Heat kernel estimates on Riemannian manifolds

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Chapter 1

The notion of heat kernel

Let (M, g) be a connected Riemannian manifold and ν be the Riemannian volume of g. Let μ be another measure such that the density $a(x) = \frac{d\mu}{d\nu}$ is a smooth positive function on M. The triple (M, g, μ) is called a *weighted manifold* (or a manifold with density).

Definition. The operator

$$\Delta_\mu u := \frac{1}{a} \operatorname{div}(a \nabla u)$$

is called the Laplace operator of (M, g, μ) or the weighted Laplace operator. Here div and ∇ are the Riemannian divergence and gradient.

If $\mu = \nu$ then Δ_μ is the Laplace-Beltrami operator of (M, g).

The operator Δ_μ is symmetric with respect to μ in the following sense: for all smooth compactly supported functions u, v on M

$$\int_M u \Delta_\mu v \, d\mu = -\int_M \langle \nabla u, \nabla v \rangle_g \, d\mu = \int_M v \Delta_\mu u \, d\mu.$$

It is possible to show that Δ_μ has the Friedrichs extension to a self-adjoint operator in $L^2(M, \mu)$, that will also be denoted by Δ_μ. This operator is non-positive definite since

$$(u, \Delta_\mu u) = -\int |\nabla u|^2 \, d\mu \leq 0.$$

Functional calculus yields that $e^{t\Delta_\mu}$ is a bounded self-adjoint operator for any $t \geq 0$. The family $\left\{e^{t\Delta_\mu}\right\}_{t \geq 0}$ is called the *heat semigroup* of (M, g, μ). For any $f \in L^2$, the function $u(t, x) := e^{t\Delta_\mu} f(x)$ is smooth in $t > 0$, $x \in M$ and satisfies in $\mathbb{R}_+ \times M$ the heat equation $\partial_t u = \Delta_\mu u$ and the initial condition $u(t, \cdot) \xrightarrow{L^2} f$ as $t \to 0+$.

Definition. A function $p_t(x, y)$, defined for $t > 0$ and $x, y \in M$ is called the *heat kernel* of M if it is the integral kernel of the operator $e^{t\Delta_\mu}$, that is, for all $f \in L^2(M, \mu)$, $t > 0$, $x \in M$,

$$e^{t\Delta_\mu} f(x) = \int_M p_t(x, y) f(y) \, d\mu(y).$$

Theorem 1. *Any weighted manifold (M, μ) possesses a unique heat kernel $p_t(x, y)$. Furthermore, it satisfies the following properties.*

- *$p_t(x, y)$ is a C^∞ function on $\mathbb{R}_+ \times M \times M$.*

- $p_t(x,y) > 0$
- $\int_M p_t(x,y)\,d\mu(y) \le 1$
- $p_t(x,y) = p_t(y,x)$.
- $p_{t+s}(x,y) = \int_M p_t(x,z)\,p_s(z,y)\,d\mu(z)$ *(the semigroup identity)*
- $p_t(x,y)$ *is a fundamental solution of the heat equation, that is, for any* $y \in M$, *the function* $u(t,x) = p_t(x,y)$ *satisfies the heat equation* $\partial_t u = \Delta_\mu u$ *in* $\mathbb{R}_+ \times M$ *and the initial condition*
$$u(t,\cdot) \to \delta_y \text{ as } t \to 0+.$$

Moreover, $p_t(x,y)$ *is the minimal positive fundamental solution.*

The heat kernel in $\mathbb{R}^n$ is given by the Gauss-Weierstrass formula

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

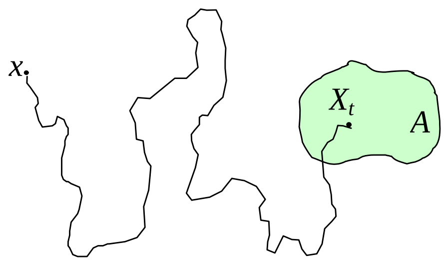
On any weighted manifold M there is a unique diffusion process $\left(\{X_t\}_{t\ge 0}, \{\mathbb{P}_x\}_{x\in M}\right)$ generated by Δ_μ (called also *Brownian motion* on M), which means that, for any bounded Borel functions f on M,

$$e^{t\Delta_u} f(x) = \mathbb{E}_x f(X_t).$$

For $f = 1_A$, where A is a Borel subset of M, we obtain the identity

$$\mathbb{P}_x(X_t \in A) = \int_A p_t(x,y)\,d\mu(y),$$

which means, that the heat kernel $p_t(x,y)$ coincides with the transition density of Brownian motion on M.



In particular, the Gauss-Weierstrass function is the transition density of Brownian motion in $\mathbb{R}^n$, that is, the normal distribution with the variance $2t$.

On general manifolds the exact formulas for the heat kernel are not possible, but one can obtain estimates of the heat kernel depending on the geometry of the manifold.

We will use the geodesic distance $d(x,y)$ on (M,g). Denote by $B(x,r)$ the geodesic ball of radius r centered at x and set $V(x,r) = \mu(B(x,r))$.

Theorem 2. *(Li and Yau '86) If* (M,g) *is a complete Riemannian manifold of non-negative Ricci curvature then the heat kernel on* (M,g) *satisfies the following estimates*

$$p_t(x,y) \asymp \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right) \qquad ((LY))$$

Here $\asymp$ means that there are $\leq$ and $\geq$ but with different values of positive constants c, C. This estimate is called the *Li-Yau estimate*. Of course, the heat kernel in $\mathbb{R}^n$ satisfies (LY) as in this case $V\left(x, \sqrt{t}\right) = \text{const}\, t^{n/2}$.

Note that for negatively curved manifolds (LY) is in general not true. For example, on the hyperbolic space $\mathbb{H}^3$ of constant negative curvature -1

$$p_t(x, y) = \frac{1}{(4\pi t)^{3/2}} \frac{r}{\sinh r} \exp\left(-\frac{r^2}{4t} - t\right)$$

where $r = d(x, y)$.

Theorem 2 is a great source of inspiration for further work on heat kernel estimates, especially on the relation between the heat kernel estimates and global geometry of M. In particular, we will see, that (LY) holds on a more general class of weighted manifolds, without curvature restriction.

Topics to be considered in this course:

1. Upper bounds of Li-Yau type
2. Two sided estimates of Li-Yau type
3. Heat kernels on manifolds with ends (main)
4. Heat kernels of some Schrödinger operators

Main references:

1. Grigor'yan A., *Heat kernel and Analysis on manifolds*, AMS-IP Studies in Advanced Mathematics **47,** 2009 (Chinese translation pending)
2. Grigor'yan A., *Heat kernels on weighted manifolds and applications*, in "The ubiquitous heat kernel", Contemporary Mathematics **398** (2006) 93-191.
3. Grigor'yan A., Saloff-Coste L., *Heat kernel on manifolds with ends*, Ann. Inst. Fourier, Grenoble **59** (2009) 1917-1997.

Chapter 2

Upper bounds of the heat kernel

Let (M, g, μ) be any weighted manifold and Ω be an open subset of M. Denote by $\lambda_1(\Omega)$ the bottom of the spectrum of $-\Delta_\mu$ in Ω, that is,

$$\lambda_1(\Omega) = \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 \, d\mu}{\int_\Omega u^2 \, d\mu}.$$

If Ω is precompact then $\lambda_1(\Omega)$ is the minimal eigenvalue of the Dirichlet problem

$$\begin{cases} \Delta_\mu u + \lambda u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

Definition. We say that (M, g, μ) satisfies the *relative Faber-Krahn inequality* if, for any ball $B(x,r) \subset M$ and any open set $\Omega \subset B(x,r)$,

$$\lambda_1(\Omega) \geq \frac{c}{r^2}\left(\frac{V(x,r)}{\mu(\Omega)}\right)^\delta \tag{(RFK)}$$

where c, δ are some positive constants.

For example, in $\mathbb{R}^n$ the classical Faber-Krahn theorem says that

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*),$$

where Ω^* is the ball of the same volume as Ω. This is a kind of isoperimetric property of Euclidean balls: among the domains with the same volume the minimal λ_1 is achieved on the balls.

If the radius of Ω^* is R, then we obtain, for $c = c(n) > 0$

$$\lambda_1(\Omega^*) = \frac{c}{R^2} = \frac{c'}{\mu(\Omega^*)^{2/n}},$$

where μ is the Lebesgue measure. Hence, we obtain

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-2/n},$$

which implies (RFK) with $\delta = 2/n$ because $V(x,r) \simeq r^n$.

One can prove that (RFK) holds on any geodesically complete Riemannian manifold with $Ricci \geq 0$. In this case there is no uniform estimate of $V(x,r)$ so that (RFK) has to be used in the general form as stated above.

One of the advantages of (RFK) is that it is stable under *quasi-isometric* changes of metric and measure. Two Riemannian metrics g', g'' on M are called quasi-isometric if $g' \simeq g''$, that is, $C^{-1}g' \leq g'' \leq Cg'$ for some constant $C > 0$. Similarly, two measures μ', μ'' on M are called quasi-isometric if $\mu' \simeq \mu''$. Two weighted manifolds (M, g', μ') and (M, g'', μ'') are called quasi-isometric if $g' \simeq g''$ and $\mu' \simeq \mu''$. It follows that $V'(x,r) \simeq V''(x,r)$ and $\lambda_1'(\Omega) \simeq \lambda_2''(\Omega)$ so that (RFK) is stable under quasi-isometry.

Definition. We say that the *volume doubling* property holds if there is a constant C such that, for all $x \in M$ and $r > 0$,

$$V(x, 2r) \leq CV(x,r). \tag{(VD)}$$

Theorem 3. *Let (M, g, μ) be a geodesically complete weighted manifold. Then the following conditions $(a), (b), (c)$ are equivalent:*

(a) (RFK) holds.

(b) (VD) holds and the heat kernel satisfies the on-diagonal upper bound

$$p_t(x,x) \leq \frac{C}{V\left(x, \sqrt{t}\right)},$$

for all $t > 0$ and $x \in M$

(c) (VD) holds and and the heat kernel satisfies the upper bound in (LY), that is,

$$p_t(x,y) \leq \frac{C}{V\left(x, \sqrt{t}\right)} \exp\left(-\frac{d^2(x,y)}{ct}\right), \tag{(LY_{\leq})}$$

for all $t > 0$ and $x, y \in M$.

As a consequence we obtain the following non-trivial fact: under (VD), heat kernel upper bound $(LY_{\leq})$ is stable under quasi-isometry.

Interestingly enough, an *upper* bound of the heat kernel implies under some conditions a *lower* bound on-diagonal.

Theorem 4. *Let (M, g, μ) be a geodesically complete weighted manifold. Assume that for* some $x \in M$ *and all $r, t > 0$*

$$V(x, 2r) \leq CV(x,r)$$

and

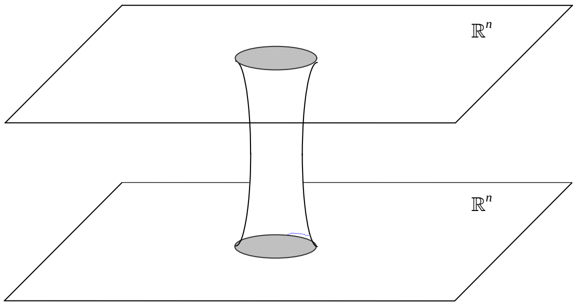
$$p_t(x,x) \leq \frac{C}{V\left(x, \sqrt{t}\right)}.$$

Then also for all $t > 0$

$$p_t(x,x) \geq \frac{c}{V\left(x, \sqrt{t}\right)}$$

for some $c > 0$.

In particular, if (RFK) is satisfied then the on-diagonal lower bound of the heat kernel holds for all $x \in M$ and $t > 0$. The question still remains under what conditions the off-diagonal Li-Yau lower bound of $p_t(x,y)$ holds. Here is an example to show that (RFK) is not enough.



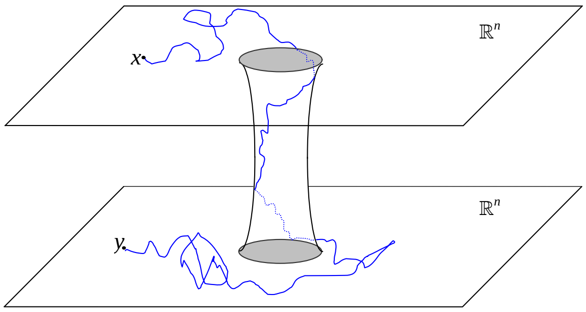
Let $M = \mathbb{R}^n \# \mathbb{R}^n$ be a connected sum of two copies of $\mathbb{R}^n$ with $n \geq 3$. On this manifold $V(x, r) \simeq r^n$ as in $\mathbb{R}^n$. One can also show that

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-2/n},$$

which implies (RFK), as we have seen above. Consequently, the upper bounds of Theorem 3 and the on-diagonal lower bound of Theorem 4 are satisfied. However, the full Li-Yau lower bound

$$p_t(x, y) \geq \frac{C}{V(x, \sqrt{t})} \exp\left(-\frac{d^2(x, y)}{ct}\right)$$

breaks down if x and y belong to different copies of $\mathbb{R}^n$ as on the picture below.



Indeed, as we will see later on, in this case, for large t,

$$p_t(x, y) \asymp \frac{C}{t^{n/2}} \left(\frac{1}{|x|^{n-2}} + \frac{1}{|y|^{n-2}}\right) \exp\left(-\frac{d^2(x, y)}{ct}\right).$$

In particular, if $|x| \simeq |y| \simeq \sqrt{t}$ then $p_t(x, y) \simeq \dfrac{1}{t^{n-1}} \ll \dfrac{1}{t^{n/2}}$ where the value $\frac{1}{t^{n/2}}$ is predicted by (LY).

A probabilistic meaning of this phenomenon is as follows: if x and y are located on different sheets and far away from the central part, then in order for Brownian motion to get from x to y, it has to go through the *bottleneck* of the central part, thus drastically reducing the (density of) probability of getting from x to y.

Chapter 3

Two-sided Gaussian estimates of heat kernel

Recall the classical Harnack inequality for harmonic functions in $\mathbb{R}^n$: if u is a positive harmonic function in a ball $B\left(z,r\right)$ in $\mathbb{R}^n$ then

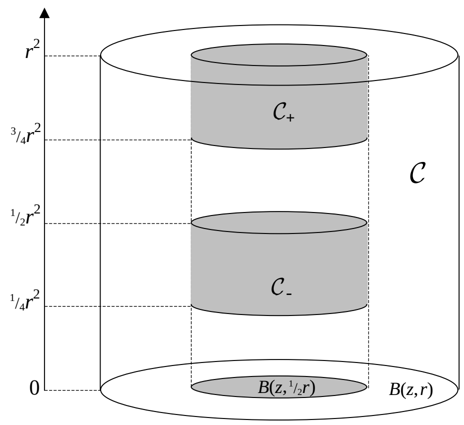
$$\sup_{B(z,r/2)} u \leq C \inf_{B(z,r/2)} u$$

where $C = C\left(n\right)$. Hadamard proved a similar property for positive solutions of the heat equation, which is now referred to as a parabolic Harnack inequality.

Definition. Let (M,g,μ) be a complete weighted manifold. We say that M satisfies the uniform *parabolic Harnack inequality* if the following is true: there exists a constant C such that, for any ball $B(z,r)$ on M, any positive solution u of the heat equation $\partial_t u = \Delta_\mu u$ in the cylinder $\mathcal{C} := (0,r^2) \times B(z,r)$ satisfies

$$\sup_{\mathcal{C}_-} u \leq C \inf_{\mathcal{C}_+} u\,, \tag{(PHI)}$$

where $\mathcal{C}_- = (\frac{1}{4}r^2, \frac{1}{2}r^2) \times B(z, \frac{1}{2}r)$, $\;\mathcal{C}_+ = (\frac{3}{4}r^2, r^2) \times B(z, \frac{1}{2}r)$.



The parabolic Harnack inequality is used in the following two deep results.

Theorem 5. *$(PHI) \Leftrightarrow (LY)$, that is, the parabolic Harnack inequality holds if and only if the heat kernel satisfies the Li-Yau estimate.*

Definition. We say that a weighted manifold satisfies the (*weak*) *Poincaré inequality* if there are constants $C > 0$ and $\varepsilon \in (0,1)$ such that, for any ball $B(z,r)$ and for any function $u \in C^1(B(z,r))$,

$$\inf_{s\in\mathbb{R}} \int_{B(z,\varepsilon r)} (u-s)^2 \, d\mu \le Cr^2 \int_{B(z,r)} |\nabla u|^2 \, d\mu. \tag{(PI)}$$

The word "weak" refers to the small coefficient ε in $B(z,\varepsilon r)$. As it is well-known, (PI) holds in $\mathbb{R}^n$ with $\varepsilon = 1$. In this case (PI) can be equivalently restated as follows:

$$\lambda_1^{(N)}(B(z,r)) \ge \frac{1}{Cr^2},$$

where $\lambda_1^{(N)}(\Omega)$ is the first non-zero Neumann eigenvalue of the Laplace operator in Ω. It is known that (PI) holds on any complete Riemannian manifold with $Ricci \ge 0$.

Theorem 6. *$(PHI) \Leftrightarrow (VD) + (PI)$, where (VD) is the volume doubling property, and (PI) is the Poincaré inequality.*

Theorems 5 and 6 can be put together as follows:

$$(PHI) \Leftrightarrow (LY) \Leftrightarrow (VD) + (PI).$$

It is easy to see that both (VD) and (PI) are stable under quasi-isometry. Consequently, we obtain an amazing conclusion: both (PHI) and (LY) are stable under quasi-isometry of weighted manifolds. For example, let us compare two parabolic equations in $\mathbb{R}^n$: the classical heat equation

$$\partial_t u = \Delta u$$

where $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ is the classical Laplace operator, and

$$\partial_t u = Lu$$

where $L = \sum_{i,j=1} \partial_{x_i}\left(a_{ij}(x)\,\partial_{x_j}\right)$ is a uniformly elliptic operator with smooth coefficients. It is easy to show that $L = \Delta_\mu$ for the weighted manifold $(\mathbb{R}^n, g, \mu)$, where $g = (a_{ij})^{-1}$ and μ is the Lebesgue measure. Since $(\mathbb{R}^n, g, \mu)$ and the canonical $\mathbb{R}^n$ are quasi-isometric, we conclude that both (PHI) and (LY) hold also for $\partial_t u = Lu$, which is highly non-trivial.

Let compare Theorems 5,6: under (VD)

$$(PHI) \Leftrightarrow (LY) \Leftrightarrow (PI)$$

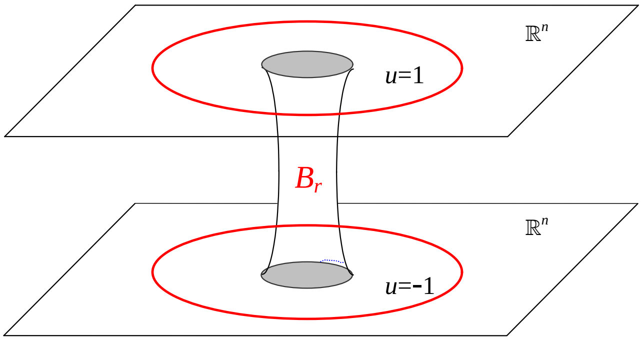
with Theorem 3:

$$(LY_{\le}) \Leftrightarrow (RFK).$$

In particular, we see that, under (VD),

$$(PI) \Rightarrow (RFK).$$

This implication can be proved directly, though. The aforementioned example of $\mathbb{R}^n \# \mathbb{R}^n$ shows that the inverse implication is not true. The fact that $M = \mathbb{R}^n \# \mathbb{R}^n$ does not satisfy (PI) can be seen directly as follows.



In the ball B_r as on the picture consider function u that is equal to 1 on the upper sheet, -1 on the lower sheet, and is between -1 and 1 on the tube connecting the sheets. Assume also that the picture and u are mirror symmetric with respect to the upper and lower sheets. Then the mean value m of u is equal to 0 and

$$\inf_{s\in\mathbb{R}} \int_{B_r} (u-s)^2 \, d\mu = \int_{B_r} (u-m)^2 \, d\mu = \int_{B_r} u^2 d\mu \simeq r^n.$$

On the other hand, ∇u vanishes on the upper and lower sheet and, hence,

$$\int_{B_r} |\nabla u|^2 \, d\mu = \text{const}\,.$$

Consequently,

$$\inf_{s\in\mathbb{R}} \int_{B_r} (u-s)^2 \, d\mu \simeq r^n \int_{B_r} |\nabla u|^2 \, d\mu$$

and (PI) fails for $n > 2$. In fact, a more subtle example shows that (PI) fails also for $n = 2$.

We see that the role of the Poincaré inequality in the equivalence

$$(LY) \Leftrightarrow (PHI) \Leftrightarrow (VD) + (PI)$$

is to prohibit *bottlenecks* on the manifold in question, like the central part in $\mathbb{R}^n \# \mathbb{R}^n$. Those bottlenecks are obstacles for obtaining off-diagonal lower estimates of the heat kernel in (LY).

Chapter 4

Parabolic and non-parabolic manifolds

Let (M, g, μ) be a weighted manifold. A function $u \in C^2(M)$ is called superharmonic if $\Delta_\mu u \leq 0$.

Definition. A weighted manifold M is called *parabolic* if any positive superharmonic function on M is constant, and *non-parabolic* otherwise.

The parabolicity is equivalent to each of the following properties, that can be regarded as equivalent definitions:

1. There exists no positive fundamental solution of $-\Delta_\mu$.
2. $\int^\infty p_t(x, y)\, dt = \infty$ for all/some $x, y \in M$.
3. Brownian motion on M is recurrent.
4. The capacity of any compact set in M is zero.

For example, $\mathbb{R}^n$ is parabolic for $n \leq 2$ and non-parabolic for $n > 2$, because $p_t(x, y) \simeq t^{-n/2}$ as $t \to \infty$.

Theorem 7. *Let M be geodesically complete. If*

$$\int^\infty \frac{r dr}{V(x, r)} = \infty \tag{(1)}$$

then M is parabolic. Conversely, if M is parabolic and satisfies (LY) then (1) *holds.*

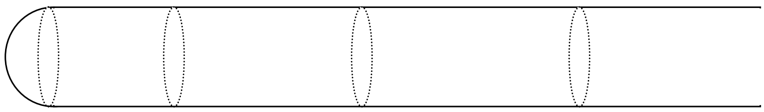
For example, if $V(x, r) \simeq r^\alpha$ then (1) is satisfies if and only if $\alpha \leq 2$.

If M is a simply connected Riemann surface then by the uniformization theorem M is conformally equivalent to either $\mathbb{S}^2$ or $\mathbb{R}^2$ or $\mathbb{H}^2$. In the first case M is called of elliptic type, in the second – of parabolic type and in the third – of hyperbolic type. It is possible to prove that if M is non-compact then M is of parabolic type as a Riemann surface if and only if M is a parabolic manifold in the above sense, which explains this terminology.

Let us give some examples of manifolds satisfying (LY), both parabolic and non-parabolic. Fix some (large) integer N, and for any $n \leq N$ consider the manifold

$$\mathcal{R}^n = \mathbb{R}^n \times \mathbb{S}^{N-n}$$

If $n = 1$ then modify this definition as follows: $\mathcal{R}^1$ is obtained from $\mathbb{R}_+ \times \mathbb{S}^{N-1}$ by closing it into a complete manifold:



The manifold $\mathcal{R}^n$ satisfies (LY) for all n while it is non-parabolic if and only $n > 2$. Fix a reference point $o \in \mathcal{R}^n$. Then, for large r,

$$V(o, r) \simeq r^n.$$

The number n is called the global dimension of $\mathcal{R}^n$ (the dimension at ∞), in contrast to N that is the usual topological (local) dimension.

Consider even general family of manifolds $\mathcal{R}^\alpha$ for all real $\alpha > 0$. Define $\mathcal{R}^\alpha$ as $\left(\mathbb{R}^N, g_\alpha\right)$ where the metric g_α is defined in polar coordinates (r, θ) by

$$g_\alpha = dr^2 + r^{2\beta} d\theta^2$$

for $r > 1$ and is Euclidean for small r, where $\beta = \frac{\alpha-1}{N-1}$. Note that $\alpha = N$ we obtain $\mathcal{R}^N = \mathbb{R}^N$. It is possible to prove that, for large r,

$$V(o, r) \simeq r^\alpha,$$

and that $\mathcal{R}^\alpha$ satisfies (LY) provided $0 < \alpha \leq N$. Clearly, $\mathcal{R}^\alpha$ is parabolic if and only if $\alpha \leq 2$.

Chapter 5

Heat kernels on manifolds with ends

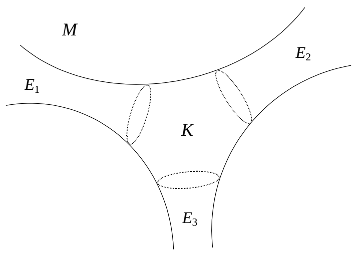
Let $M_1, ..., M_k$ be weighted manifolds. We say that a weighted manifold M is a connected sum of $M_1, ..., M_k$ and write

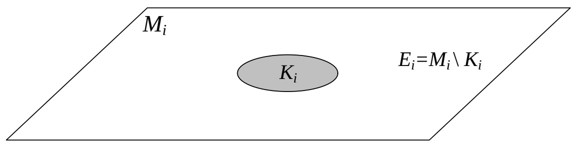
$$M = M_1 \# M_2 \# ... \# M_k$$

if there are compact sets $K \subset M$ and $K_i \subset M_i$ such that

$$M = K \sqcup E_1 \sqcup ... \sqcup E_k$$

where each set E_i is isometric to $M_i \setminus K_i$. The sets E_i are called the *ends* of M (sometimes M_i are also called ends).

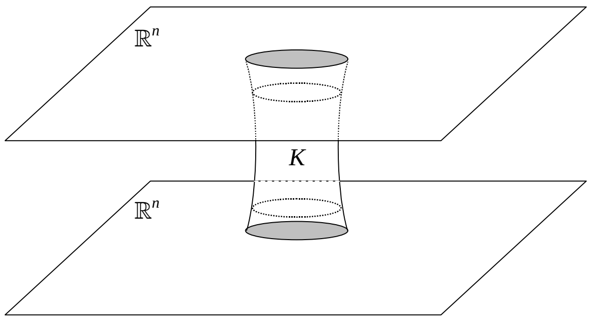




The question to be discussed here is:

Assuming that all M_i are complete and satisfy (LY), how to estimate the heat kernel on M?

As we have seen, this question is non-trivial already for the simplest non-trivial manifold with ends $\mathbb{R}^n \# \mathbb{R}^n$



The answer to this question depends on the property of the ends M_i to be parabolic or not.

Let $M_1, ..., M_k$ be complete manifolds satisfying (LY). Fix a reference point $o_i \in M_i$, set $|x| = d(x, o_i)$. Assume also that

$$V_i(o_i, r) \simeq r^{\alpha_i} \text{ for large } r.$$

Set $M = M_1 \# ... \# M_k$.We present in this setting partial estimates of the heat kernel $p_t(x, y)$ on M when x and y belong to different ends and t is large. Full estimates for all t, x, y are available as well.

Theorem 8. *(Non-parabolic case) Under the above conditions, assume that all $\alpha_i > 2$. Set*

$$\alpha = \min_{1 \le i \le k} \alpha_i \ .$$

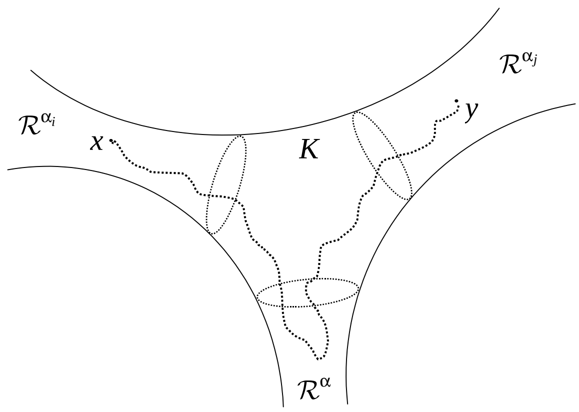
For $x \in E_i$ and $y \in E_j$ with $i \neq j$ we have, for large $|x|, |y|, t$,

$$\begin{aligned} p_t(x, y) &\asymp C\left(\frac{1}{t^{\alpha/2}|x|^{\alpha_i - 2}|y|^{\alpha_j - 2}} + \frac{1}{t^{\alpha_j/2}|x|^{\alpha_i - 2}} + \frac{1}{t^{\alpha_i/2}|y|^{\alpha_j - 2}}\right) \\ &\quad \times \exp\left(-\frac{d^2(x, y)}{ct}\right). \end{aligned} \qquad ((2))$$

For example, (2) holds for $M = \mathcal{R}^{\alpha_1} \# ... \# \mathcal{R}^{\alpha_k}$ if all $\alpha_i > 2$. It follows from (2) that, for fixed x, y,

$$p_t(x, y) \simeq \frac{1}{t^{\alpha/2}} \text{ as } t \to \infty,$$

so that the long time decay of the heat kernel is determined by the end with the *minimal* volume growth. Note for comparison, that $V\left(x, \sqrt{t}\right) \simeq t^{\max \alpha_i/2}$.



In order to get from x to y in time t, the Brownian motion X_t on M spends most time in the end $\mathcal{R}^{\alpha}$. The reason for that is, that the return probability in that end is the largest.

If α_i or α_j is equal to α then the estimate (2) simplifies:

$$p_t(x, y) \asymp C\left(\frac{1}{t^{\alpha_j/2}|x|^{\alpha_i - 2}} + \frac{1}{t^{\alpha_i/2}|y|^{\alpha_j - 2}}\right) \exp\left(-\frac{d^2(x, y)}{ct}\right).$$

In particular, in the case $M=\mathbb{R}^n\#\mathbb{R}^n$, $n>2$, we obtain

$$p_t(x,y)\asymp\frac{C}{t^{n/2}}\left(\frac{1}{|x|^{n-2}}+\frac{1}{|y|^{n-2}}\right)\exp\left(-\frac{d^2(x,y)}{ct}\right).$$

In particular, if $|x|\simeq|y|\simeq\sqrt{t}$ then $p_t(x,y)\simeq\frac{1}{t^{n-1}}\ll\frac{1}{t^{n/2}}$, which means the bottleneck effect.

In Theorems 9 and 10 we assume that all ends M_i satisfy not only (LY) but also (RCA) (see below for explanation).

Theorem 9. *(Strongly parabolic case) Assume that all* $0<\alpha_i<2$. *Set*

$$\alpha=\max_{1\le i\le k}\alpha_i\ .$$

For $x\in E_i$ *and* $y\in E_j$ *with* $i\neq j$ *we have, for large* $|x|,|y|,t,$

$$p_t(x,y)\asymp\frac{C}{t^{\alpha/2}}\exp\left(-\frac{d^2(x,y)}{ct}\right).$$

In this case the long time behavior of the heat kernel $p_t(x,y)\simeq t^{-\alpha/2}$ is determined by the end with the *maximal* volume growth. There is no bottleneck effect in this case.

Theorem 10. *(Mixed case) Assume that all* $\alpha_i\neq2$ *and there is at least one* $\alpha_i>2$. *Set*

$$\widetilde{\alpha}_i:=\begin{cases}4-\alpha_i, & \alpha_i<2\\ \alpha_i, & \alpha_i>2\end{cases}$$

and

$$\alpha:=\min_{1\le i\le k}\widetilde{\alpha}_i.$$

For $x\in E_i$ *and* $y\in E_j$ *with* $i\neq j$ *we have, for large* $|x|,|y|,t,$

$$\begin{aligned}p_t(x,y)\ &\simeq\ C\left(\frac{1}{t^{\alpha/2}|x|^{\widetilde{\alpha}_i-2}|y|^{\widetilde{\alpha}_j-2}}+\frac{1}{t^{\widetilde{\alpha}_i/2}|y|^{\widetilde{\alpha}_j-2}}+\frac{1}{t^{\widetilde{\alpha}_j/2}|x|^{\widetilde{\alpha}_i-2}}\right)\\ &\quad\times|x|^{(2-\alpha_i)_+}\ |y|^{(2-\alpha_j)_+}\ e^{-\frac{d^2(x,y)}{ct}}\end{aligned}\qquad((3))$$

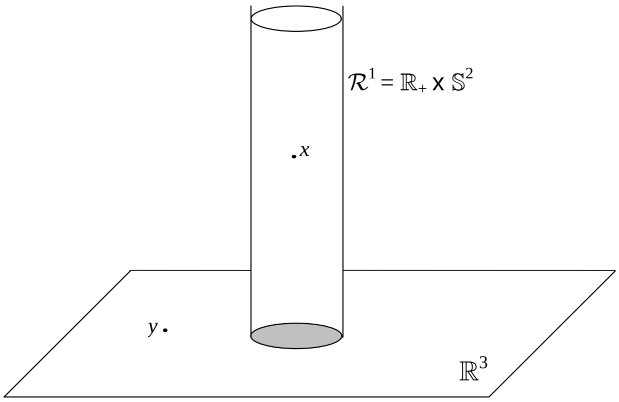
If all $\alpha_i>2$ then $\widetilde{\alpha}_i=\alpha_i$, $(2-\alpha_i)_+=0$ and (3) coincides with (2).

Observe that always $\widetilde{\alpha}_i>2$, and the minimal $\widetilde{\alpha}_i$ is determined by the value of α_i that is *nearest* to 2! Hence, the long time behavior of the heat kernel $p_t(x,y)\simeq t^{-\alpha/2}$ is determined by the end whose global dimension is nearest to 2. This rule applies also to Theorems 8,9.

There are more general theorems that provide heat kernel bounds on $M=M_1\#...\#M_k$ for arbitrary volume functions $V_i(o_i,r)$, including $V_i(o_i,r)\simeq r^2$, but the statement is quite cumbersome. Instead, we give a few examples, containing $\mathcal{R}^2$.

5.1 Some examples

$M = \mathcal{R}^1 \# \mathcal{R}^3$



In this case $\alpha_1 = 1,\ \alpha_2 = 3$ whence $\widetilde{\alpha}_1 = \widetilde{\alpha}_2 = 3$. For $x \in \mathcal{R}^1, y \in \mathcal{R}^3$ we obtain

$$\begin{aligned} p_t(x,y) &\asymp C\left(\frac{1}{t^{3/2}\,|x|} + \frac{1}{t^{3/2}\,|y|}\right)|x| \exp\left(-c\frac{d^2(x,y)}{t}\right) \\ &= \frac{C}{t^{3/2}}\left(1 + \frac{|x|}{|y|}\right)\exp\left(-\frac{d^2\,(x,y)}{ct}\right). \end{aligned}$$

For $t \to \infty$ we obtain $p_t\,(x,y) \simeq t^{-3/2}$. In the case $|y| \simeq 1$, $|x| \simeq \sqrt{t}$ we obtain $p_t\,(x,y) \simeq t^{-1} \gg t^{-3/2}$ – a kind of anti-bottleneck effect!

$M = \mathcal{R}^2 \# \mathcal{R}^3$

For $x \in \mathcal{R}^2, y \in \mathcal{R}^3$ we have

$$p_t(x,y) \asymp C\left(\frac{\log|x|}{t\log^2 t\ \ |y|} + \frac{1}{t^{3/2}}Q\,(x,t)\right)\exp\left(-\frac{d^2\,(x,y)}{ct}\right).$$

where

$$Q\,(x,t) = \frac{1}{\log|x|} + \left(1 - \frac{\log|x|}{\log\sqrt{t}}\right)_+ \simeq \begin{cases} \frac{1}{\log|x|}, & \text{if } |x| > \sqrt{t} \\ \frac{\log\left(2\sqrt{t}\right) - \log|x|}{\log\sqrt{t}}, & \text{if } |x| \le \sqrt{t}, \end{cases}$$

For $t \to \infty$ we have $p_t\,(x,y) \simeq \frac{1}{t\log^2 t}$. The bottleneck effect can be seen for $|x| \simeq |y| \simeq \sqrt{t}$ as in this case

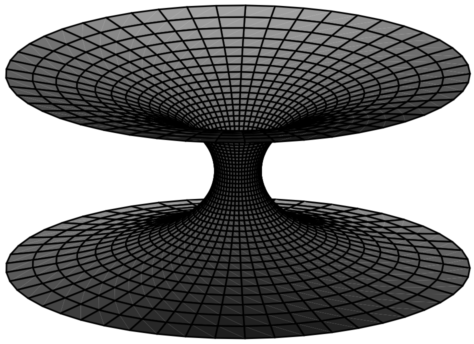
$$p_t\,(x,y) \simeq \frac{1}{t^{3/2}\log t}.$$

The anti-bottleneck effect is also present: for $|x| \simeq \sqrt{t}$ and $|y| \simeq 1$ we have

$$p_t\,(x,y) \simeq \frac{1}{t\log t}.$$

$M = \mathbb{R}^2 \# \mathbb{R}^2$

Equivalently, one can consider the catenoid:



Let x, y belong to different copies of $\mathbb{R}^2$. Then

$$p_t(x,y) \simeq \frac{C}{t}\left(Q(x,t)\frac{\log|y|}{\log|y|+\log t} + Q(y,t)\frac{\log|x|}{\log|x|+\log t} + Q(x,t)Q(y,t)\right) e^{-\frac{d^2(x,y)}{ct}}.$$

If $t \to +\infty$ then $p_t(x,y) \simeq t^{-1}$. If $|x| \geq \sqrt{t}$ and $|y| \geq \sqrt{t}$ then

$$p_t(x,y) \asymp \frac{C}{t}\left(\frac{1}{\log|x|} + \frac{1}{\log|y|}\right)\exp\left(-\frac{d^2(x,y)}{ct}\right).$$

In particular, if $|x| \simeq |y| \simeq \sqrt{t}$ then

$$p_t(x,y) \simeq \frac{1}{t\log t},$$

so that there is a "logarithmic" bottleneck.

$M = \mathcal{R}^1 \# \mathcal{R}^2$,

For $x \in \mathcal{R}^1$ and $y \in \mathcal{R}^2$ we have

$$p_t(x,y) \asymp \frac{C}{t}\left(1 + \left(\frac{|x|}{\sqrt{t}} \wedge 1\right) Q(y,t)\log t\right)\exp\left(-\frac{d^2(x,y)}{ct}\right)$$

If $|x|, |y| > \sqrt{t}$ then

$$p_t(x,y) \asymp \frac{C}{t}\exp\left(-\frac{d^2(x,y)}{ct}\right),$$

If $|x|, |y| \leq \sqrt{t}$ then

$$p_t(x,y) \simeq \frac{1}{t}\left(1 + \frac{|x|}{\sqrt{t}}\left(\log 2\sqrt{t} - \log|y|\right)\right).$$

For $t \to \infty$ we obtain $p_t(x,y) \simeq \frac{1}{t}$. If $y \simeq 1$ and $|x| \simeq \sqrt{t} \to \infty$ and then $p_t(x,y) \simeq \frac{\log t}{t}$.

$M = \mathcal{R}^1 \# \mathcal{R}^2 \# \mathcal{R}^3$.

For $x \in \mathcal{R}^1$ and $y \in \mathcal{R}^2$ we have

$$p_t(x,y) \simeq C\left(\frac{\log|y|}{t\log^2 t} + \left(\frac{|x|}{t^{3/2}} + \frac{1}{t\log^2 t}\right) Q(y,t)\right) e^{-\frac{d^2(x,y)}{ct}}.$$

In particular, for $t \to \infty$

$$p_t(x,y) \simeq \frac{1}{t\log^2 t},$$

For $|x| \simeq |y| \simeq \sqrt{t}$ we have an anti-bottleneck effect:

$$p_t(x,y) \simeq \frac{1}{t\log t}.$$

For $x \in \mathcal{R}^1$ and $y \in \mathcal{R}^3$ we have

$$p_t(x,y) \asymp C\left(\frac{1}{t^{3/2}}\left(1 + \frac{|x|}{|y|}\right) + \frac{1}{|y|\, t\log^2 t}\right) e^{-\frac{d^2(x,y)}{ct}}.$$

For $|x| \simeq |y| \simeq \sqrt{t}$ we have a bottleneck effect:

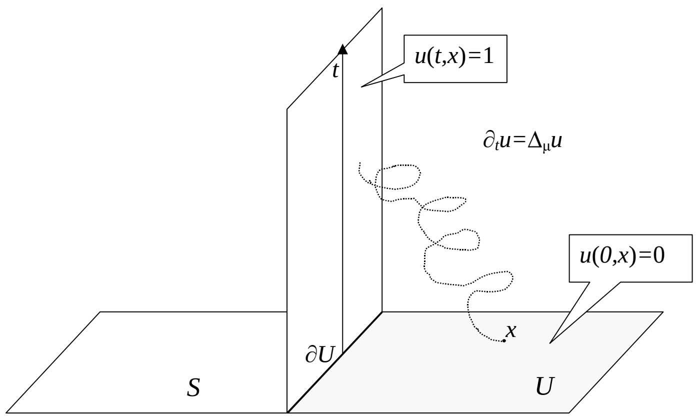
$$p_t(x,y) \simeq \frac{1}{t^{3/2}}.$$

Chapter 6

The gluing techniques for heat kernels

Let M be an arbitrary weighted manifold. For any closed set $S \subset M$, define the function $\psi_S(t,x)$ on $\mathbb{R}_+ \times M$ as the probability that Brownian motion X_t on M hits S before time t provided the starting point is x. This function is called the *hitting probability* of S. The analytic definition is as follows: if $x \in S$ then $\psi_S(t,x) = 1$, and if $x \in U := M \setminus S$ then $\psi_S(t,x)$ is the minimal positive solution of the following boundary value problem in $\mathbb{R}_+ \times U$:

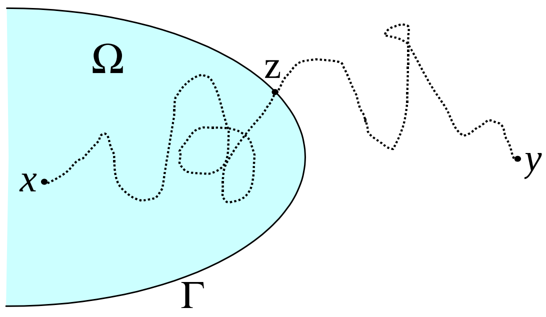
$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_\mu u \\ u|_{x\in\partial U} = 1 \\ u|_{t=0} = 0 \end{cases}$$



The purpose of the next two statements provide the upper and lower bounds of the heat kernel on M in terms of intrinsic geometric properties of some subsets.

Lemma 11. *Let Ω be an open subset of M with the boundary Γ. Then, for all $x \in \Omega$, $y \in M$, $t > 0$, the following estimate holds:*

$$p_t(x,y) \geq \psi_\Gamma\left(\frac{t}{2},x\right) \inf_{\substack{t/2\leq s\leq t \\ z\in\Gamma}} p_s(z,y) \ . \tag{(4)}$$



This estimate corresponds to the following way for Brownian motion to get from x to y in time t: first hit the boundary Γ before time $t/2$, at a random point z, then get from z to y in the remaining (random) time $s \in [t/2, t]$.

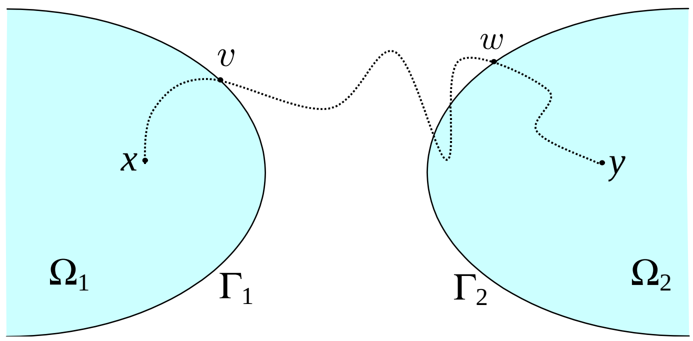
Lemma 12. *Let Ω_1 and Ω_2 be two disjoint open subsets of M with the boundaries Γ_1 and Γ_2 respectively. Then, for all $x \in \Omega_1$, $y \in \Omega_2$, and $t > 0$,*

$$p_t(x,y) \geq \psi_{\Gamma_1}(\frac{t}{4}, x)\psi_{\Gamma_2}(\frac{t}{4}, y) \inf_{s\in[t/4,t]} \inf_{v\in\Gamma_1, w\in\Gamma_2} p_s(v,w) \qquad ((5))$$

and

$$\begin{aligned} p_t(x,y) \quad &\leq \quad 2\psi_{\Gamma_1}(t,x)\psi_{\Gamma_2}(t,y) \sup_{s\in[t/4,t]} \sup_{v\in\Gamma_1, w\in\Gamma_2} p_s(v,w) \\ &+ \left(\psi_{\Gamma_1}(t,x) \sup_{s\in[t/4,t]} \psi'_{\Gamma_2}(s,y) + \psi_{\Gamma_2}(t,y) \sup_{s\in[t/4,t]} \psi'_{\Gamma_1}(s,x)\right) \qquad ((6)) \\ &\times \int_0^t \sup_{v\in\Gamma_1, w\in\Gamma_2} p_s(v,w)ds \end{aligned}$$

(here ψ'_Γ stands for $\frac{\partial}{\partial t}\psi_\Gamma$; note that $\psi'_\Gamma \geq 0$).



Chapter 7

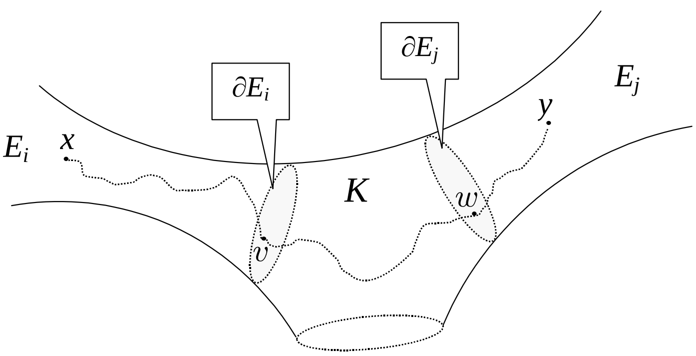
Approach to the proof of Theorems 8,9,10

Let $M = M_1\#...\#M_k$, where each M_i is complete and satisfies (LY). Let $p_t(x,y)$ be the heat kernel on M. Set $V_i(r) = V_i(o_i, r)$.

7.1 Non-parabolic case (Theorem 8)

7.1.1 Upper bounds

Let us use the estimate (6) of Lemma 12 in the following situation. Fix $i \neq j$ and set $\Omega_1 = E_i$, $\Gamma_1 = \partial E_i$, $\Omega_2 = E_j$, $\Gamma_2 = \partial \Omega_2$.



Then, for $x \in E_i$ and $y \in E_j$, we have

$$\begin{aligned} p_t(x,y) &\leq 2\psi_{\partial E_i}(t,x)\psi_{\partial E_j}(t,y) \sup_{s\in[t/4,t]} \sup_{v\in\partial E_i, w\in\partial E_j} p_s(v,w) \\ &\quad + \left(\psi_{\partial E_i}(t,x) \sup_{s\in[t/4,t]} \psi'_{\partial E_j}(s,y) + \psi_{\partial E_j}(t,y) \sup_{s\in[t/4,t]} \psi'_{\partial E_i}(s,x)\right) \qquad ((6')) \\ &\quad \times \int_0^t \sup_{v\in\partial E_i, w\in\partial E_j} p_s(v,w) ds. \end{aligned}$$

Note that the hitting probability $\psi_{\partial E_i}(t,x)$ is determined by the intrinsic properties of E_i and can be estimated if M_i satisfies (LY). Its time derivative can also be estimated.

The points v, w, used in the heat kernel $p_s(v,w)$ in the right hand side of $(6')$, belong to K. By the local Harnack inequality we have $p_s(v,w) \simeq p_s(o,o)$ for a fixed reference point $o \in K$. We prove the following upper bound

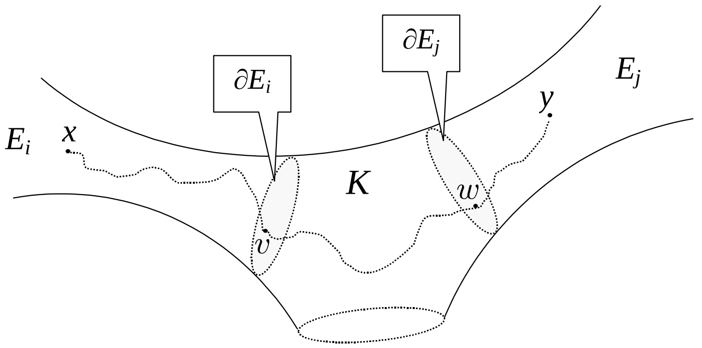
$$p_s(o,o) \leq \frac{C}{V_{\min}(\sqrt{s})}$$

where $V_{\min}(r) = \min_i V_i(r)$. This estimate is obtained by "gluing" relative Faber-Krahn inequalities on all M_i, which allows to obtain a weaker Faber-Krahn inequality on M involving the minimal volume function $V_{\min}(r)$. Inserting the on-diagonal heat kernel bound and the estimates of the hitting probabilities into $(6')$, we obtain the desired upper bound for $p_t(x,y)$.

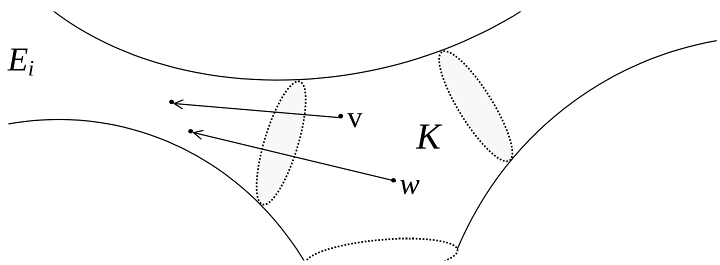
7.1.2 Lower bounds

Using estimate (5) of Lemma 12, we obtain, for $x \in E_i$, $y \in E_j$, $i \neq j$ that

$$p_t(x,y) \geq \left(\inf_{s\in[t/4,t]} \inf_{v,w\in K} p_s(v,w) \right) \psi_{\partial E_i}(t/4,x) \psi_{\partial E_j}(t/4,y). \qquad ((5'))$$



The lower bounds for the hitting probabilities can be obtained again using (LY) on each M_i. To obtain the lower bound for $p_s(v,w)$ for $v,w \in K$, let us shift them from K towards E_i using local Harnack inequality.



As soon as $v, w \in E_i$, we observe that

$$p_s(v,w) \geq p_s^{E_i}(v,w),$$

where $p_s^{E_i}$ is the Dirichlet heat kernel in E_i vanishing on ∂E_i. Using the non-parabolicity of M_i it is possible to show that p_t^E satisfies also (LY) away from ∂E_i. It follows that

$$p_s(v,w) \geq \frac{c}{V_i(o_i,\sqrt{s})},$$

for any i, which implies then

$$p_s(v,w) \geq \frac{c}{V_{\min}(\sqrt{s})}.$$

Substituting into $(5')$ gives us a part of the desired lower bound of $p_t(x,y)$. The other parts are obtained in a similar way from the estimate (4) of Lemma 11.

7.2 Parabolic case (Theorem 9)

In this case the main difficulty lies in proving the on-diagonal upper bound of the form

$$p_t(o,o) \le \frac{C}{V\left(o,\sqrt{t}\right)}, \qquad ((7))$$

where in fact $V(o,r) \simeq V_{\max}(r) := \max_i V_i(r)$. Note that in the previous cases the estimate (7) is simply not true. If (7) does hold then we obtain by Theorem 4 also a matching lower bound

$$p_t(o,o) \ge \frac{c}{V\left(o,\sqrt{t}\right)}, \qquad ((8))$$

since the function $V(o,r)$ is doubling (at fixed point o). The method of obtaining (7) involves estimating of the resolvents on each end:

$$R_\lambda^{(i)}(x,y) = \int_0^\infty e^{-t\lambda} p_t^{(i)}(x,y)\,dt$$

and a certain relation between the resolvents on M and M_i. Obtaining the resolvent on M, one can extract then the upper bound for $p_t(o,o)$.

As soon as (7) and (8) are present, the rest is done using the estimates of hitting probabilities and Lemmas 11,12, as in the proof of Theorem 8.

7.3 Mixed case (Theorem 10)

The above approach partially works but does not yield optimal upper and lower bound. The strategy in the mixed case is to reduce to the non-parabolic case by using an appropriate change of measure. Along with the weighted manifold (M,g,μ), consider a weighted manifold $(M,g,\widetilde{\mu})$, where

$$d\widetilde{\mu} = h^2\,d\mu$$

and h is a positive **harmonic** function on (M,g,μ) (that is, $\Delta_\mu h = 0$). A direct computation shows that

$$\Delta_{\widetilde{\mu}} = \frac{1}{h} \circ \Delta_\mu \circ h,$$

where h and $\frac{1}{h}$ are regarded as multiplication operators. The operator $\frac{1}{h} \circ \Delta_\mu \circ h$ is called *h-transform* of Δ_μ. The operators Δ_μ in $L^2(\mu)$ and $\Delta_{\widetilde{\mu}}$ in $L^2(\widetilde{\mu})$ are conjugate, which implies that the heat kernel $\widetilde{p}_t(x,y)$ on $(M,g,\widetilde{\mu})$ is related to the heat kernel $p_t(x,y)$ on (M,g,μ) by a simple identity:

$$p_t(x,y) = \widetilde{p}_t(x,y)\,h(x)\,h(y). \qquad ((9))$$

Hence, it suffices to estimate $h(x)$ and $\widetilde{p}_t(x,y)$.

Extending h from each end E_i to the whole manifold M_i (not necessarily as a harmonic function), one obtains in the same way a new measure $\widetilde{\mu}_i$ on M_i. A crucial observation: if $h(x) \to \infty$ as $|x| \to \infty$ on M_i then $(M_i, g_i, \widetilde{\mu}_i)$ is non-parabolic! Indeed, we have

$$\Delta_{\widetilde{\mu}} \frac{1}{h} = \frac{1}{h}\Delta_\mu 1 = 0$$

so that $\frac{1}{h}$ is harmonic on $(M,g,\widetilde{\mu})$. Hence, $\frac{1}{h}$ is harmonic outside a compact on M_i with respect to the new weighted structure $(M_i, g_i, \widetilde{\mu}_i)$, while $\frac{1}{h}(x) \to 0$ as $x \to \infty$. The presence of such a harmonic function implies the existence of a positive fundamental solution of $-\Delta_{\widetilde{\mu}}$ and, hence, the non-parabolicity.

In order to implement this approach, one needs two ingredients:

1. Constructing of a positive harmonic function on M that goes to $+\infty$ on any parabolic end, and estimating this function.

2. Verifying that if (M_i, g_i, μ_i) satisfies (LY) then also $(M_i, g_i, \widetilde{\mu}_i)$ satisfies (LY).

Part 2 is most non-trivial and amounts to the stability of the parabolic Harnack inequality under a *non-uniform* change of measure. We will return to this problem in Theorem 13 below.

Concerning Part 1. Since there is i with $\alpha_i > 2$, the connected sum M is non-parabolic. Then one proves that there exists a positive harmonic function h on M that satisfies the following properties:

1. on a non-parabolic end E_i we have $h(x) \simeq 1$;

2. on a parabolic end E_i (when $\alpha_i < 2$), we have $h(x) \simeq |x|^{2-\alpha_i}$ for large $|x|$ (this is similar to the fact that $|x|^{2-n}$ is harmonic in $\mathbb{R}^n$).

Furthermore, each end $(M_i, \widetilde{\mu}_i)$ satisfies (LY)!

If (M_i, μ_i) parabolic then the volume function on $(M_i, \widetilde{\mu}_i)$ is given by

$$\widetilde{V}_i(r) = \int_{B_i(r)} h^2(x)\, d\mu_i(x) \simeq r^{2(2-\alpha_i)} V_i(r) \simeq r^{2(2-\alpha_i)} r^{\alpha_i} \simeq r^{4-\alpha_i} = r^{\widetilde{\alpha}_i}.$$

If (M_i, μ_i) is non-parabolic, then the volume function on $(M_i, \widetilde{\mu}_i)$ is

$$\widetilde{V}_i(r) \simeq V_i(r) \simeq r^{\alpha_i} = r^{\widetilde{\alpha}_i}.$$

Recall that $\widetilde{\alpha}_i = 4 - \alpha_i$ if $\alpha_i < 2$ and $\widetilde{\alpha}_i = \alpha_i$ of $\alpha_i > 2$. Since always $\widetilde{\alpha}_i > 2$, all the ends $(M_i, \widetilde{\mu}_i)$ are non-parabolic! We apply Theorem 8 and obtain, for $x \in E_i$ and $y \in E_j$ with $i \neq j$,

$$\widetilde{p}_t(x,y) \asymp C\left(\frac{1}{t^{\alpha/2}|x|^{\widetilde{\alpha}_i-2}|y|^{\widetilde{\alpha}_j-2}} + \frac{1}{t^{\widetilde{\alpha}_i/2}|y|^{\widetilde{\alpha}_j-2}} + \frac{1}{t^{\widetilde{\alpha}_j/2}|x|^{\widetilde{\alpha}_i-2}}\right) e^{-\frac{d^2(x,y)}{ct}}$$

where $\alpha = \min_i \widetilde{\alpha}_i$. Since on each end $h(x) \simeq |x|^{(2-\alpha_i)_+}$, we obtain from (9)

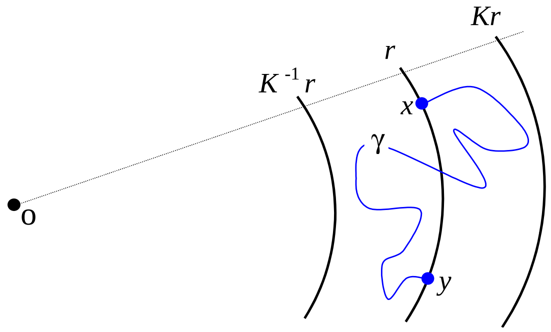
$$\begin{aligned} p_t(x,y) &\asymp C\left(\frac{1}{t^{\alpha/2}|x|^{\widetilde{\alpha}_i-2}|y|^{\widetilde{\alpha}_j-2}} + \frac{1}{t^{\widetilde{\alpha}_i/2}|y|^{\widetilde{\alpha}_j-2}} + \frac{1}{t^{\widetilde{\alpha}_j/2}|x|^{\widetilde{\alpha}_i-2}}\right) \\ &\quad \times |x|^{(2-\alpha_i)_+}|y|^{(2-\alpha_j)_+} e^{-\frac{d^2(x,y)}{ct}}, \end{aligned}$$

which finishes the proof of Theorem 10.

Chapter 8

Stability of (LY) under a non-uniform change of measure

Let (M, g) be a geodesically complete Riemannian manifold. We say that M has *relatively connected annuli* (shortly, (RCA)) if there exists a point $o \in M$ (called a *pole*) and $K > 1$ such that, for all large enough r, any two points $x, y \in M$, such that $d(o, x) = d(o, y) = r$, can be connected by a continuous path in $B(o, Kr) \setminus B(o, K^{-1}r)$.



For example, $\mathbb{R}^n$ satisfies (RCA) provided $n \geq 2$, but $\mathbb{R}^1$ does not. In fact, In Theorems 9,10 we have to assume that all ends (M_i, g_i) satisfy not only (LY) but also (RCA).

Let (M, g) satisfies (RCA) with a pole o. Set $\langle x \rangle := 1 + d(o, x)$ and $V(r) := V(o, r)$.

Theorem 13. *Let (M, g, μ) be a geodesically complete weighted manifold satisfying (RCA) and (LY). Assume also that $V(r) \simeq r^\alpha$, $\alpha > 0$. Let h be a smooth positive function on M satisfying*

$$h(x) \simeq \langle x \rangle^\beta \quad .$$

Define measure $\widetilde{\mu}$ on M by

$$d\widetilde{\mu} = h^2 d\mu.$$

If $\beta > -\alpha/2$ then $(M, g, \widetilde{\mu})$ also satisfies (LY). The latter can be equivalently stated as follows:

$$\widetilde{p}_t(x, y) \asymp \frac{C}{V\left(x, \sqrt{t}\right)\left(\langle x \rangle + \sqrt{t}\right)^\beta \left(\langle y \rangle + \sqrt{t}\right)^\beta} \exp\left(-\frac{d^2(x, y)}{ct}\right). \tag{$(\widetilde{LY})$}$$

In the proof of Theorem 10 we used Theorem 13 on each parabolic end with $\beta = 2 - \alpha_i > 0$. Theorem 13 can be stated for a general volume growth function $V(r)$ and more general functions h.

Example in $\mathbb{R}^n$.

Let $(M,g)=\mathbb{R}^n$, $n\geq 2$, so that (RCA) holds. Let μ be the Lebesgue measure, so that $(\mathbb{R}^n,\mu)$ satisfies (LY). Let h be a smooth positive function on $\mathbb{R}^n$ such that

$$h(x)\simeq \langle x\rangle^{\beta}, \quad \text{where } \langle x\rangle := 1+|x|,$$

where $\beta>-n/2$. Then the heat kernel $\widetilde{p}_t(x,y)$ of $(\mathbb{R}^n,\widetilde{\mu})$ satisfies $(\widetilde{LY})$, that is,

$$\widetilde{p}_t(x,y)\asymp \frac{C}{t^{n/2}\left(\langle x\rangle+\sqrt{t}\right)^{\beta}\left(\langle y\rangle+\sqrt{t}\right)^{\beta}}\exp\left(-\frac{|x-y|^2}{ct}\right). \qquad ((10))$$

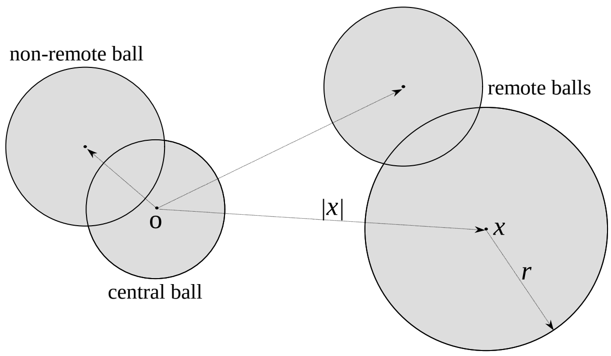
In particular, if $t\geq \langle x\rangle^2+\langle y\rangle^2$ then $\widetilde{p}_t(x,y)\simeq t^{-(n/2+\beta)}$.

Approach to the proof.

By Theorems 5 and 6, it suffices to prove that (VD) and (PI) are stable under the aforementioned change of measure. Let us first develop some tools how one can prove that (VD) and (PI) hold in all balls for a given measure μ.

Let (M,g,μ) be an arbitrary geodesically complete manifold, satisfying (RCA) with the pole o.

Definition. A ball $B(x,r)\subset M$ is called *remote* if $r<\frac{1}{2}|x|$. A ball $B(x,r)$ is called *central* if $x=o$.



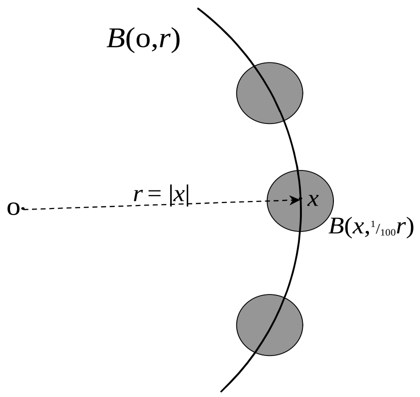
Lemma 14. *Conditions (VD), (PI) hold for for **all** ball $\Longleftrightarrow$ they hold for all **remote** and **central** balls.*

Idea: any non-remote ball can be approximated by a larger central ball.

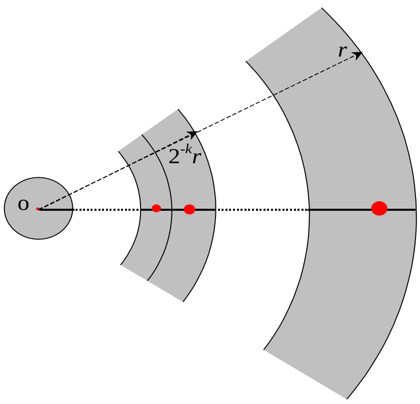
Lemma 15. *Conditions (VD), (PI) hold for **all** balls $\Longleftrightarrow$ (VD),(PI) hold for all **remote** balls and, for all $x\in M$ and $r=|x|$,*

$$V(o,r)\leq CV\left(x,\frac{1}{100}r\right). \qquad ((VC))$$

Here (VC) stands for "volume comparison". Condition (VC) implies (VD) for central balls and is used to prove (PI) in central balls (which is enough for the proof of Lemma 15).



For the proof of Lemma 15, we split $B(o,r)$ into dyadic annuli and consider them as vertices in a linear weighted graph. Then (VC) implies a *discrete* (PI) on this graph, which can be then "merged" with (PI) in each annulus, which is true because any annulus is remote.



For the proof of Theorem 13 observe that the function $h(x)$ is "nearly" constant on any remote ball, which implies that (VD), (PI) for measure $\widetilde{\mu}$ hold trivially in all remote balls. Then one shows that $\beta > -\alpha/2$ implies (VC) for measure $\widetilde{\mu}$. Hence, by Lemma 15, we obtain that (VD), (PI) for measure $\widetilde{\mu}$ hold in **all** balls, which finishes the proof.

Chapter 9

Heat kernels of Schrödinger operators

Consider an elliptic Schrödinger operator

$$L = -\Delta + \Phi(x),$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in $\mathbb{R}^n$ and Φ is a smooth function in $\mathbb{R}^n$. Denote by $p_t^{\Phi}(x,y)$ the heat kernel of L. We show how some of the above techniques can be used to prove the existence and to obtain estimates of $p_t^{\Phi}(x,y)$ for some examples of Φ.

Assume in the sequel that the equation

$$-\Delta h + \Phi h = 0$$

has a solution $h(x) > 0$ in $\mathbb{R}^n$. Consider the weighted manifold $(\mathbb{R}^n, \widetilde{\mu})$ where $d\widetilde{\mu} = h^2 d\mu$ and μ is the Lebesgue measure, and the corresponding weighted Laplacian

$$\Delta_{\widetilde{\mu}} = \frac{1}{h^2} \operatorname{div}\left(h^2 \nabla\right).$$

Denote by $\widetilde{p}_t(x,y)$ the heat kernel of $(\mathbb{R}^n, \widetilde{\mu})$.

Using $-\Delta h + \Phi h = 0$, we compute

$$\begin{aligned} \Delta_{\widetilde{\mu}} u &= \Delta u + 2\frac{\nabla h}{h}\nabla u = \frac{1}{h}\left(h\Delta u + 2\nabla h \nabla u + \Delta h\ u\right) - \Phi u \\ &= \frac{1}{h}\left(\Delta(hu) - \Phi hu\right) \end{aligned}$$

whence

$$\Delta_{\widetilde{\mu}} = \frac{1}{h} \circ (\Delta - \Phi) \circ h.$$

Hence, the operators $\Delta_{\widetilde{\mu}}$ in $L^2(\widetilde{\mu})$ and $\Delta - \Phi$ in $L^2(\mu)$ are conjugate, which implies

$$p_t^{\Phi}(x,y) = \widetilde{p}_t(x,y)\, h(x)\, h(y). \qquad ((11))$$

In order to implement this method, one should to solve the following two problems:

1. Prove the existence of a positive solution to $-\Delta h + \Phi h = 0$ and estimate $h(x)$.

2. Estimate the heat kernel $\widetilde{p}_t(x,y)$ of $(\mathbb{R}^n, \widetilde{\mu})$.

Green bounded potentials.

Let $G(x)$ denote the Green function of $-\Delta$ in $\mathbb{R}^n$, that is,

$$G(x) = \begin{cases} c_n |x|^{2-n}, & n > 2 \\ +\infty, & n \leq 2. \end{cases}$$

A potential Φ in $\mathbb{R}^n$ is called *Green bounded*, if

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} G(x-y) |\Phi(y)| \, dy < \infty.$$

Note that in $\mathbb{R}^1$ and $\mathbb{R}^2$ only $\Phi \equiv 0$ is Green bounded, whereas in $\mathbb{R}^n$ any Φ satisfying the estimate

$$|\Phi(x)| \leq C \langle x \rangle^{-(2+\varepsilon)}$$

is Green bounded, provided $\varepsilon > 0$.

One can prove that if $\Phi \geq 0$ is Green bounded then there is a solution $h(x)$ of $-\Delta h + \Phi h = 0$ that is bounded between two positive constants. In this case the operator $\Delta_{\widetilde{\mu}}$ is uniformly elliptic and his heat kernel $\widetilde{p}_t(x,y)$ satisfies (LY). Hence, the same estimate holds for $p_t^\Phi(x,y)$, that is,

$$p_t^\Phi(x,y) \asymp \frac{C}{t^{n/2}} \exp\left(-\frac{|x-y|^2}{ct}\right).$$

Hence, Green bounded potentials are *small perturbations* as far as heat kernel bounds are concerned.

Compactly supported potentials in $\mathbb{R}^2$.

Let $\Phi \geq 0$ be a non-zero potential with a compact support in $\mathbb{R}^2$. Then it is possible to show that the equation $\Delta h - \Phi h = 0$ has a positive solution h such that $h(x) \simeq \log\langle x \rangle$. By extension of Theorem 13 to such functions, we obtain

$$\widetilde{p}_t(x,y) \asymp \frac{C}{t \log\left(\langle x \rangle + \sqrt{t}\right) \log\left(\langle y \rangle + \sqrt{t}\right)} \exp\left(-\frac{|x-y|^2}{ct}\right),$$

whence

$$p_t^\Phi(x,y) = \widetilde{p}_t(x,y)\, h(x)\, h(y) \asymp \frac{C \log\langle x \rangle \log\langle y \rangle}{t \log\left(\langle x \rangle + \sqrt{t}\right) \log\left(\langle y \rangle + \sqrt{t}\right)} \exp\left(-\frac{|x-y|^2}{ct}\right).$$

In particular, in the long time range $t \geq \langle x \rangle^2 + \langle y \rangle^2$, we have

$$p_t^\Phi(x,y) \simeq \frac{\log\langle x \rangle \log\langle y \rangle}{t \log^2 t}.$$

Potential $\Phi(x) = b|x|^{-2}$ in $\mathbb{R}^n$, $n \geq 2$.

Given a real parameter β, define a function h on $\mathbb{R}^n$ by setting $h(x) = |x|^\beta$ for $|x| > 1$ and extending h positively and smoothly to $|x| \leq 1$. Then define function Φ by

$$\Phi(x) := \frac{\Delta h}{h} \text{ for all } x \in \mathbb{R}^n,$$

so that $\Delta h - \Phi h = 0$. Computing Δh, we obtain

$$\Phi(x) = \frac{b}{|x|^2} \text{ for } |x| > 1,$$

where $b = \beta^2 + (n-2)\beta$. In practice it is convenient first to give b and then find β from this quadratic equation:

$$\beta = -\frac{n}{2} + 1 + \sqrt{\left(\frac{n}{2} - 1\right)^2 + b}, \tag{(12)}$$

assuming that $b \geq -\left(\frac{n}{2} - 1\right)^2$. Obviously, we have $\beta > -n/2$ so that Theorem 13 applies. By (10) we obtain the following estimate for the heat kernel of $(\mathbb{R}^n, \widetilde{\mu})$:

$$\widetilde{p}_t(x,y) \asymp \frac{C}{t^{n/2}\left(\langle x\rangle + \sqrt{t}\right)^\beta \left(\langle y\rangle + \sqrt{t}\right)^\beta} \exp\left(-\frac{|x-y|^2}{ct}\right).$$

Using the relation $p_t^\Phi(x,y) = \widetilde{p}_t(x,y)\, h(x)\, h(y)$, we obtain

$$p_t^\Phi(x,y) \asymp \frac{C}{t^{n/2}}\left(1 + \frac{\sqrt{t}}{\langle x\rangle}\right)^{-\beta}\left(1 + \frac{\sqrt{t}}{\langle y\rangle}\right)^{-\beta} \exp\left(-\frac{|x-y|^2}{ct}\right). \tag{(13)}$$

In particular, in the most interesting long time range $t \geq \langle x\rangle^2 + \langle y\rangle^2$ we have

$$p_t^\Phi(x,y) \simeq C\frac{\langle x\rangle^\beta \langle y\rangle^\beta}{t^{n/2+\beta}},$$

where β depends on b as in (12).

Let us compare heat kernel for different examples of Φ, assuming $b, \varepsilon > 0$.

1. (*subcritical case*) If $\Phi(x) = b|x|^{-(2+\varepsilon)}$ and $n > 2$ then the long time estimate $p_t^\Phi(x,y) \simeq t^{-n/2}$ does not see Φ.
2. (*critical case*) If $\Phi(x) = b|x|^{-2}$ then the exponent $n/2 + \beta$ in the long time regime depends on b and can be arbitrarily large.
3. (*supercritical case*) If $\Phi(x) = b|x|^{-(2-\varepsilon)}$ then

$$p_t^\Phi(x,y) \asymp C\exp\left(-ct^{\frac{\varepsilon}{4+\varepsilon}}\right) \text{ as } t \to \infty.$$

Potential $\Phi(x) = b|x|^{-2}$ in $\mathbb{R}^1$.

The previous argument works only if $n \geq 2$ because of (RCA), which is not satisfied on $\mathbb{R}^1$. In fact, (LY) does not hold on $(\mathbb{R}, \widetilde{\mu})$ but one can still obtain the estimates of $\widetilde{p}_t(x,y)$ by considering $(\mathbb{R}, \widetilde{\mu})$ as a connected sum of two ends $(\mathbb{R}_+, \widetilde{\mu})$ and $(\mathbb{R}_-, \widetilde{\mu})$. Each end satisfies (LY) by Theorem 13, and the volume growth function at each end is

$$V_i(r) \simeq h^2(r)\, r \simeq r^{2\beta+1} = r^\alpha \text{ where } \alpha = 2\beta + 1.$$

Assuming $b > -\frac{1}{4}$, we obtain

$$\beta = \frac{1}{2} + \sqrt{b + \frac{1}{4}} > \frac{1}{2}$$

and, hence, $\alpha > 2$. Hence, the both ends are non-parabolic! By Theorem 8, we obtain the following estimate for the heat kernel on $(\mathbb{R}, \widetilde{\mu})$ assuming that x, y have different signs:

$$\widetilde{p}_t(x, y) \asymp \frac{C}{t^{\alpha/2}} \left(\frac{1}{\langle x \rangle^{\alpha-2}} + \frac{1}{\langle y \rangle^{\alpha-2}} \right) \exp\left(-c \frac{|x-y|^2}{t} \right),$$

which by (11) implies

$$p_t^{\Phi}(x, y) \asymp \frac{C \langle x \rangle^{\beta} \langle y \rangle^{\beta}}{t^{\beta+1/2}} \left(\frac{1}{\langle x \rangle^{2\beta-1}} + \frac{1}{\langle y \rangle^{2\beta-1}} \right) \exp\left(-c \frac{|x-y|^2}{t} \right).$$

Recall for comparison that in the case $n \geq 2$

$$p_t^{\Phi}(x, y) \simeq \frac{\langle x \rangle^{\beta} \langle y \rangle^{\beta}}{t^{n/2+\beta}}, \qquad t \to \infty,$$

whereas in the case $n = 1$ the additional term $\frac{1}{\langle x \rangle^{2\beta-1}} + \frac{1}{\langle y \rangle^{2\beta-1}}$ appears that is responsible for the bottleneck effect.

Assuming $b = -\frac{1}{4}$, we obtain $\beta = \frac{1}{2}$ and, hence, $\alpha = 2$. Hence, we obtain on $(\mathbb{R}, \widetilde{\mu})$ the same estimate as in the case $\mathbb{R}^2 \# \mathbb{R}^2$, which results in the following:

If $t \to +\infty$ then

$$p_t^{\Phi}(x, y) \simeq \frac{\langle x \rangle^{1/2} \langle y \rangle^{1/2}}{t}.$$

If $|x| \geq \sqrt{t}$ and $|y| \geq \sqrt{t}$ then

$$p_t^{\Phi}(x, y) \asymp \frac{C \langle x \rangle^{1/2} \langle y \rangle^{1/2}}{t} \left(\frac{1}{\log \langle x \rangle} + \frac{1}{\log \langle y \rangle} \right) \exp\left(-\frac{d^2(x, y)}{ct} \right).$$

In particular, if $|x| \simeq |y| \simeq \sqrt{t}$ then

$$p_t^{\Phi}(x, y) \simeq \frac{1}{\log t}.$$

Chapter 10

Heat kernel in $\mathbb{H}^3$

This topic is a bit disconnected from the main part. The purpose of its inclusion is to show a simple method of obtaining an explicit formula for the heat kernel in the hyperbolic space $\mathbb{H}^3$, using h-transform.

Claim. *For all $x, y \in \mathbb{H}^3$ and $t > 0$, the heat kernel of $\mathbb{H}^3$ is given by*

$$p_t(x,y) = \frac{1}{(4\pi t)^{3/2}} \frac{r}{\sinh r} \exp\left(-\frac{r^2}{4t} - t\right), \tag{(14)}$$

where $r = d(x,y)$.

Because of homogeneity of $\mathbb{H}^3$, it suffices to prove (14) for a fixed point $y = o$.Consider the polar coordinates (r,θ) in $\mathbb{H}^3$ centered at o. The hyperbolic metric g is given in the polar coordinates by

$$g = dr^2 + \sinh^2 r d\theta^2.$$

Denoting by $S(r)$ the boundary area of the geodesic ball $B(o,r)$ in $\mathbb{H}^3$, we obtain hence

$$S(r) = 4\pi \sinh^2 r.$$

The radial part of the Laplace operator Δ on $\mathbb{H}^3$ in the coordinates (r,θ) is as follows:

$$\begin{aligned} \Delta^{\mathrm{rad}} &= \frac{\partial^2}{\partial r^2} + \frac{S'(r)}{S(r)} \frac{\partial}{\partial r} \qquad ((15)) \\ &= \frac{\partial^2}{\partial r^2} + 2\coth r \frac{\partial}{\partial r}. \end{aligned}$$

Consider the function on $\mathbb{H}^3 \setminus \{o\}$

$$h(x) = \frac{r}{\sinh r}$$

where $r = d(x,o)$, which obviously extends to a smooth function on $\mathbb{H}^3$. Computing $\Delta^{\mathrm{rad}} h$, one easily proves that h satisfies in $\mathbb{H}^3$ the equation

$$\Delta h + h = 0. \tag{(16)}$$

Denote by μ the Riemannian measure on $\mathbb{H}^3$ and consider now a new measure $d\widetilde{\mu} = h^2 d\mu$. By (16), the heat kernels $p_t(x,y)$ of $\left(\mathbb{H}^3, \mu\right)$ and $\widetilde{p}_t(x,y)$ of $\left(\mathbb{H}^3, \widetilde{\mu}\right)$ are related by the identity

$$p_t(x,y) = \widetilde{p}_t(x,y) h(x) h(y) e^{-t}.$$

Now let us compute $\widetilde{p}_t(x,o)$. Observe that the boundary area function of the weighted manifold $\left(\mathbb{H}^3,\widetilde{\mu}\right)$ is

$$\widetilde{S}(r) = h^2(r)\,S(r) = \left(\frac{r}{\sinh r}\right)^2 4\pi \sinh^2 r = 4\pi r^2,$$

that is exactly the same as in $\mathbb{R}^3$. Since the radial part $\Delta_{\widetilde{\mu}}^{\mathrm{rad}}$ is determined by $\widetilde{S}(r)$ (like in (15)), we see that $\Delta_{\widetilde{\mu}}^{\mathrm{rad}}$ coincides with the radial part of the Laplacian in $\mathbb{R}^3$, that is,

$$\Delta_{\widetilde{\mu}}^{\mathrm{rad}} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}.$$

This means that the heat kernel $\widetilde{p}_t(x,o)$ as a function of $r = d(x,o)$ and t satisfies the same PDE as the heat kernel in $\mathbb{R}^3$, which implies that $\widetilde{p}_t(x,o)$ is given by the same formula as the heat kernel in $\mathbb{R}^3$, that is

$$\widetilde{p}_t(x,o) = \frac{1}{(4\pi t)^{3/2}} \exp\left(-\frac{r^2}{4t}\right).$$

Finally, we obtain

$$p_t(x,o) = \widetilde{p}_t(x,o)\,h(x)\,h(o)\,e^{-t} = \frac{1}{(4\pi t)^{3/2}}\frac{r}{\sinh r}\exp\left(-\frac{r^2}{4t} - t\right),$$

which was to be proved.