Heat kernel on manifolds with ends (non-parabolic case)

Alexander Grigor’yan

Based on a joint work with Laurent Saloff-Coste

1 April 2015, CUHK
1 Heat kernel

Let $M$ be a Riemannian manifold and $\Delta$ be the Laplace-Beltrami operator on $M$. Denote by $p_t(x, y)$ the heat kernel, that is, the smallest positive fundamental solution to the heat equation $\frac{\partial u}{\partial t} = \Delta u$ on $\mathbb{R}_+ \times M$. In $\mathbb{R}^n$ we have

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{|x - y|^2}{4t} \right).$$

A theorem of Li and Yau ’86 states: if $\text{Ricci}_M \geq 0$ and $M$ is geodesically complete then

$$p_t(x, y) \asymp \frac{C}{V(x, \sqrt{t})} \exp \left( -\frac{d^2(x, y)}{ct} \right). \quad (LY)$$

Here $d(x, y)$ is the geodesic distance, $V(x, r) = \mu(B(x, r))$ is the Riemannian volume of the geodesic ball $B(x, r)$, $C, c$ are positive constants, and $\asymp$ means that both $\leq$ and $\geq$ take place, but with different values of $C, c$.

Moreover, $(LY)$ holds on any geodesically complete manifold satisfying the volume doubling property and the Poincaré inequality.
2 Parabolicity and recurrence

A Riemannian manifold $M$ is called *non-parabolic* if the Laplace operator $\Delta$ on $M$ has a positive fundamental solution, and *parabolic* otherwise. The parabolicity is equivalent to the condition

$$\int_{-\infty}^{\infty} p_t(x,y) \, dt = \infty.$$ 

If $M$ satisfies $(LY)$ then this condition is equivalent to

$$\int_{-\infty}^{\infty} \frac{r \, dr}{V(x,r)} = \infty.$$ 

If in addition $V(x,r) \simeq r^\alpha$ then

parabolicity $\iff \alpha \leq 2$.

For example, in $\mathbb{R}^n$ we have $V(x,r) = c_n r^n$ so that $\mathbb{R}^n$ is parabolic if $n \leq 2$ and non-parabolic if $n > 2$. 
Consider the Brownian motion \( \{X_t\}_{t \geq 0} \) on the manifold \( M \), which by definition, has the transition density \( p_t(x, y) \): for any Borel set \( A \subset M \),

\[
\mathbb{P}_x (X_t \in A) = \int_A p_t(x, y) \, d\mu(y).
\]

The Brownian motion is called *recurrent* if the probability to hit eventually any non-empty open set is equal to 1, and *transient* otherwise. It is known that

\[
\text{recurrence} \iff \text{parabolicity}
\]

In particular, the Brownian motion is recurrent in \( \mathbb{R}^1 \) and \( \mathbb{R}^2 \), and transient in \( \mathbb{R}^n \) if \( n > 2 \).
3 Connected sum (manifold with ends)

Let $M_1, M_2, \ldots, M_k$ be a finite family of geodesically complete non-compact Riemannian manifolds. We say that a manifold $M$ is a connected sum of manifolds $M_i$ and write $M = M_1 \# M_2 \# \ldots \# M_k$ if, for some compact $K \subset M$ (called the central part of $M$), the exterior $M \setminus K$ is a disjoint union of open sets $E_1, E_2, \ldots, E_k$ (called the end of $M$), such that each $E_i$ is isometric to $M_i \setminus K_i$, for some compact $K_i \subset M_i$ (in fact, we will identify $E_i$ and $M_i \setminus K_i$). Always assume that all sets $K_i, K$ have smooth boundaries.
Here are two examples of connected sums: the catenoid and $\mathbb{R}^n \# \mathbb{R}^n$:

In what follows we always assume that each manifold $M_i$ is non-compact, geodesically complete and satisfies the Li-Yau estimate \((LY)\). The main objective of this work is to obtain the heat kernel estimates on $M = M_1 \# \ldots \# M_k$. We will restrict ourselves to the values of $p_t(x,y)$ when $t > 1$, $x \in E_i, y \in E_j$ with $i \neq j$ although estimates for general $(t,x,y)$ are available as well.
For each manifold $M_i$, $i = 1, \ldots, k$, denote by $d_i$ the geodesic distance, by $B_i(x, r)$ the geodesic balls, by $V_i(x, r)$ the volume of $B_i(x, r)$. Choose a reference point $o_i \in K_i$ and make the following additional assumption: 

*there is a constant $C > 1$ such that, for large enough $r$, the annuli $B_i(o_i, Cr) \setminus B_i(o_i, r)$ are connected sets.*

Denote

$$V_i(r) = V_i(o_i, r).$$

For example, if $M_i = \mathbb{R}^n$ then $V_i(r) = c_n r^n$. Fix some $o \in K$ and, for any $x \in M$, set $|x| = d(x, o)$.

We consider first the case when each $V_i(r)$ satisfies

$$V_i(r) \simeq r^{\alpha_i}$$

for $r > 1$. The value of $\alpha_i$ is called “dimension at $\infty$” of $M_i$. For example, the manifold

$$M_i = \mathbb{R}^{n_i} \times \text{(compact)}$$

has “dimension at $\infty$” equal to $n_i$. Fractional values of the dimension at $\infty$ can be achieved for surfaces of revolution.
4 Pure non-parabolic case: all $\alpha_i > 2$

**Theorem 1** If all $\alpha_i > 2$ then the heat kernel on $M = M_1 \# \ldots \# M_k$ satisfies the following estimate for all $t > 1$, $x \in E_i$ and $y \in E_j$ with $i \neq j$:

$$p_t(x, y) \asymp C \left( \frac{1}{t^{\alpha_j/2} |x|^{\alpha_i-2}} + \frac{1}{t^{\alpha_i/2} |y|^{\alpha_j-2}} + \frac{1}{t^{\alpha/2} |x|^{\alpha_i-2} |y|^{\alpha_j-2}} \right) e^{-\frac{d^2(x, y)}{ct}},$$

(1)

where $\alpha = \min(\alpha_1, \ldots, \alpha_k)$.

In the long time regime, when $x, y$ are fixed and $t \to \infty$, we obtain

$$p_t(x, y) \asymp t^{-\alpha/2} \quad \text{as } t \to \infty$$

where $\alpha$ is determined by the smallest end, which, therefore, is dominant.

In the case $k = 2$ the the third term in (1) is dominated by the first two and we obtain

$$p_t(x, y) \asymp C \left( \frac{1}{t^{\alpha_j/2} |x|^{\alpha_i-2}} + \frac{1}{t^{\alpha_i/2} |y|^{\alpha_j-2}} \right) e^{-\frac{d^2(x, y)}{ct}}.$$
For $M = \mathbb{R}^n \# \mathbb{R}^n$ with $n > 2$ we have $\alpha = \alpha_1 = \alpha_2 = n$ and obtain for $t \geq |x|^2 + |y|^2$

$$p_t (x, y) \simeq \frac{1}{t^{n/2}} \left( \frac{1}{|x|^{n-2}} + \frac{1}{|y|^{n-2}} \right).$$

In particular, in the medium time regime $|x|^2 \simeq |y|^2 \simeq t \to \infty$ we obtain

$$p_t (x, y) \simeq \frac{1}{t^{n-1}} \ll \frac{1}{t^{n/2}},$$

which reflects the bottleneck effect between $E_1$ and $E_2$. 
If $M$ has at least 3 ends then it can happen that $\alpha_i > \alpha$ and $\alpha_j > \alpha$. Let $\alpha = \alpha_l$. The long time asymptotic

$$p_t(x, y) \simeq t^{-\alpha/2} = t^{-\alpha_l/2}$$

means that the process $X_t$ spends most time the smallest end $E_l$, even when going from $E_i$ to $E_j$. 

![Diagram](image)
The proof of Theorem 1 consists of a number of steps.

1. Central upper bound: for $o \in K$ and $t > 1$

$$p_t (o, o) \leq C t^{-\alpha/2}.$$  \hspace{1cm} (2)

The heat kernel $p_t^{(i)} (x, y)$ on $M_i$ satisfies by \((LY)\)

$$p_t^{(i)} (o_i, o_i) \simeq t^{-\alpha_i/2},$$  \hspace{1cm} (3)

and (2) is obtained as the worst of the estimates (3).

2. Central lower bound: $p_t (o, o) \geq c t^{-\alpha/2}$, which follows from comparison with the Dirichlet heat kernel on each end.

3. Upper and lower bounds for the hitting probability $\psi_A (t, x) = \mathbb{P}_x (\tau_A \leq t)$ and its time derivative $\psi'_A (t, x)$ on each $M_i$. 

![Diagram of a random walk with a region A and points x and X_{\tau_A}](image)
4. Let $\Omega_1$ and $\Omega_2$ be two disjoint open sets on $M$. Set $\psi_i = \psi_{\partial \Omega_i}$.

Then, for all $x \in \Omega_1$ and $y \in \Omega_2$

$$p_t(x, y) \leq 2 \left[ \sup_{s \in [t/4, t]} \sup_{v \in \partial \Omega_1} p_s(v, w) \psi_1(t, x) \psi_2(t, y) \right.$$

$$+ \left[ \psi_2(t, y) \sup_{s \in [t/4, t]} \psi'_1(s, x) + \psi_1(t, x) \sup_{s \in [t/4, t]} \psi'_2(s, y) \right] \int_0^t \sup_{v \in \partial \Omega_1 \atop w \in \partial \Omega_2} p_s(v, w) ds$$

and there is also a similar lower bound for $p_t(x, y)$.

For $\Omega_1 = E_i$, $\Omega_2 = E_j$, substituting the central estimates of the heat kernel and the estimates of the hitting probability, we obtain (1).
5 Mixed case: there are \( \alpha_i < 2 \) and \( \alpha_j > 2 \)

Assuming that all \( \alpha_i \neq 2 \), set

\[
\alpha_i^* = \max (\alpha_i, 4 - \alpha_i) = \begin{cases}
\alpha_i, & \alpha_i > 2 \\
4 - \alpha_i, & \alpha_i < 2
\end{cases}
\]

(note that \( \alpha_i^* > 2 \))

**Theorem 2** Assume that, for any \( i \), we have \( V_i (r) \simeq r^{\alpha_i} \) for \( r > 1 \), where all \( \alpha_i \neq 2 \) and \( \max_i \alpha_i > 2 \). Then the heat kernel of the manifold \( M = M_1 \# M_2 \# \ldots \# M_k \) satisfies the following estimate: for all \( t > 1 \), \( x \in E_i \), \( y \in E_j \) where \( i \neq j \),

\[
p_t (x, y) \asymp C \left( \frac{1}{t^{\alpha_j^*/2} |x|^{\alpha_j^* - 2}} + \frac{1}{t^{\alpha_i^*/2} |y|^{\alpha_i^* - 2}} + \frac{1}{t^{\alpha/2} |x|^{\alpha_i^* - 2} |y|^{\alpha_j^* - 2}} \right) \\
\times |x|^{(2 - \alpha_i)_+} |y|^{(2 - \alpha_j)_+} e^{-\frac{d^2(x,y)}{ct}}
\]

(4)

where \( \alpha = \min (\alpha_1^*, \ldots, \alpha_k^*) \).

If all \( \alpha_i > 2 \) then (4) amounts to Theorem 1. A new interesting case occurs when there are \( \alpha_i < 2 \) and \( \alpha_j > 2 \).
Consider the case of two ends with $\alpha_1 = 1$ and $\alpha_2 = 3$. For example, this is the case when

$$E_1 = \mathbb{R}_+ \times S^2 \quad \text{and} \quad M_2 = \mathbb{R}^3.$$
We have
\[ \alpha_1^* = 4 - \alpha_1 = 3, \quad \alpha_2^* = \alpha_2 = 3, \quad \text{and} \quad \alpha = \min(\alpha_1^*, \alpha_2^*) = 3. \]

Hence, if \( x \in E_1 \) and \( y \in E_2 \), we obtain by (4)
\[
\begin{align*}
pt(x, y) &\asymp C \left( \frac{1}{t^{\alpha_1^*/2} |x|^{\alpha_1^*-2}} + \frac{1}{t^{\alpha_2^*/2} |y|^{\alpha_2^*-2}} \right) |x|^{(2-\alpha_i)+} |y|^{(2-\alpha_j)+} e^{-\frac{d^2(x,y)}{ct}} \\
&= \frac{C}{t^{3/2}} \left( \frac{1}{|x|} + \frac{1}{|y|} \right) |x| e^{-\frac{d^2(x,y)}{ct}}
\end{align*}
\]

In the long time regime \( t \to \infty \) and in the medium time regime \( |x|^2 \simeq |y|^2 \simeq t \to \infty \) we obtain the same behavior
\[
pt(x, y) \simeq t^{-3/2}.
\]

Therefore, there is no bottleneck effect. However, there is a new effect in the intermediate regime when \( y \) is fixed and \( |x|^2 \simeq t \to \infty \), as in this case we obtain
\[
pt(x, y) \simeq t^{-1} \gg t^{-3/2}.
\]
The proof of Theorem 2 uses Doob’s $h$-transform $M$. If at least one $M_i$ is non-parabolic, then there is a positive harmonic function $h$ on $M$ such that $h \approx 1$ on non-parabolic ends and $h \to \infty$ on parabolic ends. Consider a new measure $\tilde{\mu}$ on $M$ given by
\[
d\tilde{\mu} = h^2 d\mu,
\]
the weighted Laplace operator
\[
\tilde{\Delta} = \frac{1}{h^2} \text{div} (h^2 \text{grad}) = \frac{1}{h} \circ \Delta \circ h
\]
and the associated weighted heat kernel
\[
\tilde{p}_t (x, y) = \frac{p_t (x, y)}{h(x) h(y)}.
\]
The weighted manifold $(M, \tilde{\mu})$ has the ends $(M_i, \tilde{\mu})$ that are now all non-parabolic. Moreover, one shows that each $(M_i, \tilde{\mu})$ satisfies Li-Yau estimate. Theorem 1 can be extended in the case, which yields the estimates of $\tilde{p}_t (x, y)$. Since function $h(x)$ can be effectively estimates via $|x|$, we obtain the estimates of $p_t (x, y)$.
6 The case of general volume functions $V_i(r)$

Introduce the following functions:

$$h_i(r) = 1 + \left( \int_1^r \frac{sds}{V_i(s)} \right)_+,$$

$$\tilde{V}_i(r) = h_i^2(r)V_i(r), \quad \tilde{V}_{\min}(r) = \min_{1 \leq i \leq k} \tilde{V}_i(r),$$

$$H_i(r,t) = \frac{r^2}{\tilde{V}_i(r)} + \left( \int_r^\sqrt{t} \frac{sds}{\tilde{V}_i(s)} \right)_+.$$

**Theorem 3** If at least one of the ends is non-parabolic then the heat kernel $p_t(x,y)$ on the connected sum $M = M_1 \# M_2 \# \ldots \# M_k$ admits the following estimate: for all $t > 0$ and $x \in E_i$, $y \in E_j$, $i \neq j$:

$$p_t(x,y) \asymp C \left( \frac{H_i(|x|,t)}{\tilde{V}_j(\sqrt{t})} + \frac{H_j(|y|,t)}{\tilde{V}_i(\sqrt{t})} + \frac{H_i(|x|,t)H_j(|y|,t)}{\tilde{V}_{\min}(\sqrt{t})} \right) h_i(|x|)h_j(|y|)e^{-\frac{d^2(x,y)}{ct}}.$$  

(5)
If \( V_i(r) \simeq r^{\alpha_i} \) where \( \alpha_i \neq 2 \) then, for \( r > 1 \),

\[
h_i(r) \simeq \begin{cases} 
1, & \alpha_i > 2, \\
r^{2-\alpha_i}, & \alpha_i < 2, 
\end{cases} = r^{(2-\alpha_i)_+},
\]

whence

\[
\tilde{V}_i(r) \simeq \begin{cases} 
r^{\alpha_i}, & \alpha_i > 2, \\
r^{4-\alpha_i}, & \alpha_i < 2 
\end{cases} = r^{\alpha_i^*},
\]

\[
\tilde{V}_{\text{min}}(r) \simeq r^\alpha \quad \text{where } \alpha = \min_{1 \leq i \leq k} \alpha_i^*
\]

and \( H_i(r, t) \simeq r^{2-\alpha_i^*} \). Hence, in this case (5) is equivalent (4).

Theorem 3 allows to cover the new case \( \alpha_i = 2 \). Consider an example with \( \alpha_1 = 2, \alpha_2 = 3 \). For example, this is the case when \( M_1 = \mathbb{R}^2 \times S^1 \) and \( M_2 = \mathbb{R}^3 \). We have for large \( r \) and \( t \)

\[
V_1(r) \simeq r^2, \quad V_2(r) \simeq r^3,
\]

\[
h_1(r) \simeq \log r, \quad h_2(r) \simeq 1,
\]

\[
\tilde{V}_1(r) \simeq r^2 \log^2 r, \quad \tilde{V}_2(r) \simeq r^3, \quad \tilde{V}_{\text{min}}(r) = r^2 \log^2 r
\]
\[ H_1(r, t) \approx \frac{1}{\log^2 r} + \left( \frac{1}{2 \log r} - \frac{1}{\log t} \right)_+ , \]
\[ H_2(r, t) \approx r^{-1}. \]

Hence, for \( x \in E_1, y \in E_2, \)
\[ p_t(x, y) \leq C \left( \frac{\log |x|}{|y| t \log^2 t} + \frac{1}{t^{3/2}} \left[ \frac{1}{\log |x|} + \left( \frac{1}{2} - \frac{\log |x|}{\log t} \right)_+ \right] \right) e^{-\frac{d^2(x,y)}{ct}}. \]

In particular, in the long time regime \( t \to \infty \) we obtain
\[ p_t(x, y) \approx \frac{1}{t \log^2 t}, \]
while in the medium time regime \( |x|^2 \approx |y|^2 \approx t \to \infty \)
\[ p_t(x, y) \approx \frac{1}{t^{3/2} \log t}. \]
Consider an example with \( \alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3 \), which is the case when \( E_1 = \mathbb{R}_+ \times S^2 \), \( M_2 = \mathbb{R}^2 \times S^1 \) and \( M_3 = \mathbb{R}^3 \). Then, for large \( r \) and \( t \),
\[
V_1(r) \simeq r, \quad V_1(r) \simeq r^2, \quad V_3(r) \simeq r^3,
\]
\[
h_1(r) \simeq r, \quad h_2(r) \simeq \log r, \quad h_3(r) \simeq 1,
\]
\[
\tilde{V}_1(r) \simeq r^3, \quad \tilde{V}_2(r) \simeq r^2 \log^2 r, \quad \tilde{V}_2(r) \simeq r^3,
\]
\[
\tilde{V}_{\text{min}}(r) \simeq r^2 \log^2 r,
\]
\[
H_1(r, t) \simeq r^{-1}
\]
\[
H_2(r, t) \simeq \frac{1}{\log^2 r} + \left( \frac{1}{2 \log r} - \frac{1}{\log t} \right)_+,
\]
\[
H_3(r, t) \simeq r^{-1}.
\]
Hence, for \( x \in E_1, y \in E_2 \),
\[
p_t(x, y) \leq C \left( \frac{\log |y|}{t \log^2 t} + \left( \frac{|x|}{t^{3/2}} + \frac{1}{t \log^2 t} \right) \left[ \frac{1}{\log |y|} + \left( \frac{1}{2} - \frac{\log |y|}{\log t} \right)_+ \right] \right) e^{-\frac{a^2(x, y)}{ct}}.
\]
In particular, in the long time regime

\[ p_t(x, y) \simeq \frac{1}{t \log^2 t}, \]

in the medium time regime

\[ p_t(x, y) \simeq \frac{1}{t \log t}. \]

If \( x \in E_1 \) and \( y \in E_3 \) then

\[ p_t(x, y) \asymp C \left( \frac{1}{t^{3/2}} \left( 1 + \frac{|x|}{|y|} \right) + \frac{1}{|y| t \log^2 t} \right) e^{-\frac{d^2(x,y)}{ct}}, \]

and in the medium time regime

\[ p_t(x, y) \simeq \frac{1}{t^{3/2}}. \]

One sees that there is a bottleneck effect between \( \mathbb{R}_+ \) and \( \mathbb{R}^3 \) but not between \( \mathbb{R}_+ \) and \( \mathbb{R}^2 \).
7 One-dimensional Schrödinger operator

Consider the operator in \( \mathbb{R} \)

\[
H = -\frac{d^2}{dx^2} + \Phi(x)
\]

where \( \Phi(x) = b|x|^{-2} \) for large \( |x| \), where \( b \geq 0 \). Theorem 3 allows to obtain the heat kernel estimate for this operator.

**Theorem 4** Set

\[
\beta = \frac{1}{2} + \sqrt{\frac{1}{4} + b}.
\]

Then the heat kernel \( p_t^\Phi(x,y) \) of the operator \( H \) satisfies the estimate:

for all \( x < -1, y > 1, t \geq 1, \)

\[
p_t^\Phi(x,y) \lesssim \frac{C}{t^{1/2+\beta}} \left( \frac{|y|^\beta}{|x|^{\beta-1}} + \frac{|x|^\beta}{|y|^{\beta-1}} \right) \exp \left( -c \frac{|x-y|^2}{t} \right).
\]
In particular, if $t \to \infty$ then
\[
p_t^\Phi (x, y) \simeq \frac{C}{t^{1/2 + \beta}} \left( \frac{|y|^\beta}{|x|^{\beta-1}} + \frac{|x|^\beta}{|y|^{\beta-1}} \right) = \frac{C |x|^\beta |y|^\beta}{t^{1/2 + \beta}} \left( \frac{1}{|x|^{2\beta-1}} + \frac{1}{|y|^{2\beta-1}} \right).
\]

For comparison, for the operator $H = -\Delta + \Phi(x)$ in $\mathbb{R}^n$, $n \geq 2$, we have as $t \to \infty$
\[
p_t^\Phi (x, y) \simeq \frac{C |x|^\beta |y|^\beta}{t^{n/2 + \beta}},
\]
where
\[
\beta = -\frac{n}{2} + 1 + \sqrt{\left(\frac{n}{2} - 1 \right)^2 + b}.
\]