Heat kernel on manifolds with ends (non-parabolic case)

Alexander Grigor'yan

Based on a joint work with Laurent Saloff-Coste

1 April 2015, CUHK

1 Heat kernel

Let M be a Riemannian manifold and Δ be the Laplace-Beltrami operator on M. Denote by $p_t(x,y)$ the heat kernel, that is, the smallest positive fundamental solution to the heat equation $\frac{\partial u}{\partial t} = \Delta u$ on $\mathbb{R}_+ \times M$. In $\mathbb{R}^n$ we have

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

A theorem of Li and Yau '86 states: if $Ricci_M \geq 0$ and M is geodesically complete then

$$p_t(x,y) \asymp \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right). \tag{LY}$$

Here $d(x,y)$ is the geodesic distance, $V(x,r) = \mu(B(x,r))$ is the Riemannian volume of the geodesic ball $B(x,r)$, C, c are positive constants, and $\asymp$ means that both $\leq$ and $\geq$ take place, but with different values of C, c.

Moreover, (LY) holds on any geodesically complete manifold satisfying the volume doubling property and the Poincaré inequality.

2 Parabolicity and recurrence

A Riemannian manifold M is called *non-parabolic* if the Laplace operator Δ on M has a positive fundamental solution, and *parabolic* otherwise. The parabolicity is equivalent to the condition

$$\int^{\infty} p_t(x, y)\, dt = \infty.$$

If M satisfies (LY) then this condition is equivalent to

$$\int^{\infty} \frac{r dr}{V(x, r)} = \infty.$$

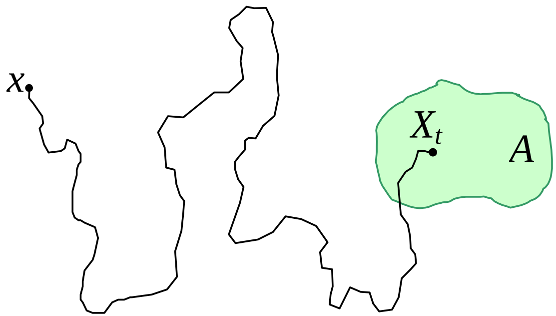
If in addition $V(x, r) \simeq r^{\alpha}$ then

$$\text{parabolicity} \;\Leftrightarrow\; \alpha \leq 2.$$

For example, in $\mathbb{R}^n$ we have $V(x, r) = c_n r^n$ so that $\mathbb{R}^n$ is parabolic if $n \leq 2$ and non-parabolic if $n > 2$.

Consider the Brownian motion $\{X_t\}_{t\geq 0}$ on the manifold M, which by definition, has the transition density $p_t(x,y)$: for any Borel set $A \subset M$,

$$\mathbb{P}_x(X_t \in A) = \int_A p_t(x,y)\, d\mu(y) .$$



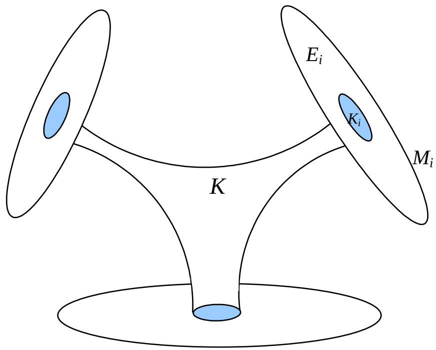
The Brownian motion is called *recurrent* if the probability to hit eventually any non-empty open set is equal to 1, and *transient* otherwise. It is known that

$$\text{recurrence} \Leftrightarrow \text{parabolicity}$$

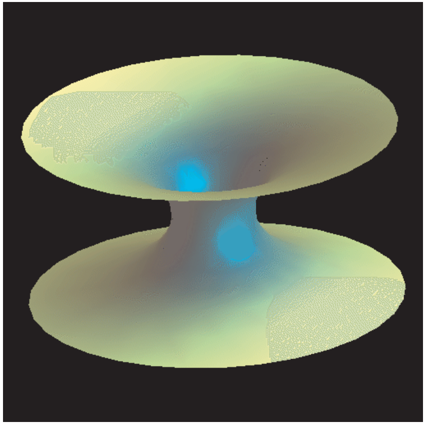
In particular, the Brownian motion is recurrent in $\mathbb{R}^1$ and $\mathbb{R}^2$, and transient in $\mathbb{R}^n$ if $n > 2$.

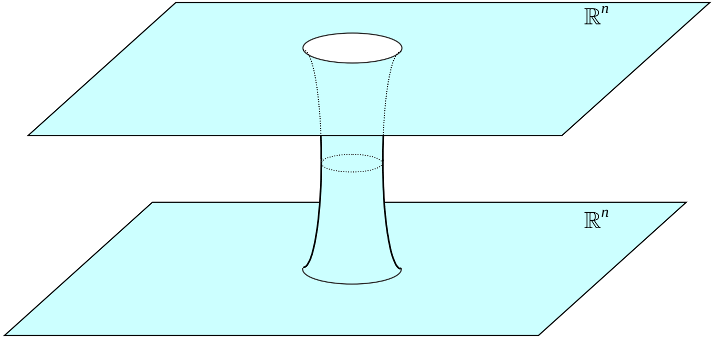
3 Connected sum (manifold with ends)

Let M_1, M_2,.., M_k be a finite family of geodesically complete non-compact Riemannian manifolds. We say that a manifold M is a *connected sum* of manifolds M_i and write $M = M_1\#M_2\#...\#M_k$ if, for some compact $K \subset M$ (called *the central part* of M), the exterior $M \setminus K$ is a disjoint union of open sets $E_1, E_2, ..., E_k$ (called the *end* of M), such that each E_i is isometric to $M_i \setminus K_i$, for some compact $K_i \subset M_i$ (in fact, we will identify E_i and $M_i \setminus K_i$). Always assume that all sets K_i, K have smooth boundaries.



Here are two examples of connected sums: the catenoid and $\mathbb{R}^n \# \mathbb{R}^n$:





In what follows we always assume that each manifold M_i is non-compact, geodesically complete and satisfies the *Li-Yau estimate* (*LY*). The main objective of this work is to *obtain the heat kernel estimates on* $M = M_1 \# ... \# M_k$. We will restrict ourselves to the values of $p_t(x, y)$ when $t > 1$, $x \in E_i, y \in E_j$ with $i \neq j$ although estimates for general (t, x, y) are available as well.

For each manifold M_i, $i = 1, ..., k$, denote by d_i the geodesic distance, by $B_i(x, r)$ the geodesic balls, by $V_i(x, r)$ the volume of $B_i(x, r)$. Choose a reference point $o_i \in \overset{o}{K}_i$ and make the following additional assumption: *there is a constant $C > 1$ such that, for large enough r, the annuli $B_i(o_i, Cr) \setminus B_i(o_i, r)$ are connected sets.*

Denote

$$V_i(r) = V_i(o_i, r).$$

For example, if $M_i = \mathbb{R}^n$ then $V_i(r) = c_n r^n$. Fix some $o \in \overset{o}{K}$ and, for any $x \in M$, set $|x| = d(x, o)$.

We consider first the case when each $V_i(r)$ satisfies

$$V_i(r) \simeq r^{\alpha_i}$$

for $r > 1$. The value of α_i is called "dimension at ∞" of M_i. For example, the manifold

$$M_i = \mathbb{R}^{n_i} \times (\text{compact})$$

has "dimension at ∞" equal to n_i. Fractional values of the dimension at ∞ can be achieved for surfaces of revolution.

4 Pure non-parabolic case: all $\alpha_i > 2$

Theorem 1 *If all $\alpha_i > 2$ then the heat kernel on $M = M_1\#...\#M_k$ satisfies the following estimate for all $t > 1$, $x \in E_i$ and $y \in E_j$ with $i \neq j$:*

$$p_t(x,y) \asymp C\left(\frac{1}{t^{\alpha_j/2}|x|^{\alpha_i-2}} + \frac{1}{t^{\alpha_i/2}|y|^{\alpha_j-2}} + \frac{1}{t^{\alpha/2}|x|^{\alpha_i-2}|y|^{\alpha_j-2}}\right)e^{-\frac{d^2(x,y)}{ct}}, \tag{1}$$

where $\alpha = \min(\alpha_1, ..., \alpha_k)$.

In the *long time* regime, when x, y are fixed and $t \to \infty$, we obtain

$$p_t(x,y) \simeq t^{-\alpha/2} \quad \text{as } t \to \infty$$

where α is determined by the smallest end, which, therefore, is dominant.

In the case $k = 2$ the the third term in (1) is dominated by the first two and we obtain

$$p_t(x,y) \asymp C\left(\frac{1}{t^{\alpha_j/2}|x|^{\alpha_i-2}} + \frac{1}{t^{\alpha_i/2}|y|^{\alpha_j-2}}\right)e^{-\frac{d^2(x,y)}{ct}}.$$

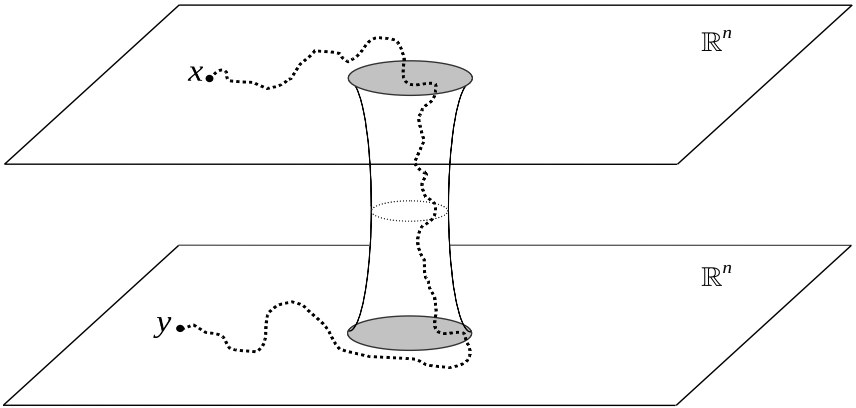
For $M = \mathbb{R}^n \# \mathbb{R}^n$ with $n > 2$ we have $\alpha = \alpha_1 = \alpha_2 = n$ and obtain for $t \geq |x|^2 + |y|^2$

$$p_t(x,y) \simeq \frac{1}{t^{n/2}} \left(\frac{1}{|x|^{n-2}} + \frac{1}{|y|^{n-2}} \right).$$

In particular, in the *medium time* regime $|x|^2 \simeq |y|^2 \simeq t \to \infty$ we obtain

$$p_t(x,y) \simeq \frac{1}{t^{n-1}} \ll \frac{1}{t^{n/2}},$$

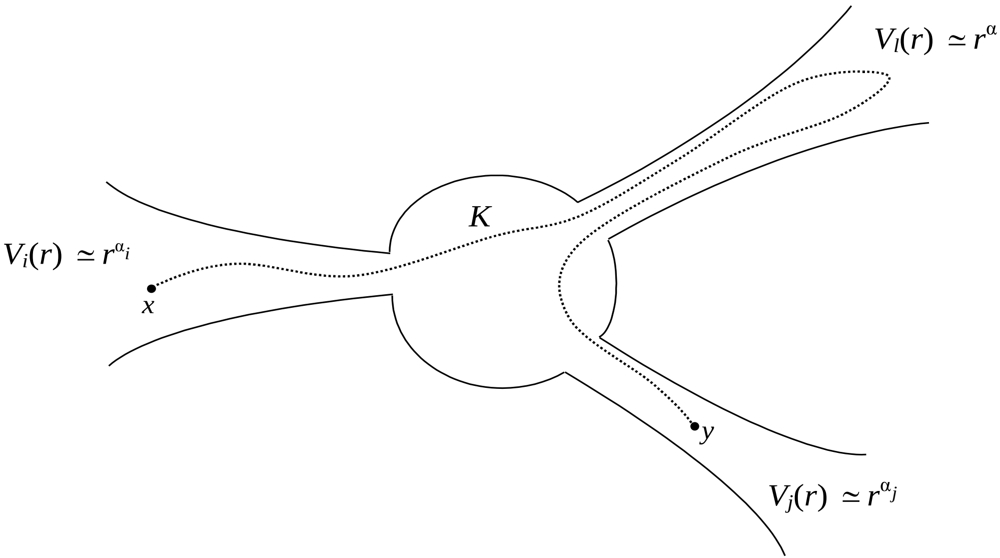
which reflects the *bottleneck effect* between E_1 and E_2.



If M has at least 3 ends then it can happen that $\alpha_i > \alpha$ and $\alpha_j > \alpha$. Let $\alpha = \alpha_l$. The long time asymptotic

$$p_t(x, y) \simeq t^{-\alpha/2} = t^{-\alpha_l/2}$$

means that the process X_t spends most time the smallest end E_l, even when going from E_i to E_j.



The proof of Theorem 1 consists of a number of steps.

1. Central upper bound: for $o \in K$ and $t > 1$

$$p_t(o,o) \leq Ct^{-\alpha/2}. \tag{2}$$

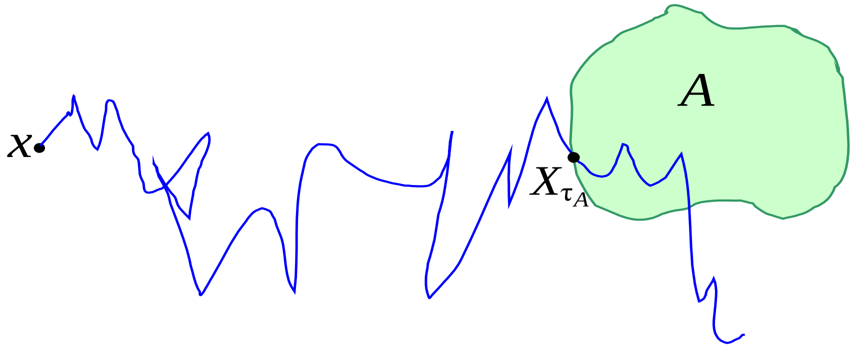
The heat kernel $p_t^{(i)}(x,y)$ on M_i satisfies by (LY)

$$p_t^{(i)}(o_i,o_i) \simeq t^{-\alpha_i/2}, \tag{3}$$

and (2) is obtained as the worst of the estimates (3).

2. Central lower bound: $p_t(o,o) \geq ct^{-\alpha/2}$, which follows from comparison with the Dirichlet heat kernel on each end.

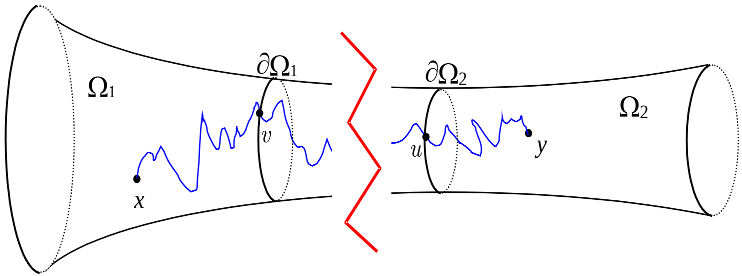
3. Upper and lower bounds for the hitting probability $\psi_A(t,x) = \mathbb{P}_x(\tau_A \leq t)$ and its time derivative $\psi'_A(t,x)$ on each M_i.



4. Let Ω_1 and Ω_2 be two disjoint open sets on M. Set $\psi_i = \psi_{\partial\Omega_i}$. Then, for all $x \in \Omega_1$ and $y \in \Omega_2$

$$\begin{aligned} p_t(x,y) \;\; &\leq \;\; 2\left[\sup_{s\in[t/4,t]} \sup_{\substack{v\in\partial\Omega_1 \\ w\in\partial\Omega_2}} p_s(v,w)\right]\psi_1(t,x)\psi_2(t,y) \\ &+\left[\psi_2(t,y)\sup_{s\in[t/4,t]}\psi_1'(s,x)+\psi_1(t,x)\sup_{s\in[t/4,t]}\psi_2'(s,y)\right]\int_0^t \sup_{\substack{v\in\partial\Omega_1 \\ w\in\partial\Omega_2}} p_s(v,w)ds \end{aligned}$$

and there is also a similar lower bound for $p_t\,(x,y)$.



For $\Omega_1 = E_i$, $\Omega_2 = E_j$, substituting the central estimates of the heat kernel and the estimates of the hitting probability, we obtain (1).

5 Mixed case: there are $\alpha_i < 2$ and $\alpha_j > 2$

Assuming that all $\alpha_i \neq 2$, set

$$\alpha_i^* = \max(\alpha_i, 4 - \alpha_i) = \begin{cases} \alpha_i, & \alpha_i > 2 \\ 4 - \alpha_i, & \alpha_i < 2 \end{cases} \quad \text{(note that } \alpha_i^* > 2\text{)}$$

Theorem 2 *Assume that, for any i, we have $V_i(r) \simeq r^{\alpha_i}$ for $r > 1$, where all $\alpha_i \neq 2$ and* $\max_i \alpha_i > 2$. *Then the heat kernel of the manifold $M = M_1 \# M_2 \# ... \# M_k$ satisfies the following estimate: for all $t > 1$, $x \in E_i$, $y \in E_j$ where $i \neq j$,*

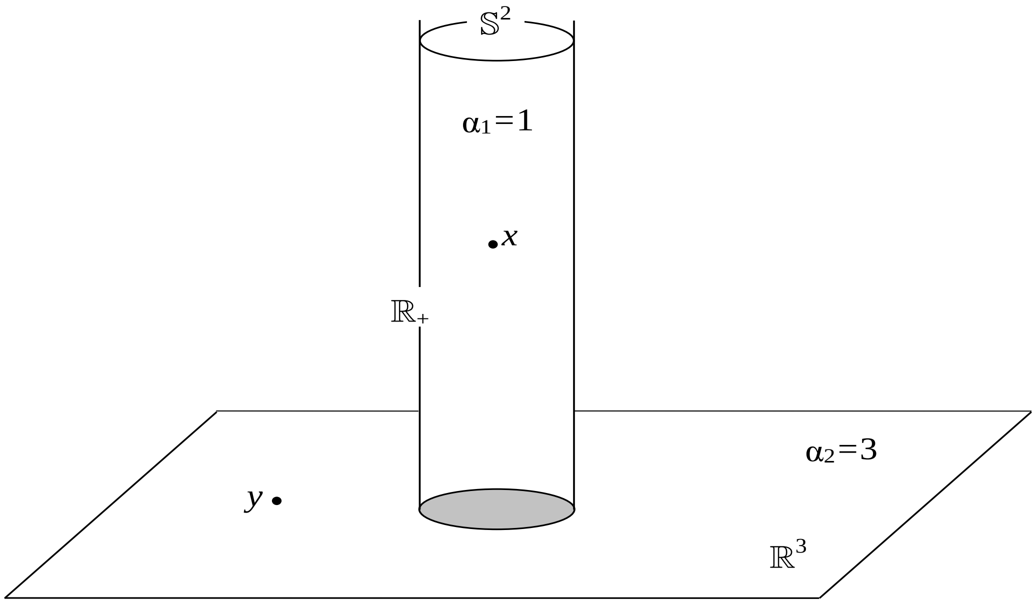
$$\begin{aligned} p_t(x, y) \quad &\asymp \quad C \left(\frac{1}{t^{\alpha_j^*/2} |x|^{\alpha_i^* - 2}} + \frac{1}{t^{\alpha_i^*/2} |y|^{\alpha_j^* - 2}} + \frac{1}{t^{\alpha/2} |x|^{\alpha_i^* - 2} |y|^{\alpha_j^* - 2}} \right) \\ &\quad \times |x|^{(2-\alpha_i)_+} |y|^{(2-\alpha_j)_+} e^{-\frac{d^2(x,y)}{ct}}, \end{aligned} \tag{4}$$

where $\alpha = \min(\alpha_1^*, ..., \alpha_k^*)$.

If all $\alpha_i > 2$ then (4) amounts to Theorem 1. A new interesting case occurs when there are $\alpha_i < 2$ and $\alpha_j > 2$.

Consider the case of two ends with $\alpha_1 = 1$ and $\alpha_2 = 3$. For example, this is the case when

$$E_1 = \mathbb{R}_+ \times \mathbb{S}^2 \quad \text{and} \quad M_2 = \mathbb{R}^3.$$



We have

$$\alpha_1^* = 4 - \alpha_1 = 3, \qquad \alpha_2^* = \alpha_2 = 3, \quad \text{and } \alpha = \min\left(\alpha_1^*, \alpha_2^*\right) = 3.$$

Hence, if $x \in E_1$ and $y \in E_2$, we obtain by (4)

$$\begin{aligned} p_t\left(x,y\right) &\asymp C\left(\frac{1}{t^{\alpha_j^*/2}\left|x\right|^{\alpha_i^*-2}} + \frac{1}{t^{\alpha_i^*/2}\left|y\right|^{\alpha_j^*-2}}\right)\left|x\right|^{(2-\alpha_i)_+}\left|y\right|^{(2-\alpha_j)_+} e^{-\frac{d^2(x,y)}{ct}} \\ &= \frac{C}{t^{3/2}}\left(\frac{1}{|x|} + \frac{1}{|y|}\right)|x|\, e^{-\frac{d^2(x,y)}{ct}} \end{aligned}$$

In the long time regime $t \to \infty$ and in the medium time regime $|x|^2 \simeq |y|^2 \simeq t \to \infty$ we obtain the same behavior

$$p_t\left(x,y\right) \simeq t^{-3/2}.$$

Therefore, there is no bottleneck effect. However, there is a new effect in the *intermediate regime* when y is fixed and $|x|^2 \simeq t \to \infty$, as in this case we obtain

$$p_t\left(x,y\right) \simeq t^{-1} \gg t^{-3/2}.$$

The proof of Theorem 2 uses Doob's h-transform M. If at least one M_i is non-parabolic, then there is a positive harmonic function h on M such that $h \simeq 1$ on non-parabolic ends and $h \to \infty$ on parabolic ends. Consider a new measure $\widetilde{\mu}$ on M given by

$$d\widetilde{\mu} = h^2 d\mu,$$

the weighted Laplace operator

$$\widetilde{\Delta} = \frac{1}{h^2} \operatorname{div} \left(h^2 \operatorname{grad} \right) = \frac{1}{h} \circ \Delta \circ h$$

and the associated weighted heat kernel

$$\widetilde{p}_t (x, y) = \frac{p_t (x, y)}{h (x) h (y)}.$$

The weighted manifold $(M, \widetilde{\mu})$ has the ends $(M_i, \widetilde{\mu})$ that are now all non-parabolic. Moreover, one shows that each $(M_i, \widetilde{\mu})$ satisfies Li-Yau estimate. Theorem 1 can be extended in the case, which yields the estimates of $\widetilde{p}_t (x, y)$. Since function $h (x)$ can be effectively estimates via $|x|$, we obtain the estimates of $p_t (x, y)$.

6 The case of general volume functions $V_i(r)$

Introduce the following functions:

$$h_i(r) = 1 + \left(\int_1^r \frac{sds}{V_i(s)} \right)_+ ,$$

$$\widetilde{V}_i(r) = h_i^2(r) V_i(r), \qquad \widetilde{V}_{\min}(r) = \min_{1 \le i \le k} \widetilde{V}_i(r),$$

$$H_i(r,t) = \frac{r^2}{\widetilde{V}_i(r)} + \left(\int_r^{\sqrt{t}} \frac{sds}{\widetilde{V}_i(s)} \right)_+ .$$

Theorem 3 *If at least one of the ends is non-parabolic then the heat kernel* $p_t(x,y)$ *on the connected sum* $M = M_1 \# M_2 \# .... \# M_k$ *admits the following estimate: for all* $t > 0$ *and* $x \in E_i$, $y \in E_j$, $i \neq j$*:*

$$p_t(x,y) \asymp C \left(\frac{H_i(|x|,t)}{\widetilde{V}_j(\sqrt{t})} + \frac{H_j(|y|,t)}{\widetilde{V}_i(\sqrt{t})} + \frac{H_i(|x|,t) H_j(|y|,t)}{\widetilde{V}_{\min}(\sqrt{t})} \right) h_i(|x|) h_j(|y|) e^{-\frac{d^2(x,y)}{ct}}. \tag{5}$$

If $V_i(r) \simeq r^{\alpha_i}$ where $\alpha_i \neq 2$ then, for $r > 1$,

$$h_i(r) \simeq \begin{cases} 1, & \alpha_i > 2, \\ r^{2-\alpha_i}, & \alpha_i < 2, \end{cases} = r^{(2-\alpha_i)_+},$$

whence

$$\widetilde{V}_i(r) \simeq \begin{cases} r^{\alpha_i}, & \alpha_i > 2, \\ r^{4-\alpha_i}, & \alpha_i < 2 \end{cases} = r^{\alpha_i^*},$$

$$\widetilde{V}_{\min}(r) \simeq r^{\alpha} \qquad \text{where } \alpha = \min_{1 \leq i \leq k} \alpha_i^*$$

and $H_i(r,t) \simeq r^{2-\alpha_i^*}$. Hence, in this case (5) is equivalent (4).

Theorem 3 allows to cover the new case $\alpha_i = 2$. Consider an example with $\alpha_1 = 2$, $\alpha_2 = 3$. For example, this is the case when $M_1 = \mathbb{R}^2 \times \mathbb{S}^1$ and $M_2 = \mathbb{R}^3$. We have for large r and t

$$V_1(r) \simeq r^2, \qquad V_2(r) \simeq r^3,$$

$$h_1(r) \simeq \log r, \quad h_2(r) \simeq 1,$$

$$\widetilde{V}_1(r) \simeq r^2 \log^2 r, \quad \widetilde{V}_2(r) \simeq r^3, \quad \widetilde{V}_{\min}(r) = r^2 \log^2 r$$

$$
\begin{aligned}
H_1(r,t) &\simeq \frac{1}{\log^2 r} + \left(\frac{1}{2\log r} - \frac{1}{\log t}\right)_+ , \\
H_2(r,t) &\simeq r^{-1}.
\end{aligned}
$$

Hence, for $x \in E_1$, $y \in E_2$,

$$
p_t(x,y) \asymp C\left(\frac{\log|x|}{|y|\, t\log^2 t} + \frac{1}{t^{3/2}}\left[\frac{1}{\log|x|} + \left(\frac{1}{2} - \frac{\log|x|}{\log t}\right)_+\right]\right) e^{-\frac{d^2(x,y)}{ct}}.
$$

In particular, in the long time regime $t \to \infty$ we obtain

$$
p_t(x,y) \simeq \frac{1}{t\log^2 t},
$$

while in the medium time regime $|x|^2 \simeq |y|^2 \simeq t \to \infty$

$$
p_t(x,y) \simeq \frac{1}{t^{3/2}\log t}.
$$

Consider an example with $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, which is the case when $E_1 = \mathbb{R}_+ \times \mathbb{S}^2$, $M_2 = \mathbb{R}^2 \times \mathbb{S}^1$ and $M_3 = \mathbb{R}^3$. Then, for large r and t,

$$V_1(r) \simeq r, \qquad V_1(r) \simeq r^2, \quad V_3(r) \simeq r^3,$$

$$h_1(r) \simeq r, \qquad h_2(r) \simeq \log r, \qquad h_3(r) \simeq 1,$$

$$\widetilde{V}_1(r) \simeq r^3, \qquad \widetilde{V}_2(r) \simeq r^2 \log^2 r, \qquad \widetilde{V}_2(r) \simeq r^3,$$

$$\widetilde{V}_{\min}(r) \simeq r^2 \log^2 r,$$

$$\begin{aligned} H_1(r,t) &\simeq r^{-1} \\ H_2(r,t) &\simeq \frac{1}{\log^2 r} + \left(\frac{1}{2\log r} - \frac{1}{\log t} \right)_+, \\ H_3(r,t) &\simeq r^{-1}. \end{aligned}$$

Hence, for $x \in E_1$, $y \in E_2$,

$$p_t(x,y) \asymp C \left(\frac{\log|y|}{t\log^2 t} + \left(\frac{|x|}{t^{3/2}} + \frac{1}{t\log^2 t} \right) \left[\frac{1}{\log|y|} + \left(\frac{1}{2} - \frac{\log|y|}{\log t} \right)_+ \right] \right) e^{-\frac{d^2(x,y)}{ct}}.$$

In particular, in the long time regime

$$p_t(x,y) \simeq \frac{1}{t \log^2 t},$$

in the medium time regime

$$p_t(x,y) \simeq \frac{1}{t \log t}.$$

If $x \in E_1$ and $y \in E_3$ then

$$p_t(x,y) \asymp C\left(\frac{1}{t^{3/2}}\left(1+\frac{|x|}{|y|}\right)+\frac{1}{|y|\, t \log^2 t}\right) e^{-\frac{d^2(x,y)}{ct}},$$

and in the medium time regime

$$p_t(x,y) \simeq \frac{1}{t^{3/2}}.$$

One sees that there is a bottleneck effect between $\mathbb{R}_+$ and $\mathbb{R}^3$ but not between $\mathbb{R}_+$ and $\mathbb{R}^2$.

7 One-dimensional Schrödinger operator

Consider the operator in $\mathbb{R}$

$$H = -\frac{d^2}{dx^2} + \Phi(x)$$

where $\Phi(x) = b|x|^{-2}$ for large $|x|$, where $b \geq 0$. Theorem 3 allows to obtain the heat kernel estimate for this operator.

Theorem 4 *Set*

$$\beta = \frac{1}{2} + \sqrt{\frac{1}{4} + b}\ .$$

Then the heat kernel $p_t^\Phi(x,y)$ of the operator H satisfies the estimate: for all $x < -1$, $y > 1$, $t \geq 1$,

$$p_t^\Phi(x,y) \asymp \frac{C}{t^{1/2+\beta}} \left(\frac{|y|^\beta}{|x|^{\beta-1}} + \frac{|x|^\beta}{|y|^{\beta-1}} \right) \exp\left(-c\frac{|x-y|^2}{t} \right).$$

In particular, if $t \to \infty$ then

$$p_t^{\Phi}(x,y) \simeq \frac{C}{t^{1/2+\beta}} \left(\frac{|y|^{\beta}}{|x|^{\beta-1}} + \frac{|x|^{\beta}}{|y|^{\beta-1}} \right) = \frac{C\,|x|^{\beta}\,|y|^{\beta}}{t^{1/2+\beta}} \left(\frac{1}{|x|^{2\beta-1}} + \frac{1}{|y|^{2\beta-1}} \right).$$

For comparison, for the operator $H = -\Delta + \Phi(x)$ in $\mathbb{R}^n$, $n \geq 2$, we have as $t \to \infty$

$$p_t^{\Phi}(x,y) \asymp \frac{C\,|x|^{\beta}\,|y|^{\beta}}{t^{n/2+\beta}},$$

where

$$\beta = -\frac{n}{2} + 1 + \sqrt{\left(\frac{n}{2} - 1\right)^2 + b}.$$