Homotopy and homology of digraphs

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Contents

1 Homology theory of digraphs

In this Section we state the basic notions of homology theory for digraphs in the form that we need in subsequent sections. This is a slight adaptation of a more general theory from [5], [7]. A dual cohomology theory was developed from a different point of view in [3], [4].

1.1 Paths and their boundaries

Let V be a finite set. For any $p \geq 0$, an *elementary* p*-path* is any (ordered) sequence $i_0, ..., i_p$ of $p+1$ vertices of V that will be denoted simply by $i_0...i_p$ or by $e_{i_0...i_p}$. Fix a commutative ring $\mathbb{K}$ with unity and denote by $\Lambda_p = \Lambda_p(V) = \Lambda_p(V, \mathbb{K})$ the free $\mathbb{K}$-module that consist of all formal $\mathbb{K}$-linear combinations of all elementary p-paths. Hence, each p-path has a form

$$v = \sum_{i_0 i_1 ... i_p} v^{i_0 i_1 ... i_p} e_{i_0 i_1 ... i_p}, \quad \text{where} \quad v^{i_0 i_1 ... i_p} \in \mathbb{K}.$$

Definition. Define for any $p \geq 1$ the *boundary operator* $\partial : \Lambda_p \to \Lambda_{p-1}$ as follows. We set

$$\partial e_{i_0 ... i_{p-1}} = \sum_{q=0}^{p} (-1)^q e_{i_0 ... \widehat{i_q} ... i_p} \tag{1}$$

where $\widehat{i_q}$ means omission of the index i_q; then extend ∂ to an arbitrary $v \in \Lambda_p$ by $\mathbb{K}$-linearity.

For example, we have

$$\begin{aligned} \partial e_{ab} &= e_b - e_a \\ \partial e_{abc} &= e_{bc} - e_{ac} + e_{ab}. \end{aligned}$$

It follows that, for a general p-path v,

$$(\partial v)^{i_0 ... i_{p-1}} = \sum_k \sum_{q=0}^{p} (-1)^q v^{i_0 ... i_{q-1} k i_q ... i_{p-1}}, \tag{2}$$

where the index k is inserted between i_{q-1} and i_q (in case $q = 0$ before i_0 and in case $q = n$ after i_{n-1}).

Set also $\Lambda_{-1} = \{0\}$ and define the operator $\partial : \Lambda_0 \to \Lambda_{-1}$ by $\partial = 0$.

Claim. $\partial^2 v = 0$ *for any* $v \in \Lambda_p$ *with* $p \geq 1$.

Proof. For $p=1$ this is trivial. For $p\geq 2$ we have by (1)

$$\begin{aligned}
\partial^2 e_{i_0\ldots i_p} &= \sum_{q=0}^{p}(-1)^q\,\partial e_{i_0\ldots\widehat{i_q}\ldots i_p}\\
&= \sum_{q=0}^{p}(-1)^q\left(\sum_{r=0}^{q-1}(-1)^r\,e_{i_0\ldots\widehat{i_r}\ldots\widehat{i_q}\ldots i_p}+\sum_{r=q+1}^{p}(-1)^{r-1}\,e_{i_0\ldots\widehat{i_q}\ldots\widehat{i_r}\ldots i_p}\right)\\
&= \sum_{0\leq r<q\leq p}(-1)^{q+r}\,e_{i_0\ldots\widehat{i_r}\ldots\widehat{i_q}\ldots i_p}-\sum_{0\leq q<r\leq p}(-1)^{q+r}\,e_{i_0\ldots\widehat{i_q}\ldots\widehat{i_r}\ldots i_p}.
\end{aligned}$$

After switching q and r in the last sum we see that the two sums cancel out, whence $\partial^2 e_{i_0\ldots i_p}=0$. This implies $\partial^2 v=0$ for all $v\in\Lambda_p$. ■

Hence, the family of $\mathbb{K}$-modules $\{\Lambda_p\}_{p\geq -1}$ with the boundary operator ∂ determines a chain complex

$$0\leftarrow\Lambda_0\leftarrow\ldots\leftarrow\Lambda_{p-1}\leftarrow\Lambda_p\leftarrow\ldots \tag{3}$$

that will be denoted by $\Lambda_*(V)=\Lambda_*(V,\mathbb{K})$.

Given a map $f:V\to V'$ between two finite sets V and V', define for any $p\geq 0$ the *induced map*

$$f_*\colon\Lambda_p(V)\to\Lambda_p(V')$$

by the rule

$$f_*\left(e_{i_0\ldots i_p}\right)=e_{f(i_0)\ldots f(i_p)},$$

extended by $\mathbb{K}$-linearity to all elements of $\Lambda_p(V)$. It follows from (1) that $\partial f_*=f_*\partial$, that is, the following diagram is commutative

$$\begin{array}{ccccccccccccc}
0 & \leftarrow & \Lambda_0(V) & \leftarrow & \Lambda_1(V) & \leftarrow & \ldots & \leftarrow & \Lambda_{p-1}(V) & \leftarrow & \Lambda_p(V) & \leftarrow & \ldots\\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \leftarrow & \Lambda_0(V') & \leftarrow & \Lambda_1(V') & \leftarrow & \ldots & \leftarrow & \Lambda_{p-1}(V') & \leftarrow & \Lambda_p(V') & \leftarrow & \ldots
\end{array}$$

where the horizontal arrows are given by ∂ and the vertical arrows are given by f_*. Hence, the map f_* is a morphism of chain complexes $\Lambda_*(V)$ and $\Lambda_*(V')$.

1.2 Regular paths

Definition. An elementary p-path $e_{i_0\ldots i_p}$ on a set V is called *regular* if $i_k\neq i_{k+1}$ for all $k=0,\ldots,p-1$, and non-regular otherwise.

Let N_p be the submodule of Λ_p that is $\mathbb{K}$-spanned by non-regular $e_{i_0\ldots i_p}$.

Claim. $\partial N_p\subset N_{p-1}$.

Proof. Since any $v\in N_p$ is a linear combination of elementary non-regular paths $e_{i_0\ldots i_p}$, it suffices to prove that if $e_{i_0\ldots i_p}$ is non-regular then $\partial e_{i_0\ldots i_p}$ is non-regular, too.

Indeed, for a non-regular path $i_0...i_p$ there exists an index k such that $i_k = i_{k+1}$. Then we have

$$\begin{aligned}\partial e_{i_0...i_p} &= e_{i_1...i_p} - e_{i_0 i_2...i_p} + ... \\ &\quad + (-1)^k e_{i_0...i_{k-1} i_{k+1} i_{k+2}...i_p} + (-1)^{k+1} e_{i_0...i_{k-1} i_k i_{k+2}...i_p} \\ &\quad + ... + (-1)^p e_{i_0...i_{p-1}}.\end{aligned} \tag{4}$$

By $i_k = i_{k+1}$ the two terms in the middle line of (4) cancel out, whereas all other terms are non-regular, whence $\partial e_{i_0...i_p} \in N_{p-1}$. ■

Consider the quotient $\mathcal{R}_p := \Lambda_p / N_p$. Since $\partial N_p \subset N_{p-1}$, the induced boundary operator

$$\partial : \mathcal{R}_p \to \mathcal{R}_{p-1} \quad (p \geq 0)$$

is well-defined. This operator ∂ is called the *regular* boundary operator. In what follows ∂ will always be the regular boundary operator.

Clearly, $\mathcal{R}_p$ is linearly isomorphic to the space of regular p-paths:

$$\mathcal{R}_p \cong \operatorname{span}_{\mathbb{K}} \left\{ e_{i_0...i_p} : i_0...i_p \text{ is regular} \right\} \tag{5}$$

For simplicity of notation, we will identify $\mathcal{R}_p$ with this space, by setting all non-regular p-paths to be equal to 0.

Then the regular ∂ satisfies (1) as before but with the following adjustments: all the terms in the right hand side of (1), that are non-regular, are set to be zero. For example, for non-regular operator $\partial : \Lambda_2 \to \Lambda_1$ we have

$$\partial e_{iji} = e_{ji} - e_{ii} + e_{ij}$$

whereas for the regular operator $\partial : \mathcal{R}_2 \to \mathcal{R}_1$

$$\partial e_{iji} = e_{ji} + e_{ij}.$$

We denote by $\mathcal{R}_* (V)$ the regular chain complex

$$0 \leftarrow \mathcal{R}_0 \leftarrow ... \leftarrow \mathcal{R}_{p-1} \leftarrow \mathcal{R}_p \leftarrow ...$$

Consider a map $f : V \to V'$ between two finite sets V and V'. We know that the induced map f_* is a morphism of chain complexes $\Lambda_* (V)$ and $\Lambda_* (V')$. If an elementary path $e_{i_0...i_p}$ is non-regular then $f_* \left(e_{i_0...i_p} \right)$ is obviously also non-regular, so that

$$f_* \left(N_p (V) \right) \subset N_p (V').$$

Therefore, f_* is well-defined on the quotient Λ_p / N_p so that we obtain the induced map

$$f_* : \mathcal{R}_p (V) \to \mathcal{R}_p (V'). \tag{6}$$

With identification (5) of $\mathcal{R}_p$ we have the following rule for the map (6):

$$f_* \left(e_{i_0...i_p} \right) = \begin{cases} e_{f(i_0)...f(i_p)}, & \text{if } f(i_0)...f(i_p) \text{ is regular}, \\ 0, & \text{if } f(i_0)...f(i_p) \text{ is non-regular}. \end{cases} \tag{7}$$

Since f_* commutes with ∂, we see that (6) provides a morphism $\mathcal{R}_*(V) \to \mathcal{R}_*(V')$ of chain complexes.

1.3 The notion of a digraph

Definition. A *directed graph (digraph)* $G = (V, E)$ is a couple of a set V, whose elements are called the *vertices*, and a subset $E \subset \{V \times V \setminus \operatorname{diag}\}$ of ordered pairs of vertices that are called (directed) *edges* or *arrows*. The fact that $(v, w) \in E$ will also be denoted by $v \to w$.

In particular, a digraph has no edges $v \to v$. We consider only finite digraphs, that is, digraphs with a finite set of vertices.

We write

$$v \overset{\rightarrow}{=} w$$

if either $v = w$ or $v \to w$.

Definition. A *morphism* from a digraph $G = (V, E)$ to a digraph $G' = (V', E')$ is a map $f : V \to V'$ such that

$$v \overset{\rightarrow}{=} w \text{ on } G \ \text{ implies } \ f(v) \overset{\rightarrow}{=} f(w) \ \text{ on } G'. \tag{8}$$

We will refer to such morphisms also as *digraphs maps* and denote them shortly by $f : G \to G'$.

The set of all digraphs with digraphs maps form a *category of digraphs*.

The condition (8) is trivially true for $v = w$ but for $v \to w$ it means that the images $f(v)$ and $f(w)$ either coincide or $f(v) \to f(w)$. In other words, each arrow in G goes to either a vertex or an arrow in G'.

1.4 Allowed and ∂-invariant paths on digraphs

Definition. Let $G = (V, E)$ be a digraph. An elementary p-path $i_0...i_p$ on V is called *allowed* if $i_k \to i_{k+1}$ for any $k = 0, ..., p-1$, and *non-allowed* otherwise. The set of all allowed elementary p-paths will be denoted by E_p.

For example, $E_0 = V$ and $E_1 = E$. Denote by $\mathcal{A}_p = \mathcal{A}_p(G)$ the free $\mathbb{K}$-module spanned by the allowed elementary p-paths, that is,

$$\mathcal{A}_p = \operatorname{span}_{\mathbb{K}} \left\{ e_{i_0...i_p} : i_0...i_p \in E_p \right\}. \tag{9}$$

The elements of $\mathcal{A}_p$ are called *allowed* p-paths. Clearly, all allowed elementary paths are regular, which implies that $\mathcal{A}_p \subset \mathcal{R}_p$.

Note that the family of modules $\mathcal{A}_*$ is in general *not* invariant for ∂. For example, in the digraph

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet}$$

the 2-path e_{123} is allowed while

$$\partial e_{123} = e_{23} - e_{13} + e_{12}$$

is non-allowed.

Consider the following submodules of $\mathcal{A}_p$

$$\Omega_p \equiv \Omega_p(G) := \{v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1}\}. \tag{10}$$

Claim. *Ω_* is ∂-invariant, that is, $\partial\Omega_p \subset \Omega_{p-1}$ for all $p \geq 0$.*

Proof. Indeed, $v \in \Omega_p$ implies $\partial v \in \mathcal{A}_{p-1}$ and $\partial(\partial v) = 0 \in \mathcal{A}_{p-2}$, whence $\partial v \in \Omega_{p-1}$. ■

The elements of Ω_p are called *∂-invariant* p-paths. Hence, we obtain a chain complex $\Omega_* = \Omega_*(G) = \Omega_*(G, \mathbb{K})$:

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots$$

By construction we have $\Omega_0 = \mathcal{A}_0$ and $\Omega_1 = \mathcal{A}_1$, while in general $\Omega_p \subset \mathcal{A}_p$.

Let us define for any $p \geq 0$ the homologies of the digraph G with coefficients from $\mathbb{K}$ by

$$H_p(G, \mathbb{K}) = H_p(G) := H_p(\Omega_*(G)) = \ker \partial|_{\Omega_p} \big/ \operatorname{Im} \partial|_{\Omega_{p+1}}.$$

The elements of $\ker \partial|_{\Omega_p}$ are called closed paths and the elements of $\operatorname{Im} \partial|_{\Omega_{p+1}}$ are called exact.

Let us note that homology groups $H_p(G)$ (as well as the modules $\Omega_p(G)$) can be computed directly by definition using simple tools of linear algebra, in particular, those implemented in modern computational software.

Example. Fix $n \geq 3$. Denote by S_n a digraph with the vertex set $V_{S_n} = \{0, ..., n-1\}$ and with the set of edges E_{S_n} that contains for any $i \in V_{S_n}$ exactly one of the edges $i \to i+1$, $i+1 \to i$ (where $n \equiv 0$), and no other edge. We refer to S_n as a *cycle digraph.*

The following 1-path on S_n

$$\varpi = \sum_{\{i \in S_n : i \to i+1\}} e_{i(i+1)} - \sum_{\{i \in S_n : i+1 \to i\}} e_{(i+1)i} \tag{11}$$

lies in $\Omega_1(S_n)$ and is closed. We will refer to ϖ as a *standard* 1-path on S_n. It is possible to show that ϖ generates the space of all closed 1-paths in $\Omega_1(S_n)$, which is therefore one-dimensional. The homology group $H_1(S_n, \mathbb{K})$ is, hence, generated by the homology class $[\varpi]$, provided this class is non-trivial. One can show that $[\varpi] = 0$ if and only if S_n is isomorphic to one of the following two digraphs:

$$\text{a triangle} \quad \begin{array}{ccc} & \overset{1}{\bullet} & \\ \nearrow & & \searrow \\ {}^{0}\bullet & \longrightarrow & \bullet^{2} \end{array} \qquad \text{or a square} \quad \begin{array}{ccc} {}^{1}\bullet & \longrightarrow & \bullet^{2} \\ \uparrow & & \uparrow \\ {}^{0}\bullet & \longrightarrow & \bullet^{3} \end{array}, \tag{12}$$

so that in this case $H_1(S_n, \mathbb{K}) = \{0\}$. In the case of triangle, ϖ is the boundary of the 2-path $e_{012} \in \Omega_2$, and, in the case of square, ϖ is the boundary of $e_{012} - e_{032} \in \Omega_2$.

If S_n is neither triangle nor square, then $[\varpi]$ is a generator of $H_1(S_n, \mathbb{K}) \cong \mathbb{K}$.

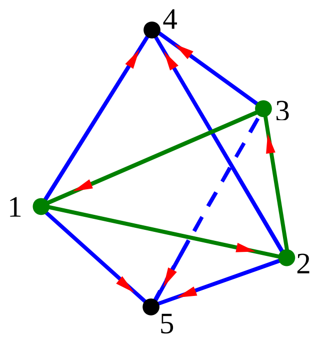


Figure 1: Planar digraph with a nontrivial homology group H_2

Example. Consider a digraph G as on Fig. 1.

A direct computation shows that $H_1(G, \mathbb{K}) = \{0\}$ and $H_2(G, \mathbb{K}) \cong \mathbb{K}$, where a generating element of $H_2(G)$ is

$$e_{124} + e_{234} + e_{314} - (e_{125} + e_{235} + e_{315}) \tag{13}$$

(this path is closed but not exact). It is easy to see that G is a planar graph but nevertheless its second homology group is non-zero. This shows that the digraph homologies "see" some non-trivial intrinsic dimensions of digraphs that are not necessarily related to embedding properties.

Proposition 1 *Let G be any finite digraph. Then any $\omega \in \Omega_2(G, \mathbb{Z})$ can be represented as a linear combination of the ∂-invariant 2-paths of following three types:*

1. *e_{iji} with $i \to j \to i$ (a double edge in G);*
2. *e_{ijk} with $i \to j \to k$ and $i \to k$ (a triangle as a subgraph of G);*
3. *$e_{ijk} - e_{imk}$ with $i \to j \to k$, $i \to m \to k$, $i \not\to k$, $i \neq k$ (a square as a subgraph of G).*

Proof. Since the 2-path ω is allowed, it can be represented as a sum of elementary 2-path e_{ijk} with $i \to j \to k$ multiplied with $+1$ or -1. If $k = i$ then e_{ijk} is a double edge. If $i \neq k$ and $i \to k$ then e_{ijk} is a triangle. Subtracting from ω all double edges and triangles, we can assume that ω has no such terms any more. Then, for any term e_{ijk} in ω we have $i \neq k$ and $i \not\to k$. Fix such a pair i, k and consider any vertex j with $i \to j \to k$. The 1-path $\partial\omega$ is the sum of 1-paths of the form

$$\partial e_{ijk} = e_{ij} - e_{ik} + e_{jk}.$$

Since $\partial\omega$ is allowed but e_{ik} is not allowed, the term e_{ik} should cancel out after we sum up all such terms over all possible j. Therefore, the number of j such that e_{ijk} enters ω with coefficient $+1$ is equal to the number of j such that e_{ijk} enters in ω with the coefficient -1. Combining the pair with $+1$ and -1 together, we obtain that ω is the sum of the terms of the third type (squares). ■

Proposition 2 *Let G and G' be two digraphs, and $f\colon G\to G'$ be a digraph map. Then, for any $p\geq 0$,*

$$f_*\left(\Omega_p(G)\right)\subset\Omega_p\left(G'\right), \tag{14}$$

where f_* is the induced map (6). Moreover, the map

$$f_*:\Omega_p(G)\to\Omega_p\left(G'\right)$$

is a morphism of the chain complexes

$$\Omega_*(G)\to\Omega_*(G')$$

and, consequently, a homomorphism of homology groups

$$H_*(G)\to H_*(G')$$

that will also be denoted by f_*.

Proof. Let us first show that

$$f_*\left(\mathcal{A}_p(G)\right)\subset\mathcal{A}_p\left(G'\right).$$

It suffices to prove that if $e_{i_0\ldots i_p}$ is allowed on G then $f_*\left(e_{i_0\ldots i_p}\right)$ is allowed on G'. Indeed, if $f(i_0)\ldots f(i_p)$ is non-regular then we have by (7) that $f_*\left(e_{i_0\ldots i_p}\right)=0\in\mathcal{A}_p(G')$. If $f(i_0)\ldots f(i_p)$ is regular then $f(i_k)\neq f(i_{k+1})$ for all $k=0,\ldots,p-1$. Since $i_k\to i_{k+1}$ on G, by the definition of a digraph map we have either $f(i_k)\to f(i_{k+1})$ on G' or $f(i_k)=f(i_{k+1})$. Since the second possibility is excluded, we obtain $f(i_k)\to f(i_{k+1})$ for all k, whence it follows that $f_*\left(e_{i_0\ldots i_p}\right)=e_{f(i_0)\ldots(i_p)}$ is allowed on G'.

Now let us prove (14). For any $v\in\Omega_p(G)$ we have by (10) $v\in\mathcal{A}_p(G)$ and $\partial v\in\mathcal{A}_{p-1}(G)$, whence

$$f_*(v)\in\mathcal{A}_p\left(G'\right)\ \text{ and }\ \partial\left(f_*(v)\right)=f_*(\partial v)\in\mathcal{A}_{p-1}\left(G'\right),$$

which implies $f_*(v)\in\Omega_p(G')$ and, hence, (14). Since f_* commutes with ∂, we see that f_* is a morphism of $\Omega_*(G)$ and $\Omega_*(G')$.

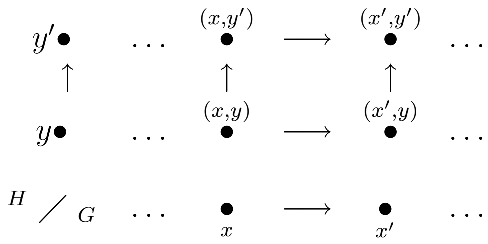
Any morphism of chain complexes induces canonically homomorphism of homology groups because f_* maps closed paths to closed ones and exact paths to exact ones. ■

1.5 Cylinders

Definition. For two digraphs $G=(V_G,E_G)$ and $H=(V_H,E_H)$ define the *Cartesian product* $G\boxdot H$ as a digraph with the set of vertices $V_G\times V_H$ and with the set of edges as follows: for $x,x'\in V_G$ and $y,y'\in V_H$, we have $(x,y)\to(x',y')$ in $G\boxdot H$ if and only if

$$\text{either }\ x'=x\text{ and }y\to y',\ \text{ or }\ x\to x'\text{ and }\ y=y',$$

as is shown on the following diagram:



For any digraph G consider its product $G \boxdot I$ with the digraph $I = ({}^0\bullet \rightarrow \bullet^1)$.

Definition. The digraph $G \boxdot I$ is called the *cylinder* over G and will be denoted by $\widehat{G}$.

We shall put the hat $\widehat{}$ over all notation related to $\widehat{G}$, for example, $\widehat{\mathcal{R}}_p := \mathcal{R}_p(\widehat{G})$ and $\widehat{\Omega}_p := \Omega_p(\widehat{G})$.

By the definition of Cartesian product, the set of vertices of $\widehat{G}$ is $\widehat{V} = V \times \{0,1\}$ and the edges of $\widehat{G}$ are given by the rules:

1. $(x,0) \rightarrow (x,1)$ for all $x \in V_G$ (vertical edges);
2. $(x,a) \rightarrow (y,a)$ for all $x \rightarrow y$ in G and $a \in \{0,1\}$ (horizontal edges).

The digraph $\widehat{G}$ consists of two subgraphs $G \times 0$ and $G \times 1$ both being isomorphic to G. So, we identify $G \times 0$ with G, that is, put $(x,0) \equiv x$ for all $x \in V$ and denote $G \times 1$ by G' using the notation $(x,1) =: x'$ for all $x \in V$. Then, for $x \in V$, we have $x \rightarrow x'$, and, for all $x, y \in V$, we have

$$x \rightarrow y \text{ in } \widehat{G} \Leftrightarrow x \rightarrow y \text{ in } G \Leftrightarrow x' \rightarrow y' \text{ in } G'.$$

Define the operation of *lifting* paths from G to $\widehat{G}$ as follows. Fix $p \geq 0$. If $v = e_{i_0 \ldots i_p}$ then $\widehat{v}$ is a $(p+1)$-path in $\widehat{G}$ defined by

$$\widehat{e_{i_0 \ldots i_p}} = \sum_{k=0}^{p} (-1)^k \, e_{i_0 \ldots i_k i'_k \ldots i'_p}. \tag{15}$$

By $\mathbb{K}$-linearity this definition extends to all $v \in \Lambda_p$, thus giving $\widehat{v} \in \widehat{\Lambda}_{p+1}$. In the case $p = -1$ define lifting of $v = 0 \in \Lambda_{-1}(G)$ by $\widehat{v} = 0 \in \widehat{\Lambda}_0$.

For example, we have

$$\begin{aligned} \widehat{e_a} &= e_{aa'} \\ \widehat{e_{ab}} &= e_{aa'b'} - e_{abb'} \\ \widehat{e_{abc}} &= e_{aa'b'c'} - e_{abb'c'} + e_{abcc'} \end{aligned}$$

(see Fig. 2).

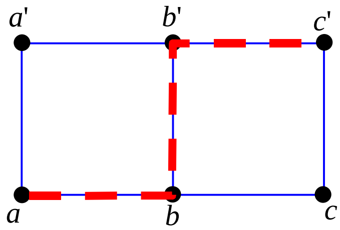


Figure 2: The component $e_{abb'c'}$ of the 3-path $\widehat{e_{abc}}$

If $i_0...i_p$ is allowed in G then, for any k, the path $i_0...i_k i_k'...i_p'$ is allowed in $\widehat{G}$:

$$\begin{array}{ccccccccccccc} & & & \dots & \overset{i_k'}{\bullet} & \longrightarrow & \overset{i_{k+1}'}{\bullet} & \longrightarrow & \dots & \longrightarrow & \overset{i_p'}{\bullet} & & \\ & & & & \uparrow & & \uparrow & & & & & & , \\ \overset{i_0}{\bullet} & \longrightarrow & \dots & \longrightarrow & \overset{i_k}{\bullet} & \longrightarrow & \overset{i_{k+1}}{\bullet} & & \dots & & & & \end{array}$$

Hence, for any $v \in \mathcal{A}_p$ we have $\widehat{v} \in \widehat{\mathcal{A}}_{p+1}$. Below we will prove that if $v \in \Omega_p$ then $\widehat{v} \in \widehat{\Omega}_{p+1}$.

For any path v in G define its image v' in G' by

$$\left(e_{i_0...i_p}\right)' = e_{i_0'...i_p'}.$$

Lemma 3 *(*Product rule*) For any p-path v on G with $p \geq 0$*

$$\partial \widehat{v} + \widehat{\partial v} = v' - v. \tag{16}$$

Proof. It suffices to prove (16) for $v = e_{i_0...i_p}$. For $p = 0$ set $v = e_i$ so that $\partial v = 0$ and $\widehat{v} = e_{ii'}$ whence

$$\partial \widehat{v} + \widehat{\partial v} = e_{i'} - e_i + 0 = v' - v.$$

For $p \geq 1$ we have

$$\begin{aligned} \partial \widehat{v} &= \sum_{k=0}^{p} (-1)^k \, \partial e_{i_0...i_k i_k'...i_p'} \\ &= \sum_{k=0}^{p} (-1)^k \left[\sum_{l=0}^{l} (-1)^l \, e_{i_0...\widehat{i_l}...i_k i_k'...i_p'} + \sum_{l=k}^{p} (-1)^{l+1} \, e_{i_0...i_k i_k'...\widehat{i_l'}...i_p'} \right] \\ &= \sum_{0 \leq l \leq k \leq p} (-1)^{k+l} \, e_{i_0...\widehat{i_l}...i_k i_k'...i_p'} + \sum_{0 \leq k \leq l \leq p} (-1)^{k+l+1} \, e_{i_0...i_k i_k'...\widehat{i_l'}...i_p'} \end{aligned}$$

and

$$\begin{aligned}\widehat{\partial v} &= \left(\sum_{l=0}^{p}(-1)^l e_{i_0\ldots\widehat{i_l}\ldots i_p}\right)^{\frown} \\ &= \sum_{l=0}^{p}(-1)^l\left[\sum_{k=l+1}^{p}(-1)^{k-1} e_{i_0\ldots\widehat{i_l}\ldots i_k i'_k\ldots i'_p} + \sum_{k=0}^{l-1}(-1)^k e_{i_0\ldots i_k i'_k\ldots\widehat{i'_l}\ldots i'_p}\right] \\ &= \sum_{0\le l<k\le p}(-1)^{k+l-1} e_{i_0\ldots\widehat{i_l}\ldots i_k i'_k\ldots i'_p} + \sum_{0\le k<l\le p}(-1)^{k+l} e_{i_0\ldots i_k i'_k\ldots\widehat{i'_l}\ldots i'_p}.\end{aligned}$$

We see that in the sum $\partial\widehat{v}+\widehat{\partial v}$ all the terms with $k\neq l$ cancel out and we obtain

$$\begin{aligned}\partial\widehat{v}+\widehat{\partial v} &= \sum_{k=0}^{p} e_{i_0\ldots i_{k-1}i'_k\ldots i'_p} - \sum_{k=0}^{p} e_{i_0\ldots i_k i'_{k+1}\ldots i'_p} \\ &= e_{i'_0\ldots i'_p} - e_{i_0\ldots i_p} = v'-v.\end{aligned}$$

■

Proposition 4 *If $v\in\Omega_p$ then $\widehat{v}\in\widehat{\Omega}_{p+1}$.*

Proof. By definition, the condition $v\in\Omega_p$ is equivalent to $v\in\mathcal{A}_p$ and $\partial v\in\mathcal{A}_{p-1}$. We know already that in this case $\widehat{v}\in\mathcal{A}_{p+1}$, and we need to prove that $\partial\widehat{v}\in\widehat{\mathcal{A}}_p$. Indeed, if $v\in\mathcal{A}_p$ and $\partial v\in\mathcal{A}_{p-1}$ then v' and $\widehat{\partial v}$ belong to $\widehat{\mathcal{A}}_p$ whence it follows from (16) that also $\partial\widehat{v}\in\widehat{\mathcal{A}}_p$. ■

Example. The cylinder over the digraph $I=({}^0\bullet\rightarrow\bullet^1)$ is a square

$$\begin{array}{ccc} {}^2\bullet & \longrightarrow & \bullet^3 \\ \uparrow & & \uparrow \\ {}^0\bullet & \longrightarrow & \bullet^1 \end{array} \tag{17}$$

where $0'=2$ and $1'=3$. Lifting a ∂-invariant 1-path $e_{01}\in\Omega_1$ we obtain a ∂-invariant 2-path on the square:

$$\widehat{e_{01}} = e_{00'1'} - e_{011'} = e_{023} - e_{013}.$$

The cylinder over the square (17) is a 3-cube that is shown in Fig. 3, where we take $i'=i+4$.

Lifting the ∂-invariant 2-path $v=e_{023}-e_{013}$ we obtain a ∂-invariant 3-path on the 3-cube:

$$\begin{aligned}\widehat{v} &= e_{00'2'3'} - e_{022'3'} + e_{0233'} - \left(e_{00'1'2'} - e_{011'2'} + e_{0133'}\right) \\ &= e_{0467} - e_{0267} + e_{0237} - e_{0457} + e_{0157} - e_{0137}.\end{aligned}$$

Defining further n-cube as the cylinder over $(n-1)$-cube, we see that n-cube possesses a ∂-invariant n-path that is a lifting of a ∂-invariant $(n-1)$-path from $(n-1)$-cube and that is an alternating sum of $n!$ elementary terms. One can show that this n-path generates Ω_n on n-cube (see [7]).

All homology groups of the n-cube are trivial as it will be shown below.

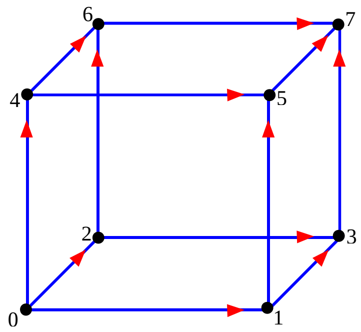


Figure 3: 3-cube

2 Homotopy theory of digraphs

In this Section we introduce a homotopy theory of digraphs and establish the relations between this theory and the homology theory of digraphs, following [6]. A similar homotopy theory for undirected graphs was earlier developed in [1].

2.1 The notion of homotopy

Fix $n \geq 0$. Denote by I_n any digraph whose the set of vertices is $\{0, 1, \ldots, n\}$ and the set of edges contains exactly one of the edges $i \to (i+1)$, $(i+1) \to i$ for any $i = 0, 1, \ldots, n-1$, and no other edges. A digraph I_n is called *a line digraph*.

Denote by $\mathcal{I}_n$ the set of all line digraphs I_n. Clearly, there is only one digraph in $\mathcal{I}_0$ – the one-point digraph. There are two digraphs in $\mathcal{I}_1$: the digraph I with the edge $(0 \to 1)$ and the digraph I^- with the edge $(1 \to 0)$.

Definition. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two digraphs. Two digraph maps $f, g \colon G \to H$ are called *homotopic* if there exists a line digraph $I_n \in \mathcal{I}_n$ with $n \geq 1$ and a digraph map

$$F \colon G \boxdot I_n \to H$$

such that

$$F|_{G\times 0} = f \quad \text{and} \quad F|_{G\times n} = g. \tag{18}$$

In this case we shall write $f \simeq g$. The map F is called a *homotopy* between f and g.

In the case $n = 1$ we refer to the map F as an *one-step homotopy* between f and g and write $f \overset{\text{1-step}}{\simeq} g$. In this case the identities (18) become

$$F|_{G\times 0} = f \quad \text{and} \quad F|_{G\times 1} = g,$$

which determine F uniquely. The requirement, that $F : G \boxdot I_1 \to H$ is a digraph map, can be understood as follows. For I_1 there are only two choices: $I_1 = I = (0 \to 1)$ or $I_1 = I^{-1} = (0 \leftarrow 1)$. Consider the case $I_1 = I$. Then $G \boxdot I_1$ is the cylinder $\widehat{G}$ considered in the previous section. Identifying G with the union two sheets G and G' as before, we see that $F|_G = f$ and $F|_{G'} = g$ are digraph maps. For the vertical edges $x \to x'$ of $\widehat{G}$ we have

$$F(x) = f(x) \quad \text{and} \quad F(x') = g(x)$$

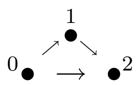
so that the requirement $F(x) \overrightarrow{=} F(x')$ becomes

$$f(x) \overrightarrow{=} g(x) \ \text{ in } H.$$

Combining with the case $I_1 = I^-$, we obtain that f and g are one-step homotopic if and only if

$$\boxed{\text{either } f(x) \overrightarrow{=} g(x) \text{ for all } x \in V_G \text{ or } \ g(x) \overrightarrow{=} f(x) \text{ for all } x \in V_G.} \tag{19}$$

Example. Consider the digraphs

$G =$  and $H = {}^a\bullet \to \bullet^b$

and the mappings $f, g : V_G \to V_H$ given by the table:

$x \in V_G$	0	1	2
$f(x)$	a	a	b
$g(x)$	a	b	b

It is easy to see that both f and g are digraph maps from G to H. Moreover, f and g are one-step homotopic, because $f(x) \overrightarrow{=} g(x)$ for all $x \in V_G$.

It follows from definition of homotopy, that $f, g : G \to H$ are homotopic if and only if there is a finite sequence of digraph maps $f = f_0, f_1, ..., f_n = g$ from G to H such that f_k and f_{k+1} are one-step homotopic. It is obvious that the relation "$\simeq$" is an equivalence relation on the set of all digraph maps from G to H.

Definition. Two digraphs G and H are called *homotopy equivalent* if there exist digraph maps

$$f : G \to H, \quad g : H \to G \tag{20}$$

such that

$$f \circ g \simeq \mathrm{id}_H, \qquad g \circ f \simeq \mathrm{id}_G\,. \tag{21}$$

In this case we shall write $G \simeq H$. The maps f and g as in (21) are called *homotopy inverses* of each other.

A digraph G is called *contractible* if $G \simeq \{*\}$ where $\{*\}$ is a single vertex digraph. It follows from definition that a digraph G is contractible if and only if there is a digraph map $h : G \to G$ such that the image of h consists of a single vertex and $h \simeq \mathrm{id}_G$. Indeed, with $H = \{*\}$ the mapping f in (20) is trivial and $f \circ g = \mathrm{id}_H$ for any choice of g. The mapping $g \circ f : G \to G$ is any digraph map $h : G \to G$ whose

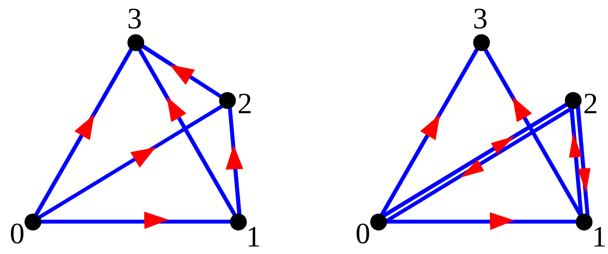


Figure 4: Star-like digraphs

image consists of a single vertex. Hence, we are left only to satisfy the requirement $h \simeq \mathrm{id}_G$.

Example. A digraph G is called *star-like* (resp. inverse star-like) if there is a vertex $a \in V_G$ such that $a \to x$ (resp. $x \to a$) for all $x \in V_G \setminus \{a\}$. if G is a (inverse) star-like digraph, then the map $h : G \to G$ given by $h(x) = a$ for all $x \in V_G$ is one-step homotopic to id_G. Indeed, for all $x \in V_G$ we have

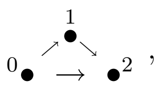
$$h(x) = a \overrightarrow{=} x = \mathrm{id}_G(x)$$

so that h and id_G satisfy (19) and, hence, $h \simeq \mathrm{id}_G$. Therefore, G is contractible.

For example, consider the *digraph-simplex* of dimension n, which is a digraph G with the set of vertices $\{0, 1, \dots, n\}$ and the set of edges given by the condition

$$i \to j \iff i < j$$

Then G is star-like and, hence, G is contractible. The digraph-simplex of dimension 1 is ${}^0\bullet \to \bullet^1$, the digraph-simplex of dimension 2 is the triangle



the digraph-simplex of dimension 3 is shown on the left panel on Fig. 4. In particular, the triangle is contractible. Another star-like digraph is shown on the right panel of Fig. 4.

2.2 Retraction

A (induced) sub-digraph H of a digraph G is a digraph whose set of vertices is a subset of that of G and the edges of H are all those edges of G whose adjacent vertices belong to H.

Definition. Let G be a digraph and H be its sub-digraph. A *retraction* of G onto H is a digraph map $r : G \to H$ such that $r|_H = \mathrm{id}_H$.

Let $r : G \to H$ be a retraction and let $i : H \to G$ be the natural inclusion map. By definition of retraction we have $r \circ i = \mathrm{Id}_H$. Therefore, if

$$i \circ r \simeq \mathrm{id}_G, \tag{22}$$

then i and r are homotopy inverses and we obtain that $G \simeq H$. A retraction $r : G \to H$ with the property (22) is called a *deformation retraction*.

Proposition 5 *Let $r : G \to H$ be a retraction of a digraph G onto a sub-digraph H such that*

$$\text{either } x \overrightarrow{=} r(x) \text{ for all } x \in V_G \quad \text{or} \quad r(x) \overrightarrow{=} x \text{ for all } x \in V_G. \tag{23}$$

Then r is a deformation retraction and, consequently, the digraphs G and H are homotopy equivalent.

Proof. Set $f = \mathrm{id}_G$ and $g = i \circ r$. For any $x \in V_G$ we have $f(x) = x$ and $g(x) = i \circ r(x) = r(x)$. The condition (23) means that f and g satisfy (19), whence $f \overset{\text{1-step}}{\simeq} g$. Hence, we obtain (22) and, consequently, $G \simeq H$. ■

Example. Let us show that the square

$$G = \begin{array}{ccc} {}^2\bullet & \longrightarrow & \bullet^3 \\ \uparrow & & \uparrow \\ {}^0\bullet & \longrightarrow & \bullet^1 \end{array}$$

is also contractible. It suffices to show that $G \simeq H$ where H is the following subgraph

$$H = \ {}^0\bullet \longrightarrow \bullet^1 .$$

Consider a retraction $r : G \to H$ given by

$$r(0) = r(2) = 0 \text{ and } r(1) = r(3) = 1.$$

Clearly, it satisfies $r(x) \overrightarrow{=} x$ for all $x \in V_G$ and we conclude by Proposition 5 that $G \simeq H$. Since H is contractible, we obtain that G is also contractible.

Example. For any $n \geq 1$, consider the n-dimensional cube

$$I^n = \underbrace{I \boxdot I \boxdot \cdots \boxdot I}_{n \text{ times}}$$

For example, I^2 is the square from (12) and I^3 is a 3-cube shown on Fig. 3. As in the previous example, one constructs an obvious deformation retraction of I^n onto I^{n-1} thus proving that $I^n \simeq I^{n-1}$. By induction we obtain that all cubes I^n are contractible.

2.3 Homotopy preserves homologies

Now we can prove the main result about connections between homotopy and homology theories for digraphs.

Theorem 6 *Let G, H be two digraphs.*

(i) Let $f, g : G \to H$ be two digraph maps. If $f \simeq g$ then the induced maps

$$f_* : H_p(G) \to H_p(H) \quad \textit{and} \quad g_* : H_p(G) \to H_p(H)$$

of the homology groups are identical, that is, $f_* = g_*$ in homologies.

(ii) If the digraphs G and H are homotopy equivalent, then all their homology groups are isomorphic. Furthermore, if the homotopical equivalence of G and H is provided by the digraph maps (20) then their induced maps f_* and g_* provide mutually inverse isomorphisms of the homology groups of G and H.

Proof. (i) Let $F : G \boxdot I_n \to H$ be a homotopy between f and g. It suffices to treat the case $n = 1$ as the general case then follows by induction. Let $I_1 = I = (0 \to 1)$ so that $G \boxdot I_1 = G \boxdot I = \widehat{G}$ (the case $I_1 = I^-$ can be treated similarly). The maps f and g induce morphisms of chain complexes

$$f_*, g_* \colon \Omega_*(G) \to \Omega_*(H),$$

and F induces a morphism

$$F_* \colon \Omega_*(\widehat{G}) \to \Omega_*(H).$$

As before, we identify G with $G \times 0$ and set $G' = G \times 1$. For any path $v \in \Omega_*(G)$ considering as a path in $\widehat{G}$ we have $F_*(v) = f_*(v)$ and $F_*(v') = g_*(v')$.

In order to prove that f_* and g_* induce the identical homomorphisms $H_*(G) \to H_*(H)$, it suffices by [9, Theorem 2.1, p.40] to construct a chain homotopy between the chain complexes $\Omega_*(G)$ and $\Omega_*(H)$, that is, the $\mathbb{K}$-linear mappings

$$L_p : \Omega_p(G) \to \Omega_{p+1}(H)$$

such that

$$\partial L_p + L_{p-1}\partial = g_* - f_*$$

(note that all the terms here are mapping from $\Omega_p(G)$ to $\Omega_p(H)$) as on the following diagram:

$$\begin{array}{ccccc}
\Omega_{p-1}(G) & \overset{\partial}{\longleftarrow} & \Omega_p(G) & \longleftarrow & \Omega_{p+1}(G) \\
 & \overset{L_{p-1}}{\searrow} & \downarrow^{f_*} \downarrow^{g_*} & \overset{L_p}{\searrow} & \\
\Omega_{p-1}(H) & \longleftarrow & \Omega_p(H) & \underset{\partial}{\longleftarrow} & \Omega_{p+1}(H)
\end{array}$$

Let us define the mapping L_p as follows

$$L_p(v) = F_*(\widehat{v}) \quad \text{for any } v \in \Omega_p(G),$$

where $\widehat{v} \in \Omega_{p+1}(\widehat{G})$ is the lifting of v to the graph $\widehat{G}$ defined in Section 1.5. Using $\partial F_* = F_* \partial$ (see Proposition 2) and the product rule (16) of Lemma 3, we obtain

$$
\begin{aligned}
(\partial L_p + L_{p-1}\partial)(v) &= \partial(F_*(\widehat{v})) + F_*(\widehat{\partial v}) \\
&= F_*(\partial \widehat{v}) + F_*(\widehat{\partial v}) \\
&= F_*(\partial \widehat{v} + \widehat{\partial v}) \\
&= F_*(v' - v) \\
&= g_*(v) - f_*(v) .
\end{aligned}
$$

(ii) Let $f : G \to H$ and $g : H \to G$ be digraph maps such that

$$
f \circ g \simeq \mathrm{id}_H, \qquad g \circ f \simeq \mathrm{id}_G . \tag{24}
$$

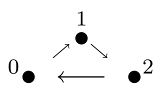
Then they induce the following mappings

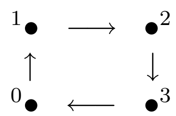
$$
H_p(G) \xrightarrow{f_*} H_p(H) \xrightarrow{g_*} H_p(G) \xrightarrow{f_*} H_p(H) .
$$

By (i) and (24) we have $f_* \circ g_* = \mathrm{id}$ and $g_* \circ f_* = \mathrm{id}$, which implies that f_* and g_* are mutually inverse isomorphisms of $H_p(G)$ and $H_p(H)$. ■

Example. If a digraph G is contractible, then all the homology groups of G are trivial (that is, are those of $\{*\}$, that is, $H_0 \cong \mathbb{K}$ and $H_p = \{0\}$ for all $p \geq 1$). For example, all homology groups of star-like digraphs are trivial; in particular, this is the case for all digraph simplexes, including triangle. Also, all cubes are contractible and, hence, have all trivial homology groups.

Example. Let S_n be a cycle digraph. If S_n is triangle or square then S_n is contractible as was shown above. If S_n is neither triangle nor square then as we know, $H_1(S_n, \mathbb{K}) \cong \mathbb{K}$ and, hence, S_n is not contractible by Theorem 6. In particular, this is always the case when $n \geq 5$. Here are other examples of non-contractible cycles with $n = 3, 4$:



and


,

Let us show that two cycles S_n and S_m with $n \neq m$ are not homotopy equivalent, except for the case when one of them is a triangle and the other is a square. Assume that S_n and S_m with $n < m$ are homotopy equivalent. Then by Theorem 6 there is a digraph map $f : S_n \to S_m$ such that $f_* : H_1(S_n) \to H_1(S_m)$ is an isomorphism. If homology groups $H_1(S_n)$ and $H_1(S_m)$ are not isomorphic then we are done. If they are isomorphic, then they are isomorphic to $\mathbb{K}$. Let $\varpi_n \in \Omega_1(S_n)$ be the generator of closed 1-paths on S_n and $\varpi_m \in \Omega_1(S_m)$ be the generator of closed 1-paths on S_n, as in (11). Then $[\varpi_n]$ generates $H_1(S_n)$, $[\varpi_m]$ generates $H_1(S_m)$, and we should have

$$
f_*([\varpi_n]) = \alpha [\varpi_m]
$$

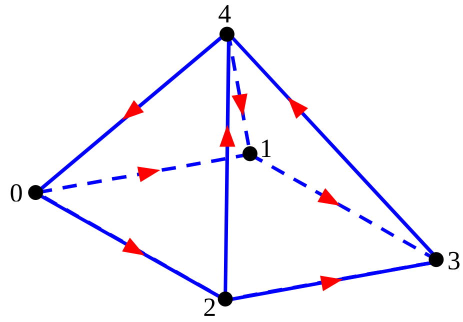


Figure 5: The digraph admits a deformation retraction onto a subgraph $\{1, 3, 4\}$

for some non-zero constant $\alpha \in \mathbb{K}$. Consequently, we obtain

$$f_* (\varpi_n) = \alpha \varpi_m,$$

which is impossible because f cannot be surjective by $n < m$, whereas ϖ_m uses all the vertices of S_m.

Example. Consider the digraph G as on Fig. 5.

Consider also its sub-digraph H with the vertex set $V_H = \{1, 3, 4\}$ and a retraction $r : G \to H$ given by $r(0) = 1$, $r(2) = 3$ and $r|_H = \text{id}$. Since $x \overrightarrow{=} r(x)$ for all x, by Proposition 5, we conclude that r is a deformation retraction, whence $G \simeq H$. Consequently, we obtain $H_1 (G, \mathbb{K}) \cong H_1 (H, \mathbb{K}) \cong \mathbb{K}$ and $H_p (G, \mathbb{K}) = \{0\}$ for $p \geq 2$.

Example. Let a, b be two vertices of a digraph G such that either $a \to b$ or $b \to a$. Denote by H the digraph that is obtained from G by removing a vertex a with all adjacent edges. Assume that the map $r : V_G \to V_H$ given by

$$r(a) = b \quad \text{and} \quad r|_H = \text{id}_H$$

is a digraph map. We claim that in this case $G \simeq H$. Indeed, r is a retraction from G to H. If $a \to b$ then r satisfies $x \overrightarrow{=} r(x)$ and if $b \to a$ then $r(x) \overrightarrow{=} x$ for all $x \in V_G$. By Proposition 5 r is a deformation retraction, whence we obtain that $G \simeq H$. Consequently, all homology groups of G and H are the same. This is very similar to the results about transformations of simplicial complexes by simple homotopy (see, for example, [2]).

The requirement that r is a digraph map is equivalent to the following condition.

$$\forall c \in V_G \setminus \{a, b\} \qquad a \to c \Rightarrow b \to c \quad \text{and} \quad a \leftarrow c \Rightarrow b \leftarrow c. \tag{25}$$

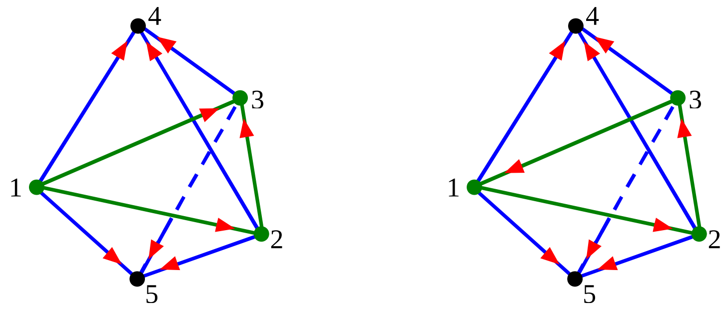
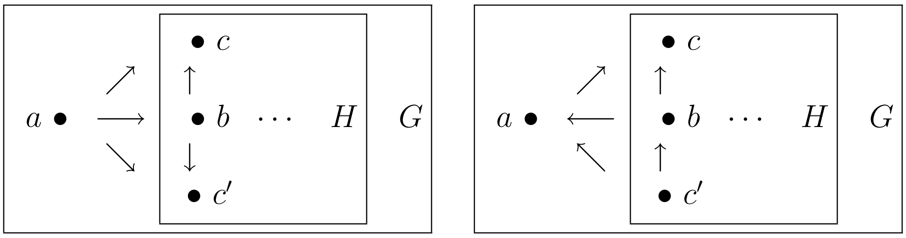
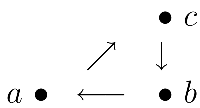


Figure 6: The left digraph is contractible while the right one is not.

Two examples when (25) is satisfied are shown in the following diagram:



On the contrary, the digraph G on following diagram



does not satisfy (25). Moreover, this digraph is not homotopy equivalent to subgraph $H = \left({}^{c}\bullet \rightarrow \bullet^{b} \right)$ since G and H have different homology group H_1.

The digraph on the left panel of Fig. 6 is contractible as one can successively remove the vertices $5, 4, 3, 2$ each time satisfying (25).

The digraph on the right panel of Fig. 6 is different from the left one only by the direction of the edge between 1 and 3, but it is not contractible as its H_2 group is non-trivial (cf. (13)).

Consider one more example: the digraph G on Fig. 7.

Removing successively the vertices $A, B, 8, 9, 6, 7$, which each time satisfy (25), we obtain a digraph H with $V_H = \{0, 1, 2, 3, 4, 5\}$ that is homotopy equivalent to G and, in particular, has the same homologies as G. The digraph H is shown in two ways on Fig. 8. Clearly, the second representation of this graph is reminiscent of an octahedron.

It is possible to show that $H_p(H, \mathbb{K}) = \{0\}$ for $p = 1$ and $p > 2$ while $H_2(H, \mathbb{K}) \cong \mathbb{K}$. It follows that the same is true for the homology groups of G. Furthermore, it is possible to show that $H_2(G, \mathbb{K})$ is generated by the following 2-path

$$\omega = e_{024} - e_{025} - e_{034} + e_{035} - e_{124} + e_{125} + e_{134} - e_{135},$$

that determines a 2-dimensional "hole" in G given by the octahedron H. Note that on Fig. 7 this octahedron is hardy visible.

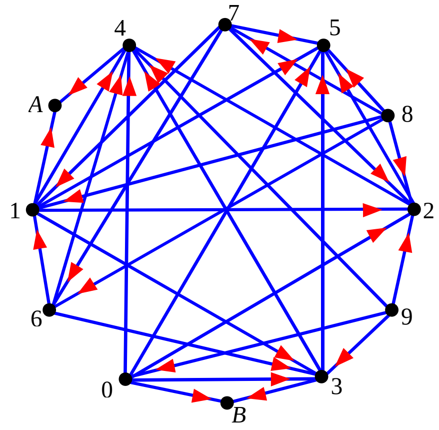


Figure 7: Digraph G whose H_2 group is generated by an octahedron

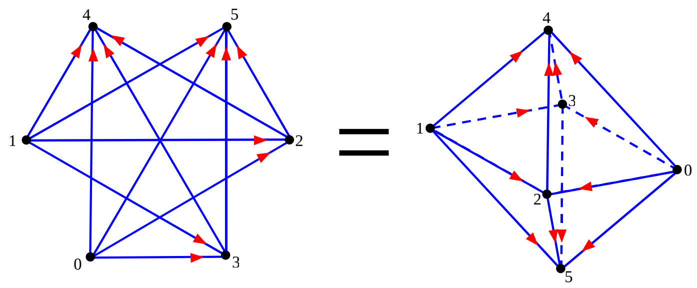


Figure 8: Two representations of the digraph H

2.4 Cylinder of a map

Let us give some further examples of homotopy equivalent digraphs.

Definition. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two digraphs and f be a digraph map from G to H. The *cylinder* C_f of f is the digraph with the set of vertices $V_{C_f} = V_G \sqcup V_H$ and with the set of edges E_{C_f} that consists of all the edges from E_G and E_H as well as of the edges of the form $x \to f(x)$ for all $x \in V_G$.

The inverse cylinder C_f^- is defined in the same way except that the edge $x \to f(x)$ is replaced by $f(x) \to x$.

For example, for $f = \mathrm{id}_G$ we have $C_f = G \boxdot I$ where $I = ({}^0\bullet \longrightarrow \bullet^1)$ and $C_f^- = G \boxdot I^-$ where $I^- = ({}^0\bullet \longleftarrow \bullet^1)$.

Example. Let G be the digraph with vertices $\{0, 1, 2, 3, 4, 5\}$ and H is be the digraph with vertices $\{a, b, c\}$ as on Fig. 9. Consider the digraph map $f : G \to H$ given by $f(0) = f(1) = a$, $f(2) = f(3) = b$ and $f(4) = f(5) = c$. The cylinder C_f of f is shown on Fig. 9.

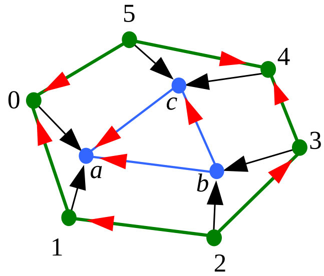


Figure 9: The cylinder of the map

Claim. *For any digraph $f : G \to H$*

$$\mathrm{C}_f \simeq H \simeq \mathrm{C}_f^- .$$

Proof. Indeed, the projection $p : \mathrm{C}_f \to H$ defined by

$$p(x) = \begin{cases} x, & x \in V_H, \\ f(x), & x \in V_G, \end{cases}$$

is by Proposition 5 a deformation retraction of C_f onto H, whence it follows that $\mathrm{C}_f \simeq H$. The case of the inverse cylinder C_f^- is similar. ∎

3 Fundamental group of a digraph

In this Section we define fundamental group of a digraph and describe theirs basic properties. A *based* digraph G^* is a digraph G with a fixed base vertex $* \in V_G$. A based digraph map $f : G^* \to H^*$ is a digraph map $f : G \to H$ such that $f(*) = *$. Any line digraph $I_n \in \mathcal{I}_n$ will always be considered as a based digraph with the base point 0.

3.1 C-homotopy and π_1

Definition. A *loop* in a digraph G is any digraph map $\phi : I_n \to G$ with $\phi(0) = \phi(n)$. A *based loop* on a based digraph G^* is a loop $\phi : I_n \to G^*$, such that $\phi(0) = \phi(n) = *$.

Definition. A digraph map $h : I_n \to I_m$ is called *shrinking* if $h(0) = 0$, $h(n) = m$, and $h(i) \le h(j)$ whenever $i \le j$ (that is, if h as a function from $\{0, ..., n\}$ to $\{0, ..., m\}$ is monotone increasing).

It follows from the definition that $h(i) \le h(i+1) \le h(i) + 1$. The existence of a shrinking map $h : I_n \to I_m$ implies $m \le n$. If $n = m$ then h is a bijection.

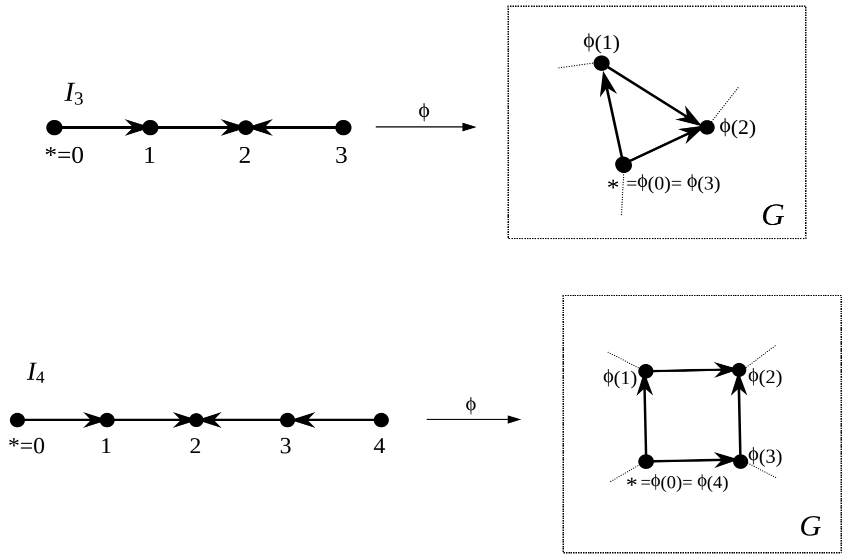


Figure 10: Two examples of based loops

Definition. Consider two based loops

$$\phi\colon I_n \to G^* \quad \text{and} \quad \psi\colon I_m \to G^*.$$

We say that there is *one-step direct C-homotopy* from ϕ to ψ if there exists a shrinking map $h : I_n \to I_m$ such that the map $F : V_{C_h} \to V_G$ given by

$$F|_{I_n} = \phi \quad \text{and} \quad F|_{I_m} = \psi, \tag{26}$$

is a digraph map from C_h to G. If F is a digraph map from C_h^- to G then we refer to an one-step *inverse* C-homotopy.

Remark. The requirement that F is a digraph map is equivalent to the condition

$$\phi(i) \overrightarrow{=} \psi(h(i)) \quad \text{for all } i \in I_n. \tag{27}$$

In turn, (27) implies that the digraph maps ϕ and $\psi \circ h$ (from I_n to G) satisfy (19), which yields $\phi \simeq \psi \circ h$.

If $n = m$ then $h = \mathrm{id}_{I_n}$ and an one-step C-homotopy is a homotopy.

Example. An example of one-step direct C-homotopy is shown in Fig. 11.

Note that the images of the loops ϕ and ψ on Fig. 11 are not homotopic as digraphs because they are cycles of different lengths 5 and 3. Nevertheless, the loops ϕ and ψ are one-step C-homotopic.

Definition. Two based loops ϕ, ψ in G^* are called *C-homotopic* if there exists a finite sequence $\{\phi_k\}_{k=0}^{l}$ of based loops in G^* such that $\phi_0 = \phi$, $\phi_l = \psi$ and, for any

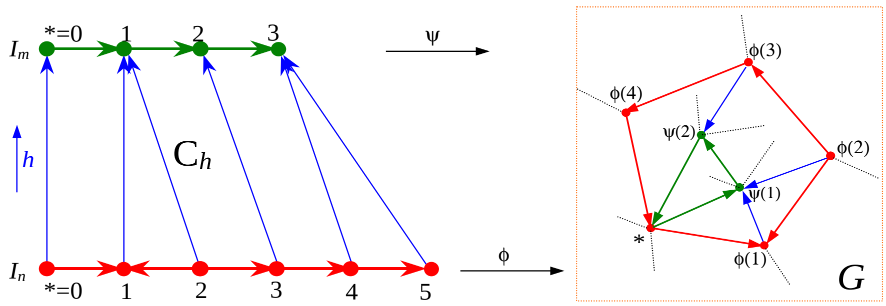


Figure 11: The loops $\phi : I_5 \to G$ and and $\psi : I_3 \to G$ are C-homotopic. Note that $\phi(0) = \phi(5) = * = \psi(0) = \psi(3)$.

$k = 0, ..., l-1$, the loops ϕ_k and ϕ_{k+1} are related by an one-step direct or inverse C-homotopy (either from ϕ_k to ϕ_{k+1} or from ϕ_{k+1} to ϕ_k). We write in this case $\phi \overset{C}{\simeq} \psi$.

Clearly, the C-homotopy is an equivalence relation. The C-homotopy class of a based loop ϕ will be denoted by $[\phi]$. We say that a loop ϕ is *C-contractible* if $\phi \overset{C}{\simeq} e$, that is, $[\phi] = [e]$.

Definition. Denote by $\pi_1(G^*)$ the set of all equivalence classes $[\phi]$ for all bases loops ϕ in G^*.

Example. A *triangular loop* is a loop $\phi : I_3 \to G^*$ with $I_3 = (0 \to 1 \to 2 \leftarrow 3)$.

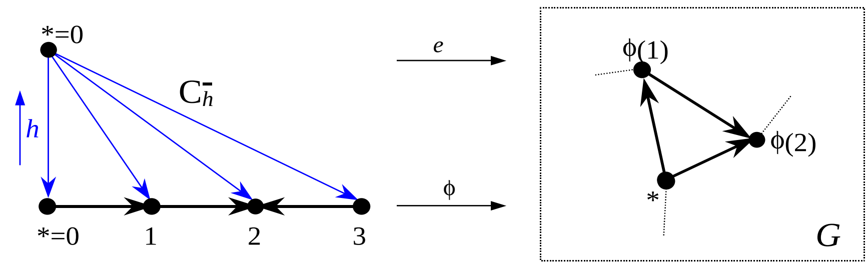


Figure 12: A triangular loop ϕ is C-contractible.

The triangular loop is C-contractible because the following shrinking map

$$h : I_3 \to I_0, \quad h(k) = 0 \text{ for all } k = 0, ..., 3,$$

provides an inverse one-step C-homotopy between ϕ and e (see Fig. 12).

Example. A *square loop* is a loop $\phi : I_4 \to G$ with $I_4 = (0 \to 1 \to 2 \leftarrow 3 \leftarrow 4)$.The square loop can be C-contracted to e in two steps as is shown on Fig. 13.

In the case $n \geq 5$, a loop $\phi : I_n \to G^*$ does not have to be C-contractible, which is the case, for example, if ϕ is the natural map $I_n \to S_n$.

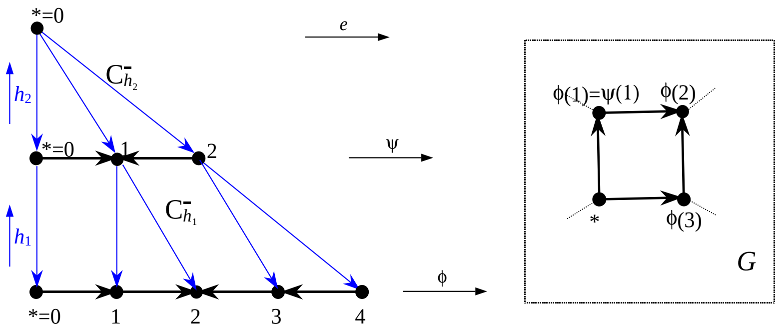


Figure 13: A square loop ϕ is C-contractible. Note that $\phi(0) = \phi(4) = \psi(0) = \psi(2) = *$.

3.2 Local description of C-homotopy

We prove here an equivalent description of C-homotopy using images of the loops.

Lemma 7 *Let a, b be two vertices in a digraph G such that either $a = b$ or $a \to b \to a$. Then loop $\phi : I_n \to G^*$, such that $\phi(i) = a$, $\phi(i+1) = b$, and $i \to i+1$ in I_n, is C-homotopic to a loop $\phi' : I'_n \to G^*$ where I'_n is obtained from I_n by changing one edge $i \to i+1$ to $i+1 \to i$ and $\phi'(j) = \phi(j)$ for all $j = 0, ..., n$.*

Proof. A C-homotopy between ϕ and ϕ' is constructed in two one-step inverse C-homotopies as is shown on the following diagram:

$$\begin{array}{ccccccc} \phi' : I_{n'} \to G & ... & \underset{a}{i} & \leftarrow & \underset{b}{i+1} & ... & \\ & & \downarrow & \searrow & & \searrow & \\ \psi : I_{n+1} \to G & ... & \underset{a}{i} & \to & \underset{a}{i+1} & \leftarrow & \underset{b}{i+2}... \\ & & \uparrow & & \uparrow & \nearrow & \\ \phi : I_n \to G & ... & \underset{a}{i} & \to & \underset{b}{i+1} & ... & \end{array}$$

The subscript under each element of the line digraph indicates the value of the loop on this element. ∎

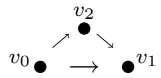
Any digraph map ϕ: $I_n \to G$ determines a sequence $\theta_\phi = \{v_i\}_{i=0}^n$ of vertices of G by $v_i = \phi(i)$. By the definition of a digraph map, we have for any $i = 0, ..., n-1$ one of the following relations:

$$v_i = v_{i+1}, \quad v_i \to v_{i+1}, \quad v_{i+1} \to v_i. \tag{28}$$

We consider the sequence θ_ϕ as a *word* over the alphabet V_G. Any sequence $\{v_i\}_{i=0}^n$ that satisfies (28) is the word of some digraph map $\phi : I_n \to G$.

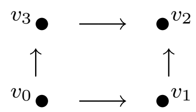
Theorem 8 *Two based loops $\phi : I_n \to G^*$ and $\psi : I_m \to G^*$ are C-homotopic if and only if the word θ_ψ can be obtained from θ_ϕ by a finite sequence of the following transformations (or their inverses):*

(i) ...abc... ↦ ...ac... where (a, b, c) is any permutation of a triple (v_0, v_1, v_2) of vertices forming a triangle in G:



(and the dots "..." denote the unchanged parts of the words).

(ii) ...abc... ↦ ...adc... where (a, b, c, d) is any cyclic permutation (or an inverse cyclic permutation) of a quadruple (v_0, v_1, v_2, v_3) of vertices forming a square in G:



(iii) ...abcd... ↦ ...ad... where (a, b, c, d) is as in (ii).

(iv) ...aba... ↦ ...a... if $a \to b$ or $b \to a$.

(v) ...aa... ↦ ...a...

Proof. Let us first show that if $\theta_\phi = \theta_\psi$ then $\phi \overset{C}{\simeq} \psi$. If, for any edge $i \to i+1$ (or $i \leftarrow i+1$) in I_n we have also $i \to i+1$ (resp. $i \leftarrow i+1$) in I_m then $I_n = I_m$ and $\phi = \psi$ (although $n = m$, the line digraphs I_n and I_m could a priori be different elements of $\mathcal{I}_n$). Assume that, for some i, we have $i \to i+1$ in I_n but $i \leftarrow i+1$ in I_m. Then, by Lemma 7, we can change the edge $i \to i+1$ in I_n to $i \leftarrow i+1$ while staying in the same C-homotopy class of ϕ. Arguing by induction, we obtain $\phi \overset{C}{\simeq} \psi$.

We write $\theta_\phi \sim \theta_\psi$ if θ_ψ can be obtained from θ_ϕ by a finite sequence of transformations $(i) - (v)$ (or inverses to them). Let us show that $\theta_\phi \sim \theta_\psi$ implies that $\phi \overset{C}{\simeq} \psi$. For that we construct for each of the transformations $(i) - (v)$ a C-homotopy between ϕ and ψ.

(i) Assume that $a \to c$ (the case $c \to a$ is similar). Then either $b \to c$ or $a \to b$ (otherwise we would have got $a \to c \to b \to a$ which is excluded by a triangle hypothesis). The C-homotopies in the both cases are shown on the diagram:

$$\begin{array}{cccccc} I_m & \dots & a & \to & c & \dots \\ & & | & \diagdown & & \diagdown \\ I_n & \dots & a & - & b & \to & c\dots \end{array} \qquad \begin{array}{cccccc} I_m & \dots & a & \to & c & \dots \\ & & | & & | & \diagdown \\ I_n & \dots & a & \to & b & - & c\dots \end{array}$$

Each position here corresponds to a vertex in a cylinder C_h or C_h^- (that is, in I_n or I_m) and shows its image (a, b or c) under the map ϕ resp. ψ. The arrows and undirected segments shows the edges in the cylinder C_h or C_h^- (in particular, horizontal arrows and segments show the edges in I_n and I_m). The undirected segments, such as $a - b$ and $c - b$, should be given directions matching those on the digraph G.

(ii) Assume $a \to d$ and $b \to c$. Then we have the following C-homotopy:

$$\begin{array}{cccccccc} I_n & \dots & a & - & b & \to & c & \dots \\ & & | & & | & & | & \diagdown \\ I_{n+1} & \dots & a & - & a & \to & d & - & c\dots \end{array}$$

which shows that the loops with the words $...abc...$ and $...aadc...$ are C-homotopic. Then we use the transformation $...aa...$ to $...a...$ as in (v). Other cases are treated similarly.

(iii) Assume $a \to d$. Then we have $b \to c$, and the C-homotopy is shown on the diagram:

$$
\begin{array}{ccccccccc}
I_m & ... & & ... & a & \to & d & ... & \\
 & & & \diagup & | & & | & \diagdown & \\
I_n & ... & a & - & b & \to & c & - & d
\end{array}
$$

Note that if $a \to b$ then also $d \to c$, and if $b \to a$ then also $c \to d$.

(iv) Assuming $a \to b$ we obtain the following C-homotopy:

$$
\begin{array}{cccccc}
I_m & ... & & ... & a & ... \\
 & & & \swarrow & \downarrow & \searrow \\
I_n & ... & a & \to & b & \leftarrow \; a...
\end{array}
$$

(v) Here is the required C-homotopy:

$$
\begin{array}{cccccc}
I_m & ... & & ... & a & ... \\
 & & & \nearrow & \uparrow & \\
I_n & ... & a & - & a & ...
\end{array}
$$

Before we go to the second half of the proof, observe that the transformation

$$...abc... \mapsto ...ac... \tag{29}$$

of words is possible not only in the case when a, b, c come from a triangle as in (i) but also when a, b, c form a *degenerate triangle*, that is, when there are identical vertices among a, b, c while distinct vertices among a, b, c are connected by an edge. Indeed, in the case $a = b$ we have by (v)

$$abc = aac \sim ac,$$

in the case $a = c$ we have by (iv) and (v)

$$abc = aba \sim a \sim ac,$$

and in the case $b = c$ by (v)

$$abc = acc \sim ac.$$

Now let us prove that $\phi \overset{C}{\simeq} \psi$ implies $\theta_\phi \sim \theta_\psi$. It suffices to assume that there exists an one-step direct C-homotopy from ϕ to ψ given by a shrinking map $h : I_n \to I_m$. Set

$$\theta_\phi = a_0 a_1 ... a_n \quad \text{and} \quad \theta_\psi = b_0 b_1 ... b_m$$

where $a_i, b_j \in V_G$ and $a_0 = b_0 = a_n = b_m = *$. For any $i = 0, ..., n$ set $j = h(i)$ and consider two words

$$A_i = a_0 a_1 ... a_i b_j \quad \text{and} \quad B_i = b_0 b_1 ... b_j.$$

We will prove by induction in i that $A_i \sim B_i$ for all $i = 0, ..., n$. If this is already known, then for $i = n$ we have $j = m$ and

$$a_0 a_1 ... a_n b_m \sim b_0 b_1 ... b_m.$$

Since $a_n b_m = ** \sim * = a_n$, it follows that $\theta_\phi \sim \theta_\psi$.

For $i = 0$ we have $A_0 = a_0 b_0 = ** \sim * = b_0 = B_0$. Assuming that $A_i \sim B_i$, let us prove that $A_{i+1} \sim B_{i+1}$. Let us consider a structure of the cylinder C_h over the edge between i and $i+1$ in I_n. Set as before $j = h(i)$ and consider two cases.

Case 1. $h(i+1) = j$. In this case we have $B_i = B_{i+1}$ and the following structure in C_h:

$$\begin{array}{ccccc} & & b_j & & \\ & \nearrow & & \nwarrow & \\ a_i & & - & & a_{i+1} \end{array}.$$

Note that each arrow on C_h transforms either to an arrow between the vertices of G or to a vertex. Then we obtain by (29) and by the induction hypothesis that

$$A_{i+1} = a_0 a_1 ... a_{i-1} \underbrace{a_i a_{i+1} b_j} \sim a_0 a_1 ... a_{i-1} \underbrace{a_i b_j} = A_i \sim B_i = B_{i+1}.$$

Case 2. $h(i+1) = j+1$. Then we have the following fragment of C_h:

$$\begin{array}{ccc} b_j & - & b_{j+1} \\ \uparrow & & \uparrow \\ a_i & - & a_{i+1} \end{array}. \tag{30}$$

Let us show that in this case

$$a_i a_{i+1} b_{j+1} \sim a_i b_j b_{j+1}. \tag{31}$$

Indeed, if all the vertices $a_i, a_{i+1}, b_j, b_{j+1}$ are distinct, then they form a square and (31) follows by transformation (ii). Consider various cases of equal vertices in the diagram (30).

In the case $a_{i+1} = b_j$ (31) is an equality, and in the case $a_i = b_{j+1}$ the relation (31) follows by transformation (iv):

$$a_i a_{i+1} b_{j+1} \sim a_i = b_{j+1} \sim a_i b_j b_{j+1}.$$

In the case $a_i = b_j$ the triple a_i, a_{i+1}, b_{j+1} is a triangle or a degenerate triangle, and we obtained from (29) and (v)

$$a_i a_{i+1} b_{j+1} \sim a_i b_{j+1} \sim a_i a_i b_{j+1} = a_i b_j b_{j+1},$$

and the case $a_{i+1} = b_{j+1}$ is similar. Finally, if $a_i = a_{i+1}$ then similarly by (v) and (29) we obtain

$$a_i a_{i+1} b_{j+1} = a_i a_i b_{j+1} \sim a_i b_{j+1} \sim a_i b_j b_{j+1},$$

and the case $b_j = b_{j+1}$ is similar.

It follows from (31) that

$$A_{i+1} = a_0 a_1 ... a_{i-1} \underbrace{a_i a_{i+1} b_{j+1}} \sim a_0 a_1 ... a_{i-1} \underbrace{a_i b_j b_{j+1}} = A_i b_{j+1} \sim B_i b_{j+1} = B_{i+1},$$

which proves the induction step. ■

Remark. Note that the transformation (iii) was not used in the second half of the proof, so (iii) is logically not necessary in the statement of Theorem 8. Note also that (iii) can be obtained as composition of (ii) and (iv) as follows:

$$abcd \sim adcd \sim ad.$$

However, in applications it is still convenient to be able to use (iii).

Example. 1. A triangular loop ϕ on Fig. 12 is contractible because for $a = \phi(0), b = \phi(1), c = \phi(2)$ we have

$$\theta_\phi = abca \sim aca \sim a,$$

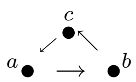
where we have used transformations (i) and (iv).

2. A square loop ϕ on Fig. 13 is contractible because if a, b, c, d are vertices of the square then

$$\theta_\phi = abcda \sim ada \sim a,$$

where we have used (iii) and (iv).

3. Consider a cyclic digraph S_3 as follows



with the base vertex $* = a$ and a loop $\phi : I_3 \to S_3$ where

$$I_3 = (0 \to 1 \to 2 \to 3)$$

and $\phi(0) = \phi(3) = a$, $\phi(1) = b$, $\phi(2) = c$. We have $\theta_\phi = abca$. It is clear that this word does not allow any of the transformations of Theorem 8, which implies that ϕ is not C-contractible.

4. Consider the loops ϕ and ψ on Fig. 11, that were proved above to be C-homotopic. It is shown on Fig. 14 how to transform θ_ϕ to θ_ψ using the word transformations of Theorem 8.

3.3 Group structure in π_1

For any $I_n \in \mathcal{I}_n$ define a line digraph $\hat{I}_n \in \mathcal{I}_n$ as follows:

$$i \to j \text{ in } \hat{I}_n \Leftrightarrow (n-i) \to (n-j) \text{ in } I_n.$$

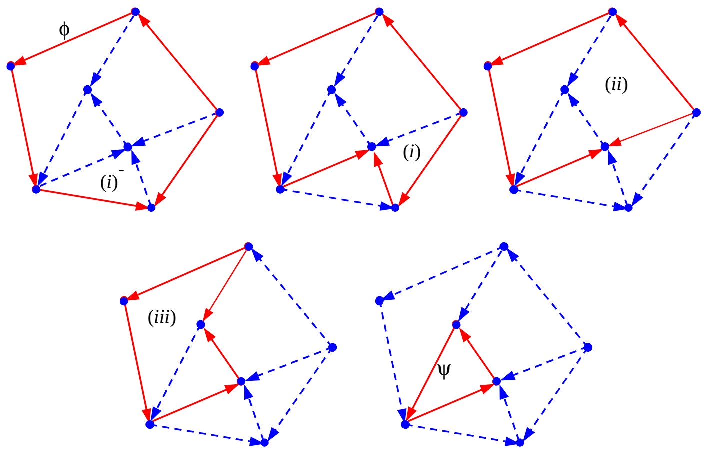


Figure 14: Transforming a 5-cycle θ_ϕ to a 3-cycle θ_ψ using successively $(i)^-$ (the inverse of (i)), $(i)\,,(ii)$ and $(iii)\,.$

For any two line digraphs I_n and I_m, define the line digraph $I_n \vee I_m \in \mathcal{I}_{n+m}$ that is obtained from I_n and I_m by identification of the vertices $n \in I_n$ and $0 \in I_m$.

Definition. (i) For any digraph map $\phi : I_n \to G$ define its *inversion* by

$$\begin{aligned} \hat{\phi} &: \hat{I}_n \to G \\ \hat{\phi}(i) &= \phi(n-i). \end{aligned}$$

(ii) For two digraph maps $\phi : I_n \to G$ and $\psi : I_m \to G$ with $\phi(n) = \psi(0)$ define their *concatenation* by

$$\begin{aligned} \phi \vee \psi &: I_n \vee I_m \to G \\ \phi \vee \psi(i) &= \begin{cases} \phi(i), & 0 \le i \le n \\ \psi(i-n), & n \le i \le n+m. \end{cases} \end{aligned}$$

Clearly, if ϕ is a based loop then $\hat{\phi}$ is also a based loop. If ϕ and ψ are based loops then $\phi \vee \psi$ is always defined and is also a based loop.

Now we can define a product in $\pi_1(G^*)$ as follows.

Definition. For any two based loops ϕ, ψ in G^* define the product of the equivalence classes $[\phi]$ and $[\psi]$ by

$$[\phi] \cdot [\psi] = [\phi \vee \psi]. \tag{32}$$

Lemma 9 *The product in $\pi_1(G^*)$ is well defined.*

Proof. Let ϕ, ϕ', ψ, ψ' be loops of G^* and let

$$\phi \overset{C}{\simeq} \phi', \quad \psi \overset{C}{\simeq} \psi'. \tag{33}$$

We must prove that

$$\phi \vee \psi \overset{C}{\simeq} \phi' \vee \psi'. \tag{34}$$

It suffices to consider only the case when the both C-homotopies in (33) are one-step C-homotopies. Then we have

$$\phi \vee \psi \overset{C}{\simeq} \phi' \vee \psi$$

because one-step C-homotopy between ϕ and ϕ' easily extends to that between $\phi \vee \psi$ and $\phi' \vee \psi$. In the same way we obtain

$$\phi' \vee \psi \overset{C}{\simeq} \phi' \vee \psi',$$

whence (34) follows. ■

Lemma 10 *For any loop $\phi \colon I_n \to G^*$ we have $\phi \vee \hat{\phi} \overset{C}{\simeq} e$.*

Proof. Let $\theta_\phi = v_0...v_n$. Then $\theta_{\hat{\phi}} = v_n...v_0$ and

$$\theta_{\phi \vee \hat{\phi}} = v_0...v_{n-1}v_nv_{n-1}...v_0.$$

Using successively the transformations $aba \mapsto a$ and $aa \mapsto a$ of Theorem 8, we obtain that $\theta_{\phi \vee \hat{\phi}} \sim *$ whence $\phi \vee \hat{\phi} \overset{C}{\simeq} e$ follows. ■

Theorem 11 *Let G, H be digraphs.*

(i) The set $\pi_1(G^*)$ with the product (32) and neutral element $[e]$ is a group. It will be referred to as the fundamental group of a digraph G^*.

(ii) Any based digraph map $f : G^* \to H^*$ induces a group homomorphism

$$\begin{aligned} f &: \pi_1(G^*) \to \pi_1(H^*) \\ f([\phi]) &= [f \circ \phi], \end{aligned}$$

which depends only on homotopy class of f.

(iii) Let G, H be connected. If $G \simeq H$ then the fundamental groups $\pi_1(G^*)$ and $\pi_1(H^*)$ are isomorphic (for any choice of the base vertices).

Proof. (i) This follows from Lemmas 9 and 10, since the product in $\pi_1(G^*)$ satisfies the associative law, the class $[e] \in \pi_1(G^*)$ satisfies the definition of a neutral element, and $[\hat{\phi}]$ is the inverse of $[\phi]$ for any $[\phi] \in \pi_1(G^*)$.

(ii) Let $\phi : I_n \to G^*$ and $\psi : I_m \to G^*$ be C-homotopic. Let us show that $f \circ \phi$ and $f \circ \psi$ are C-homotopic in H^*. It suffices to prove this for one-step C-homotopy, for example, for direct C-homotopy. In this case there is a shrinking map $h : I_n \to I_m$ such that

$$\phi(i) \overrightarrow{=} \psi(h(i)) \quad \text{for all } i \in I_n.$$

It follows that

$$f(\phi(i)) \overrightarrow{\underline{=}} f(\psi(h(i)))$$

that is,

$$f \circ \phi \overset{C}{\simeq} f \circ \psi.$$

Hence, the map f is well defined on $\pi_1(G^*)$.

The map $f : \pi_1(G^*) \to \pi_1(H^*)$ is a homomorphism because

$$f([e]) = [f \circ e] = [e]$$

and, for any two loops ϕ, ψ in G^*,

$$\begin{aligned} f([\phi] \cdot [\psi]) &= f([\phi \vee \psi]) = [f \circ (\phi \vee \psi)] \\ &= [(f \circ \phi) \vee (f \circ \psi)] \\ &= f([\phi]) \cdot f([\psi]). \end{aligned}$$

If f and g are homotopic then also $f \circ \phi \simeq g \circ \phi$, whence $f \circ \phi \overset{C}{\simeq} g \circ \phi$ and, hence, $f([\phi]) = g([\phi])$.

(iii) Let $f : G \to H$ and $g : H \to G$ be homotopy inverses maps, that is,

$$f \circ g \simeq \mathrm{id}_H \quad \text{and} \quad g \circ f \simeq \mathrm{id}_G. \tag{35}$$

Consider a special case when $f(*) = *$ and $g(*) = *$ (the general case requires some additional argument). By (ii) we have group homomorphisms

$$\pi_1(G^*) \xrightarrow{f} \pi_1(H^*) \xrightarrow{g} \pi_1(G^*) \xrightarrow{f} \pi_1(H^*).$$

It follows from (35) and (ii) that on this diagram

$$f \circ g = \mathrm{id}_{\pi_1(H^*)} \quad \text{and} \quad g \circ f = \mathrm{id}_{\pi_1(G^*)},$$

which implies that f and g are mutual inverses and, hence, isomorphisms. ■

3.4 Application to graph coloring

An an illustration of the theory of digraph homotopy, we give here a new proof of the classical lemma of Sperner, using the notion the fundamental group and C-homotopy.

Consider a triangle ABC on the plane $\mathbb{R}^2$ and its triangulation T. The set of vertices of T is colored with three colors $1, 2, 3$ in such a way that

- the vertices A, B, C are colored with $1, 2, 3$ respectively;
- each vertex on any side of ABC is colored with one of the two colors of the endpoints of the side (see Fig. 15).

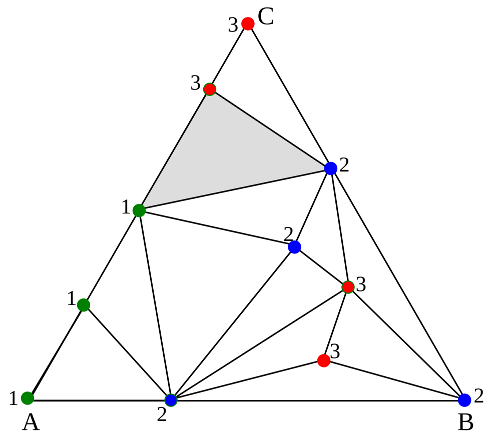


Figure 15: A Sperner coloring

The classical lemma of Sperner says that then there exists in T a 3-color triangle, that is, a triangle, whose vertices are colored with the three different colors.

To prove this, let us first modify the triangulation T so that there are no vertices on the sides AB, AC, BC except for A, B, C. Indeed, if X is a vertex on AB then we move X a bit inside the triangle ABC. This gives rise to a new triangle in the triangulation T that is formed by X and its former neighbors, say Y and Z, on the edge AB (while keeping all other triangles). However, since all X, Y, Z are colored with two colors, no 3-color triangle emerges after that move. By induction, we remove all the vertices from the sides of ABC.

The triangulation T can be regarded as a graph. Let us make it into a digraph G by choosing the direction on the edges as follows. If the vertices a, b are connected by an edge in T then choose direction between a, b using the colors of a, b and the following rule:

$$\begin{array}{c} 1 \rightarrow 2, \quad 2 \rightarrow 3, \quad 3 \rightarrow 1 \\ 1 \leftrightarrows 1, \quad 2 \leftrightarrows 2, \quad 3 \leftrightarrows 3 \end{array} \tag{36}$$

Assume now that there is no 3-color triangle in T. Then each triangle from T looks in G like

$$\begin{array}{ccc} & \bullet & \\ \nearrow & & \nwarrow \\ \bullet & \leftrightarrows & \bullet \end{array} \quad \text{or} \quad \begin{array}{ccc} & \bullet & \\ \swarrow & & \searrow \\ \bullet & \leftrightarrows & \bullet \end{array} \quad \text{or} \quad \begin{array}{ccc} & \bullet & \\ \nearrow\swarrow & & \searrow\nwarrow \\ \bullet & \leftrightarrows & \bullet \end{array} \quad ,$$

in particular, each of them contains a triangle in the sense of Theorem 8.

Consider a 3-loop $\phi : I_3 \rightarrow G^*$ with the word $\theta_\phi = ABCA$. Using the transformation (ii) of Theorem 8 and the partition of G into the triangle digraphs, we can contract the word $ABCA$ to an empty word. Hence, $\phi \overset{C}{\sim} e$.

Consider the cycle digraph S_3 with the vertices a, b, c as follows

$$\begin{array}{ccc} & c_3 & \\ \swarrow & & \nwarrow \\ a_1 & \longrightarrow & b_2 \end{array} \tag{37}$$

where the vertex a is colored by 1, b by 2 and c by 3. Define a map $f : G \to S_3$ by the rule that $f(x)$ has the same color in S_3 as x in G.

By the choice of directions on the edges of G, f is a digraph map. The loop $f \circ \phi$ on S_3 has the word

$$\theta_{f \circ \phi} = abca,$$

which is not contractible on S_3 as we have seen above. However, by Theorem 11, f induces homomorphism of $\pi_1(G)$ to $\pi_1(S_3)$. Therefore, $\phi \overset{C}{\simeq} e$ implies that also $f \circ \phi \overset{C}{\simeq} e$, which contradicts the previous observation.

4 Hurewicz theorem

One of our main results is the following discrete version of Hurewicz theorem.

Theorem 12 *For any based connected digraph G^* we have an isomorphism*

$$\pi_1(G^*) / [\pi_1(G^*), \pi_1(G^*)] \cong H_1(G, \mathbb{Z})$$

where $[\pi_1(G^*), \pi_1(G^*)]$ is a commutator subgroup.

Proof. The proof is similar to that in the classical algebraic topology [8, p.166]. For any based loop $\phi\colon I_n \to G^*$ of a digraph G^*, define a 1-path $\chi(\phi)$ on G as follows: $\chi(\phi) = 0$ for $n = 0, 1, 2$, and for $n \geq 3$

$$\chi(\phi) = \sum_{\{i: i \to i+1\}} e_{\phi(i)\phi(i+1)} - \sum_{\{i: i+1 \to i\}} e_{\phi(i+1)\phi(i)}, \tag{38}$$

where the summation index i runs from 0 to $n-1$. It is easy to see that the 1-path $\chi(\phi)$ is allowed and closed and, hence, determines a homology class $[\chi(\phi)] \in H_1(G, \mathbb{Z})$. Let us first prove that, for any two based loops $\phi\colon I_n \to G^*$ and $\psi\colon I_m \to G^*$,

$$\phi \overset{C}{\simeq} \psi \quad \Rightarrow \quad [\chi(\phi)] = [\chi(\psi)]. \tag{39}$$

Note that any based loop with $n \leq 2$ is C-homotopic to trivial. For $n \geq 3$, it is sufficiently to check (39) assuming that $\phi \overset{C}{\simeq} \psi$ is given by an one-step direct C-homotopy with a shrinking map $h : I_n \to I_m$. Set

$$\phi' := \psi \circ h : I_n \to G^*$$

and observe that by (38) $\chi(\phi') = \chi(\psi)$. It remains to show that $[\chi(\phi)] = [\chi(\phi')]$.

By Remark 3.1 the digraph maps ϕ and ϕ', acting from I_n to G, are homotopic. Denote by S_n the digraph that is obtained from I_n by identification of the vertices 0 and n (that is, S_n is a cycle digraph as defined before). Then φ and ϕ' can be regarded as digraph maps from S_n to G, and they are again homotopic as such.

Consider the standard homology class $[\varpi] \in H_1(S_n)$ given by (11). Comparing (11) and (38), we see that

$$\phi_*(\varpi) = \chi(\varphi) \quad \text{and} \quad \phi'_*(\varpi) = \chi(\phi').$$

On the other hand, by Theorem 6 we have $[\phi_*(\varpi)] = [\phi'_*(\varpi)]$, which finishes the proof of (39).

Hence, χ determines a map

$$\chi_*\colon \pi_1(G^*) \to H_1(G,\mathbb{Z}), \quad \chi_*[\phi] = [\chi(\phi)].$$

The map χ_* is a group homomorphism because, for based loops ϕ, ψ and the neutral element $[e] \in \pi_1(G^*)$, we have $\chi_*([e]) = 0$ and

$$\begin{aligned}\chi_*([\phi]\cdot[\psi]) &= \chi_*([\phi \vee \psi]) = [\chi(\phi \vee \psi)] \\ &= [\chi(\phi) + \chi(\psi)] = [\chi(\phi)] + [\chi(\psi)] = \chi_*([\phi]) + \chi_*([\psi]).\end{aligned}$$

Since the group $H_1(G,\mathbb{Z})$ is abelian, it follows that

$$[\pi_1(G^*), \pi_1(G^*)] \subset \operatorname{Ker}\chi_*.$$

Now let us prove that χ_* is an epimorphism. Define a *standard loop* on G as a finite sequence $v = \{v_k\}_{k=0}^n$ of vertices of G such that $v_0 = v_n$ and, for any $k = 0, ..., n-1$, either $v_k \to v_{k+1}$ or $v_{k+1} \to v_k$. For a standard loop v define an 1-path

$$\varpi_v = \sum_{\{k: v_k \to v_{k+1}\}} e_{v_k v_{k+1}} - \sum_{\{k: v_{k+1} \to v_k\}} e_{v_k v_{k+1}} \tag{40}$$

and observe that ϖ_v is allowed and closed. The 1-paths of the form (40) will be referred to as standard paths. Consider an arbitrary closed 1-path

$$w = \sum_k n_k e_{i_k j_k} \in \Omega_1(G,\mathbb{Z}).$$

Since $\partial w = 0$ and $\partial e_{ij} = e_j - e_i$, the path w can be represented as a finite sum of standard paths. Hence, in order to prove that χ_* is an epimorphism, it suffices to show that any standard 1-path ϖ_v is in the image of χ. Note that the standard loop v determines naturally a based loop $\phi : I_n \to G^{v_0}$ by $\phi(i) = v_i$. Since the digraph G is connected, there exists a based path $f : I_s \to G^*$ with $f(s) = v_0$. Thus we obtain a based loop

$$f \vee \phi \vee \hat{f} : I_{2s+n} \to G^*.$$

It follows directly from our construction, that $\chi(f \vee \phi \vee \hat{f}) = \varpi_v$, and hence χ_* is an epimorphism.

We are left to prove that

$$\operatorname{Ker}\chi_* \subset [\pi_1(G^*), \pi_1(G^*)].$$

For that we need to prove that, for any loop $\phi : I_n \to G^*$, if $\chi_*([\phi]) = 0 \in H_1(G,\mathbb{Z})$, then $[\phi]$ lies in the commutator $[\pi_1(G^*), \pi_1(G^*)]$. In the case $n \leq 2$ any loop ϕ is C-homotopic to the trivial loop. Assuming in the sequel $n \geq 3$, we use the word $\theta_\phi = v_0 v_1 ... v_n$ where $v_i = \phi(i)$.

Consider first the case, when $\chi(\phi) = 0 \in \Omega_1(G)$. Since the digraph G is connected, for any vertex v_i there exists a based digraph map $\psi_i\colon I_{p_i} \to G^*$ with $\psi_i(p_i) =$

v_i. If $v_i = v_j$ for some i, j then we make sure to choose ψ_i and ψ_j identical. For $i = 0$ and $i = n$ choose ψ_i to be trivial loop $e : I_0 \to G^*$. For any $i = 0, ..., n-1$ define the digraph maps $\phi_i \colon I^{\pm} \to G$ by the conditions $\phi_i(0) = v_i, \phi_i(1) = v_{i+1}$ and consider the following loop

$$\gamma = \psi_0 \vee \phi_0 \vee \hat{\psi}_1 \vee \psi_1 \vee \phi_1 \vee \hat{\psi}_2 \vee \psi_2 \vee \phi_2 \vee \cdots \vee \hat{\psi}_{n-1} \vee \psi_{n-1} \vee \phi_{n-1} \vee \psi_n \quad (41)$$

(see Fig. 16).

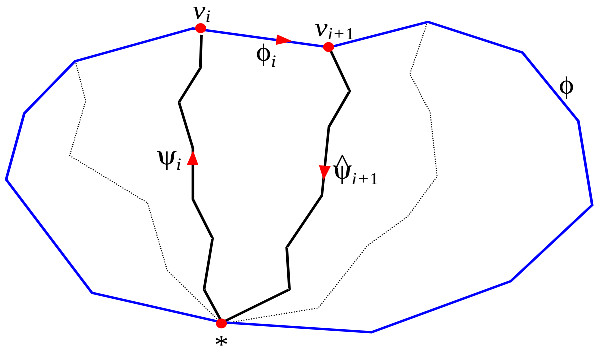


Figure 16: Loop $\psi_i \vee \phi_i \vee \hat{\psi}_{i+1}$

Using transformation (iv) of Theorem 8 (similarly to the proof of Lemma 10), we obtain that

$$\gamma \overset{C}{\simeq} \phi_0 \vee \phi_1 \vee ... \vee \phi_{n-1} = \phi.$$

On the other hand, it follows from (41) that

$$[\gamma] = \prod_{i=0}^{n-1} \left[\psi_i \vee \phi_i \vee \hat{\psi}_{i+1}\right]$$

Consider for some $i = 0, ..., n-1$, such that $i \to i+1$, the vertices $a = v_i$ and $b = v_{i+1}$. If $a = b$ then the loop $\psi_i \vee \phi_i \vee \hat{\psi}_{i+1}$ is C-homotopic to e. Assume $a \neq b$, so that $a \to b$. Then the term e_{ab} is present in the right hand side of the identity (38) defining $\chi(\phi)$. Due to $\chi(\phi) = 0$, the term e_{ab} should cancel out with $-e_{ab}$ in the right hand side of (38). Therefore, there exists $j = 0, ..., n-1$ such that $j+1 \to j$, $v_{j+1} = a$ and $v_j = b$. It follows that

$$\psi_j \vee \phi_j \vee \hat{\psi}_{j+1} = \psi_{i+1} \vee \hat{\phi}_i \vee \hat{\psi}_i,$$

and that the loops

$$\left[\psi_i \vee \phi_i \vee \hat{\psi}_{i+1}\right] \text{ and } \left[\psi_j \vee \phi_j \vee \hat{\psi}_{j+1}\right] \quad (42)$$

are mutually inverse. Therefore, $[\gamma]$ is a product of pairs of mutually inverse loops, which implies that $[\gamma] = [\phi]$ lies in the commutator of π_1.

Now consider the general case, when $\chi(\phi) \in \Omega_1(G)$ is exact, that is, $\chi(\phi) = \partial\omega$ for some $\omega \in \Omega_2(G)$. Recall that by Proposition 1 any 2-path $\omega \in \Omega_2$ can be represented in the form

$$\omega = \sum_{j=1}^{N} \kappa_j \sigma_j$$

where $N \in \mathbb{N}$, $\kappa_l = \pm 1$ and σ_l is one of the following 2-paths: a double edge, a triangle, a square. Further proof goes by induction in N. In the case $N = 0$ we have $\omega = 0$ which was already considered above.

In the case $N \geq 1$ choose an arbitrary index $i = 0, ..., n-1$ such that the vertices $a = \phi(i)$ and $b = \phi(i+1)$ are distinct. Assume for certainty that $i \to i+1$ and, hence, $a \to b$ (the case $i+1 \to i$ can be handled similarly). Then e_{ab} enters $\chi(\phi)$ with the coefficient 1. Since

$$\chi(\phi) = \partial\omega = \sum_{j=1}^{N} \kappa_j \partial\sigma_j,$$

there exists σ_l such that $\partial\sigma_l$ contains a term $\kappa_l e_{ab}$. Fix this l and define a new loop ϕ' as follows.

If σ_l is a double edge a, b, a, then consider a loop ϕ' that is obtained from $\phi : I_n \to G^*$ by changing one edge $i \to i+1$ in I_n to $i \to i+1$. Then by Lemma 7 we have $\phi' \overset{C}{\simeq} \phi$.

Let σ_l be a triangle with the vertices a, b, c. Noticing that

$$\theta_\phi = ...ab...$$

consider a loop ϕ' such that

$$\theta_{\phi'} = ...acb...$$

(see Fig. 17).

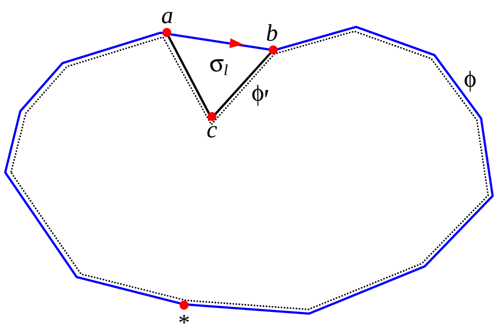


Figure 17: Loops ϕ and ϕ' in the case when σ_l is a triangle.

If σ_l is a square with the vertices a, b, c, d, then we define a loop ϕ' so that

$$\theta_{\phi'} = ...adcb.$$

By Theorem 8, we have in the both cases $\phi' \overset{C}{\simeq} \phi$ and, hence, $[\phi'] = [\phi]$.

By construction, $\chi(\phi')$ contains no longer the term e_{ab}. On the other hand, we will prove below that, for some $\kappa = \pm 1$,

$$\chi(\phi') = \chi(\phi) - \kappa\partial\sigma_l. \tag{43}$$

Comparing the coefficients in front of e_{ab} in the both parts of (43), we obtain the identity $0 = 1 - \kappa\kappa_l$ whence $\kappa = \kappa_l$. It follows from (43) with $\kappa = \kappa_l$ that

$$\chi(\phi') = \chi(\phi) - \partial(\kappa_l\sigma_l) = \partial\omega - \partial(\kappa_l\sigma_l) = \partial\omega',$$

where

$$\omega' = \sum_{j \neq l} c_j\sigma_j.$$

By the inductive hypothesis we conclude that $[\phi']$ lies in the commutator $[\pi_1(G^*), \pi_1(G^*)]$, whence the same for $[\phi]$ follows.

We are left to prove the identity (43). If σ_l is a double edge a, b, a then

$$\chi(\phi') - \chi(\phi) = -e_{ba} - e_{ab} = -\partial e_{aba} = -\partial\sigma_l.$$

If σ_l is a triangle

$$\begin{array}{ccc} & c & \\ \diagup & & \diagdown \\ a & \longrightarrow & b \end{array}$$

then we obtain a cycle digraph S_3 with the vertices a, b, c, and if σ_l is a square

$$\begin{array}{ccc} d & \longrightarrow & c \\ | & & | \\ a & \longrightarrow & b \end{array}$$

then we obtain a cycle digraph S_4 with the vertices a, b, c, d. Let ϖ be the standard 1-path on S_3 in the first case and that on S_4 in the second case (see (11)). Then it is easy to see that

$$\chi(\phi) - \chi(\phi') = \varpi,$$

and (43) follows from the observation that $\partial\sigma_l = \pm\varpi$. ■

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