}$$

Let V be any chart containing p, with coordinates $x^1, ..., x^n$. Let g^x be the matrix of the metric tensor $\mathbf{g}$ in the coordinates $x^1, ..., x^n$ and g^y be the matrix of $\mathbf{g}$ in another coordinate system $y^1, ..., y^n$ in U (yet to be defined).We know that

$$g^y = J^T g^x J,$$

where $J = \left(\frac{\partial x^i}{\partial y^j}\right)$ is the Jacobi matrix. It is well known from linear algebra that any quadratic form can be brought to a diagonal form by a linear change of the variables. The quadratic form $\xi \mapsto g_{ij}^x(p)\,\xi^i\xi^j$ is positive definite and, hence, can be transform to the diagonal form $(\eta^1)^2 + ... + (\eta^n)^2$ by a linear change $\xi^i = A_j^i \eta^j$, where A is a numerical non-singular matrix. This implies that

$$A^T g^x(p)\, A = \mathrm{id}\,.$$

Defining the new coordinates y^j by the linear equations $x^i = A^i_j y^j$, we obtain that $J(p) = A$ and, hence, $g^y(p) = \text{id}$. By continuity, the matrix g^y is close enough to id in a small enough neighborhood U of p. More precisely, by choosing U small enough, we can ensure that the matrix g^y_{ij} satisfies the conditions (73). We are left to rename $y^1, ..., y^n$ back to $x^1, ..., x^n$.

Note that the conditions (73) mean that all the eigenvalues of the matrix $g_{ij}(x)$ are located in the interval $[C^{-2}, C^2]$. Then the same is true for the eigenvalues of the inverse matrix $g^{ij}(x)$. Hence, the inequalities (73) hold also for g^{ij} in place of g_{ij}.

Hence, let U be a chart as in the above Claim. By shrinking further U, we can assume that U is a ball in the coordinates $x^1, ..., x^n$ centered at p. Then, for any two points $x, y \in U$, the straight line segment between x, y is also contained in U. By (73), the Riemannian length of this segment is bounded by $C|x-y|$, which implies that

$$d(x,y) \le C|x-y|. \tag{93}$$

Let now f be a Lipschitz function on M with the Lipschitz constant L. In a chart U as above, we have

$$|f(x) - f(y)| \le Ld(x,y) \le CL|x-y|,$$

so that f is Lipschitz with a Lipschitz constant CL in the Euclidean metric in U. Hence, we conclude that f has the weak gradient $\nabla_{\mathbf{e}} f = \left(\frac{\partial f}{\partial x^1}, ..., \frac{\partial f}{\partial x^n}\right)$ in the Euclidean metric $\mathbf{e} = \mathbf{g}_{\mathbb{R}^n}$ and

$$|\nabla_{\mathbf{e}} f| \le CL \text{ a.e..} \tag{94}$$

The Riemannian weak gradient $\nabla_{\mathbf{g}} f$ is given then by

$$(\nabla_{\mathbf{g}} f)^k = g^{ki} \frac{\partial f}{\partial x^i},$$

and

$$|\nabla_{\mathbf{g}} f|^2 = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}.$$

Using the above Claim for g^{ij} we obtain

$$|\nabla_{\mathbf{g}} f|^2 \le C^2 \sum_{i=1}^n \left(\frac{\partial f}{\partial x^i}\right)^2 \le C^2 (CL)^2 \text{ a.e.}$$

that is, in U,

$$|\nabla_{\mathbf{g}} f| \le C^2 L \text{ a.e..} \tag{95}$$

Since M can be covered by a countable family of such charts U, (**??**) holds also in M. Finally, since $C > 1$ was arbitrary, we obtain $|\nabla_{\mathbf{g}} f| \le L$ a.e., which finishes the proof.

Blatt 12. Abgabe bis 23.01.2026

Die mit *markierten Aufgaben sind zusätzlich und werden korrigiert
Die mit **markierten Aufgaben sind zusätzlich und werden nicht korrigiert.

In all questions, $(M, \mathbf{g}, \mu)$ is a weighted manifold, Ω is a precompact open subset of M, $\{\lambda_k\}_{k=1}^{\infty}$ is the sequence of the Dirichlet eigenvalues of $\Delta = \Delta_{\mathbf{g},\mu}$ in Ω in the increasing order, and $\{v_k\}$ is the sequence of the corresponding eigenfunctions that forms an orthonormal basis in $L^2(\Omega)$.

73. Recall that any function $u \in L^2(\Omega)$ admits an eigenfunction expansion

$$u = \sum_{k=1}^{\infty} a_k v_k \ , \tag{96}$$

where $a_k \in \mathbb{R}$ and the series converges in $L^2(\Omega)$.

(a) Prove that if $u \in W_0^1(\Omega)$ then the series (74) converges also in $W^1(\Omega)$ and

$$\|u\|_{W^1}^2 = \sum_{k=1}^{\infty} (\lambda_k + 1)\, a_k^2. \tag{97}$$

Hint. Use the Parseval identity in $L^2(\Omega)$ and in $W_0^1(\Omega)$.

(b) Prove that if $u \in W_0^1(\Omega)$ and $\Delta u \in L^2(\Omega)$ then

$$\Delta u = -\sum_{k=1}^{\infty} \lambda_k a_k v_k, \tag{98}$$

where the series converges in $L^2(\Omega)$.

Solution. (a) For any $u \in L^2(\Omega)$ we have an expansion

$$u = \sum_{k=1}^{\infty} a_k v_k \tag{99}$$

where $a_k = (u, v_k)$ and the series converges in $L^2(\Omega)$. We know that $\{v_k\}$ is also an orthogonal basis in $W_0^1(\Omega)$. If $u \in W_0^1(\Omega)$ then u can be expanded as

$$u = \sum_{k=1}^{\infty} b_k v_k$$

where the series converges in $W_0^1(\Omega)$. In particular, this series converges in $L^2(\Omega)$, which implies that $b_k = a_k$. Hence, if $u \in W_0^1(\Omega)$ then the series (76) converges also in $W_0^1(\Omega)$. It follows by the Parseval identity in L^2

$$\|u\|_{L^2}^2 = \sum_{k=1}^{\infty} a_k^2 \|v_k\|_{L^2}^2 = \sum_{k=1}^{\infty} a_k^2$$

and in W_0^1:

$$\|u\|_{W^1}^2 = \sum_{k=1}^{\infty} a_k^2 \|v_k\|_{W^1}^2 .$$

Since

$$\|v_k\|_{W^1}^2 = \|\nabla v_k\|_{\vec{L}^2}^2 + \|v_k\|_{L^2}^2 = \lambda_k + 1,$$

we obtain

$$\|u\|_{W^1}^2 = \sum_{k=1}^{\infty} (\lambda_k + 1) a_k^2.$$

(b) Let $f = -\Delta u \in L^2(\Omega)$. Then f allows an L^2 expansion

$$f = \sum_{k=1}^{\infty} b_k v_k$$

where

$$b_k = (f, v_k)_{L^2} = (\nabla u, \nabla v_k)_{\vec{L}^2} ,$$

where we have used the definition of the weak Δ and $v_k \in W_0^1(\Omega)$. Since v_k is an eigenfunction and $u \in W_0^1(\Omega)$, we have

$$(\nabla v_k, \nabla u)_{\vec{L}^2} = \lambda_k (v_k, u)_{L^2} = \lambda_k a_k$$

which implies that

$$b_k = \lambda a_k,$$

which finishes the proof.

74. *(Variational property of the bottom eigenvalue)* For any non-zero function $u \in W^1(\Omega)$, consider its *Rayleigh quotient*:

$$\mathcal{R}(u) := \frac{\|\nabla u\|_{\vec{L}^2}^2}{\|u\|_{L^2}^2}.$$

Prove that the bottom Dirichlet eigenvalue $\lambda_1(\Omega)$ satisfies the following identity:

$$\lambda_1(\Omega) = \min_{u \in W_0^1(\Omega) \setminus \{0\}} \mathcal{R}(u) = \inf_{u \in \mathcal{D}(\Omega) \setminus \{0\}} \mathcal{R}(u) . \tag{100}$$

Hint. Use Exercise 5(f)v.

Remark. The notation $\lambda_1(\Omega)$ for any open set Ω was defined in Exercise 5(f)v as $\lambda_1(\Omega) = \inf_{u \in \mathcal{D}(\Omega) \setminus \{0\}} \mathcal{R}(u)$. The identity (77) shows that, for precompact Ω, this notation is consistent with the notation $\lambda_1(\Omega)$ for the bottom Dirichlet eigenvalue.

Solution. Using notation of Exercise (5(f)v), we have

$$\|u\|_{L^2}^2 = \sum_{k=1}^{\infty} a_k^2$$

and, by (**??**),

$$\|\nabla u\|_{\vec{L}^2}^2 = \|u\|_{W^1}^2 - \|u\|_{L^2}^2 = \sum_{k=1}^{\infty} \lambda_k a_k^2.$$

Hence, for any $u \in W_0^1(\Omega) \setminus \{0\}$,

$$\mathcal{R}(u) = \frac{\|\nabla u\|_{\vec{L}^2}^2}{\|u\|_{L^2}^2} = \frac{\sum_{k=1}^{\infty} \lambda_k a_k^2}{\sum_{k=1}^{\infty} a_k^2} \geq \lambda_1.$$

On the other hand, for $u = v_1$ we obtain

$$\mathcal{R}(v_1) = \frac{\|\nabla v_1\|_{\vec{L}^2}^2}{\|v_1\|_{L^2}^2} = \lambda_1,$$

which implies that

$$\min_{u \in W_0^1(\Omega) \setminus \{0\}} \mathcal{R}(u) = \lambda_1.$$

Hence, we have proved the first equality in (77).

The second equality follows from the fact that $\mathcal{R}(u)$ is a continuous functional in $W^1(\Omega) \setminus \{0\}$ and that $\mathcal{D}(\Omega)$ is dense in $W_0^1(\Omega)$.

75. Let a function $f \in L^2(\Omega)$ have an eigenfunction expansion

$$f = \sum_{k=1}^{\infty} a_k v_k.$$

(a) Prove that, for any $\alpha > 0$,

$$R_\alpha f = \sum_{k=1}^{\infty} \frac{1}{\alpha + \lambda_k} a_k v_k. \tag{101}$$

Hint. Use Exercise 5(f)v.

(b) Using (78), prove the following *resolvent identity* for all $\alpha, \beta > 0$:

$$R_\alpha - R_\beta = (\beta - \alpha) R_\alpha R_\beta. \tag{102}$$

Solution. (*a*) For the function $u = R_\alpha f$ we know that $u \in W_0^1(\Omega)$ and

$$\Delta u = \alpha u - f \in L^2(\Omega). \tag{103}$$

Let

$$u = \sum_{k=1}^{\infty} b_k v_k$$

where the series converges in $L^2(\Omega)$. By Exercise 5(f)v, we have

$$\Delta u = -\sum_{k=1}^{\infty} \lambda_k b_k v_k.$$

Substituting into (79), we obtain

$$-\sum_{k=1}^{\infty}\lambda_k b_k v_k=\alpha\sum_{k=1}^{\infty}b_k v_k-\sum_{k=1}^{\infty}a_k v_k,$$

whence we obtain the identity for any k:

$$-\lambda_k b_k=\alpha b_k-a_k$$

and

$$b_k=\frac{a_k}{\alpha+\lambda_k},$$

which was to be proved.

(b) By (78) we have

$$(R_\alpha-R_\beta)\,f=\sum_{k=1}^{\infty}\left(\frac{1}{\alpha+\lambda_k}-\frac{1}{\beta+\lambda_k}\right)a_k v_k$$

and

$$(\beta-\alpha)\,R_\alpha R_\beta=\sum_{k=1}^{\infty}\frac{(\beta-\alpha)}{(\alpha+\lambda_k)\,(\beta+\lambda_k)}a_k v_k.$$

The right hand sides of these equalities are identically equal because

$$\frac{1}{\alpha+\lambda}-\frac{1}{\beta+\lambda}=\frac{(\beta-\alpha)}{(\alpha+\lambda)\,(\beta+\lambda)},$$

whence (**??**) follows.

76. * Let $f\in L^2\,(\Omega)$.

(a) Prove that $\alpha R_a f\xrightarrow{L^2}f$ as $\alpha\to+\infty$.
Hint. Use Exercise 5(f)v.

(b) Prove that if in addition $f\in W_0^1\,(\Omega)$ and $\Delta f\in L^2\,(\Omega)$ then, for all $\alpha>0$,

$$\|\alpha R_\alpha f-f\|_{L^2}\leq\frac{1}{\alpha}\|\Delta f\|_{L^2}.$$

Hint. Use Exercise 5(f)v.

Solution. Let

$$f=\sum_{k=1}^{\infty}a_k v_k.$$

Then by Exercise 5(f)v(a)

$$R_\alpha f=\sum_{k=1}^{\infty}\frac{a_k}{\alpha+\lambda_k}v_k,$$

where the both series converge in L^2. Hence,

$$\alpha R_\alpha f - f = \sum_{k=1}^{\infty} \left(\frac{\alpha a_k}{\alpha + \lambda_k} - a_k \right) v_k = -\sum_{k=1}^{\infty} \frac{\lambda_k}{\alpha + \lambda_k} a_k v_k.$$

By the Parseval identity, we have

$$\|\alpha R_\alpha f - f\|_{L^2}^2 = \sum_{k=1}^{\infty} \left(\frac{\lambda_k}{\alpha + \lambda_k} \right)^2 a_k^2. \tag{104}$$

Clearly, we have for any k

$$\left(\frac{\lambda_k}{\alpha + \lambda_k} \right)^2 \to 0 \text{ as } \alpha \to \infty.$$

We need to prove that the whole sum (**??**) goes to 0 as $\alpha \to \infty$. For that, fix $\varepsilon > 0$ and find N such that

$$\sum_{k=N}^{\infty} a_k^2 < \varepsilon.$$

Then

$$\begin{aligned} \|\alpha R_\alpha f - f\|_{L^2}^2 &= \sum_{k=1}^{N} \left(\frac{\lambda_k}{\alpha + \lambda_k} \right)^2 a_k^2 + \sum_{k=N}^{\infty} \left(\frac{\lambda_k}{\alpha + \lambda_k} \right)^2 a_k^2 \\ &\leq \sum_{k=1}^{N} \left(\frac{\lambda_k}{\alpha + \lambda_k} \right)^2 a_k^2 + \varepsilon, \end{aligned}$$

because $\left(\frac{\lambda_k}{\alpha + \lambda_k} \right)^2 \leq 1$. Now letting $\alpha \to \infty$ and noticing that the finite sum goes to 0, we obtain that

$$\limsup_{\alpha \to \infty} \|\alpha R_\alpha f - f\|_{L^2}^2 \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lim_{\alpha \to \infty} \|\alpha R_\alpha f - f\|_{L^2}^2 = 0,$$

which was to be proved.

If $f \in W_0^1(\Omega)$ and $\Delta f \in L^2(\Omega)$ then by Exercise 5(f)v

$$\Delta f = -\sum_{k=1}^{\infty} \lambda_k a_k v_k,$$

and, hence,

$$\|\Delta f\|_{L^2}^2 = \sum_{k=1}^{\infty} \lambda_k^2 a_k^2.$$

Since $\lambda_k \geq 0$ and, hence,

$$\frac{\lambda_k}{\alpha + \lambda_k} \leq \frac{\lambda_k}{\alpha},$$

it follows from (**??**) that

$$\|\alpha R_\alpha f - f\|_{L^2}^2 \leq \frac{1}{\alpha^2}\sum_{k=1}^{\infty}\lambda_k^2 a_k^2 = \frac{1}{\alpha^2}\|\Delta f\|_{L^2}^2,$$

which finishes the proof.

77. Prove that there exists a positive constant c_n such that, for any ball B_R of radius R in $\mathbb{R}^n$,

$$\lambda_1(B_R) = \frac{c_n}{R^2}.$$

Hint. Using the variational property (77) of Exercise 5(f)v, prove first that

$$\lambda_1(B_R) = R^{-2}\lambda_1(B_1).$$

You can assume without loss of generality that the ball B_R is centered at the origin of $\mathbb{R}^n$.

Remark. Letting $R \to \infty$ and using Exercise 5(f)v we conclude that $\lambda_1(\mathbb{R}^n) = 0$.

Solution. Observe the following:

(i) $c := \lambda_1(B_1) > 0$ by a Theorem from lectures.

(ii) $\lambda_1(B_R) = R^{-2}\lambda_1(B_1)$. Indeed, let $f \in \mathcal{D}(B_R)$. Then the function $\widetilde{f}(x) = f(Rx)$ belongs to $\mathcal{D}(B_1)$, and we have

$$\left\|\widetilde{f}\right\|_{L^2}^2 = \int_{B_R} f(Rx)^2\,dx = \int_{B_1} f(y)^2 R^{-n}dy = R^{-n}\|f\|_{L^2}^2$$

and

$$\nabla\widetilde{f}(x) = R\nabla f(Rx)$$

whence

$$\left\|\nabla\widetilde{f}\right\|_{L^2}^2 = R^2\int_{B_1}|\nabla f|(Rx)^2\,dx = R^{2-n}\|\nabla f\|_{L^2}^2.$$

It follows that

$$\mathcal{R}\left(\widetilde{f}\right) = \frac{\left\|\nabla\widetilde{f}\right\|_{L^2}^2}{\left\|\widetilde{f}\right\|_{L^2}^2} = R^{-2}\mathcal{R}(f).$$

Taking infimum in all $f \in \mathcal{D}(B_R)$, we obtain $\lambda_1(B_R) = R^{-2}\lambda_1(B_1)$. Hence, we conclude that $\lambda_1(B_R) = c/R^2$ where $c = \lambda_1(B_1) > 0$ depends only on n.

78. ** Prove that, for any geodesic ball $B_R = B(x_0, R)$ on an arbitrary connected weighted manifold M,

$$\lambda_1(B_R) \leq \frac{4}{R^2}\frac{\mu(B_R)}{\mu\left(B_{R/2}\right)}.$$

Solution. Consider the distance function

$$\rho(x) = d(x, x_0)$$

and set

$$f(x) = (R - \rho(x))_+ .$$

Since ρ is the Lipschitz function with the Lipschitz constant 1, f is also a Lipschitz function on M with the Lipschitz constant 1. Since $f(x) = 0$ on ∂B_R, it follows that $f \in W_0^1(B_R)$. We have

$$\mathcal{R}(f) = \frac{\int_{B_R} |\nabla f|^2 d\mu}{\int_{B_R} f^2 d\mu}.$$

Since $|\nabla f| \le 1$ and $f \ge \frac{R}{2}$ in $B\left(x_0, \frac{R}{2}\right)$, we obtain

$$\mathcal{R}(f) \le \frac{\mu(B_R)}{\left(\frac{R}{2}\right)^2 \mu\left(B_{R/2}\right)},$$

whence the claim follows.

79. ** Let M be connected. Fix a point $x_0 \in M$ and consider the function $\rho(x) = d(x, x_0)$.

   (a) Prove that $\rho \in W_{loc}^1(M)$ and $\|\nabla \rho\|_{\vec{L}^\infty} \le 1$.
   Hint. Use Exercise 5(f)v.

   (b) Assume ρ has in Ω the weak Laplacian $\Delta\rho$, and that $\Delta\rho$ satisfies in Ω the inequality $\Delta\rho \ge a$, where a is a positive constant. Prove that

   $$\lambda_1(\Omega) \ge \frac{a^2}{4}.$$

   Hint. First prove that $h(\Omega) \ge a$ where $h(\Omega)$ is the Cheeger constant from Exercise 5(f)v, and then use the Cheeger inequality.

Solution. (a) Function ρ is continuous and, hence, in $L_{loc}^2(M)$. The function ρ is Lipschitz with the Lipschitz constant 1 because by the triangle inequality

$$|\rho(x) - \rho(y)| = |d(x, x_0) - d(y, x_0)| \le d(x, y).$$

Hence, by Exercise 5(f)v, $\nabla\rho$ exists weakly and satisfies $\|\nabla\rho\|_{\vec{L}^\infty} \le 1$. Hence, $\nabla\rho \in \vec{L}_{loc}^2(M)$ and $\rho \in W_{loc}^1(M)$.

(b) For any non-negative $\varphi \in \mathcal{D}(\Omega)$, we have

$$\int_\Omega \Delta\rho\, \varphi d\mu \ge a \int_\Omega \varphi d\mu.$$

By the definition of the weak Laplacian, we have

$$\int_\Omega \Delta\rho\, \varphi d\mu = -\int_\Omega \langle \nabla\rho, \nabla\varphi \rangle d\mu,$$

whence

$$-\int_\Omega \langle \nabla\rho, \nabla\varphi \rangle d\mu \ge a \int_\Omega \varphi d\mu$$

Since

$$-\int_{\Omega}\langle\nabla\rho,\nabla\varphi\rangle\,d\mu\leq\int_{\Omega}|\nabla\rho|\,|\nabla\varphi|\,d\mu\leq\|\nabla\rho\|_{L^{\infty}}\int_{\Omega}|\nabla\varphi|\,d\mu\leq\int_{\Omega}|\nabla\varphi|\,d\mu,$$

we obtain

$$\int_{\Omega}|\nabla\varphi|\,d\mu\geq a\int_{\Omega}\varphi d\mu.$$

By the definition of the Cheeger constant, we conclude that $h\left(\Omega\right)\geq a$. By the Cheeger inequality, we obtain that $\lambda_1\left(\Omega\right)\geq\frac{a^2}{4}$.

Remark. Although in the definition of the Cheeger constant one uses also signed functions $\varphi\in\mathcal{D}\left(\Omega\right)$, one can see from the solution of Exercise 5(f)v that it is enough to restrict the definition (71) of h to non-negative φ, which is needed for the present solution.

80. ** Prove that

$$\lambda_1\left(\mathbb{H}^n\right)\geq\frac{\left(n-1\right)^2}{4}.$$

Hint. Use Exercise 5(f)v and the polar coordinates in $\mathbb{H}^n$.

Solution. By Exercise 5(f)v, it suffices to prove that, for any precompact open set $\Omega\subset\mathbb{H}^n$,

$$\lambda_1\left(\Omega\right)\geq\frac{\left(n-1\right)^2}{4}.$$

By the translation invariance of $\mathbb{H}^n$ (see Exercise 5(f)v), we can assume that the origin o lies outside $\overline{\Omega}$. Then Ω is contained in the domain of the polar coordinates. In the polar coordinate system, we have

$$\Delta_{\mathbb{H}^n}=\frac{\partial^2}{\partial r^2}+\left(n-1\right)\coth r\frac{\partial}{\partial r}+\frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}}.$$

Applying this to the function $\rho=r=d\left(x,o\right)$ (note that r is smooth in Ω), we obtain

$$\Delta r=\left(n-1\right)\coth r\geq n-1,$$

Solution. By Exercise 5(f)v we conclude that $\lambda_1\left(\Omega\right)\geq\frac{\left(n-1\right)^2}{4}$, which finishes the proof.

81. ** Prove that

$$\lambda_1\left(\mathbb{H}^n\right)=\frac{\left(n-1\right)^2}{4}.$$

Hint. It suffices to find for any $\varepsilon>0$ a function $f\in W_0^1\left(\mathbb{H}^n\right)$ such that

$$\mathcal{R}\left(f\right)\leq\frac{\left(n-1\right)^2}{4}+\varepsilon.$$

Look for f in the form $f\left(x\right)=e^{-cr}$ where r is the polar radius. You may use without proof the fact that if $f\in W^1\left(M\right)$ and $f\left(x\right)\to0$ as $x\to\infty$ then $f\in W_0^1\left(M\right)$.

Solution. In the view of Exercise 5(f)v, we need to prove that $\lambda_1(\mathbb{H}^n) \le \frac{(n-1)^2}{4}$. For that, it suffices to find for any $\varepsilon > 0$ a function $f \in W_0^1(\mathbb{H}^n)$ such that

$$\mathcal{R}(f) < \frac{(n-1)^2}{4} + \varepsilon.$$

Recall that the area function on $\mathbb{H}^n$ is $S(r) = \omega_n \sinh^{n-1} r$. Since $\sinh r \le e^r$, it follows that

$$S(r) \le \omega_n e^{(n-1)r}.$$

Using r as a polar radius, consider on $\mathbb{H}^n$ the function $f(x) = e^{-\frac{1}{2}ar}$. Assuming that $a > n-1$, we obtain

$$\|f\|_{L^2}^2 = \int_{\mathbb{H}^n} f^2 d\mu = \int_0^\infty f(r)^2 S(r)\, dr \le \omega_n \int_0^\infty e^{-ar} e^{(n-1)r} dr < \infty,$$

whence $f \in L^2(M)$. Since function r has weak derivative bounded by 1, we have

$$\nabla f = \nabla e^{-\frac{1}{2}ar} = \frac{1}{2} a f \nabla r,$$

whence

$$\int_M |\nabla f|^2 d\mu = \frac{a^2}{4} \int_M f^2 |\nabla r|^2 d\mu \le \frac{a^2}{4} \int_M f^2 d\mu. \tag{105}$$

Hence, we see that $f \in W^1(M)$ and $\mathcal{R}(f) \le a^2/4$. Since $f(x) \to 0$ as $x \to \infty$, we conclude that $f \in W_0^1(M)$.

By (**??**) we obtain $\mathcal{R}(f) \le \frac{a^2}{4}$ whence also $\lambda_1(\mathbb{H}^n) \le \frac{a^2}{4}$. Since a was any number $> n-1$, we conclude that $\lambda_1(\mathbb{H}^n) \le \frac{(n-1)^2}{4}$, which finishes the proof.

Blatt 13. Abgabe bis 30.01.2026

Die mit *markierten Aufgaben sind zusätzlich und werden korrigiert

In all questions, $(M, \mathbf{g}, \mu)$ is a weighted manifold, Ω is a precompact open subset of M, $\{\lambda_k(\Omega)\}_{k=1}^{\infty}$ is the sequence of the Dirichlet eigenvalues of $\Delta = \Delta_{\mathbf{g},\mu}$ in Ω in the increasing order, and $\{v_k\}$ is the sequence of the corresponding eigenfunctions that forms an orthonormal basis in $L^2(\Omega)$.

82. Let M be a compact connected manifold. Set $\Omega = M$. Prove that $\lambda_1(\Omega) = 0$ and $\lambda_2(\Omega) > 0$. In other words, 0 is a simple eigenvalue of Δ in Ω.

    Hint. You need to prove that if v is an eigenfunction of Δ in Ω with the eigenvalue 0 then $v = \text{const}$.

    Solution. Since function $v(x) \equiv 1$ belongs to $C_0^\infty(M) = C_0^\infty(M)$ and $\Delta v = 0$, it follows that the constant function is an eigenfunction of Δ in Ω with the eigenvalue 0. Hence, $\lambda_1(\Omega) = 0$. Let us verify that the eigenvalue $\lambda_1 = 0$ is simple. Indeed, if $v \in W_0^1(\Omega)$ is another eigenfunction with the eigenvalue 0, that is, $\Delta v = 0$ then we obtain that $\|\nabla v\|_{\vec{L}^2}^2 = \lambda_1 \|v\|_{L^2}^2 = 0$ whence $\nabla v = 0$. Since Ω is connected and $v \in C^\infty(\Omega)$, it follows that $v = \text{const}$. Hence, only constant is the eigenfunction of $\lambda_1 = 0$, that is, $\lambda_1 = 0$ is a simple eigenvalue, which implies that $\lambda_2(\Omega) > 0$.

83. Recall that the *Rayleigh quotient* of a non-zero function $u \in W^1(\Omega)$ is defined by

$$\mathcal{R}(u) := \frac{\|\nabla u\|_{\vec{L}^2}^2}{\|u\|_{L^2}^2}.$$

    Assume that, for a function $f \in W_0^1(\Omega) \setminus \{0\}$,

$$\mathcal{R}(f) = \lambda_1(\Omega). \tag{106}$$

    Prove that f is the eigenfunction of Δ in Ω with the eigenvalue $\lambda_1(\Omega)$.

    Remark. We know that if v is an eigenfunction of Δ in Ω with the eigenvalue λ then $\mathcal{R}(v) = \lambda$. The above claim says that the converse statement is also true provided $\lambda = \lambda_1(\Omega)$ (but it is not true for higher eigenvalues).

    Hint. Set $\lambda = \lambda_1(\Omega)$ and recall that by Exercise 5(f)v,

$$\lambda = \inf_{u \in W_0^1(\Omega)\setminus\{0\}} \mathcal{R}(u).$$

    Hence, for any $\varphi \in W_0^1(\Omega)$ and any $t \in \mathbb{R}$, we have

$$\mathcal{R}(f + t\varphi) \geq \lambda = \mathcal{R}(f).$$

    Use this inequality with $t \to 0$ to deduce that $(\nabla f, \nabla \varphi) - \lambda(f, \varphi) = 0$, which will imply the claim.

    Solution. Denote for simplicity $\lambda = \lambda_1(\Omega)$ and observe that, for any $\varphi \in W_0^1(\Omega)$ and real t, we have

$$\mathcal{R}(f + t\varphi) \geq \lambda = \mathcal{R}(f),$$

that is

$$\frac{\|\nabla (f+t\varphi)\|^2}{\|f+t\varphi\|^2} \geq \lambda = \frac{\|\nabla f\|^2}{\|f\|^2},$$

which implies

$$\|\nabla (f+t\varphi)\|^2 - \lambda\|f+t\varphi\|^2 \geq 0 = \|\nabla f\|^2 - \lambda\|f\|^2.$$

We have

$$\|\nabla f + t\nabla\varphi\|^2 = \|\nabla f\|^2 + 2t(\nabla f, \nabla\varphi) + t^2\|\nabla\varphi\|^2$$

and

$$\|f+t\varphi\|^2 = \|f\|^2 + 2(f,\varphi) + t^2\|\varphi\|^2,$$

whence

$$\|\nabla (f+t\varphi)\|^2 - \lambda\|f+t\varphi\|^2 = 2t\left((\nabla f, \nabla\varphi) - \lambda (f,\varphi)\right) + t^2\left(\|\nabla\varphi\|^2 - \lambda\varphi^2\right).$$

Since the left hand side is non-negative for all real t, the linear in t term in the right hand side must vanish, that is

$$(\nabla f, \nabla\varphi) - \lambda (f,\varphi) = 0.$$

This equality means that f is the eigenfunction of Δ in Ω with the eigenvalue λ.

84. Let M be a compact connected manifold and $\Omega = M$. Let u be a solution of the mixed problem for the heat equation in $\mathbb{R}_+ \times \Omega$ with the initial function $f \in L^2(\Omega)$. Prove that, for any $t > 0$,

$$\int_\Omega u(t,\cdot)d\mu = \int_\Omega f d\mu. \tag{107}$$

Hint. Use Exercise 5(f)v and expansions of f and $u(t,\cdot)$ in the eigenfunction basis $\{v_k\}_{k=1}^\infty$.

Solution. By Exercise 5(f)v we know that $\lambda_1(\Omega) = 0$, $\lambda_2(\Omega) > 0$ and $v_1 = \text{const}$. We would like to have all eigenfunctions $\{v_k\}$ normalized in $L^2(\Omega)$, in particular, we have

$$v_1(x) = \frac{1}{\sqrt{\mu(\Omega)}}.$$

Observe that since $\mu(\Omega) < \infty$, we have $L^2(\Omega) \subset L^1(\Omega)$ so that $\int_\Omega f d\mu$ is well defined. Let

$$f = \sum_{k=1}^\infty a_k v_k.$$

Then

$$\int_\Omega f d\mu = (f,1)_{L^2} = \sum_{k=1}^\infty a_k (v_k, 1)_{L^2}.$$

Since $v_k \perp v_1$ for $k > 1$, we obtain that $(v_k, 1) = 0$ for $k > 1$, and

$$(v_1, 1) = \int_\Omega v_1 d\mu = \sqrt{\mu(\Omega)}.$$

Hence,

$$\int_\Omega f d\mu = a_1 \sqrt{\mu(\Omega)}.$$

Since

$$u(t,\cdot) = \sum_{k=1}^{\infty} e^{-\lambda_k t} a_k v_k,$$

we obtain in the same way that

$$\int_\Omega u(t,\cdot)\, d\mu = e^{-\lambda_1 t} a_1 \sqrt{\mu(\Omega)}.$$

Since $\lambda_1 = 0$, we obtain (81).

85. *(Product rule for L^2-derivatives)* Let I be an interval in $\mathbb{R}$ and $u(t), v(t) : I \to L^2(\Omega)$ be differentiable paths. Prove that

$$\frac{d}{dt}(u,v) = (u, \frac{dv}{dt}) + (\frac{du}{dt}, v),$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$.

Solution. We have

$$\begin{aligned}\frac{(u(t+\varepsilon), v(t+\varepsilon)) - (u(t), v(t))}{\varepsilon} &= \left(u(t+\varepsilon), \frac{v(t+\varepsilon) - v(t)}{\varepsilon}\right) \\ &\quad + \left(\frac{u(t+\varepsilon) - u(t)}{\varepsilon}, v(t)\right).\end{aligned}$$

When $\varepsilon \to 0$, we have $u(t+\varepsilon) \to u(t)$ and

$$\frac{v(t+\varepsilon) - v(t)}{\varepsilon} \to v'(t) \quad \text{and} \quad \frac{u(t+\varepsilon) - u(t)}{\varepsilon} \to u'(t),$$

where all the convergencies are in L^2-norm . Since the inner product is a continuous functional of the both arguments, we obtain

$$\frac{(u(t+\varepsilon), v(t+\varepsilon)) - (u(t), v(t))}{\varepsilon} \to (u(t), v'(t)) + (u'(t), v(t)),$$

which was to be proved.

86. * (*Chain rule for L^2-derivatives*) Let $u(t) : I \to L^2(\Omega)$ be a differentiable path. Consider a function $\psi \in C^1(\mathbb{R})$ such that

$$\psi(0) = 0 \text{ and } \sup|\psi'| < \infty. \tag{108}$$

Prove that the path $\psi(u(t))$ is also differentiable in $t \in I$ and

$$\frac{d\psi(u)}{dt} = \psi'(u) \frac{du}{dt}.$$

Solution. The condition (82) implies that $|\psi(t)| \leq C|t|$ whence it follows that $\psi(u(t))$ is also in $L^2(\Omega)$. Fix $t \in I$. Denoting

$$r(s) := \frac{u(t+s) - u(t)}{s}$$

and $u' = \frac{du}{dt}$, we have by hypothesis

$$r(s) \xrightarrow{L^2} u'(t) \quad \text{as } s \to 0. \tag{109}$$

We need to prove that

$$\frac{\psi(u(t+s)) - \psi(u(t))}{s} \xrightarrow{L^2} \psi'(u)\, u' \quad \text{as } s \to 0. \tag{110}$$

It suffices to show that for any sequence $s_k \to 0$, there is a subsequence along which (**??**) holds.

By the mean value theorem, we have

$$\begin{aligned} \psi(u(t+s)) - \psi(u(t)) &= \psi(u(t) + sr(s)) - \psi(u(t)) \\ &= \psi'(u(t) + \xi sr(s))\, sr(s) \end{aligned}$$

where $\xi = \xi(s,x) \in (0,1)$. Therefore,

$$\begin{aligned} \frac{\psi(u(t+s)) - \psi(u(t))}{s} - \psi'(u)\, u' &= [\psi'(u(t) + \xi sr(s)) - \psi'(u(t))]\, u'(t) \\ &\quad + \psi'(u(t) + \xi sr(s))\, [r(s) - u'(t)] \end{aligned}$$

and, hence,

$$\begin{aligned} &\left\| \frac{\psi(u(t+s)) - \psi(u(t))}{s} - \psi'(u)\, u' \right\|_{L^2} \\ \leq\ &\left(\int_\Omega |\psi'(u(t) + \xi sr(s)) - \psi'(u(t))|^2 |u'(t)|^2 d\mu \right)^{1/2} + \sup|\psi'|\, \|r(s) - u(t)\|_{L^2} \end{aligned} \tag{111}$$

When $s \to 0$, the second term in (83) tends to 0 by (**??**). Let us show that, for any sequence $s_k \to 0$, there is a subsequence along which the first term in (83) tends to 0. The sequence of functions $s_k r(s_k)$ tends to 0 in L^2 because the norms $\|r(s)\|_{L^2}$ remain bounded as $s \to 0$. Therefore, there is a subsequence s_{k_i}, which will be renumbered by $\{s_k\}$, along which $s_k r(s_k, \cdot) \to 0$ a.e. Since $\xi_k := \xi(s_k)$ is bounded, we also have $\xi_k s_k r(s_k) \to 0$ a.e., and by the continuity of ψ',

$$\psi'(u(t) + \xi_k s_k r(s_k)) \to \psi'(u(t)) \text{ a.e.}$$

Hence, the function under the integral sign in (83) tends to 0 almost everywhere. Since this function is bounded for all s by the integrable function $4C^2|u'|^2$, we conclude by the dominated convergence theorem that the integral in (83) tends to 0, which finishes the proof.