HÖLDER ESTIMATES OF HEAT KERNELS FOR JUMP TYPE DIRICHLET FORMS

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Abstract. Consider a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ without killing part on metric measure spaces (M,d,μ) satisfying doubling condition. We obtain the (local) Hölder estimates of (Dirichlet) heat kernels of $(\mathcal{E},\mathcal{F})$ under the assumption: generalized capacity condition, Poincaré inequality, and the tail estimate of jump kernel with a parameter $q\in[2,\infty]$. In the case when μ is Ahlfors-regular, the range of the parameter q can be relaxed to $q\in[1,\infty]$.

Contents

1. Introduction and main results

Let (M,d) be a locally compact separable metric space and let μ be a Radon measure on M with full support. A triple (M,d,μ) is called a *metric measure space*. Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on $L^2:=L^2(M,\mu)$. Let $\{P_t\}_{t>0}$ be the heat semigroup in L^2 associated with $(\mathcal{E},\mathcal{F})$, that is, $P_t=e^{t\mathcal{L}},\ t>0$, where $\mathcal{L}$ is the generator of $(\mathcal{E},\mathcal{F})$.

Note that P_t is a bounded self-adjoint operator in L^2. If, for any $t>0$, the operator P_t has an integral kernel then the latter will be denoted by $p_t(x,y)$ and will be referred to as the *heat kernel* of $(\mathcal{E},\mathcal{F})$. The heat kernel coincides with the transition density of the Hunt process associated with $(\mathcal{E},\mathcal{F})$.

By the general theory of Dirichlet form, each function in $\mathcal{F}$ has a quasi continuous version (see [5, Theorem 2.1.7, p. 75])). We always use its quasi continuous version for each function in $\mathcal{F}$.

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For a non-empty open subset Ω of M, let $C_0(\Omega)$ denote the space of all continuous functions with compact supports contained in Ω. Let $\mathcal{F}(\Omega)$ be a vector space defined by

$$\mathcal{F}(\Omega) = \text{ the closure of } \mathcal{F} \cap C_0(\Omega) \text{ in the norm of } \sqrt{\mathcal{E}(\cdot) + \|\cdot\|_{L^2}^2},$$

where $\mathcal{E}(u) := \mathcal{E}(u,u)$. By the theory of Dirichlet forms, $(\mathcal{E}, \mathcal{F}(\Omega))$ is a regular Dirichlet form on $L^2(\Omega)$ if $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(M,\mu)$ (see, for example, [5, Theorem 4.4.3]). In this case, denote the heat semigroup of $(\mathcal{E}, \mathcal{F}(\Omega))$ by $\{P_t^\Omega\}_{t>0}$. The integral kernel of $\{P_t^\Omega\}_{t>0}$ (should it exist) is denoted by $p_t^\Omega(x,y)$ and is referred to as the *heat kernel* of $(\mathcal{E}, \mathcal{F}(\Omega))$ or the *Dirichlet heat kernel* of $(\mathcal{E}, \mathcal{F})$ in Ω.

In this paper, we are concerned with the Hölder estimates of the (Dirichlet) heat kernel of $(\mathcal{E}, \mathcal{F})$ under some mild assumptions.

For any $x \in M$ and $r > 0$, denote an open metric ball by

$$B(x,r) := \{y \in M : d(y,x) < r\}$$

and its volume by

$$V(x,r) := \mu(B(x,r)).$$

We also use the following notation. For any set $U \subset M$ and $r > 0$, denote by U_r by the open r-neighborhood of U, that is

$$U_r = \bigcup_{x \in U} B(x,r).$$

1.1. **Main result for Ahlfors regular measures.** In this subsection, we assume that $(\mathcal{E}, \mathcal{F})$ is of pure jump type, that is, it is determined by a Radon measure j on $(M \times M) \setminus \text{diag}$ (called *jump measure*),

$$\mathcal{E}(u,v) = \iint_{M \times M} (u(x) - u(y))(v(x) - v(y)) dj(x,y), \quad u, v \in \mathcal{F}. \tag{1.1}$$

Fix a number $\alpha > 0$, we assume that the metric measure space (M, d, μ) is α-regular, that is, the following condition (V) is satisfied.

Definition 1.1. We say that condition (V) is satisfied if for any metric open ball $B(x,r)$ with $x \in M$ and $r > 0$,

$$V(x,r) \simeq r^\alpha. \tag{1.2}$$

Here and in the rest of the paper, $\simeq$ means the ration of both sides are bounded from above and below by two uniform constants.

Fix another number $\beta > 0$, we assume that conditions $(\mathrm{TJ})_\beta$ and $(\mathrm{PI})_\beta$ are satisfied.

Definition 1.2. We say that condition $(\mathrm{TJ})_\beta$ is satisfied if there exists a kernel $J : M \times \mathcal{B}(M) \mapsto [0, \infty]$ such that $dj(x,y) = J(x,dy)d\mu(x)$ and, for all $x \in M$ and $r > 0$,

$$J(x, B(x,r)^c) \leq \frac{C}{r^\beta}, \tag{1.3}$$

where $C \in [0, \infty)$ is a constant independent of x, r.

For a measurable function u and a measurable set A, let u_A denote the mean of the function u over A, that is,

$$u_A := \frac{1}{\mu(A)} \int_A u d\mu =: \fint_A u d\mu,$$

whenever the integral makes sense. For any ball $B = B(x,r)$ and any $a > 0$, set

$$aB := B(x, ar).$$

Definition 1.3 (Poincaré inequality)**.** We say that the *Poincaré inequality* (PI)$_\beta$ holds if there exist constants $C>0$ and $\kappa\in(0,1]$ such that, for any ball $B:=B(x_0,r)$ with $x_0\in M$, $r>0$ and for any function $u\in\mathcal{F}\cap L^\infty$,

$$\int_{\kappa B}|u-u_{\kappa B}|^2d\mu\le Cr^\beta\iint_{B\times B}(u(x)-u(y))^2dj(x,y), \tag{1.4}$$

where $C\in[0,\infty)$ is a constant independent of x_0,r,u.

Let $U\subset M$ be an open set, A be a Borel subset of U and $\overline{\kappa}\ge 1$ be a number. A $\overline{\kappa}$*-cutoff function* of the pair (A,U) is any function ϕ in $\mathcal{F}$ such that

- $0\le\phi\le\overline{\kappa}$ μ-a.e. in M;
- $\phi\ge 1$ μ-a.e. in A;
- $\phi=0$ μ-a.e. in U^c.

We denote by $\overline{\kappa}$-cutoff(A,U) the collection of all $\overline{\kappa}$-cutoff functions of the pair (A,U).

Define a function space $\mathcal{F}'$ by

$$\mathcal{F}':=\{v+a: v\in\mathcal{F},\ a\in\mathbb{R}\},$$

that is, $\mathcal{F}'$ is a vector space that contains $\mathcal{F}$ and constants.

Definition 1.4. We say that condition (Gcap)$_\beta$ is satisfied if there exist two numbers $\overline{\kappa}\ge1, C>0$ such that, for any $u\in\mathcal{F}'\cap L^\infty$ and for any pair of concentric balls $B_0:=B(x_0,R)$, $B:=B(x_0,R+r)$ with $x_0\in M$ and $0<R<R+r<\infty$, there exists $\phi\in\overline{\kappa}$-cutoff(B_0,B) such that

$$\mathcal{E}(u^2\phi,\phi)\le\frac{C}{r^\beta}\int_B u^2d\mu. \tag{1.5}$$

We remark that the function ϕ in (Gcap)$_\beta$ may depend on u, but the constants $\overline{\kappa},C$ are independent of u,B_0,B.

Theorem 1.5. *Let $(\mathcal{E},\mathcal{F})$ be the regular Dirichlet form in (1.1). Assume that (V), (TJ)$_\beta$, (PI)$_\beta$ and (Gcap)$_\beta$ are satisfied. For any non-empty open set $\Omega\subset M$, the Dirichlet heat kernel $p_t^\Omega(x,y)$ of $(\mathcal{E},\mathcal{F})$ in Ω exists and is jointly continuous in $(x,y,t)\in\Omega\times\Omega\times(0,\infty)$. Besides, it satisfies the following estimates.*

(1) *(On-diagonal upper estimates) For any $x,y\in\Omega$ and $t>0$, we have*

$$p_t^\Omega(x,y)\le\frac{C}{t^{\alpha/\beta}}. \tag{1.6}$$

(2) *(Local Hölder continuity) For any non-empty open subset $U\subset\Omega$ and $r>0$ with $U_r\subset\Omega$, and for all $x,x'\in U$, $y\in\Omega$ and $t\ge s>0$,*

$$\left|p_t^\Omega(x,y)-p_s^\Omega(x',y)\right|\le\frac{C}{t^{\alpha/\beta}}\left(\frac{d(x,x')}{t^{1/\beta}\wedge r}\right)^\lambda+\frac{C(t-s)}{s^{\alpha/\beta+1}}, \tag{1.7}$$

where the constants $\lambda\in(0,1)$ and $C>0$ depend only on the constants in the hypotheses.

In particular, when $\Omega=M$, we have for all $x,x',y\in M$, $t\ge s>0$,

$$\left|p_t(x,y)-p_s(x',y)\right|\le\frac{C}{t^{\alpha/\beta}}\left(\frac{d(x,x')}{t^{1/\beta}}\right)^\lambda+\frac{C(t-s)}{s^{\alpha/\beta+1}}. \tag{1.8}$$

Remark 1.6. Theorem 1.5 is also true for the Dirichlet form in (1.11), and its proof remains the same.

Moreover, one can also consider the localized versions of (1.6), (1.7) and (1.8). Indeed, Let $\overline{R}\in(0,\operatorname{diam}M]$. Suppose that the lower bound in (V) is true for $r<\overline{R}$ and each radius r in conditions (PI)$_\beta$ and (Gcap)$_\beta$ is restricted to $r<\overline{R}$. In this case, each term $t^{1/\beta}$ and $s^{1/\beta}$ in (1.6)-(1.8) shall be replaced by $t^{1/\beta}\wedge\overline{R}$ and $s^{1/\beta}\wedge\overline{R}$ respectively. For example, (1.8) should be as follows:

$$\left|p_t(x,y)-p_s(x',y)\right|\le\frac{C}{t^{\alpha/\beta}\wedge\overline{R}^\alpha}\left(\frac{d(x,x')}{t^{1/\beta}\wedge\overline{R}}\right)^\lambda+\frac{C}{s^{\alpha/\beta}\wedge\overline{R}^\alpha}\frac{t-s}{s}.$$

In fact, we have obtained a similar result in [1, Lemma 12.1, p. 447] for ultrametric spaces, while in the present paper we consider general metric spaces. In [8, Lemma 7.1, p. 737], we obtained a similar result to (1.6) and (1.7) but only for *bounded* open set Ω with coefficients depending on $\operatorname{diam}\Omega$, so that the result can not be extended to *unbounded* Ω.

1.2. **Main result for doubling measures.** In this subsection, we generalize Theorem 1.5 from Ahlfors regular measures to doubling measures, and from pure jump type Dirichlet forms to any regular Dirichlet forms without killing part.

Recall that any regular Dirichlet form $(\mathcal{E},\mathcal{F})$ in L^2 admits the following unique *Beurling-Deny decomposition* (cf. [5, Theorem 3.2.1 and Theorem 4.5.2]):

$$\mathcal{E}(u,v)=\mathcal{E}^{(L)}(u,v)+\mathcal{E}^{(J)}(u,v)+\mathcal{E}^{(K)}(u,v), \tag{1.9}$$

where $\mathcal{E}^{(L)}$ is the *local part* (or *diffusion part*) associated with a unique Radon measure $d\Gamma^{(L)}$ (the notions $\mathcal{E}^{(L)}(u,v), d\Gamma^{(L)}(u,v)$ are instead denoted by $\mathcal{E}^{(c)}(u,v), \frac{1}{2}d\mu^c_{\langle u,v\rangle}$ respectively in [5, see Eq. (3.2.22) on p. 126]):

$$\mathcal{E}^{(L)}(u,v)=\int_M d\Gamma^{(L)}(u,v),$$

$\mathcal{E}^{(J)}$ is the jump part associated with a unique Radon measure j defined on $M\times M\setminus \operatorname{diag}$:

$$\mathcal{E}^{(J)}(u,v)=\iint_{M\times M\setminus\operatorname{diag}}(u(x)-u(y))(v(x)-v(y))dj(x,y), \tag{1.10}$$

and finally, $\mathcal{E}^{(K)}$ is the *killing part*. We assume in this subsection that

$$\mathcal{E}(u,v)=\mathcal{E}^{(L)}(u,v)+\mathcal{E}^{(J)}(u,v). \tag{1.11}$$

That is, $\mathcal{E}^{(K)}\equiv 0$. In this subsection, we also assume that there exists a non-negative measurable function $J: M\times M\mapsto\mathbb{R}$ such that

$$dj(x,y)=J(x,y)d\mu(x)d\mu(y).$$

Definition 1.7 (Volume doubling condition)**.** We say that a measure μ on (M,d) satisfies the condition (VD) if there exists a constant $C\geq 1$ such that, for all $x\in M$ and all $r>0$,

$$V(x,2r)\leq CV(x,r). \tag{1.12}$$

Condition (VD) implies that $0<V(x,r)<\infty$ for all $r>0$. It is known that condition (VD) is equivalent to the following: there exists a positive number α such that, for all $x,y\in M$ and all $0<r\leq R<\infty$,

$$\frac{V(x,R)}{V(y,r)}\leq C\left(\frac{d(x,y)+R}{r}\right)^{\alpha}, \tag{1.13}$$

where constant C can be taken the same as in (VD).

Fix throughout in this subsection a parameter $\overline{R}\in(0,\operatorname{diam}M]$, where $\operatorname{diam}M$ is the diameter of M.

Definition 1.8 (Reverse volume doubling condition)**.** We say that μ satisfies the condition (RVD) if there exist positive numbers C,α' such that, for all $x\in M$ and for all $0<r\leq R<\overline{R}$,

$$\frac{V(x,R)}{V(x,r)}\geq C^{-1}\left(\frac{R}{r}\right)^{\alpha'}. \tag{1.14}$$

Let us fix a *scaling function* $W: M\times[0,\infty]\to[0,\infty]$ such that, for each $x\in M$, the function $W(x,\cdot)$ is strictly increasing, and $W(x,0)=0$, $W(x,\infty)=\infty$. Assume also that there exist three positive numbers C,β_1,β_2 $(\beta_1\leq\beta_2)$ such that, for all $0<r\leq R<\infty$ and for all $x,y\in M$ with $d(x,y)\leq R$,

$$C^{-1}\left(\frac{R}{r}\right)^{\beta_1}\leq\frac{W(x,R)}{W(y,r)}\leq C\left(\frac{R}{r}\right)^{\beta_2}. \tag{1.15}$$

The readers can refer to [9, Example 2.1] for examples of W that depends on the space variable.

For $x \in M$, let ω_x be the inverse function of $t \mapsto W(x,t)$. Clearly, we have by (1.15) that, for all $x \in M$ and all $0 < r \le R < \infty$

$$C^{-1}\left(\frac{R}{r}\right)^{1/\beta_2} \le \frac{\omega_x(R)}{\omega_x(r)} \le C\left(\frac{R}{r}\right)^{1/\beta_1}, \tag{1.16}$$

where $\omega_x(\cdot)$ is the inverse function of $W(x,\cdot)$ for every $x \in M$.

For a given number $1 \le q \le \infty$, let q' be the *Hölder conjugate* of q, that is,

$$q' := \frac{q}{q-1}$$

so that $q' = 1$ if $q = \infty$, and $q' = \infty$ if $q = 1$.

Definition 1.9 (L^q-tail estimate of jump kernel)**.** For $q \in [1,\infty]$, we say that condition $(\mathrm{TJ}_q)_W$ is satisfied if $(\mathcal{E},\mathcal{F})$ has the jump kernel $J(x,y)$ such that, for all $x \in M$ and $R > 0$,

$$\|J(x,\cdot)\|_{L^q(B(x,R)^c)} \le \frac{C}{V(x,R)^{1/q'}\,W(x,R)}, \tag{1.17}$$

where $C \in [0,\infty)$ is a constant independent of x, R.

Note that $(\mathrm{TJ}_\infty)_W$ is equivalent to that the jump kernel J satisfies the pointwise upper bound:

$$J(x,y) \le \frac{C}{V(x,d(x,y))W(x,d(x,y))}.$$

The readers can refer to [9, Section 7] for examples of the jump kernel J that satisfies $(\mathrm{TJ}_q)_W$ for some $q \in [1,\infty)$, but does not satisfy the above pointwise upper bounds.

For a Borel measurable subset $U \subset M$ and $u \in \mathcal{F}'$, define the *energy measure* $d\Gamma_U(u)$ by

$$d\Gamma_U(u)(x) := d\Gamma^{(L)}(u,u)(x) + \int_M \mathbf{1}_U(y)(u(x)-u(y))^2 dj(x,y). \tag{1.18}$$

Definition 1.10 (Poincaré inequality)**.** We say that the *Poincaré inequality* $(\mathrm{PI})_W$ holds if there exist constants $C > 0$ and $\kappa \in (0,1]$ such that, for any ball $B := B(x_0,R)$ with $0 < R < \overline{R}$ and for any function $u \in \mathcal{F} \cap L^\infty$,

$$\int_{\kappa B} |u - u_{\kappa B}|^2 d\mu \le CW(x_0,R)\int_B d\Gamma_B(u), \tag{1.19}$$

where Γ_B is defined in (1.18).

Definition 1.11 (Generalized capacity condition)**.** We say that condition $(\mathrm{Gcap})_W$ is satisfied if there exist two numbers $\overline{\kappa} \ge 1, C > 0$ such that, for any $u \in \mathcal{F}' \cap L^\infty$ and for any pair of concentric balls $B_0 := B(x_0,R)$, $B := B(x_0,R+r)$ with $x_0 \in M$ and $0 < R < R+r < \overline{R}$, there exists $\phi \in \overline{\kappa}$-cutoff$(B_0,B)$ such that

$$\mathcal{E}(u^2\phi,\phi) \le \sup_{x\in B}\frac{C}{W(x,r)}\int_B u^2 d\mu. \tag{1.20}$$

Recall that for a non-empty open subset U of M, the heat kernel (should it exist) of $(\mathcal{E},\mathcal{F}(U))$ is denoted by $\{p_t^U\}_{t>0}$.

The following is our main result.

Theorem 1.12. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Assume that conditions* (VD), (RVD), $(\mathrm{Gcap})_W$, $(\mathrm{PI})_W$ *and* $(\mathrm{TJ}_q)_W$ *for some $q \in [2,+\infty]$ are all satisfied. Let Ω be any non-empty open subset of M. Then the Dirichlet heat kernel $p_t^\Omega(x,y)$ is locally Hölder continuous. Moreover, for each $t > 0$ and $x \in \Omega$,*

$$p_t^\Omega(x,x) \le \frac{C}{V(x,\omega_x(t)\wedge\overline{R})}, \tag{1.21}$$

and, for any non-empty open subset $U \subset \Omega$ and $r > 0$ with $U_r \subset \Omega$, and for all $x, x' \in U$, $y \in \Omega$ and $t \geq s > 0$,

$$\left|p_t^\Omega(x,y) - p_s^\Omega(x',y)\right| \leq \left(\frac{1}{\sqrt{V(x,\omega_x(t\wedge r^\beta)\wedge\overline{R})}} + \frac{1}{\sqrt{V(x',\omega_{x'}(s\wedge r^\beta)\wedge\overline{R})}}\right) \cdot \frac{C}{\sqrt{V(y,\omega_y(s)\wedge\overline{R})}}\left(\left(\frac{d(x,x')}{\omega_x(t\wedge r^\beta)\wedge\overline{R}}\right)^\lambda + \frac{t-s}{s}\right), \tag{1.22}$$

where the constants $\lambda \in (0,1)$ and $C > 0$ depend only on the constants in the hypotheses.

In particular, when $\Omega = M$, we have for all $x, x', y \in M$ and $t \geq s > 0$,

$$\left|p_t(x,y) - p_s(x',y)\right| \leq \left(\frac{1}{\sqrt{V(x,\omega_x(t)\wedge\overline{R})}} + \frac{1}{\sqrt{V(x',\omega_{x'}(s)\wedge\overline{R})}}\right) \cdot \frac{C}{\sqrt{V(y,\omega_y(s)\wedge\overline{R})}}\left(\left(\frac{d(x,x')}{\omega_x(t)\wedge\overline{R}}\right)^\lambda + \frac{t-s}{s}\right). \tag{1.23}$$

Remark 1.13. Note that in the case when $W(x,r) = r^\beta$ for some $\beta > 0$ and $\overline{R} = \operatorname{diam} M = \infty$, conditions $(\mathrm{PI})_W$ and $(\mathrm{Gcap})_W$ become $(\mathrm{PI})_\beta$ and $(\mathrm{Gcap})_\beta$ respectively.

Since Theorem 1.5 is also true for the Dirichlet form in (1.11) (see Remark 1.6), in the rest of the paper, we will always refer to $(\mathcal{E},\mathcal{F})$ as defined in (1.11).

Hölder estimates for heat kernels in doubling spaces were also obtained by Chen, Kumagai and Wang under appropriate assumptions but assuming that $W(x,t) = W(t)$ and that the jump kernel $J(x,y)$ satisfies the following pointwise upper bound (cf. [3])

$$J(x,y) \leq \frac{C}{V(x,d(x,y))W(d(x,y))},$$

that is, $(\mathrm{TJ}_\infty)_W$, which is of course much stronger than $(\mathrm{TJ}_q)_W$ for $q \in [2,\infty]$. In [4, Theorem 2.14], Cho and Kim obtained the Hölder estimates for *bounded* caloric functions, whereas heat kernels may be unbounded. So, the main novelties of our results are as follows:

- The jump kernel is required to satisfy a rather weaker assumption $(\mathrm{TJ}_q)_W$ for $q \in [2,\infty]$ in Theorem 1.12, or $(\mathrm{TJ})_\beta$ in Theorem 1.5. In particular, our setting covers rather singular cases;
- The set Ω can be any open set, including $\Omega = M$;
- There is no restriction on the boundedness of heat kernels.

The structure of this paper. In Section 2, we introduce a new metric d_* on (M,d) such that under this new metric, the scaling function $W(x,r)$ in each condition $(\mathrm{TJ}_q)_W$, $(\mathrm{PI})_W$ and $(\mathrm{Gcap})_W$ becomes $W_*(x,r) := r^\beta$ for some $\beta > 0$. In Section 3, we investigate various oscillation inequalities of harmonic functions and heat semigroups, where the condition $(\mathrm{TJ}_2)_W$ is explicitly used. It follows from these results that each $P_t^\Omega g$ has a (locally) Hölder continuous version. In Section 4, we construct (locally) Hölder continuous (Dirichlet) heat kernels by using the (local) Hölder estimates of $P_t^\Omega g$'s. We will firstly prove Theorem 1.12 in Subsection 4.1 and then Theorem 1.5 in Subsection 4.2.

Notation. Letters c, C, C', C_1, C_2, etc. are used to denote positive numbers, depending on the constants in the hypotheses, whose values may change at each occurrence.

2. Change of metric

Suppose that (VD) and (RVD) are satisfied in (M,d,μ). In [11], the authors introduced a new metric d_* on M with the following properties: under this new metric d_*, the measure μ still retains the doubling property (or the reverse doubling property), while the scaling function $W(x,R)$ becomes independent of the point x. This type of change of metric was first used by Kigami in [14]. Let us recall the construction and some properties of the new metric.

For any $x, y \in M$, set $W(x,y) := W(x, d(x,y))$. Let

$$D(x,y) := W(x,y) + W(y,x),$$

Clearly, the quantity $D(x,y) = 0$ if and only if $x = y$, and is symmetric: $D(x,y) = D(y,x)$. The following proposition shows that $D(x,y)$ is a quasi-metric on M.

Proposition 2.1 ([11, Proposition 6.1]). *There exists a constant $C_1 \geq 1$ such that for all x, y, z in M,*

$$D(x,y) \leq C_1(D(x,z) + D(z,y)).$$

Consequently, there exist two constants β, $C_2 > 0$ and a metric d_* on M such that

$$C_2^{-1} d_*(x,y)^\beta \leq D(x,y) \asymp W(x,y) \leq C_2 d_*(x,y)^\beta \tag{2.1}$$

for all x, y in M.

In the rest of the paper, the parameter β will be always referred to as the constant from Theorem 1.5 in the case when $W(x,r) = r^\beta$, and the constant from Proposition 2.1 in the general case $W(x,r)$ satisfying (1.15).

We have by (2.1)

$$L^{-1} d_*(x,y) \leq W(x,y)^{1/\beta} \leq L d_*(x,y), \quad x, y \in M, \tag{2.2}$$

for some constant $L \geq 1$.

For any $r > 0$, let

$$B_*(x,r) := \{y \in M : d_*(y,x) < r\}$$

be an open ball under the new metric d_*. Recall that for any $x \in M$, the function ω_x is the inverse function of $t \mapsto W(x,t)$.

Proposition 2.2 ([11, Proposition 6.2]). *There exists a number L_0 with $L_0 \geq L^2 > 1$, where $L > 1$ is the same constant as in (2.2), such that the following properties are true.*

(1) *For all $x \in M$ and all $r > 0$,*

$$B_*(x, L_0^{-1} r) \subset B(x, \omega_x(L^{-\beta} r^\beta)) \subset B_*(x,r). \tag{2.3}$$

(2) *For all $x \in M$ and all $R > 0$,*

$$B(x, L_0^{-1} R) \subset B_*(x, L^{-1} W(x,R)^{1/\beta}) \subset B(x,R). \tag{2.4}$$

Consequently, a subset of M is open under the metric d_* if and only if it is also open under the original metric d.

In the rest of the paper, we will always use the notation d_* to denote the new metric defined in Proposition 2.1.

For any $x \in M$ and any $r > 0$, let $V_*(x,r)$ be the volume of a ball $B_*(x,r)$ under the metric d_*, that is,

$$V_*(x,r) := \mu(B_*(x,r)).$$

Note that $\overline{R}$ is the diameter of M in [11], while in this paper, it can be smaller than diam M.

Proposition 2.3 ([8, Proposition 4.4]). *Assume that (VD) is satisfied. Then the following statements are true.*

(1) *Condition (VD_*) holds true: there exists a constant $C > 0$ such that, for all $x \in M$ and all $r > 0$,*

$$V_*(x, 2r) \leq C V_*(x,r). \tag{2.5}$$

Consequently, there exists a constant $\alpha_* > 0$ such that for all $x, y \in M$ and all $0 < s \le r$ with $d_*(x,y) \le r$,

$$\frac{V_*(x,r)}{V_*(y,s)} \le C\left(\frac{r}{s}\right)^{\alpha_*}.$$

(2) *Assume in addition that (RVD) is satisfied. Then condition (RVD$_*$) holds true: there exists a constant $\alpha'_* > 0$ such that for all $x \in M$ and all $0 < s \le r < W(x, \overline{R})^{1/\beta}$,*

$$\frac{V_*(x,r)}{V_*(x,s)} \ge C^{-1}\left(\frac{r}{s}\right)^{\alpha'_*}. \tag{2.6}$$

Definition 2.4. We say that condition (TJ$^*_q$) is satisfied for some $1 \le q \le \infty$, if there exists a non-negative function J such that

$$dj(x,y) = J(x,y)d\mu(y)d\mu(x) \ \text{ in } M \times M,$$

and, for any $x \in M$ and any $r > 0$,

$$\|J(x,\cdot)\|_{L^q(B_*(x,r)^c)} \le \frac{C}{V_*(x,r)^{1/q'} r^\beta}, \tag{2.7}$$

where $q' = \frac{q}{q-1}$ and $C \in [0,\infty)$ is independent of x, r.

It is proved in [11, Proposition 7.4(3)] that for any $q \ge 1$,

$$(\text{VD}) + (\text{TJ}_q)_W \Rightarrow (\text{TJ}^*_q). \tag{2.8}$$

3. Oscillation inequalities

In this section, we show the Hölder continuity of the heat solution, including the harmonic function.

3.1. **Harmonic functions and Green operator.**

Definition 3.1. Let Ω be an open subset of M. We say that a function $u \in \mathcal{F}'$ is *subharmonic* (resp. *superharmonic*) in Ω if

$$\mathcal{E}(u,\varphi) \le 0 \quad (\text{resp. } \mathcal{E}(u,\varphi) \ge 0) \tag{3.1}$$

for any $0 \le \varphi \in \mathcal{F}(\Omega)$. A function $u \in \mathcal{F}'$ is called *harmonic* in Ω if it is both subharmonic and superharmonic in Ω.

We introduce condition (OSL$_*$) that is called the *oscillation lemma* for harmonic functions on a ball. This condition says that any harmonic function is locally uniformly Hölder continuous.

Definition 3.2 (Oscillation lemma)**.** We say that condition (OSL$_*$) holds if there exist three positive constants $\sigma_* \in (0,1)$ and γ, C such that for any ball $B_r := B_*(x_0, r)$ with $r \in (0, \sigma_* W(x_0, \overline{R})^{1/\beta})$ and any function $u \in \mathcal{F}' \cap L^\infty$, which is harmonic in B_r, we have for any $\rho \in (0, r]$,

$$\operatorname*{osc}_{B_\rho} u \le C\left(\frac{\rho}{r}\right)^\gamma \left(r^\beta \operatorname*{esup}_{x \in \frac{3}{4}B_r} \int_{B_r^c} |u(y)| J(x,y)dy + \|u\|_{L^\infty(B_r)} \right). \tag{3.2}$$

We mention that constants σ_*, γ and C are independent of B_r, u, ρ.

It follows from [8, Lemma 5.19, p. 727] and [10, Proposition 3.1] that for all $q \in [1,\infty]$,

$$(\text{VD}) + (\text{RVD}) + (\text{Gcap})_W + (\text{PI})_W + (\text{TJ}_q)_W \Rightarrow (\text{OSL}_*). \tag{3.3}$$

Let $\Omega \subset M$ be a non-empty open set. Note that if

$$G^\Omega 1 := \int_0^\infty P_t^\Omega \mathbf{1}_\Omega dt \in L^\infty(\Omega)$$

then G^Ω can be extended to a bounded operator on $L^2(\Omega)$ that satisfies the identity $G^\Omega = (\mathcal{L}^\Omega)^{-1}$, see for example [13, Lemma 3.2, p. 1232][1]. The function $G^\Omega 1$ is called the *mean exit time* from the set Ω.

[1]Although this lemma was stated for the local Dirichlet form, its proof also holds for any regular Dirichlet form.

Definition 3.3. We say that condition ($E^*_\le$) holds, if there exist two constants δ_*, $C > 0$ such that for all balls $B_* := B(x_0, r)$ of radius $r < \delta_* W(x_0, \overline{R})^{1/\beta}$,

$$\|G^{B_*}1\|_{L^\infty} \le Cr^\beta.$$

It follows from [8, Proposition 4.7, Eq. (4.33) and Proposition 4.13] and Proposition 2.3 that

$$(\mathrm{VD}) + (\mathrm{RVD}) + (\mathrm{PI})_W \Rightarrow (\mathrm{E}^*_\le). \tag{3.4}$$

3.2. **Oscillation inequality for solutions of Poisson equation.** We study the oscillation of the weak solution of the Poisson-type equation on some domain by using (3.3). This property will be used to show the Hölder continuity of the heat kernel later on.

For a non-empty open set $\Omega \subset M$ and $f \in L^2(\Omega)$, we say that a function $u \in \mathcal{F}$ solves weakly the equation (called the *Poisson-type* equation)

$$\mathcal{L}u = f \ \text{ in } \Omega, \tag{3.5}$$

if for any $\phi \in \mathcal{F}(\Omega)$,

$$\mathcal{E}(u, \phi) = (f, \phi).$$

The following proposition is stated in [8, Proposition 6.6, p. 729].

Proposition 3.4 ([8, Proposition 6.6, p. 729])**.** *Assume that $u \in \mathcal{F}$ solves weakly the equation (3.5) for some $f \in L^2(\Omega)$. Let B be a non-empty open subset of Ω.*

(1) *If $v \in \mathcal{F}$ solves weakly the equation $\mathcal{L}v = f$ in B, then $u - v$ is harmonic in B.*
(2) *If $\|G^B 1\|_{L^\infty} < \infty$, then $u - G^B f$ is harmonic in B.*

The following gives the oscillation of the weak solution of the Poisson-type equation. Recall that

$$q' = \frac{q}{q-1},$$

is the Hölder conjugate of q.

Lemma 3.5. *Assume that conditions ($E^*_\le$), (TJ^*_q) for $q \in [1, \infty]$, (OSL_*) are all satisfied. Let $C_0 \ge 1$ and Ω be any open subset of M containing a ball $B_* := B_*(x_0, r)$ of radius $r \in (0, C_0 W(x_0, \overline{R})^{1/\beta})$. If the function $u \in \mathcal{F}(\Omega) \cap L^\infty$ solves weakly the equation (3.5) for $f \in L^2(\Omega) \cap L^\infty(B_*)$, then for any $0 < \rho \le r$*

$$\underset{B_*(x_0,\rho)}{\mathrm{osc}}\, u \le C \left(\frac{\rho}{r}\right)^\gamma \left(\frac{\|u\|_{L^{q'}(\Omega)}}{V_*(x_0, r)^{1/q'}} + \|u\|_{L^\infty(B_*)} \right) + Cr^\beta \|f\|_{L^\infty(B_*)}, \tag{3.6}$$

where γ is the constant from condition (OSL_*) and C is independent of $B_*, u, \Omega, \rho, f, \overline{R}$. Consequently,

$$(\mathrm{VD}) + (\mathrm{RVD}) + (\mathrm{TJ}_q) + (\mathrm{Gcap})_W + (\mathrm{PI})_W \Rightarrow \textit{ inequality } (3.6). \tag{3.7}$$

Proof. Let δ_* be the constant from ($E^*_\le$) and σ_* be the constant from (OSL_*). We consider two cases.

Case 1. $r < (\delta_* \wedge \sigma_*) W(x_0, \overline{R})^{1/\beta}$. By condition ($E^*_\le$), we have

$$\|G^{B_*}1\|_{L^\infty} \le Cr^\beta.$$

In particular, we have $\|G^{B_*}1\|_{L^\infty} < \infty$. From this, we see that

$$\|G^{B_*}f\|_{L^\infty(B_*)} \le \|G^{B_*}1\|_{L^\infty} \|f\|_{L^\infty(B_*)} \le Cr^\beta \|f\|_{L^\infty(B_*)}. \tag{3.8}$$

Consider the function

$$v := u - G^{B_*} f.$$

Clearly, we see that $v \in \mathcal{F}(\Omega) \cap L^\infty$. By Proposition 3.4, the function v is harmonic in B_*. It follows from condition (OSL_*) that for any $0 < \rho \le r$

$$\underset{B_*(x_0,\rho)}{\mathrm{osc}}\, v \le C \left(\frac{\rho}{r}\right)^\gamma \left(r^\beta T_{\frac{3}{4}B_*, B_*}(|v|) + \|v\|_{L^\infty(B_*)} \right). \tag{3.9}$$

Since $u=0$ in Ω^c and $G^{B_*}f=0$ in B_*^c, we obtain by Hölder inequality, (VD$_*$) and condition (TJ$_{q'}^*$) that

$$\begin{aligned}
T_{\frac{3}{4}B_*,B_*}(v) &\le \operatorname*{esup}_{x\in\frac{3}{4}B_*}\int_{B_*^c}(|u(y)|+|G^{B_*}f(y)|)J(x,y)d\mu(y)\\
&\le \operatorname*{esup}_{x\in\frac{3}{4}B_*}\left(\int_{B_*^c}|u(y)|^{q'}d\mu(y)\right)^{1/q'}\left(\int_{B_*^c}J(x,y)^q d\mu(y)\right)^{1/q} \\
&\le \|u\|_{L^{q'}(\Omega)}\operatorname*{esup}_{x\in\frac{3}{4}B_*}\left(\int_{B_*(x,r/4)^c}J(x,y)^q d\mu(y)\right)^{1/q}\\
&\le \frac{C\|u\|_{L^{q'}(\Omega)}}{V_*(x,r/4)^{1/q'}(r/4)^\beta}\\
&\le \frac{C'\|u\|_{L^{q'}(\Omega)}}{V_*(x_0,r)^{1/q'}(r/4)^\beta}. \qquad \text{(by (VD}_*\text{))}
\end{aligned} \tag{3.10}$$

Substituting this into (3.9), and using (VD$_*$) and the fact that

$$\|v\|_{L^\infty(B_*)}\le \|u\|_{L^\infty(B_*)}+\|G^{B_*}f\|_{L^\infty(B_*)},$$

we obtain

$$\operatorname*{osc}_{B_*(x_0,\rho)} v\le C\left(\frac{\rho}{r}\right)^\gamma\left(\frac{\|u\|_{L^{q'}(\Omega)}}{V_*(x_0,r)^{1/q'}}+\|u\|_{L^\infty(B_*)}+\|G^{B_*}f\|_{L^\infty(B_*)}\right). \tag{3.11}$$

Therefore, we conclude by (3.11) and (3.8) that

$$\begin{aligned}
\operatorname*{osc}_{B_*(x_0,\rho)} u &\le \operatorname*{osc}_{B_*(x_0,\rho)} v+\operatorname*{osc}_{B_*(x_0,\rho)} G^{B_*}f\\
&\le C'\left(\frac{\rho}{r}\right)^\gamma\left(\frac{C\|u\|_{L^{q'}(\Omega)}}{V_*(x_0,r)^{1/q'}}+\|u\|_{L^\infty(B_*)}+\|G^{B_*}f\|_{L^\infty(B_*)}\right)+2\|G^{B_*}f\|_{L^\infty(B_*)}\\
&\le C''\left(\frac{\rho}{r}\right)^\gamma\left(\frac{\|u\|_{L^{q'}(\Omega)}}{V_*(x_0,r)^{1/q'}}+\|u\|_{L^\infty(B_*)}\right)+Cr^\beta\|f\|_{L^\infty(B_*)},
\end{aligned}$$

thus showing (3.6).

Case 2. $(\delta_*\wedge\sigma_*)W(x_0,\overline{R})^{1/\beta}\le r<C_0W(x_0,\overline{R})^{1/\beta}$ with $C_0\ge 1$ when $\overline{R}<\infty$.

If $\rho<\frac{1}{3}(\delta_*\wedge\sigma_*)W(x_0,\overline{R})^{1/\beta}$, then, applying the result in Case 1 for $r':=\frac{1}{2}(\delta_*\wedge\sigma_*)W(x_0,\overline{R})^{1/\beta}$ and using (VD$_*$), we obtain that

$$\begin{aligned}
\operatorname*{osc}_{B_*(x_0,\rho)} u &\le C\left(\frac{\rho}{\frac{1}{2}(\delta_*\wedge\sigma_*)W(x_0,\overline{R})^{1/\beta}}\right)^\gamma\left(\frac{\|u\|_{L^{q'}(\Omega)}}{V_*(x_0,r')^{1/q'}}+\|u\|_{L^\infty(B_*(x_0,r'))}\right)\\
&\quad +C\left(\frac{1}{2}(\delta_*\wedge\sigma_*)W(x_0,\overline{R})^{1/\beta}\right)^\beta\|f\|_{L^\infty(B_*(x_0,r'))}\\
&\le (2(\delta_*\wedge\sigma_*)^{-1}C_0)^{\gamma+\alpha_*/2}C'\left(\frac{\rho}{C_0W(x_0,\overline{R})^{1/\beta}}\right)^\gamma\left(\frac{\|u\|_{L^{q'}(\Omega)}}{V_*(x_0,r)^{1/q'}}+\|u\|_{L^\infty(B_*)}\right)\\
&\quad +C'r^\beta\|f\|_{L^\infty(B_*)}\\
&\le C''\left(\frac{\rho}{r}\right)^\gamma\left(\frac{\|u\|_{L^{q'}(\Omega)}}{V_*(x_0,r)^{1/q'}}+\|u\|_{L^\infty(B_*)}\right)+C'r^\beta\|f\|_{L^\infty(B_*)},
\end{aligned}$$

which is (3.6).

If $\rho \geq \frac{1}{3}(\delta_* \wedge \sigma_*)W(x_0, \overline{R})^{1/\beta}$, then, $\frac{r}{\rho} \leq \frac{3C_0}{\delta_* \wedge \sigma_*}$. Hence, by the fact that $B_*(x_0, \rho) \subset B_*$, we have

$$\operatorname*{osc}_{B_*(x_0,\rho)} u = \left(\frac{r}{\rho}\right)^{\gamma} \left(\frac{\rho}{r}\right)^{\gamma} \operatorname*{osc}_{B_*(x_0,\rho)} u \leq 2\left(\frac{3C_0}{(\delta_* \wedge \sigma_*)}\right)^{\gamma} \left(\frac{\rho}{r}\right)^{\gamma} \|u\|_{L^\infty(B_*)},$$

which implies (3.6).

Finally, the implication (3.7) follows directly from (3.3) by using the facts that (VD) + (RVD) + (PI) $\Rightarrow$ ($\mathrm{E}^*_{\leq}$) (see (3.4)) and (TJ_2) $\Rightarrow$ (TJ^*_2) (see (2.8)). □

3.3. **Estimates for heat semigroup solutions.** We derive the $L^\infty(B)$-estimate of the heat semigroup solutions for a ball B. Note that by [8, Remark 4.16, p. 704], (VD) + (RVD) + (PI) implies the so-called Faber-Krahn inequality. Then, under conditions (VD), (RVD), (PI), $(\mathrm{Gcap})_W$ and $(\mathrm{TJ}_q)_W$ for some $q \in [2, +\infty]$, it is proved in [10, Theorem 2.10, Proposition 3.1 and Lemma 6.2] that there exists a constant $C > 0$ such that for any $x \in M$ and any $0 < t < W(x, \overline{R})$,

$$\|P_t g\|_{L^\infty(B(x, \frac{1}{2}\omega_x(t)))} \leq \frac{C}{\sqrt{V(x, \omega_x(t))}} \|g\|_{L^2}, \quad g \in L^2, \tag{3.12}$$

and, it is proved in [10, Theorem 2.10, Remark 6.8 and Corollary 6.4] that there exists a constant $C > 0$ such that for any $t > 0$ and for any ball $B := B(x, R)$ of radius $R > 0$

$$\|P_t g\|_{L^\infty(B)} \leq \frac{C}{\sqrt{V(x, R)}} \left(\frac{R}{\overline{R}} \vee 1\right)^{\frac{\alpha}{2}} \left(\frac{W(x, R)}{t} + 1\right)^{\frac{\alpha}{2\beta_1}} \|g\|_{L^2}, \quad g \in L^2. \tag{3.13}$$

Moreover, it is shown in [10, Remark 6.8] that, under (VD), (3.12) and (3.13) are equivalent.

In the following lemma, we give an alternative equivalence of (3.12).

Lemma 3.6. *Suppose that (VD) and (3.12) are satisfied. Then, there exists a constant $C > 0$ such that for any $t > 0$ and $B := B(x, R)$ with $x \in M$ and $R > 0$,*

$$\|P_t g\|_{L^\infty(B)} \leq \frac{C}{\sqrt{V(x, (\omega_x(t) \wedge \overline{R}) \vee R)}} \left(\frac{R}{\overline{R}} \vee 1\right)^{\frac{\alpha}{2}} \left(\frac{W(x, R)}{t} + 1\right)^{\frac{\alpha}{2\beta_1}} \|g\|_{L^2}, \quad g \in L^2. \tag{3.14}$$

If $\overline{R} = \operatorname{diam} M$, then the above inequality can be improved further as follows:

$$\|P_t g\|_{L^\infty(B)} \leq \frac{C}{\sqrt{V(x, (\omega_x(t) \vee R) \wedge \overline{R})}} \left(\frac{W(x, R \wedge \overline{R})}{t} + 1\right)^{\frac{\alpha}{2\beta_1}} \|g\|_{L^2}, \quad g \in L^2. \tag{3.15}$$

Proof. Fix $x \in M$, $R > 0$ and $t > 0$, set $B := B(x, R)$. We divide the proof of (3.14) into four cases.

Case 1: $t < W(x, \overline{R})$ and $R < \frac{1}{2}\omega_x(t)$. In this case, (3.14) following directly from (3.12) since

$$(\omega_x(t) \wedge \overline{R}) \vee R = \omega_x(t), \quad R < \overline{R} \quad \text{and} \quad W(x, R) < t.$$

Case 2: $t < W(x, \overline{R})$ and $\frac{1}{2}\omega_x(t) \leq R < \overline{R}$. In this case, we have

$$(\omega_x(t) \wedge \overline{R}) \vee R = \omega_x(t) \vee R \leq 2R,$$

and then, by (VD),

$$V(x, R) \geq cV(x, 2R) \geq cV(x, (\omega_x(t) \wedge \overline{R}) \vee R).$$

Hence, (3.14) following from (3.13) and the above inequality.

Case 3: $t \geq W(x, \overline{R})$ and $R < \overline{R}$ (when $\overline{R} < \infty$). In this case, we set $t_0 := \frac{1}{2}W(x, \overline{R})$. Since (3.14) is true for all $t < W(x, \overline{R})$ and $R < \overline{R}$ by Cases 1 and 2, we can apply (3.14) for t_0 and R and obtain that

$$\begin{aligned}\|P_t g\|_{L^\infty(B)} &= \|P_{t_0} P_{t-t_0} g\|_{L^\infty(B)} \\ &\leq \frac{C}{\sqrt{V(x, (\omega_x(t_0) \wedge \overline{R}) \vee R)}} \left(\frac{R}{\overline{R}} \vee 1\right)^{\frac{\alpha}{2}} \left(\frac{W(x, R)}{t_0} + 1\right)^{\frac{\alpha}{2\beta_1}} \|P_{t-t_0} g\|_{L^2}, \quad g \in L^2.\end{aligned}$$

Moveover, we have $\|P_{t-t_0}g\|_{L^2}\le\|g\|_{L^2}$ and by (1.16),

$$(\omega_x(t_0)\wedge\overline{R})\vee R\ge c((\omega_x(2t_0)\wedge\overline{R})\vee R)=c(\overline{R}\vee R)=c(\omega_x(t)\wedge\overline{R})\vee R),$$
$$\frac{W(x,R)}{t_0}\le\frac{W(x,\overline{R})}{2^{-1}W(x,\overline{R})}=2\le\frac{W(x,R)}{t}+2.$$

Combining the above four inequalities, we obtain (3.14) in this case.

Case 4: $t>0$ and $R\ge\overline{R}$ (when $\overline{R}<\infty$). Let $B_z:=B(z,\frac{1}{2}\overline{R})$ for any $z\in B=B(x,R)$. Since (3.14) is true for $t>0$ and B_z by the Cases 1-3, we can apply (3.14) for t and B_z and obtain that

$$\begin{aligned}\|P_tg\|_{L^\infty(B_z)}&\le\frac{C}{\sqrt{V(z,(\omega_z(t)\wedge\overline{R})\vee(2^{-1}\overline{R}))}}\left(\frac{W(z,\frac{1}{2}\overline{R})}{t}+1\right)^{\frac{\alpha}{2\beta_1}}\|g\|_{L^2}\\&\le\frac{C}{\sqrt{V(z,2^{-1}\overline{R})}}\left(\frac{W(z,\overline{R})}{t}+1\right)^{\frac{\alpha}{2\beta_1}}\|g\|_{L^2},\quad g\in L^2.\end{aligned}$$

Moveover, since $d(x,z)<R$ and $\overline{R}\le R$, by (VD) and the left inequality in (1.15), we have

$$\frac{1}{V(z,2^{-1}\overline{R})}=\frac{1}{V(x,R)}\frac{V(x,R)}{V(z,2^{-1}\overline{R})}\le\frac{C}{V(x,R)}\left(\frac{R}{\overline{R}}\right)^\alpha=\frac{C}{V(x,(\omega_x(t)\wedge\overline{R})\vee R)}\left(\frac{R}{\overline{R}}\right)^\alpha,$$
$$W(z,\overline{R})\le cW(x,R).$$

Combining the above three inequalities, we obtain

$$\|P_tg\|_{L^\infty(B_z)}\le\frac{C}{\sqrt{V(x,(\omega_x(t)\wedge\overline{R})\vee R)}}\left(\frac{R}{\overline{R}}\right)^{\frac{\alpha}{2}}\left(\frac{W(x,R)}{t}+1\right)^{\frac{\alpha}{2\beta_1}}\|g\|_{L^2},\quad g\in L^2.$$

Since B can be covered by at most countable balls like B_z, (3.14) follows from the above inequality.

It remains to prove (3.15). Note that if $R>\overline{R}$ when $\overline{R}=\operatorname{diam}M<\infty$, then $B(z,R)=M$ for all $z\in M$. Then, we have $B=B(x,R)=B(x,R\wedge(2\overline{R}))$. Hence, applying (3.14) for $B(x,R\wedge(2\overline{R}))$, we obtain,

$$\begin{aligned}\|P_tg\|_{L^\infty(B)}&=\|P_tg\|_{L^\infty(B(x,R\wedge(2\overline{R})))}\\&\le\frac{C}{\sqrt{V(x,(\omega_x(t)\wedge\overline{R})\vee(R\wedge(2\overline{R})))}}\\&\qquad\cdot\left(\frac{R\wedge(2\overline{R})}{\overline{R}}\vee1\right)^{\frac{\alpha}{2}}\left(\frac{W(x,R\wedge(2\overline{R}))}{t}+1\right)^{\frac{\alpha}{2\beta_1}}\|g\|_{L^2},\quad g\in L^2.\end{aligned}$$

Note that

$$(\omega_x(t)\wedge\overline{R})\vee(R\wedge(2\overline{R}))\ge(\omega_x(t)\wedge\overline{R})\vee(R\wedge\overline{R})=(\omega_x(t)\vee R)\wedge\overline{R},$$

and by the right inequality in (1.15),

$$W(x,R\wedge(2\overline{R}))\le W(x,2(R\wedge\overline{R}))\le cW(x,R\wedge\overline{R}).$$

Combining the above three inequalities, we obtain (3.15).

If $R<\overline{R}=\operatorname{diam}M<\infty$, (3.15) follows directly from (3.14). □

Remark 3.7. In the proof of Theorem 1.12, we will not use (3.15). We prove it for its own independent interest and for its possible future use.

We rewrite (3.14) and (3.15) under the new metric d_*.

Corollary 3.8. *Assume that (VD) and (3.12) are satisfied. Let $g \in L^2$. Then, there exists a constant $C > 0$ such that for any $t > 0$ and $B_* := B_*(x, r)$ with $x \in M$ and $r > 0$,*

$$\|P_t g\|_{L^\infty(B_*)} \le \frac{C}{\sqrt{V_*(x, (t \wedge W(x, \overline{R}))^{1/\beta} \vee r)}} \left(\frac{\omega_x(r^\beta)}{\overline{R}} \vee 1\right)^{\frac{\alpha}{2}} \left(\frac{r^\beta}{t} + 1\right)^{\frac{\alpha}{2\beta_1}} \|g\|_{L^2}. \tag{3.16}$$

Proof. Fix $t > 0$, $x \in M$ and $r > 0$. By (2.3) with r replaced by Lr and $L((t \wedge W(x, \overline{R}))^{1/\beta} \vee r)$ respectively, we have

$$B_*(x, L_0^{-1} Lr) \subset B(x, \omega_x(r^\beta)), \tag{3.17}$$

and

$$B_*(x, L_0^{-1} L((t \wedge W(x, \overline{R}))^{1/\beta} \vee r)) \subset B(x, \omega_x((t \wedge W(x, \overline{R})) \vee r^\beta)). \tag{3.18}$$

Combining the above two inequalities and applying (3.14) with $R = \omega_x(r^\beta)$, we obtain

$$\begin{aligned}
\|P_t g\|_{L^\infty(B_*(x, L_0^{-1} Lr))} &\le \|P_t g\|_{L^\infty(B(x,R))} \qquad \text{(by (3.17))} \\
&\le \frac{C}{\sqrt{V(x, (\omega_x(t) \wedge \overline{R}) \vee \omega_x(r^\beta))}} \\
&\quad \cdot \left(\frac{\omega_x(r^\beta)}{\overline{R}} \vee 1\right)^{\frac{\alpha}{2}} \left(\frac{W(x, \omega_x(r^\beta))}{t} + 1\right)^{\frac{\alpha}{2\beta_1}} \|g\|_{L^2} \\
&= \frac{C}{\sqrt{V(x, \omega_x((t \wedge W(x, \overline{R})) \vee r^\beta))}} \\
&\quad \cdot \left(\frac{\omega_x(r^\beta)}{\overline{R}} \vee 1\right)^{\frac{\alpha}{2}} \left(\frac{r^\beta}{t} + 1\right)^{\frac{\alpha}{2\beta_1}} \|g\|_{L^2} \\
&\le \frac{C}{\sqrt{V_*(x, L_0^{-1} L((t \wedge W(x, \overline{R}))^{1/\beta} \vee r))}} \qquad \text{(by (3.18))} \\
&\quad \cdot \left(\frac{\omega_x(r^\beta)}{\overline{R}} \vee 1\right)^{\frac{\alpha}{2}} \left(\frac{r^\beta}{t} + 1\right)^{\frac{\alpha}{2\beta_1}} \|g\|_{L^2}, \quad g \in L^2.
\end{aligned}$$

Finally, the inequality (3.16) follows by replacing r by $L_0 L^{-1} r$ in the above inequality and using (VD) and (1.16). □

Lemma 3.9. *Let U and Ω be two open subsets of M. Let $g \in L^2(\Omega)$. Then for all $t > 0$,*

$$\|\partial_t P_t^\Omega g\|_{L^\infty(U)} \le \frac{4}{t} \|P_{t/2}^\Omega\|_{L^2 \to L^\infty(U)} \|P_{t/4}^\Omega g\|_{L^2(\Omega)}, \tag{3.19}$$

where $\partial_t P_t^\Omega g$ is the Fréchet derivative of the $L^2(\Omega)$-valued function $t \mapsto P_t^\Omega g$. Moreover, for all $t \ge s > 0$,

$$\|P_t^\Omega g - P_s^\Omega g\|_{L^\infty(U)} \le \frac{4(t-s)}{s} \|P_{s/2}^\Omega\|_{L^2 \to L^\infty(U)} \|P_{s/4}^\Omega g\|_{L^2(\Omega)}. \tag{3.20}$$

Proof. Since

$$P_t^\Omega g = P_r^\Omega P_{t-r}^\Omega g$$

for any $t > 0$ and any $r \in (0, t)$, we have

$$\partial_t (P_t^\Omega g) = P_r^\Omega (\partial_t P_{t-r}^\Omega g).$$

From this and using the following general inequality (see [13, Lemma 5.4])

$$\|\partial_t (P_{t-r}^\Omega g)\|_{L^2(\Omega)} \le \frac{2}{t-r} \|P_{(t-r)/2}^\Omega g\|_{L^2(\Omega)} \quad \text{for any } r \in (0, t), \tag{3.21}$$

we obtain that

$$\|\partial_t (P_t^\Omega g)\|_{L^\infty(U)} = \|P_r^\Omega \partial_t (P_{t-r}^\Omega g)\|_{L^\infty(U)} \le \|P_r^\Omega\|_{L^2 \to L^\infty(B_*)} \|\partial_t (P_{t-r}^\Omega g)\|_{L^2(\Omega)}$$

$$\leq \|P_r^\Omega\|_{L^2\to L^\infty(U)} \cdot \frac{2}{t-r}\|P_{(t-r)/2}^\Omega g\|_{L^2(\Omega)}.$$

Setting $r = t/2$ in the above inequality, we obtain (3.19).

It remains to show (3.20). For simplicity, let $t > s \geq 2\tau > 0$. Then

$$\begin{aligned}\|P_t^\Omega g - P_s^\Omega g\|_{L^\infty(U)} &= \|P_\tau^\Omega(P_{t-\tau}^\Omega g - P_{s-\tau}^\Omega g)\|_{L^\infty(U)}\\ &\leq \|P_\tau^\Omega\|_{L^2\to L^\infty(U)}\|P_{t-\tau}^\Omega g - P_{s-\tau}^\Omega g\|_{L^2(\Omega)}.\end{aligned}$$

By (3.21), we see that

$$\begin{aligned}\|P_{t-\tau}^\Omega g - P_{s-\tau}^\Omega g\|_{L^2(\Omega)} &= \Big\|\int_{s-\tau}^{t-\tau}\partial_\xi(P_\xi^\Omega g)d\xi\Big\|_{L^2(\Omega)}\\ &\leq \int_{s-\tau}^{t-\tau}\frac{2}{\xi}\|P_{\xi/2}^\Omega g\|_{L^2(\Omega)}d\xi\\ &\leq (t-s)\frac{2}{\tau}\|P_{\tau/2}^\Omega g\|_{L^2(\Omega)}.\end{aligned}$$

Therefore, combining the above two inequalities, we conclude that

$$\|P_t^\Omega g - P_s^\Omega g\|_{L^\infty(U)} \leq \|P_\tau^\Omega\|_{L^2\to L^\infty(U)}\frac{2(t-s)}{\tau}\|P_{\tau/2}^\Omega g\|_{L^2(\Omega)},$$

thus showing (3.20) by letting $\tau = \frac{s}{2}$. □

3.4. **Hölder continuity of heat semigroup solutions.** We derive that the heat semigroup solutions are locally Hölder continuous.

Lemma 3.10. *Let $\Omega \subset M$ be a non-empty open set, and let $u(x,t) = P_t^\Omega g(x)$ for $g \in L^2(\Omega)$. Assume that conditions (VD), (RVD), $(\text{Gcap})_W$, $(\text{TJ}_q)_W$ for some $q \in [2,+\infty]$, $(\text{PI})_W$ are all satisfied. Then, for any $x_0 \in M$ and $r > 0$ so that $B_* := B_*(x_0,r) \subset \Omega$, and for any $t > 0$ and $\rho > 0$ so that $\rho \leq (t \wedge W(x_0,\overline{R}))^{1/\beta} \wedge r$, we have*

$$\underset{B_*(x_0,\rho)}{\operatorname{osc}} u(\cdot,t) \leq \frac{C}{\sqrt{V_*(x_0,(t\wedge W(x_0,\overline{R}))^{1/\beta}\wedge r)}}\left(\frac{\rho}{(t\wedge W(x_0,\overline{R}))^{1/\beta}\wedge r}\right)^\theta \|g\|_{L^2(\Omega)}, \tag{3.22}$$

where C is a positive number depending only on constants in the hypothesis, and θ is given by

$$\theta = \frac{\gamma\beta}{\gamma+\beta+\alpha_*/2}. \tag{3.23}$$

Here γ is the same as in Lemma 3.5.

In particular, if $\Omega = M$, then

$$\underset{B_*(x_0,\rho)}{\operatorname{osc}} u(\cdot,t) \leq \frac{C}{\sqrt{V_*(x_0,(t\wedge W(x_0,\overline{R}))^{1/\beta})}}\left(\frac{\rho}{(t\wedge W(x_0,\overline{R}))^{1/\beta}}\right)^\theta \|g\|_{L^2}. \tag{3.24}$$

Proof. By standard approximating arguments, it suffices to consider the case when $g \in L^2(\Omega)\cap L^\infty$. In this case, we also have that $u(\cdot,t) \in L^\infty$ for all $t > 0$.

For any $t > 0$, the function $u(\cdot,t)$ belongs to $\text{dom}(\mathcal{L}^\Omega)$, is Fréchet differentiable with respect to t in $L^2(\Omega)$, and satisfies weakly

$$\partial_t u(\cdot,t) = -\mathcal{L}^\Omega u(\cdot,t),$$

that is, for any $\phi \in \mathcal{F}(\Omega)$ and $t > 0$,

$$\mathcal{E}(u(\cdot,t),\phi) = (\mathcal{L}^\Omega u(\cdot,t),\phi) = -(\partial_t u(\cdot,t),\phi).$$

By (3.19) (with $U = B_*$) and (3.16), we have $\partial_t u(\cdot,t) \in L^\infty(B_*)$ for any $t > 0$.

Let

$$\rho \leq \tau := (t \wedge W(x_0,\overline{R}))^{1/\beta} \wedge r \leq r$$

and

$$r' \in [\rho, \tau] \tag{3.25}$$

be a number to be specified later on. Note that under the assumption (3.25),

$$\omega_{x_0}((r')^\beta) \le t \quad \text{and} \quad (r')^\beta \le t. \tag{3.26}$$

Applying Lemma 3.5 (with the case when $q = 2$) for the ball $B_*(x, r') \subset \Omega$ and then using the fact that $\|u\|_{L^2(\Omega)} \le \|g\|_{L^2(\Omega)}$, we have for any $t > 0$

$$\begin{aligned}\operatorname*{osc}_{B_*(x_0,\rho)} u(\cdot,t) &\le C\left(\left(\frac{\rho}{r'}\right)^\gamma \left(\frac{\|u\|_{L^2(\Omega)}}{\sqrt{V_*(x_0,r')}} + \|u\|_{L^\infty(B_*(x_0,r'))}\right) + (r')^\beta \|\partial_t u\|_{L^\infty(B_*(x_0,r'))}\right)\\ &\le C\left(\left(\frac{\rho}{r'}\right)^\gamma \left(\frac{\|g\|_{L^2(\Omega)}}{\sqrt{V_*(x_0,r')}} + \|u\|_{L^\infty(B_*(x_0,r'))}\right) + (r')^\beta \|\partial_t u\|_{L^\infty(B_*(x_0,r'))}\right).\end{aligned}$$

Moreover, we have, by (3.16) and (3.26),

$$\begin{aligned}\|u\|_{L^\infty(B_*(x_0,r'))} \le& \frac{C}{\sqrt{V_*(x_0,(t\wedge W(x_0,\overline{R}))^{1/\beta} \vee r')}}\left(\frac{\omega_{x_0}((r')^\beta)}{\overline{R}} \vee 1\right)^{\frac{\alpha}{2}}\left(\frac{(r')^\beta}{t}+1\right)^{\frac{\alpha}{2\beta_1}}\|g\|_{L^2(\Omega)}\\ \le& \frac{C}{\sqrt{V_*(x_0,r')}}\left(\frac{\omega_{x_0}((r')^\beta)}{\overline{R}} \vee 1\right)^{\frac{\alpha}{2}}\left(\frac{(r')^\beta}{t}+1\right)^{\frac{\alpha}{2\beta_1}}\|g\|_{L^2(\Omega)}\\ \le& \frac{C}{\sqrt{V_*(x_0,r')}}\|g\|_{L^2(\Omega)}.\end{aligned}$$

By (VD$_*$) and (3.25), we have

$$\frac{1}{\sqrt{V_*(x_0,r')}} = \frac{1}{\sqrt{V_*(x_0,\tau)}}\frac{\sqrt{V_*(x_0,\tau)}}{\sqrt{V_*(x_0,r')}} \le \frac{c}{\sqrt{V_*(x_0,\tau)}}\left(\frac{\tau}{r'}\right)^{\alpha_*/2}.$$

By (3.19) (with $U = B_*(x_0, r')$), (3.16) and (3.26), we have

$$\begin{aligned}\|\partial_t u\|_{L^\infty(B_*(x_0,r'))} \le& \frac{C}{\sqrt{V_*(x_0,(t\wedge W(x_0,\overline{R}))^{1/\beta} \vee r')}}\left(\frac{\omega_{x_0}((r')^\beta)}{\overline{R}} \vee 1\right)^{\frac{\alpha}{2}}\left(\frac{(r')^\beta}{t}+1\right)^{\frac{\alpha}{2\beta_1}}\frac{\|g\|_{L^2(\Omega)}}{t}\\ \le& \frac{C}{\sqrt{V_*(x_0,\tau)}}\left(\frac{\omega_{x_0}((r')^\beta)}{\overline{R}} \vee 1\right)^{\frac{\alpha}{2}}\left(\frac{(r')^\beta}{t}+1\right)^{\frac{\alpha}{2\beta_1}}\frac{\|g\|_{L^2(\Omega)}}{t}\\ \le& \frac{C}{\sqrt{V_*(x_0,\tau)}}\frac{\|g\|_{L^2(\Omega)}}{t}.\end{aligned}$$

Combining the above four formulas, we obtain

$$\begin{aligned}\operatorname*{osc}_{B_*(x_0,\rho)} u(\cdot,t) &\le \frac{C}{\sqrt{V_*(x_0,\tau)}}\|g\|_{L^2(\Omega)}\left(\frac{\rho^\gamma \tau^{\alpha_*/2}}{(r')^{\gamma+\alpha_*/2}}\right) + \frac{C}{\sqrt{V_*(x_0,\tau)}}\|g\|_{L^2(\Omega)}\frac{(r')^\beta}{t}\\ &\le \frac{C}{\sqrt{V_*(x_0,\tau)}}\|g\|_{L^2(\Omega)}\left(\frac{\rho^\gamma \tau^{\alpha_*/2}}{(r')^{\gamma+\alpha_*/2}} + \left(\frac{r'}{\tau}\right)^\beta\right).\end{aligned} \tag{3.27}$$

Now choose r' such that

$$\frac{\rho^\gamma \tau^{\alpha_*/2}}{(r')^{\gamma+\alpha_*/2}} = \left(\frac{r'}{\tau}\right)^\beta,$$

that is,

$$r' = \rho^{\frac{\gamma}{\gamma+\beta+\alpha_*/2}}\tau^{\frac{\beta+\alpha_*/2}{\gamma+\beta+\alpha_*/2}}.$$

With this choice of r', we have

$$r' = \rho^{\frac{\gamma}{\gamma+\beta+\alpha_*/2}}\tau^{\frac{\beta+\alpha_*/2}{\gamma+\beta+\alpha_*/2}} \ge \rho^{\frac{\gamma}{\gamma+\beta+\alpha_*/2}}\rho^{\frac{\beta+\alpha_*/2}{\gamma+\beta+\alpha_*/2}} = \rho,$$

$$r' = \rho^{\frac{\gamma}{\gamma+\beta+\alpha_*/2}} \tau^{\frac{\beta+\alpha_*/2}{\gamma+\beta+\alpha_*/2}} \leq \tau^{\frac{\gamma}{\gamma+\beta}} \tau^{\frac{\beta+\alpha_*/2}{\gamma+\beta+\alpha_*/2}} = \tau \leq r,$$

which shows that the assumption (3.26) is fulfilled.

Substituting this r' into (3.27), we obtain

$$\begin{aligned} \operatorname*{osc}_{B_*(x_0,\rho)} u(\cdot,t) &\leq \frac{C}{\sqrt{V_*(x_0,\tau)}} \left(\frac{r'}{\tau}\right)^{\beta} \|g\|_{L^2(\Omega)} \\ &\leq \frac{C}{\sqrt{V_*(x_0,\tau)}} \left(\frac{\rho}{\tau}\right)^{\frac{\gamma\beta}{\gamma+\beta+\alpha_*/2}} \|g\|_{L^2(\Omega)}, \end{aligned}$$

which is exactly (3.22).

Finally, (3.24) follows directly from (3.22) by passing to the limit as $r \to \infty$. □

For any set $U \subset M$ and $r > 0$, denote by U_r^* by the open r-neighborhood of U, that is

$$U_r^* = \bigcup_{x \in U} B_*(x,r). \tag{3.28}$$

The following gives the locally Hölder continuity of the heat semigroup solutions in an open subset.

Lemma 3.11. *Let $u(x,t) = P_t^{\Omega} g(x)$ for $g \in L^2(\Omega)$. Assume that conditions (VD), (RVD), $(\mathrm{Gcap})_W$, $(\mathrm{TJ}_q)_W$ for some $q \in [2,+\infty]$, $(\mathrm{PI})_W$ are all satisfied. Then the following properties are true.*

- (a) *For any $t > 0$, the function $u(\cdot,t)$ has a locally Hölder continuous version $\widetilde{u}(\cdot,t)$ in Ω with the Hölder exponent θ. Moreover, the function $\widetilde{u}(x,t)$ is jointly continuous in $(x,t) \in \Omega \times (0,\infty)$.*
- (b) *For any open subset U of Ω and any $r > 0$ with $U_r^* \subset \Omega$, we have for all $t > 0$ and all $x \in U$, $x' \in M$ with $2d_*(x,x') < (t \wedge W(x,\overline{R}))^{1/\beta} \wedge r$,*

$$\left|\widetilde{u}(x,t) - \widetilde{u}(x',t)\right| \leq \frac{C}{\sqrt{V_*(x,(t \wedge W(x,\overline{R}))^{1/\beta} \wedge r)}} \left(\frac{d_*(x,x')}{(t \wedge W(x,\overline{R}))^{1/\beta} \wedge r}\right)^{\theta} \|g\|_{L^2(\Omega)}. \tag{3.29}$$

Here θ is given by (3.23), and the constant C depends only on constants in the hypotheses. In particular, in the case when $\Omega = M$, the function $\widetilde{u}(x,t)$ is jointly continuous in $(x,t) \in M \times (0,\infty)$ and satisfies for all $x, x' \in M$ and $t > 0$ with $2d_*(x,x') < (t \wedge W(x,\overline{R}))^{1/\beta}$,

$$\left|\widetilde{u}(x,t) - \widetilde{u}(x',t)\right| \leq \frac{C}{\sqrt{V_*(x,(t \wedge W(x,\overline{R}))^{1/\beta})}} \left(\frac{d_*(x,x')}{(t \wedge W(x,\overline{R}))^{1/\beta}}\right)^{\theta} \|g\|_{L^2}. \tag{3.30}$$

Proof. (a) By a standard argument, it follows from (3.22) that for any fixed $t > 0$, $u(\cdot,t)$ has a locally Hölder continuous version $\widetilde{u}(\cdot,t)$.

On the other hand, by (3.20) (with $U = B_*$) and (3.16), we have for any ball $B_* := B_*(x_0,r)$ with $B_* \subset \Omega$ and for all $t > s \geq \tau > 0$,

$$\begin{aligned} \sup_{x \in B_*} |\widetilde{u}(x,t) - \widetilde{u}(x,s)| \leq{} & \frac{C(t-s)}{\sqrt{V_*(x_0,(\tau \wedge W(x_0,\overline{R}))^{1/\beta} \vee r)}} \left(\frac{\omega_{x_0}(r^\beta)}{\overline{R}} \vee 1\right)^{\frac{\alpha}{2}} \\ & \cdot \left(\frac{r^\beta}{\tau} + 1\right)^{\frac{\alpha}{2\beta_1}} \frac{\|g\|_{L^2(\Omega)}}{\tau}, \end{aligned} \tag{3.31}$$

from which, we see that the function $t \mapsto \widetilde{u}(x,t)$ is continuous in $t \in (0,\infty)$ uniformly in $x \in B_*$. Since the function $x \mapsto \widetilde{u}(x,t)$ is continuous in $x \in B_*$, we conclude that $\widetilde{u}(x,t)$ is jointly continuous in $(x,t) \in B_* \times (0,\infty)$, and hence in $(x,t) \in \Omega \times (0,\infty)$ since $B_* \subset \Omega$ is arbitrary.

(b) Note that for all $x \in U$, $B_*(x,r) \subset \Omega$. If

$$\rho < \tau := (t \wedge W(x,\overline{R}))^{1/\beta} \wedge r,$$

we obtain from Lemma 3.10,

$$\operatorname*{osc}_{B_*(x,\rho)} \widetilde{u}(\cdot,t) \le \frac{C}{\sqrt{V_*(x,\tau)}} \left(\frac{\rho}{\tau}\right)^{\theta} \|g\|_{L^2(\Omega)}.$$

For any $x' \in M$ with $2d_*(x,x') < \tau = (t \wedge W(x,\overline{R}))^{1/\beta} \wedge r$, we choose $\varepsilon < \frac{\tau - 2d_*(x,x')}{2}$. Then, taking $\rho = 2d_*(x,x') + \varepsilon$ in the above inequality, we have

$$|\widetilde{u}(x,t) - \widetilde{u}(x',t)| \le \operatorname*{osc}_{B_*(x,\rho)} \widetilde{u}(\cdot,t) \le \frac{C}{\sqrt{V_*(x,\tau)}} \left(\frac{2d_*(x,x')+\varepsilon}{\tau}\right)^{\theta} \|g\|_{L^2(\Omega)}, \tag{3.32}$$

which implies (3.29) by letting $\varepsilon \to \infty$ since ε is arbitrary.

Finally, (3.30) follows from (3.29) by passing to the limit as $r \to \infty$.

□

4. Hölder continuity of the heat kernel

Let Ω be a non-empty open subset of M. Recall that P_t^{Ω} and $p_t^{\Omega}(x,y)$ denote the heat semigroup and the heat kernel of the Dirichlet form $(\mathcal{E},\mathcal{F}(\Omega))$, respectively. In this subsection, we shall show that the heat kernel $p_t^{\Omega}(x,y)$ is locally Hölder continuous. In particular, when $\Omega = M$, the global heat kernel is Hölder continuous.

4.1. Proof of Theorem 1.12.

Lemma 4.1. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Assume that conditions (VD), (RVD), $(\mathrm{Gcap})_W$, $(\mathrm{PI})_W$ and $(\mathrm{TJ}_q)_W$ for some $q \in [2,+\infty]$ are all satisfied. Let Ω be any non-empty open subset of M. Then the Dirichlet heat kernel $p_t^{\Omega}(x,y)$ is locally Hölder continuous. Moreover, for each $t > 0$ and $x \in \Omega$,*

$$p_t^{\Omega}(x,x) \le \frac{C}{V_*(x,(t \wedge W(x,\overline{R}))^{1/\beta})}, \tag{4.1}$$

and, for any non-empty open subset $U \subset \Omega$ and $r > 0$ with $U_r^* \subset \Omega$, and for all $x, x' \in U$, $y \in \Omega$ and $t \ge s > 0$ with $2d_*(x,x') < (t \wedge W(x,\overline{R}))^{1/\beta} \wedge r$,

$$\begin{aligned} \left|p_t^{\Omega}(x,y) - p_s^{\Omega}(x',y)\right| \le &\left(\frac{1}{\sqrt{V_*(x,(t\wedge W(x,\overline{R}))^{1/\beta} \wedge r)}} + \frac{1}{\sqrt{V_*(x',(s\wedge W(x',\overline{R}))^{1/\beta})}}\right) \\ &\cdot \frac{C}{\sqrt{V_*(y,(s \wedge W(y,\overline{R}))^{1/\beta})}} \left(\left(\frac{d_*(x,x')}{(t\wedge W(x,\overline{R}))^{1/\beta} \wedge r}\right)^{\theta} + \frac{t-s}{s}\right), \end{aligned} \tag{4.2}$$

where $\theta \in (0,1)$ is defined in (3.23) and $C > 0$ depends only on the constants in the hypotheses.

In particular, when $\Omega = M$, we have for all $x, x', y \in M$ and $t \ge s > 0$ with $2d_*(x,x') < (t \wedge W(x,\overline{R}))^{1/\beta}$,

$$\begin{aligned} \left|p_t(x,y) - p_s(x',y)\right| \le &\left(\frac{1}{\sqrt{V_*(x,(t\wedge W(x,\overline{R}))^{1/\beta})}} + \frac{1}{\sqrt{V_*(x',(s\wedge W(x',\overline{R}))^{1/\beta})}}\right) \\ &\cdot \frac{C}{\sqrt{V_*(y,(s \wedge W(y,\overline{R}))^{1/\beta})}} \left(\left(\frac{d_*(x,x')}{(t\wedge W(x,\overline{R}))^{1/\beta}}\right)^{\theta} + \frac{t-s}{s}\right). \end{aligned} \tag{4.3}$$

Proof. Fix an open subset U of Ω and fix a number $r > 0$ with $U_r^* \subset \Omega$ (where U_r^* is defined in (3.28)). By Lemma 3.11 and (3.29), for any $g \in L^2(\Omega)$ and $t > 0$, the function $P_t^{\Omega} g$ has a locally

Hölder continuous version, which we denote still by $P_t^\Omega g$, that is, for all $x \in U$, $x' \in M$ and all $t > 0$ with $2d_*(x,x') < (t \wedge W(x,\overline{R}))^{1/\beta} \wedge r$,

$$\left|P_t^\Omega g(x) - P_t^\Omega g(x')\right| \leq \frac{C}{\sqrt{V_*(x,(t\wedge W(x,\overline{R}))^{1/\beta}\wedge r)}}\left(\frac{d_*(x,x')}{(t\wedge W(x,\overline{R}))^{1/\beta}\wedge r}\right)^\theta \|g\|_{L^2(\Omega)}. \tag{4.4}$$

where $C > 0$ is independent of Ω, U, r, t, x, x', g.

By (3.16), we have for all $t > 0$ and all $x \in B_*(x_0,\overline{r})$ with $x_0 \in M$ and $\overline{r} > 0$,

$$\begin{aligned}\left|P_t^\Omega g(x)\right| &\leq \|P_t g\|_{L^\infty(B_*(x_0,\overline{r}))} \\ &\leq \frac{C}{\sqrt{V_*(x_0,(t\wedge W(x_0,\overline{R}))^{1/\beta}\vee\overline{r})}}\left(\frac{\omega_{x_0}(\overline{r}^\beta)}{\overline{R}}\vee 1\right)^{\frac{\alpha}{2}}\left(\frac{\overline{r}^\beta}{t}+1\right)^{\frac{\alpha}{2\beta_1}}\|g\|_{L^2(\Omega)}.\end{aligned} \tag{4.5}$$

On the other hand, for all $t > s > 0$ and all $x \in B_*(x_0,\overline{r})$ with $x_0 \in M$ and $\overline{r} > 0$, by (3.20) (with $U = B_*(x_0,\overline{r})$) and (3.16), we have

$$\begin{aligned}\left|P_t^\Omega g(x) - P_s^\Omega g(x)\right| \leq{}& \frac{C(t-s)}{\sqrt{V_*(x_0,(s\wedge W(x_0,\overline{R}))^{1/\beta}\vee\overline{r})}}\left(\frac{\omega_{x_0}(\overline{r}^\beta)}{\overline{R}}\vee 1\right)^{\frac{\alpha}{2}} \\ &\cdot\left(\frac{\overline{r}^\beta}{s}+1\right)^{\frac{\alpha}{2\beta_1}}\frac{\|g\|_{L^2(\Omega)}}{s}.\end{aligned} \tag{4.6}$$

Since $P_t^\Omega f$ is continuous, using [7, Theorem 4.3 and Remark 4.5], it follows from (4.5) that the followings are true (see also [6, Lemma 5.13, p. 506]):

(1) the Dirichlet heat kernel p_t^Ω exists pointwise for $(x,y,t) \in \Omega\times\Omega\times(0,\infty)$;
(2) both $p_t^\Omega(x,\cdot)$ and $p_t^\Omega(\cdot,x)$ are continuous in Ω for every $t > 0$ and every $x \in \Omega$;
(3) the following inequality holds:

$$\sup_{x\in B_*(x_0,r)} p_t^\Omega(x,x) \leq \frac{C}{\sqrt{V_*(x_0,(t\wedge W(x_0,\overline{R}))^{1/\beta}\vee r)}}\left(\frac{\omega_{x_0}(r^\beta)}{\overline{R}}\vee 1\right)^{\frac{\alpha}{2}}\left(\frac{r^\beta}{t}+1\right)^{\frac{\alpha}{2\beta_1}},$$

which implies (4.1) by taking $x_0 = x$ and $r < (t\wedge W(x,\overline{R}))^{1/\beta}$.

We are to show (4.2). Indeed, fix a point $y \in \Omega$. Setting $g = p_t^\Omega(\cdot,y)$ in (4.4), we obtain from (4.4) and (4.1) that for all $x \in U$, $x' \in M$, and $t > 0$ with $2d_*(x,x') < (t\wedge W(x,\overline{R}))^{1/\beta}\wedge r$,

$$\begin{aligned}\left|p_{2t}^\Omega(x,y) - p_{2t}^\Omega(x',y)\right| \leq{}& \frac{C}{\sqrt{V_*(x,(t\wedge W(x,\overline{R}))^{1/\beta}\wedge r)}}\frac{1}{\sqrt{V_*(y,(t\wedge W(y,\overline{R}))^{1/\beta})}} \\ &\cdot\left(\frac{d_*(x,x')}{(t\wedge W(x,\overline{R}))^{1/\beta}\wedge r}\right)^\theta.\end{aligned}$$

Renaming $2t$ by t in the above inequality and using (VD$_*$), we obtain

$$\begin{aligned}\left|p_t^\Omega(x,y) - p_t^\Omega(x',y)\right| \leq{}& \frac{C}{\sqrt{V_*(x,(t\wedge W(x,\overline{R}))^{1/\beta}\wedge r)}}\frac{1}{\sqrt{V_*(y,(t\wedge W(y,\overline{R}))^{1/\beta})}} \\ &\cdot\left(\frac{d_*(x,x')}{(t\wedge W(x,\overline{R}))^{1/\beta}\wedge r}\right)^\theta.\end{aligned}$$

Moreover, for $x' \in U$, $y \in \Omega$ and $t \geq s > 0$, applying (4.6) with t replaced by $t - \frac{s}{2}$, s by $\frac{s}{2}$, x_0 by x', x by x', $\overline{r}$ by $(s\wedge W(x',\overline{R}))^{1/\beta}$ and $g = p_{s/2}^\Omega(\cdot,y)$, and using (VD$_*$), we obtain

$$\left|p_t^\Omega(x',y) - p_s^\Omega(x',y)\right| \leq \frac{C(t-s)}{\sqrt{V_*(x',(s\wedge W(x',\overline{R}))^{1/\beta})}}\left(\frac{\omega_{x'}(W(x',\overline{R}))}{\overline{R}}\vee 1\right)^{\frac{\alpha}{2}}$$

$$\cdot \left(\frac{s\wedge W(x',\overline{R})}{s}+1\right)^{\frac{\alpha}{2\beta_1}} \frac{\|p^{\Omega}_{s/2}(\cdot,y)\|_{L^2(\Omega)}}{s}$$
$$\leq \frac{C}{\sqrt{V_*(x',(s\wedge W(x',\overline{R}))^{1/\beta})}}\frac{1}{\sqrt{V_*(y,(s\wedge W(y,\overline{R}))^{1/\beta})}}\frac{t-s}{s}.$$

Adding up the above two inequalities, we obtain (4.2).

Finally, (4.3) follows from (4.2) by passing to the limit as $r\to\infty$. □

Proof of Theorem 1.12. We need only to rewrite the inequalities in Lemma 4.1 in the original metric d. And we only prove (1.23) since both (1.21) and (1.22) can be proved similarly.

We can choose $\eta\in(0,1)$ small enough such that by (2.2) and the left inequality in (1.15), if $d(x,x')<\eta(\omega_x(t)\wedge\overline{R})$, then,

$$\frac{d_*(x,x')}{(t\wedge W(x,\overline{R}))^{1/\beta}} = \left(\frac{d_*(x,x')^{\beta}}{W(x,\omega_x(t)\wedge\overline{R})}\right)^{1/\beta} \leq c\left(\frac{W(x,d(x,x'))}{W(x,\omega_x(t)\wedge\overline{R})}\right)^{1/\beta}$$
$$\leq c'\left(\frac{d(x,x')}{\omega_x(t)\wedge\overline{R}}\right)^{\beta_1/\beta} < c''\eta^{\beta_1/\beta}<\frac{1}{2}.$$

On the other hand, by (2.3) with $r=(t\wedge W(x,\overline{R}))^{1/\beta}$, (1.16) and (VD), we obtain that

$$V_*(x,(t\wedge W(x,\overline{R}))^{1/\beta})\geq cV(x,\omega_x(L^{-\beta}(t\wedge W(x,\overline{R}))))\geq c'V(x,\omega_x(t)\wedge\overline{R}).$$

Similarly, we also have

$$V_*(x',(s\wedge W(x,\overline{R}))^{1/\beta})\geq c'V(x',\omega_x(s)\wedge\overline{R}),$$
$$V_*(y,(s\wedge W(y,\overline{R}))^{1/\beta})\geq c'V(y,\omega_y(s)\wedge\overline{R}).$$

Plugging the above four inequalities into (4.3), we obtain for all $x,x',y\in M$, $t\geq s>0$ with $d(x,x')<\eta(\omega_x(t)\wedge\overline{R})$,

$$\left|p_t(x,y)-p_s(x',y)\right| \leq \left(\frac{1}{\sqrt{V(x,\omega_x(t)\wedge\overline{R})}}+\frac{1}{\sqrt{V(x',\omega_{x'}(s)\wedge\overline{R})}}\right)$$
$$\cdot\frac{C}{\sqrt{V(y,\omega_y(s)\wedge\overline{R})}}\left(\left(\frac{d(x,x')}{\omega_x(t)\wedge\overline{R}}\right)^{\theta\beta_1/\beta}+\frac{t-s}{s}\right),$$

which is (1.23) with $\lambda=\theta\beta_1/\beta=\frac{\gamma\beta_1}{\gamma+\beta+\alpha_*/2}$.

In the case when $d(x,x')\geq\eta(\omega_x(t)\wedge\overline{R})$, (1.23) follows directly from (1.21) and semigroup property. □

4.2. **Proof of Theorem 1.5.** One can not directly apply Theorem 1.12 to obtain Theorem 1.5, because the condition $(\mathrm{TJ})_\beta$ in the hypothesis of Theorem 1.5 is much weaker than $(\mathrm{TJ}_q)_W$ for some $q\in[2,+\infty]$ in the hypothesis of Theorem 1.12. However, since most part of the proof of Theorem 1.5 is the same to that of Theorem 1.12, we only mention the differences between their proofs.

We again start from the oscillation lemma for harmonic functions. Indeed, it follows from [8, Lemma 5.19, p. 727] that under conditions (V), $(\mathrm{TJ})_\beta$, $(\mathrm{PI})_\beta$ and $(\mathrm{Gcap})_\beta$, we also obtain the oscillation inequality (3.2). That is, there exist three positive constants $\sigma\in(0,1)$ and γ, C such that for any ball $B_r:=B(x_0,r)$ with $r>0$ and any function $u\in\mathcal{F}'\cap L^\infty$, which is harmonic in B_r, we have for any $\rho\in(0,r]$,

$$\operatorname*{osc}_{B_\rho} u\leq C\left(\frac{\rho}{r}\right)^{\gamma}\left(r^{\beta}\operatorname*{esup}_{x\in\frac{3}{4}B_r}\int_{B_r^c}|u(y)|J(x,dy)+\|u\|_{L^\infty(B_r)}\right). \tag{4.7}$$

Using this inequality, we have the following lemma.

Lemma 4.2. *Assume that conditions* (V), $(\mathrm{TJ})_\beta$, $(\mathrm{PI})_\beta$ *and* $(\mathrm{Gcap})_\beta$ *are all satisfied. Let* $C_0 \geq 1$ *and* Ω *be any open subset of* M *containing a ball* $B := B(x_0, r)$ *of radius* $r > 0$*. If the function* $u \in \mathcal{F}(\Omega) \cap L^\infty$ *solves weakly the equation* (3.5) *for* $f \in L^2(\Omega) \cap L^\infty(B)$*, then for any* $0 < \rho \leq r$

$$\operatorname*{osc}_{B(x_0,\rho)} u \leq C \left(\frac{\rho}{r}\right)^\gamma \|u\|_{L^\infty(\Omega)} + Cr^\beta \|f\|_{L^\infty(B)}, \tag{4.8}$$

where γ *is the constant from* (4.7) *and* C *is independent of* B, u, Ω, ρ, f.

Proof. Firstly, it follows from [8, Proposition 4.7, Eq. (4.33) and Proposition 4.13] that under conditions (V) and $(\mathrm{PI})_\beta$, the condition $(\mathrm{E}^*_\leq)$ holds for the case when $W(x,r) = r^\beta$, that is, there exist two constants δ, $C > 0$ such that for all balls $B := B(x_0, r)$ of radius $r > 0$,

$$\left\|G^B 1\right\|_{L^\infty} \leq Cr^\beta.$$

Then, one can repeat the proof of Lemma 3.5 (for $q = 1$) by using the above inequality, (4.7) and $(\mathrm{TJ})_\beta$ instead of $(\mathrm{E}^*_\leq)$, (OSL_*) and (TJ^*_1) respectively. Note that (3.10) (for $q = 1$ and $q' = \infty$) is still true under condition $(\mathrm{TJ})_\beta$ even if the jump kernel $J(x,y)$ does not exist. Hence, following the proof of Lemma 3.5 (for $q = 1$), we can obtain

$$\operatorname*{osc}_{B(x_0,\rho)} u \leq C \left(\frac{\rho}{r}\right)^\gamma \left(\|u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(B)}\right) + Cr^\beta \|f\|_{L^\infty(B)},$$

which implies (4.8) since $B \subset \Omega$. □

We also need the ultra-contractivity of the semigroup P_t (a similar inequality to (3.13)). Indeed, one can following the proof of [1, Lemmas 5.2 and 5.5][2] to obtain that, under conditions (V) and $(\mathrm{PI})_\beta$,

$$\|P_t\|_{L^2 \to L^\infty} \leq Ct^{-\alpha/(2\beta)}, \quad t > 0, \tag{4.9}$$

where the constant C depends only on the constants in conditions (V) and $(\mathrm{PI})_\beta$ (see also [2, Theorem 2.1] and [12, Lemma 3.7]). Moreover, by the symmetry and duality of P_t, we have

$$\|P_t\|_{L^1 \to L^2} = \|P_t\|_{L^2 \to L^\infty} \leq Ct^{-\alpha/(2\beta)}, \quad t > 0. \tag{4.10}$$

Proof of Theorem 1.5. Firstly, we need to apply (4.9) and Lemma 4.2 instead of (3.14) and Lemma 3.5 respectively to obtain a similar oscillation result to Lemma 3.10.

Indeed, let $\Omega \subset M$ be a non-empty open set, and let $u(x,t) = P^\Omega_t g(x) \in L^2 \cap L^\infty$ for $t > 0$ and $g \in L^2(\Omega) \cap L^\infty$. Let $B := B(x_0, r)$ with $x_0 \in M$ and $r > 0$ such that

$$B \subset \Omega.$$

For any $t > 0$, note that the function $u(\cdot, t)$ satisfies weakly

$$\partial_t u(\cdot, t) = -\mathcal{L}^\Omega u(\cdot, t),$$

where $\partial_t u(\cdot, t)$ is Fréchet derivative of $u(\cdot, t)$ with respect to t in $L^2(\Omega)$. By (4.9) and (3.19) (with $U = M$) , we have $\partial_t u(\cdot, t) \in L^\infty(M)$ for any $t > 0$.

Fix

$$\rho \leq \tau := t^{1/\beta} \wedge r \leq r$$

and let

$$r' \in [\rho, \tau] \tag{4.11}$$

be a number to be specified later on. Applying (4.8), for the ball $B(x_0, r') \subset \Omega$, we have for any $t > 0$

$$\operatorname*{osc}_{B(x_0,\rho)} u(\cdot, t) \leq C \left(\frac{\rho}{r'}\right)^\gamma \|u\|_{L^\infty(\Omega)} + (r')^\beta \|\partial_t u\|_{L^\infty(\Omega)}.$$

Moreover, we have, by (4.9) and (4.10),

$$\|u\|_{L^\infty(\Omega)} \leq \|P_{t/2}\|_{L^1 \to L^2} \cdot \|P_{t/2}\|_{L^2 \to L^\infty} \cdot \|g\|_{L^1(\Omega)} \leq Ct^{-\alpha/\beta} \|g\|_{L^1(\Omega)}.$$

[2]Although these two lemmas are proved on ultrametric spaces, their proofs still work for general metric spaces

By (3.19) (with $U = \Omega$) and (4.10), we have

$$\begin{aligned}\|\partial_t u\|_{L^\infty(\Omega)} &\le \frac{4}{t}\|P_{t/2}\|_{L^2\to L^\infty}\cdot\|P_{t/4}\|_{L^1\to L^2}\|g\|_{L^1(\Omega)}\\ &\le Ct^{-\alpha/\beta-1}\|g\|_{L^1(\Omega)}.\end{aligned}$$

Combining the above three formulas, we obtain

$$\begin{aligned}\operatorname*{osc}_{B(x_0,\rho)} u(\cdot,t) &\le Ct^{-\alpha/\beta}\|g\|_{L^1(\Omega)}\left(\left(\frac{\rho}{r'}\right)^\gamma + \frac{(r')^\beta}{t}\right)\\ &\le Ct^{-\alpha/\beta}\|g\|_{L^1(\Omega)}\left(\left(\frac{\rho}{r'}\right)^\gamma + \left(\frac{r'}{\tau}\right)^\beta\right). \qquad (4.12)\end{aligned}$$

Now choose r' such that

$$\left(\frac{\rho}{r'}\right)^\gamma = \left(\frac{r'}{\tau}\right)^\beta,$$

that is,

$$r' = \rho^{\frac{\gamma}{\gamma+\beta}}\tau^{\frac{\beta}{\gamma+\beta}}.$$

With this choice of r', we have

$$\begin{aligned}r' &= \rho^{\frac{\gamma}{\gamma+\beta}}\tau^{\frac{\beta}{\gamma+\beta}} \ge \rho^{\frac{\gamma}{\gamma+\beta}}\rho^{\frac{\beta}{\gamma+\beta}} = \rho,\\ r' &= \rho^{\frac{\gamma}{\gamma+\beta}}\tau^{\frac{\beta}{\gamma+\beta}} \le \tau^{\frac{\gamma}{\gamma+\beta}}\tau^{\frac{\beta+}{\gamma+\beta}} = \tau \le r,\end{aligned}$$

which shows that the assumption (4.11) is fulfilled.

Substituting this r' into (4.12), we obtain

$$\operatorname*{osc}_{B(x_0,\rho)} P_t^\Omega g \le Ct^{-\alpha/\beta}\left(\frac{r'}{\tau}\right)^\beta \|g\|_{L^1(\Omega)} \le Ct^{-\alpha/\beta}\left(\frac{\rho}{\tau}\right)^{\frac{\gamma\beta}{\gamma+\beta}}\|g\|_{L^1(\Omega)}.$$

Using the above two inequalities instead of (3.22) and following the proof of Lemma 3.11, we obtain that for all $t > 0$, each $P_t^\Omega g$ has a locally Hölder continuous version, still denoted by $P_t^\Omega g$. Moreover, the followings are true.

(1). For any $t > 0$, the function $P_t^\Omega g$ is jointly continuous in $(x,t) \in \Omega\times(0,\infty)$.

(2). For any open subset U of Ω and any $r > 0$ with $U_r \subset \Omega$, we have for all $t > 0$ and all $x \in U$, $x' \in M$ with $2d(x,x') < t^{1/\beta}\wedge r$,

$$\left|P_t^\Omega g(x) - P_t^\Omega g(x')\right| \le Ct^{-\alpha/\beta}\left(\frac{d(x,x')}{t^{1/\beta}\wedge r}\right)^{\frac{\gamma\beta}{\gamma+\beta}}\|g\|_{L^1(\Omega)}. \qquad (4.13)$$

Moreover, using (3.20) with $U = M$ and (4.10), we have

$$\begin{aligned}\|P_t^\Omega g - P_s^\Omega g\|_{L^\infty} &\le \frac{4(t-s)}{s}\|P_{s/2}^\Omega\|_{L^2\to L^\infty}\|P_{s/4}^\Omega\|_{L^1\to L^2}\|g\|_{L^1(\Omega)}\\ &\le \frac{C(t-s)}{s}\cdot\frac{1}{s^{\alpha/\beta}}\cdot\|g\|_{L^1(\Omega)}.\end{aligned}$$

Finally, using the above two inequalities, and the following ultra-contractivity of P_t^Ω (by (4.9) and (4.10)):

$$\|P_t\|_{L^1\to L^\infty} \le \|P_{t/2}\|_{L^1\to L^2}\cdot\|P_{t/2}\|_{L^2\to L^\infty} \le Ct^{-\alpha/\beta}, \quad t>0,$$

and following the proof of Lemma 4.1, we can finish the proof of Theorem 1.5 with

$$\lambda := \frac{\gamma\beta}{\gamma+\beta}.$$

□

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