Heat kernels on manifolds with parabolic ends

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Renmin University, October 10, 2015

Based on a joint work with S.Ishiwata and L.Saloff-Coste

Li-Yau estimate

Let M be geodesically complete non-compact Riemannian manifold. Denote by $d(x,y)$ the geodesic distance, $B(x,r)$ geodesic ball of radius r centered at x, and set $V(x,r) = \mu(B(x,r))$, where μ is the Riemannian measure. Let $p_t(x,y)$ be the heat kernel of M.

Theorem 1 *(Li and Yau '86) If Ricci$_M \geq 0$ then*

$$p_t(x,y) \asymp \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right) \tag{LY}$$

Here $\asymp$ means that there are $\leq$ and $\geq$ but with different values of positive constants c, C. This estimate (LY) holds also on a more general class of manifolds described below.

Definition. We say that M satisfies *volume doubling condition* if for all $x \in M$ and $r > 0$

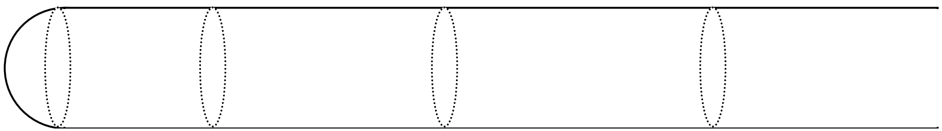
$$V(x,2r) \leq CV(x,r). \tag{VD}$$

Definition. We say that M satisfies the *(weak) Poincaré inequality* if there are constants $C > 0$ and $\varepsilon \in (0,1)$ such that, for any ball $B(x,r)$ and for any function $u \in C^1(B(x,r))$,

$$\inf_{s\in\mathbb{R}} \int_{B(x,\varepsilon r)} (u-s)^2\, d\mu \le Cr^2 \int_{B(x,r)} |\nabla u|^2\, d\mu. \tag{PI}$$

Theorem 2 *$(LY) \Leftrightarrow (VD) + (PI)$.*

Let us give some examples of manifolds satisfying (LY). Fix an integer $D \geq 2$ and for any $2 \le n \le D$ consider manifold $\mathcal{R}^n := \mathbb{R}^n \times \mathbb{S}^{D-n}$. For $n = 1$ manifold $\mathcal{R}^1$ is obtained from $\mathbb{R}_+ \times \mathbb{S}^{D-1}$ by closing it into a complete manifold:



Then $\mathcal{R}^n$ satisfies (LY) for all $n \geq 1$. Note that, for large r, $V(o,r) \simeq r^n$, where $o \in \mathcal{R}^n$ is a fixed reference point $o \in \mathcal{R}^n$.

More generally, define $\mathcal{R}^\alpha$ for any real $\alpha > 0$ as $\left(\mathbb{R}^D, g_\alpha\right)$ where the metric g_α is determined in the polar coordinates (r, θ) by

$$g_\alpha = dr^2 + r^{2\beta} d\theta^2$$

for $r > 1$ and g_α is Euclidean for small r, where

$$\beta = \frac{\alpha - 1}{N - 1}.$$

For example, for $\alpha = N$ we obtain $\mathcal{R}^D = \mathbb{R}^D$. It is easy to verify that, for large r,

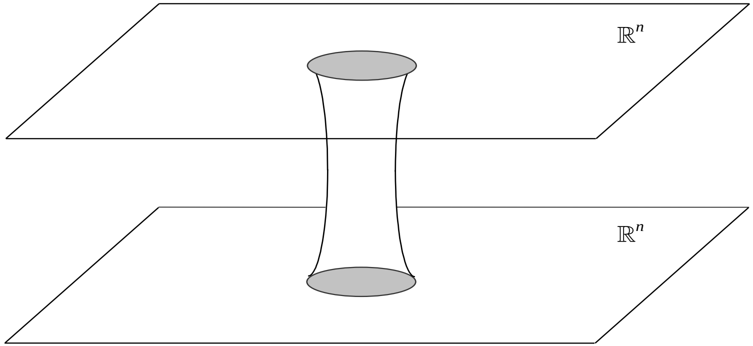
$$V\left(o, r\right) \simeq r^\alpha.$$

Moreover, $\mathcal{R}^\alpha$ satisfies (LY) provided $0 < \alpha \leq D$.

The number α is called "the dimension at ∞" of $\mathcal{R}^\alpha$, while D is the topological "local" dimension of $\mathcal{R}^\alpha$.

Example with bottleneck

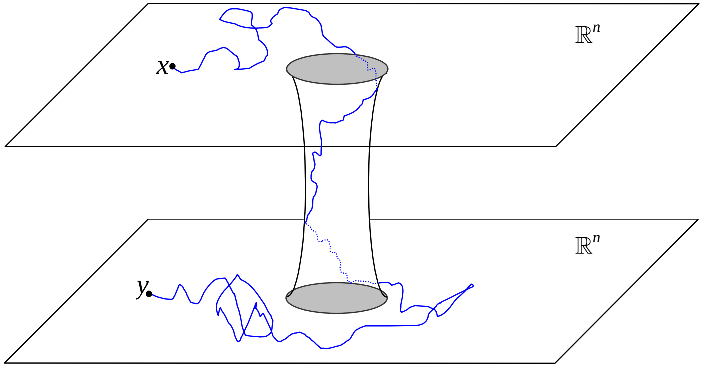
Let $M = \mathbb{R}^n \# \mathbb{R}^n$ be a connected sum of two copies of $\mathbb{R}^n$ with $n \geq 3$. On this manifold $V(x,r) \simeq r^n$ as in $\mathbb{R}^n$.



The heat kernel on M satisfies upper bound of (LY) but the lower bound

$$p_t(x,y) \geq \frac{C}{t^{n/2}} \exp\left(-\frac{d^2(x,y)}{ct}\right)$$

breaks down if x and y belong to different copies of $\mathbb{R}^n$ as on the picture below.



Indeed, as we will see later on, in this case, for large t,

$$p_t(x,y) \asymp \frac{C}{t^{n/2}} \left(\frac{1}{|x|^{n-2}} + \frac{1}{|y|^{n-2}} \right) \exp\left(-\frac{d^2(x,y)}{ct} \right). \tag{1}$$

In particular, if $|x| \simeq |y| \simeq \sqrt{t}$ then $p_t(x,y) \simeq \frac{1}{t^{n-1}} \ll \frac{1}{t^{n/2}}$ where the value $\frac{1}{t^{n/2}}$ is predicted by (LY).

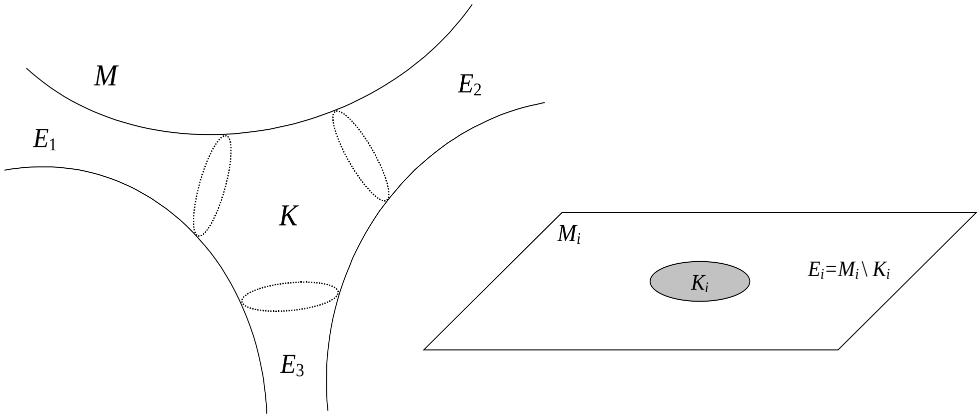
Probabilistic meaning: for Brownian motion getting from x to y is hard as it has to go through the *bottleneck* of the central part, thus significantly reducing transition probability.

Manifolds with ends

Let $M_1, ..., M_k$ and M be complete non-compact Riemannian manifolds. We say that M is a connected sum of $M_1, ..., M_k$ and write

$$M = M_1 \# M_2 \# ... \# M_k$$

if $M = K \sqcup E_1 \sqcup ... \sqcup E_k$, where $K \subset M$ is compact and each E_i is isometric to an exterior domain in M_i. The sets E_i are called the *ends* of M (sometimes M_i are also referred to as ends).



The question to be discussed here is:

Assuming that all M_i are complete and satisfy (LY),
how to estimate the heat kernel on $M = M_1 \# M_2 \# ... \# M_k$?

For example, how to estimate the heat kernel on $M = \mathcal{R}^{\alpha_1} \# \mathcal{R}^{\alpha_2} \# ... \# \mathcal{R}^{\alpha_k}$?

Or even on $M = \mathbb{R}^n \# \mathbb{R}^n$?

The estimate in the case $n \geq 3$ was stated in (1), but the case $n = 2$ is more complicated.

The answer to the above question depends on the property of the ends M_i to be parabolic or not, which will be discussed on the next page.

Parabolic and non-parabolic manifolds

Definition. A Riemannian manifold M is called *parabolic* if any positive superharmonic function on M is constant, and *non-parabolic* otherwise.

The parabolicity is equivalent to each of the following properties, that can be regarded as equivalent definitions:

1. There exists no positive fundamental solution of $-\Delta$.
2. $\int^{\infty} p_t(x,y)\, dt = \infty$ for all/some $x, y \in M$.
3. Brownian motion on M is recurrent.

For example, $\mathbb{R}^n$ is parabolic for $n \leq 2$ and non-parabolic for $n > 2$.

Theorem 3 *Let M be geodesically complete and satisfy (LY). Then M is parabolic if and only if for all/some $x \in M$*

$$\int^{\infty} \frac{r dr}{V(x,r)} = \infty. \tag{2}$$

For example, if $V(x,r) \simeq r^{\alpha}$ then (2) is satisfies if and only if $\alpha \leq 2$. In particular, $\mathcal{R}^{\alpha}$ is parabolic if and only if $\alpha \leq 2$.

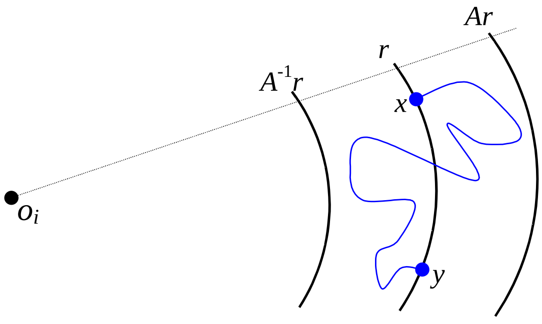
Heat kernels on manifolds with ends

Let $M_1, \ldots, M_k$ be complete non-compact manifolds satisfying (LY). Fix a reference point $o_i \in M_i$ and set $|x| = d_i(x, o_i)$. Assume for simplicity that

$$V_i(o_i, r) \simeq r^{\alpha_i} \text{ for large } r.$$

(results for general functions V_i are available as well).

In the case if M_i is parabolic, we assume in addition that M_i has "*relatively connected annuli*": there is $A > 1$ such that, for all large r and all x, y with $|x| = |y| = r$, the points x, y can be connected by a curve in the annulus $B_i(o_i, Ar) \setminus B_i(o_i, A^{-1}r)$. Clearly, all $\mathcal{R}^\alpha$ have this property (but not $\mathbb{R}^1$).



Set $M = M_1 \# ... \# M_k$. We present in this setting partial estimates of the heat kernel $p_t(x,y)$ on M when x and y belong to different ends E_i and E_j, respectively, and $|x|, |y|, t$ are large. Estimates for all t, x, y are available as well.

Non-parabolic case

Theorem 4 (AG and L.Saloff-Coste '09) *Under the above conditions, assume that all $\alpha_i > 2$ (that is, all M_i are non-parabolic). Set*

$$\alpha = \min_{1 \leq i \leq k} \alpha_i \ .$$

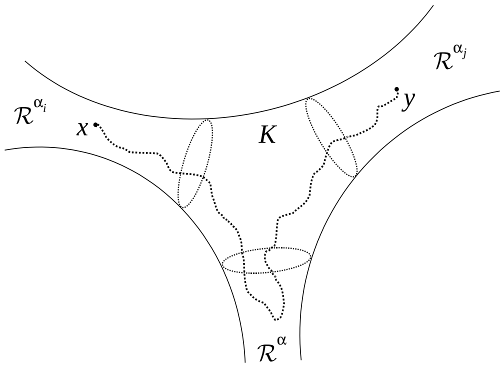
For $x \in E_i$ and $y \in E_j$ with $i \neq j$ we have

$$p_t(x,y) \asymp C \left(\frac{1}{t^{\alpha/2} |x|^{\alpha_i - 2} |y|^{\alpha_j - 2}} + \frac{1}{t^{\alpha_j/2} |x|^{\alpha_i - 2}} + \frac{1}{t^{\alpha_i/2} |y|^{\alpha_j - 2}} \right) e^{-\frac{d^2(x,y)}{ct}}. \qquad (3)$$

In the *long time regime*, that is, for fixed x, y and for $t \to \infty$, we obtain from (3) that $p_t(x,y) \simeq t^{-\alpha/2}$.

Hence, the long time decay of p_t is determined by the *minimal* volume growth exponent $\min \alpha_i$. Note for comparison, that $V(x,r) \simeq r^{\max \alpha_i}$.

For example, (3) holds for $M=\mathcal{R}^{\alpha_1}\#...\#\mathcal{R}^{\alpha_k}$ if all $\alpha_i>2$. Probabilistic meaning for $p_t(x,y)\simeq t^{-\alpha/2}$: in order to get from x to y in time t, Brownian motion on M spends most time in the *smallest* end $\mathcal{R}^{\alpha}$. The reason for that is, that the return probability in that end is the largest.



Consider also the *medium time regime* when $|x|\simeq|y|\simeq\sqrt{t}\to\infty$. In this case (3) implies $p_t(x,y)\simeq t^{-\left(\frac{\alpha_i+\alpha_j}{2}-1\right)}\ll t^{-\alpha/2}$, which we refer to as a bottleneck effect. In the case $M=\mathbb{R}^n\#\mathbb{R}^n$, $n>2$, (3) implies (1), that is,

$$p_t(x,y)\asymp\frac{C}{t^{n/2}}\left(\frac{1}{|x|^{n-2}}+\frac{1}{|y|^{n-2}}\right)e^{-\frac{d^2(x,y)}{ct}}.$$

Mixed case

Theorem 5 *Assume that all* $\alpha_i \neq 2$ *and there is* $\alpha_l > 2$*. Set*

$$\widetilde{\alpha}_i := \begin{cases} 4 - \alpha_i, & \alpha_i < 2 \\ \alpha_i, & \alpha_i > 2 \end{cases}$$

and

$$\alpha := \min_{1 \leq i \leq k} \widetilde{\alpha}_i.$$

For $x \in E_i$ *and* $y \in E_j$ *with* $i \neq j$ *we have*

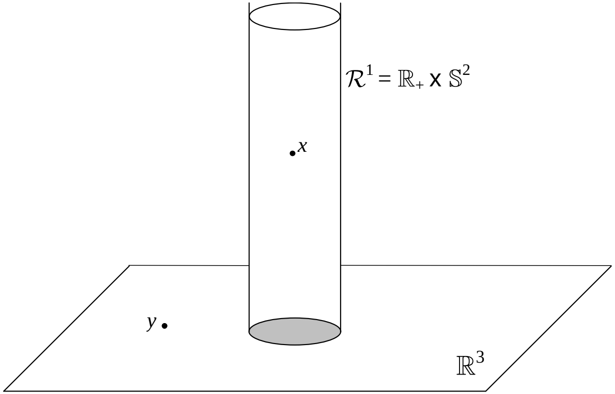
$$p_t(x,y) \simeq C \left(\frac{1}{t^{\alpha/2} |x|^{\widetilde{\alpha}_i - 2} |y|^{\widetilde{\alpha}_j - 2}} + \frac{1}{t^{\widetilde{\alpha}_i/2} |y|^{\widetilde{\alpha}_j - 2}} + \frac{1}{t^{\widetilde{\alpha}_j/2} |x|^{\widetilde{\alpha}_i - 2}} \right) \times |x|^{(2-\alpha_i)_+} \; |y|^{(2-\alpha_j)_+} \; e^{-\frac{d^2(x,y)}{ct}} \tag{4}$$

Observe that always $\widetilde{\alpha}_i > 2$, and the minimal $\widetilde{\alpha}_i$ is determined by the value of α_i that is *nearest* to 2!

Hence, the long time decay of the heat kernel $p_t(x,y) \simeq t^{-\alpha/2}$ **is determined by the nearest to 2 value of** α_i**.**

This rules applies also to Theorem 4 where the nearest to 2 exponent α_i is the minimal one. As we will see below, this rules is valid also in the parabolic case.

As an example, consider $M = \mathcal{R}^1 \# \mathcal{R}^3$, where $x \in \mathcal{R}^1$ and $y \in \mathcal{R}^3$.



In this case $\alpha_1 = 1,\ \alpha_2 = 3$ whence $\widetilde{\alpha}_1 = \widetilde{\alpha}_2 = 3$. It follows from (4) that

$$p_t(x,y) \asymp \frac{C}{t^{3/2}} \left(1 + \frac{|x|}{|y|}\right) e^{-\frac{d^2(x,y)}{ct}}.$$

For $t \to \infty$ we obtain $p_t(x,y) \simeq t^{-3/2}$. In the case $|y| \simeq 1$, $|x| \simeq \sqrt{t} \to \infty$ we obtain $p_t(x,y) \simeq t^{-1} \gg t^{-3/2}$ – a kind of anti-bottleneck effect!

Parabolic case

In the next two theorems we state our main result, obtained by AG, S.Ishiwata and L.Saloff-Coste in 2015.

Theorem 6 (Subcritical case) *Assume that* $0 < \alpha_i < 2$ *for all* $i = 1, ..., k$ *and set*

$$\alpha = \max_{1 \le i \le k} \alpha_i \ .$$

For $x \in E_i$ *and* $y \in E_j$ *with* $i \neq j$ *we have*

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/2}} e^{-\frac{d^2(x,y)}{ct}}.$$

In this case the long time behavior of the heat kernel $p_t(x, y) \simeq t^{-\alpha/2}$ is determined by the *maximal* volume growth exponent α_i, which is again nearest to 2. There is no bottleneck effect in this case.

In the next statement we use the following notation:

$$Q(x, t) = \frac{1}{\ln|x|} + \frac{1}{\ln t}\left(\ln \frac{\sqrt{t}}{|x|}\right)_+ \simeq \begin{cases} \frac{1}{\ln|x|}, & \text{if } |x| \ge \sqrt{t} \\ \frac{1}{\ln t} \ln \frac{e\sqrt{t}}{|x|}, & \text{if } |x| \le \sqrt{t}, \end{cases}$$

Theorem 7 (Critical case) *Assume that $0 < \alpha_i \leq 2$ for all $i = 1, ..., k$ and that $\alpha_l = 2$ for some l. For $x \in E_i$ and $y \in E_j$ with $i \neq j$ the following is true:*

(a) *If $\alpha_i < 2$ and $\alpha_j < 2$ then in the case $|x| + |y| \geq \sqrt{t}$*

$$p_t(x, y) \asymp \frac{C \ln t}{t} e^{-\frac{d^2(x,y)}{ct}},$$

and in the case $|x| + |y| < \sqrt{t}$

$$p_t(x, y) \asymp \frac{C}{t} \left(1 + \ln t \ \left[\left(\frac{|x|}{\sqrt{t}} \right)^{2-\alpha_i} + \left(\frac{|y|}{\sqrt{t}} \right)^{2-\alpha_j} \right] \right).$$

(b) *If $\alpha_i = 2$ and $\alpha_j < 2$ then*

$$p_t(x, y) \asymp \frac{C}{t} \left(1 + Q(x, t) \ln t \ \left(\frac{|y|}{|y| + \sqrt{t}} \right)^{2-\alpha_j} \right) e^{-\frac{d^2(x,y)}{ct}}.$$

In particular, if $|x|, |y| \geq \sqrt{t}$ then

$$p_t(x, y) \asymp \frac{C}{t} \left(1 + \frac{\ln t}{\ln |x|} \right) e^{-\frac{d^2(x,y)}{ct}}$$

and if $|x|, |y| \leq \sqrt{t}$ *then*

$$p_t(x,y) \asymp \frac{C}{t}\left(1 + \ln\frac{e\sqrt{t}}{|x|}\left(\frac{|y|}{\sqrt{t}}\right)^{2-\alpha_j}\right).$$

(c) *If* $\alpha_i = \alpha_j = 2$ *then*

$$p_t(x,y) \asymp \frac{C}{t}\left(Q(x,t)\,Q(y,t) + Q(x,t)\frac{\ln|y|}{\ln|y| + \ln t} + Q(y,t)\frac{\ln|x|}{\ln|x| + \ln t}\right)e^{-\frac{d^2(x,y)}{ct}}.$$

In particular, if $|x|, |y| \geq \sqrt{t}$ *then*

$$p_t(x,y) \asymp \frac{C}{t}\left(\frac{1}{\ln|x|} + \frac{1}{\ln|y|}\right)e^{-\frac{d^2(x,y)}{ct}},$$

and if $|x|, |y| \leq \sqrt{t}$ *then*

$$p_t(x,y) \asymp \frac{C}{t\ln^2 t}\left(\ln\frac{e\sqrt{t}}{|x|}\ln\frac{e\sqrt{t}}{|y|} + \ln|y|\ln\frac{e\sqrt{t}}{|x|} + \ln|x|\ln\frac{e\sqrt{t}}{|y|}\right).$$

Note that in the setting of Theorem 7 the long time behavior of the heat kernel is simple:

$$p_t(x,y) \simeq \frac{1}{t} \simeq \frac{1}{V(o,\sqrt{t})}$$

and is determined by the value $\alpha_l = 2$, which is again the nearest to 2 volume growth exponent.

In the medium time regime $|x| \simeq |y| \simeq \sqrt{t} \to \infty$ we have the following.

In the case (a), that is, $\alpha_i, \alpha_j < 2$:

$$p_t(x,y) \simeq \frac{\ln t}{t}.$$

In the case (b), that is, $\alpha_i = 2$, $\alpha_j < 2$:

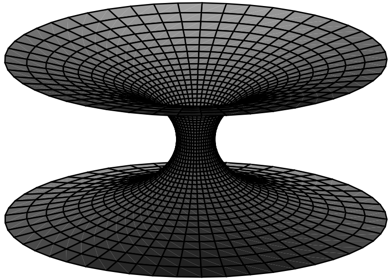
$$p_t(x,y) \simeq \frac{1}{t}.$$

In the case (c), that is, $\alpha_i = \alpha_j = 2$:

$$p_t(x,y) \simeq \frac{1}{t \ln t}.$$

Some examples

Let $M=\mathbb{R}^2\#\mathbb{R}^2$.
This manifold is equivalent to the catenoid. Let x,y belong to the different sheets.



Then by Theorem 7(c) we have

$$p_t(x,y)\simeq\frac{C}{t}\left(Q(x,t)Q(y,t)+Q(x,t)\frac{\ln|y|}{\ln|y|+\ln t}+Q(y,t)\frac{\ln|x|}{\ln|x|+\ln t}\right)e^{-\frac{d^2(x,y)}{ct}}.$$

If $t\to+\infty$ then $p_t(x,y)\simeq t^{-1}$. If $|x|\geq\sqrt{t}$ and $|y|\geq\sqrt{t}$ then

$$p_t(x,y)\asymp\frac{C}{t}\left(\frac{1}{\ln|x|}+\frac{1}{\ln|y|}\right)e^{-\frac{d^2(x,y)}{ct}}.$$

In particular, if $|x|\simeq|y|\simeq\sqrt{t}$ then $p_t(x,y)\simeq\frac{1}{t\ln t}$.

Let $M=\mathcal{R}^1\#\mathcal{R}^2$. By Theorem 7($b$) we obtain, for $x\in\mathcal{R}^1$ and $y\in\mathcal{R}^2$,

$$p_t(x,y)\asymp\frac{C}{t}\left(1+\ln t\frac{|x|}{|x|+\sqrt{t}}Q(y,t)\right)e^{-\frac{d^2(x,y)}{ct}}$$

If $|x|\,,|y|>\sqrt{t}$ then

$$p_t(x,y)\asymp\frac{C}{t}e^{-\frac{d^2(x,y)}{ct}},$$

If $|x|\,,|y|\leq\sqrt{t}$ then

$$p_t(x,y)\simeq\frac{1}{t}\left(1+\frac{|x|}{\sqrt{t}}\ln\frac{e\sqrt{t}}{|y|}\right).$$

For $t\to\infty$ we obtain

$$p_t\left(x,y\right)\simeq t^{-1}.$$

If $y\simeq 1$ and $|x|\simeq\sqrt{t}\to\infty$ then

$$p_t\left(x,y\right)\simeq\frac{\ln t}{t}.$$

Let $M=\mathcal{R}^2\#\mathcal{R}^3$. This is a mixed case, that is covered by extension of Theorem 5. It yields the following estimate for $x\in\mathcal{R}^2$ and $y\in\mathcal{R}^3$:

$$p_t(x,y)\asymp C\left(\frac{\ln|x|}{t\ln^2 t\ \ |y|}+\frac{1}{t^{3/2}}Q\left(x,t\right)\right)e^{-\frac{d^2(x,y)}{ct}}.$$

For $t\to\infty$ we have

$$p_t\left(x,y\right)\simeq\frac{1}{t\ln^2 t}.$$

For $|x|\simeq|y|\simeq\sqrt{t}\to\infty$ we obtain

$$p_t\left(x,y\right)\simeq\frac{1}{t^{3/2}\ln t},$$

so that there is a bottleneck effect. For $|y|\simeq 1$ and $|x|\simeq\sqrt{t}\to\infty$ we obtain

$$p_t\left(x,y\right)\simeq\frac{1}{t\ln t},$$

that is, an anti-bottleneck effect.

Let $M=\mathcal{R}^1\#\mathcal{R}^2\#\mathcal{R}^3$. For $x\in\mathcal{R}^1$ and $y\in\mathcal{R}^2$ we have

$$p_t(x,y)\simeq C\left(\frac{\ln|y|}{t\ln^2 t}+\left(\frac{|x|}{t^{3/2}}+\frac{1}{t\ln^2 t}\right)Q(y,t)\right)e^{-\frac{d^2(x,y)}{ct}}.$$

In particular, for $t\to\infty$

$$p_t(x,y)\simeq\frac{1}{t\ln^2 t},$$

For $|x|\simeq|y|\simeq\sqrt{t}$ we have an anti-bottleneck effect:

$$p_t(x,y)\simeq\frac{1}{t\ln t}.$$

For $x\in\mathcal{R}^1$ and $y\in\mathcal{R}^3$ we have

$$p_t(x,y)\asymp C\left(\frac{1}{t^{3/2}}\left(1+\frac{|x|}{|y|}\right)+\frac{1}{|y|\,t\ln^2 t}\right)e^{-\frac{d^2(x,y)}{ct}}.$$

For $|x|\simeq|y|\simeq\sqrt{t}$ we have a bottleneck effect:

$$p_t(x,y)\simeq\frac{1}{t^{3/2}}.$$

Approach to the proof

The following approach works when all ends are non-parabolic (Theorem 4) and when all ends are parabolic (Theorems 6, 7).

In the both cases one starts with estimates for $p_t(o,o)$ where $o \in K$ is a fixed reference point. In the non-parabolic case one uses Faber-Krahn type inequalities to obtain upper bound of $p_t(o,o)$. Li-Yau upper bound for the heat kernel $p_t^{(i)}$ on M_i implies certain FK inequality on M_i. The "weakest" of FK inequalities across all ends M_i gives a FK inequality on M, which implies then the upper bound of $p_t(o,o)$, which matches the weakest upper bound among all $p_t^{(i)}(o_i,o_i)$.

For the lower bounds of $p_t(x,y)$ one uses inequality $p_t(x,y) \geq p_t^{E_i}(x,y)$, where $p_t^{E_i}$ is the Dirichlet heat kernel in E_i. By non-parabolicity of M_i, $p_t^{E_i}(x,y)$ satisfies (LY) away from ∂E_i, which then implies the lower bound of $p_t(o,o)$ that matches the strongest lower bound among all $p_t^{(i)}(o_i,o_i)$.

To obtain estimates for $p_t(x,y)$ for arbitrary x,y one uses the *hitting probability.* For any closed set $S \subset M$, define the function $\psi_S(t,x)$ on $\mathbb{R}_+ \times M$ as the probability that Brownian motion on M hits S before time t provided the starting point is x. In fact, $\psi_S(t,x)$ solves in $\mathbb{R}_+ \times S^c$ the heat equation with the initial condition $\psi_S(0,\cdot) = 0$ and the boundary condition $\psi_S(t,\cdot) = 1$ on ∂S.

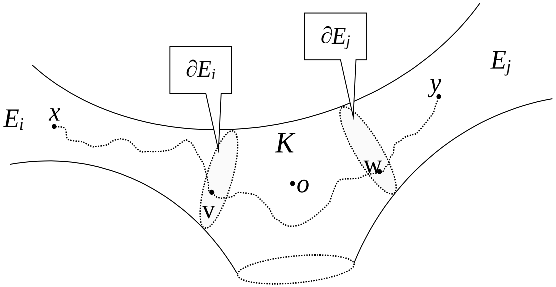
Then the following inequality is true for $x \in E_i$ and $y \in E_j$ with $i \neq j$:

$$
\begin{aligned}
p_t(x,y) \ \leq \ & 2\psi_{\partial E_i}(t,x)\psi_{\partial E_j}(t,y) \sup_{s\in[t/4,t]} \sup_{v\in\partial E_i, w\in\partial E_j} p_s(v,w) \\
& + \left(\psi_{\partial E_i}(t,x) \sup_{s\in[t/4,t]} \psi'_{\partial E_j}(s,y) + \psi_{\partial E_j}(t,y) \sup_{s\in[t/4,t]} \psi'_{\partial E_i}(s,x) \right) \\
& \times \int_0^t \sup_{v\in\partial E_i, w\in\partial E_j} p_s(v,w)\,ds,
\end{aligned}
$$

and there is a similar lower bound.

Note that $\psi_{\partial E_i}$ depends only on the intrinsic geometry of M_i and can be estimated using (LY) on M_i.

By local Harnack inequality $p_s(v,w)$ can be estimated via $p_s(o,o)$, which gives desired estimates for $p_t(x,y)$



In the parabolic case this scheme works except for the crucial upper bound for $p_t(o,o)$. Indeed, the FK method gives the upper bound of $p_t(o,o)$ using the smallest volume growth exponent α_i whereas in the parabolic case we expect to use the largest exponent α_i, that is, we need a stronger upper bound.

In fact, in the parabolic case we prove the following upper bound:

$$p_t(o,o) \le \frac{C}{V\left(o,\sqrt{t}\right)}, \tag{5}$$

using a new method involving the *resolvents* on each end:

$$R_\lambda^{(i)}(x,y) = \int_0^\infty e^{-t\lambda} p_t^{(i)}(x,y)\, dt,$$

where $\lambda > 0$. The parabolicity of M_i implies that $R_\lambda^{(i)}(x,y) \to \infty$ as $\lambda \to 0$, and the rate of increase of $R_\lambda^{(i)}(x,y)$ as $\lambda \to 0$ is related to the rate of decay of $p_t^{(i)}(x,y)$ as $t \to \infty$.

One shows that the resolvent $R_\lambda(x,y)$ on M satisfies a certain integral equation involving as coefficients $R_\lambda^{(i)}(x,y)$. This allows to estimate the rate of growth of $R_\lambda(x,y)$ as $\lambda \to 0$ and then to recover the upper bound (5). In the critical case one has to involve also the estimates of $\frac{\partial}{\partial\lambda} R_\lambda(x,y)$.

Once the upper bound (5) is known, it implies automatically the matching lower bound

$$p_t(o,o) \geq \frac{c}{V\left(o,\sqrt{t}\right)}$$

by a theorem of AG and T.Coulhon '97.