Hierarchical Schrödinger type operators: the case of locally bounded potentials *

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Abstract

The goal of this paper is the spectral analysis of the Schrödinger type operator $H = L + V$, the perturbation of the Taibleson-Vladimirov multiplier $L = \mathfrak{D}^{\alpha}$ by a potential V. Assuming that V belonges to a certain class of potentials we show that the discrete part of the spectrum of H may contain negative energies, it also appears in the spectral gaps of L. We will split the spectrum of H in two parts: high energy part containing eigenvalues which correspond to the eigenfunctions located on the support of the potential V, and low energy part which lies in the spectrum of certain bounded Schrödinger type operator acting on the Dyson hierarchical lattice.

We pay special attention to the class of sparse potentials. In this case we obtain precise spectral asymptotics for H provided the sequence of distances between locations tends to infinity fast enough.

We also obtain certain results concerning localization theory for H subject to (non-ergodic) random potential V. Examples illustrate our approach.

Contents

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1 Introduction

The spectral theory of nested fractals similar to the Sierpinski gasket, i.e. the spectral theory of the corresponding Laplacians, is well understood. It has several important features: Cantor-like structure of the essential spectrum and, as result, the large number of spectral gaps, presence of infinite number of eigenvalues each of which has infinite multiplicity and compactly supported eigenstates, non-regularly varying at infinity heat kernels which contain an oscillating in $\log t$ scale terms etc., see [16], [12] and [7].

The spectral properties mentioned above occur in the very precise form for the Taibleson-Vladimirov Laplacian $\mathfrak{D}^{\alpha}$, the operator of fractional derivative of order α. This operator can be introduced in several different forms (say, as L^2-multiplier in the p-adic analysis setting, see [39]) but we select the geometric approach [13], [28], [27], [3], [4], [5] and [6].

1.1 The Dyson hierarchical model

Let us fix an integer $p \geq 2$ and consider the family $\{\Pi_r : r \in \mathbb{Z}\}$ of partitions of the set $X = [0, +\infty[$ such that each Π_r consists of all intervals $I = [kp^r, (k+1)p^r[$, $k = 0, 1, \ldots$. We call r the rank of the partition Π_r (respectively, the rank of the interval $I \in \Pi_r$). Each interval of rank r is the union of p disjoint intervals of rank $(r-1)$. Each point $x \in X$ belongs to a certain interval $I_r(x)$ of rank r, and intersection of all intervals $I_r(x)$ is $\{x\}$.

Definition 1.1 *Let $\mathcal{B}$ be the family of all intervals $[kp^r, (k+1)p^r[$.* The hierarchical distance $\mathrm{d}(x, y)$ *is defined as the length $|I|$ of the minimal interval $I \in \mathcal{B}$ which contains both x and y.*

It is easy to see that the function $(x, y) \to \mathrm{d}(x, y)$ is non-degenerate, symmetric and for arbitrary x, y and z,

$$\mathrm{d}(x, y) \leq \max\{\mathrm{d}(x, z), \mathrm{d}(z, y)\},$$

i.e. $\mathrm{d}(x, y)$ is an *ultrametric* on X. It has the following properties:

- The ultrametric $\mathrm{d}(x, y)$ majorizes the Euclidean metric $|x - y|$ but these two metrics are not equivalent. Indeed, by the very definition, $\mathrm{d}(x, y) \geq |x - y|$ for all $x, y \in X$ whereas $\mathrm{d}(1 - \varepsilon, 1) = p$ for all $0 < \varepsilon < 1$.

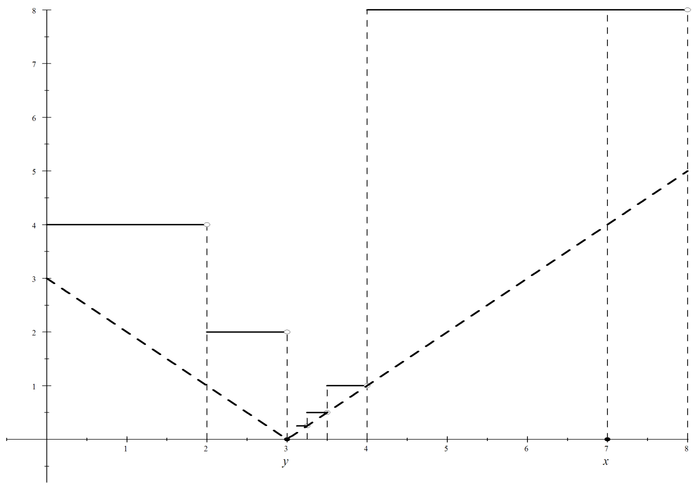

Figure 1: Comparison of two metrics: $d(x,y) \geq |x-y|$

- (X, d) is a complete locally compact non-compact and separable metric space. In this metric space the set of all open balls coincides with the set of all intervals $I \in \mathcal{B}$. Next property says that (X, d) is a totally disconnected metric space.[1]

- Each open ball B in (X, d) is a closed compact set, each point x of B can be regarded as its center, any two balls either do not intersect or one is a subset of another etc.

- The Borel σ-algebra generated by the ultrametric $\mathrm{d}(x,y)$ coincides with the classical Borel σ-algebra generated by the Euclidean metric.

Definition 1.2 *Let us fix a parameter* $\kappa \in]0,1[$. The hierarchical Laplacian L *we introduce following [30] as a linear combination of "elementary Laplacians"*

$$(Lf)(x) = \sum_{r=-\infty}^{+\infty} (1-\kappa)\kappa^{r-1} \left(f(x) - \frac{1}{|I_r(x)|} \int\limits_{I_r(x)} f dm \right). \tag{1.1}$$

The series in (1.1) diverges in general but it is finite and belongs to $L^2(X,m)$ for any $f \in L^2(X,m)$ which takes constant values on intervals of any fixed rank r.

The operator L admits a complete system of compactly supported eigenfunctions. Indeed, let I be an interval of rank r, and $I_1, I_2, ..., I_p$ be its subintervals of rank $r-1$. Set $\lambda(r) = \kappa^{r-1}$ and consider p functions

$$f_{I_i} = \frac{1_{I_i}}{|I_i|} - \frac{1_I}{|I|},\ i = 1, 2, ..., p.$$

Each function f_{I_i} belongs to the domain of the operator L and satisfies the equation

$$Lf_{I_i} = \lambda(r) f_{I_i}.$$

[1] In particular, (X, d) is homeomorphic to the punctured Cantor set $\{0,1\}^{\aleph_0} \backslash \{o\}$, see a survey on totally disconnected metric spaces in [5, Section 1].

The eigenspace $\mathcal{H}(I) = \mathrm{span}\{f_{I_i}\}$ has dimension $p-1$ because $\sum_i f_{I_i} = 0$. The eigenspaces $\mathcal{H}(I)$ and $\mathcal{H}(I')$ are orthogonal provided $I \neq I'$, and

$$L^2(X,m) = \bigoplus_{r\in\mathbb{Z}} \left(\bigoplus_{I\in\Pi_r} \mathcal{H}(I) \right).$$

In particular, L is an essentially self-adjoint operator having a pure point spectrum. Clearly each eigenvalue $\lambda(r) = \kappa^{r-1}$ has infinite multiplicity, whence $Spec(L)$ coincides with its essential part $Spec_{ess}(L)$.

We shell see below that writing $\kappa = p^{-\alpha}$ the operator L can be identified with the Taibleson-Vladimirov operator $\mathfrak{D}^\alpha$, the operator of fractional derivative of order α defined as L^2-multiplier in the p-adic analysis setting [41], [21].

According to [4] the operator L can be represented as a hypersingular integral operator acting in $L^2(0,\infty)$,

$$Lf(x) = \int_0^\infty (f(x) - f(y))\, J(x,y)dy$$

where

$$J(x,y) = \frac{\kappa^{-1}-1}{1-\kappa/p} \cdot \frac{1}{\mathrm{d}(x,y)^{1+\alpha}}.$$

The Markov semigroup $(e^{-tL})_{t>0}$ is symmetric and admits a continuous heat kernel $p(t,x,y)$. [2] The function $p(t,x,y)$ can be estimated as follows

$$p(t,x,y) \asymp \frac{t}{[t^{1/\alpha} + \mathrm{d}(x,y)]^{1+\alpha}}. \quad [3] \tag{1.2}$$

The function $p(t) := p(t,x,x)$ does not depend on x. By [30, Proposition 2.3], it can be represented in the form

$$p(t) = t^{-1/\alpha}\mathcal{A}(\log_p t), \tag{1.3}$$

where $\mathcal{A}(\tau)$ is a continuous non-constant α-periodic function, see also an extended version of this result in [7]. In particular, in contrary to the classical case (symmetric stable densities), the function $t \to p(t)$ does not vary regularly.

There are already several publications on the hierarchical Laplacian acting on a general ultrametric measure space (X,d,m), see [2], [1], [28], [27], [3], [4], [5], [6]. By the general theory developed in [3], [4] and [5], any hierarchical Laplacian L acts in $L^2(X,m)$ as essentially self-adjoint operator having a pure point spectrum. This operator can be represented in the form

$$Lf(x) = \int_X (f(x) - f(y))J(x,y)dm(y). \tag{1.4}$$

[2]The function $(x,y) \to p(t,x,y)$ is continuous (and even locally Lipschitz continuous) w.r.t. the ultrametric $\mathrm{d}(x,y)$ but it is discontinuous w.r.t. the Euclidean metric $|x-y|$

[3] We write $f \asymp g$ if the ratio f/g is bounded from above and from below by positive constants for a specified range of variables. We write $f \sim g$ if the ratio f/g tends to identity.

The Markov semigroup $(e^{-tL})_{t>0}$ admits with respect to m a continuous transition density $p(t,x,y)$. It turns out that in terms of certain (intrinsically related to L) ultrametric d_*,

$$J(x,y) = \int_0^{1/d_*(x,y)} N(x,\tau)d\tau, \tag{1.5}$$

$$p(t,x,y) = t \int_0^{1/d_*(x,y)} N(x,\tau)\exp(-t\tau)d\tau, \tag{1.6}$$

where $N(x,\tau)$ is the so-called *spectral function* related to L (will be defined later).

1.2 Outline

Let us describe the main body of the paper. In Section 2 we introduce the notion of homogeneous hierarchical Laplacian L and list its basic properties: the spectrum of the operator L is pure point, all eigenvalues of L have infinite multiplicity and compactly supported eigenfunctions, the heat kernel $p(t,x,y)$ exists and it is a continuous function having certain asymptotic properties etc. As a special example we consider the case $X=\mathbb{Q}_p$, the ring of p-adic numbers endowed with its standard ultrametric $d(x,y)=|x-y|_p$ and the normed Haar measure m. The hierarchical Laplacian L in our example coincides with the Taibleson-Vladimirov operator $\mathfrak{D}^\alpha$, the operator of fractional derivative of order α, see [39], [41], and [21]. The most complete source for the basic definitions and facts related to the p-adic analysis is [20] and [38].

The Schrödinger type operator $H=L+V$ with hierarchical Laplacian L was studied in [14], [28], [30], [31], [10], [25], [26] (the hierarchical lattice of Dyson) and in [41], [40], [21] (the field of p-adic numbers). In the next sections we consider the Schrödinger type operator acting on a homogeneous ultrametric space X. We assume that the potential V is of the form $V=\sum\sigma_i 1_{B_i}$, where B_i are balls which belong to a fixed horocycle $\mathcal{H}$ (i.e. all B_i have the same diameter). The main aim here is to study the set $Spec(H)$. Under certain assumptions on V (e.g. $V(x)\to 0$ at infinity ϖ etc.) we conclude that the set $Spec(H)$ is pure point (with possibly infinite number of limit points). We split the set $Spec(H)$ in two disjoint parts: the first part consists of the point $\lambda=0$ and the eigenvalues of the operator L which correspond to the horocycle $\mathcal{H}$ (with compactly supported eigenfunctions) and the second part is the closure of a countably infinite set Ξ of eigenvalues of the operator H (with non-compactly supported eigenfunctions). In the case of sparse potential V, i.e. when $d(B_i,B_j)\to\infty$ fast enough we specify the structure of the set Ξ. In this connection we would like to mention here pioneering works of S. Molchanov [28], D. Krutikov [23], [24], and N. Kochubei [21].

In the last section we consider the potential V of the form $V=\sum\sigma_i(\omega)1_{B_i}$, where $\sigma_i(\omega)$, $\omega\in(\Omega,\mathcal{F},P)$, are i.i.d. random variables, and embark on the localization theory. More precisely, we show that if the sequence of (non-random) distances $d(B_i,B_j)$ between locations tend to infinity fast enough then the spectrum of H is pure point for P-a.a. $\omega\in\Omega$.

In the case when X is discrete, L is the Dyson Laplacian, B_i are singletons and V is ergodic the *localization theorem* appeared first in the paper of Molchanov [28] ($\sigma_i(\omega)$

are Cauchy random variables) and later (under more general assumptions on $\sigma_i(\omega)$) in the papers of Kritchevski [26] and [25]. The proof of this theorem is based on the self-similarity of H. This approach is not applicable to the case of (random) sparse potentials.

The proof of the localization theorem for (random) sparse potentials presented in this paper is based on the abstract form of Simon-Wolff criterion [37] for pure point spectrum, technique of fractional moments, decoupling lemma of Molchanov and Borel-Cantelli type arguments, see [1], [27].

2 Preliminaries

2.1 Homogeneous ultrametric space

Let (X,d) be an ultrametric space. Recall that a metric d is called an *ultrametric* if it satisfies the ultrametric inequality

$$d(x,y)\leq \max\{d(x,z),d(z,y)\},$$

that is stronger than the usual triangle inequality. Henceforth we assume that the ultrametric space (X,d) is separable, non-compact and *proper*, that is, each d-ball is a compact set.

For any $x\in X$ and $r\geq 0$ let $B_r(x)=\{y\in X:d(x,y)\leq r\}$ be a closed ball. The basic consequence of the ultrametric property is that $B_r(x)$ is an open set for any $r>0$. Moreover, each point $y\in B_r(x)$ can be regarded as its center, any two balls of the same radius are either disjoint or identical etc. See a survey part in paper [5, Section 1] and references therein.

To any ultrametric space (X,d) one can associate in a standard fashion a tree $\mathcal{T}$. The vertices of the tree are metric balls, the boundary $\partial\mathcal{T}$ can be identified with the one-point compactification $X\cup\{\varpi\}$. We refer to [5, Section 1] for a treatment of the association between ultrametric space (X,d) and the tree $\mathcal{T}$ of its metric balls.

Definition 2.1 *Let m be a Radon measure on X. The triple (X,d,m) we call a* homogeneous ultrametric measure space *if the group of isometries of (X,d) acts transitively on X and if the measure m is invariant w.r.t. the action of this group.*

The following remarkable result is due to M. Del Muto and A. Figà-Talamanca, see paper [11, Section 2].

Theorem 2.2 *Any homogeneous ultrametric measure space (X,d,m) can be identified with certain locally compact Abelian group $\mathfrak{G}$ equipped with a translation invariant ultrametric $\mathfrak{d}$ and the Haar measure $\mathfrak{m}$.*

For example, the set $X=[0,+\infty[$ equipped with the ultrametric structure generated by p-adic intervals can be identified with $\mathbb{Q}_p$, the ring of p-adic numbers.

The identification in Theorem 2.2 is not unique. One possible way to define the identification is to choose the sequence $\{a_n\}$ of forward degrees associated with the tree of balls $\mathcal{T}$. This sequence is two-sided if X is non-compact and perfect, it is one-sided if X is compact and perfect, or if X is discrete. In the 1st case we identify X with Ω_a, the ring of a-adic numbers, in the 2nd case with $\Delta_a \subset \Omega_a$, the ring of a-adic integers, and in the 3rd case with the discrete group $[\Omega_a : \Delta_a] \simeq \mathbb{Z}_{a_1} \oplus \mathbb{Z}_{a_2} \oplus ...$, the week sum of cyclic groups $\mathbb{Z}_{a_n}$. We refer the reader to [17] for the comprehensive treatment of special groups Ω_a, Δ_a and $\mathbb{Z}_{a_1} \oplus \mathbb{Z}_{a_2} \oplus ....$

2.2 Homogeneous hierarchical Laplacian

Let (X, d, m) be a homogeneous ultrametric measure space. Let $\mathcal{B}$ be the set of all open balls, $\mathcal{B}(x) \subset \mathcal{B}$ the set of all balls centred at x, and $C : \mathcal{B} \to (0, \infty)$ a function satisfying the following conditions

1. $m(A) = m(B) \Longleftrightarrow C(A) = C(B)$,
2. $\lambda(B) := \sum_{T \in \mathcal{B}:\ B \subseteq T} C(T) < \infty$,
3. $\sup_{B \in \mathcal{B}(x)} \lambda(B) = \infty$ for any non-isolated x.

The class of functions $C(B)$ satisfying these conditions is reach enough. For example, let us fix $\alpha > 0$ and for any two nearing neighboring balls $B \subset B'$ set

$$C(B) = m(B)^{-\alpha} - m(B')^{-\alpha},$$

then

$$\lambda(B) = m(B)^{-\alpha}.$$

Definition 2.3 *Let $C : \mathcal{B} \to (0, \infty)$ be as above, we define* the homogeneous hierarchical Laplacian L *pointwise as*

$$Lf(x) := \sum_{B \in \mathcal{B}(x)} C(B) \left(f(x) - \frac{1}{m(B)} \int_B f dm \right). \tag{2.1}$$

Let $\mathcal{D}$ be the set of all compactly supported locally constant functions. [4] The series in (2.1) diverges in general but for $f \in \mathcal{D}$ it belongs to $C_\infty(X) \cap L^2(X, m)$. Moreover, the operator L admits a complete in $L^2(X, m)$ system of eigenfunctions

$$f_B = \frac{\mathbf{1}_B}{m(B)} - \frac{\mathbf{1}_{B'}}{m(B')}, \tag{2.2}$$

where the couple $B \subset B'$ runs over all nearest neighboring balls. The eigenvalue corresponding to f_B is the number $\lambda(B')$ defined at condition 2.

In particular, we conclude that $L : \mathcal{D} \to L^2(X, m)$ is an essentially self-adjoint operator. Each eigenvalue $\lambda(B)$ has infinite multiplicity so $Spec(L)$ is pure point and coincides with its essential part.

The intrinsic ultrametric $d_*(x, y)$ is defined as follows

$$d_*(x, y) := \begin{cases} 0 & \text{when} \quad x = y \\ 1/\lambda(x \curlywedge y) & \text{when} \quad x \neq y \end{cases}, \tag{2.3}$$

[4] The set $\mathcal{D}$ is a dense subset in each of the Banach spaces $C_\infty(X)$ and $L^p(X, m)$, $1 \le p < \infty$.

where $x \curlywedge y$ is the minimal ball containing both x and y. In particular, for any open ball B, we have

$$\lambda(B) = \frac{1}{\operatorname{diam}_*(B)}. \tag{2.4}$$

The spectral function $\tau \to N(\tau)$, see equation (1.5), is defined as a left-continuous step-function having jumps at the points $\lambda(B)$, and

$$N(\lambda(B)) = 1/m(B).$$

The volume function $V(r)$ is defined by setting $V(r) = m(B)$ where the ball B has d_*-radius r. It is easy to see that

$$N(\tau) = 1/V(1/\tau). \tag{2.5}$$

The Markov semigroup $P_t = e^{-tL}$ admits a continuous density $p(t,x,y)$ with respect to m, we call it *the heat kernel.* The function $p(t,x,y)$ can be represented in the form given by equation (1.6). Respectively, the Markov generator L admits the representation given by equations (1.4) and (1.5).

The resolvent operator $(L + \lambda \mathrm{I})^{-1}, \lambda > 0$, admits a continuous strictly positive kernel $\mathcal{R}(\lambda, x, y)$ with respect to the measure m. The resolvent operator is well defined for $\lambda = 0$, i.e. the Markov semigroup $(P_t)_{t>0}$ is transient, if and only if for some (equivalently, for all) $x \in X$ the function $\tau \to 1/V(\tau)$ is integrable at ∞. Its kernel $\mathcal{R}(x,y) := \mathcal{R}(0,x,y)$, called also *the Green function*, is of the form

$$\mathcal{R}(x,y) = \int_r^{+\infty} \frac{d\tau}{V(\tau)},\ r = d_*(x,y). \tag{2.6}$$

Under certain Tauberian conditions equation (2.6) takes the form

$$\mathcal{R}(x,y) \asymp \frac{r}{V(r)},\ r = d_*(x,y). \tag{2.7}$$

For all these facts we refer the reader to [3], [4], and [5].

2.3 Subordination

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing homeomorphism. For any two nearest neighboring balls $B \subset B'$ we define

$$C(B) = \Phi\left(1/m(B)\right) - \Phi\left(1/m(B')\right). \tag{2.8}$$

The following properties hold true:

(i) $\lambda(B) = \Phi\left(1/m(B)\right)$. In particular, the hierarchical Laplacians L_Φ and L_{Id} are related by the equation $L_\Phi = \Phi(L_{Id})$.[5]

(ii) $d_*(x,y) = 1/\Phi\left(1/m(x \curlywedge y)\right)$,

[5] In the case $\Phi(\tau)$ is a Bernstein function the association $L_\Phi = \Phi(L_{Id})$ has been studied in the well-known Bochner's subordination theory [15].

(iii) $V(r) \le 1/\Phi^{-1}(1/r)$. Moreover, $V(r) \asymp 1/\Phi^{-1}(1/r)$ whenever both Φ and Φ^{-1} are doubling and $m(B') \le cm(B)$ for some $c > 0$ and all neighboring balls $B \subset B'$. In turn, this yields the heat kernel estimates

$$p(t,x,y) \asymp t \cdot \min\left\{\frac{1}{t}\Phi^{-1}\left(\frac{1}{t}\right), \frac{1}{m(x \curlywedge y)}\Phi\left(\frac{1}{m(x \curlywedge y)}\right)\right\}, \tag{2.9}$$

2.4 L^2-multipliers

As a special case of the general construction consider $X = \mathbb{Q}_p$, the ring of p-adic numbers equipped with its standard ultrametric $d(x,y) = |x-y|_p$. Remined that the ultrametric space $(\mathbb{Q}_p, d)$ and the ultrametric space $([0,\infty), \mathrm{d})$ with non-Euclidean d (the Dyson's model) are isometric.

Let m be the normed Haar measure on the Abelian group $\mathbb{Q}_p$ and $\mathcal{F} : f \to \widehat{f}$ the Fourier transform acting in $L^2(\mathbb{Q}_p, m)$. It is known, see [38], [41], [21], that $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ is a bijection.

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing homeomorphism. The self-adjoint operator $\Phi(\mathfrak{D})$ we define as L^2-multiplier, that is,

$$\widehat{\Phi(\mathfrak{D})f}(\xi) = \Phi(|\xi|_p)\widehat{f}(\xi), \quad \xi \in \mathbb{Q}_p. \tag{2.10}$$

By [4, Theorem 3.1], $\Phi(\mathfrak{D})$ is a homogeneous hierarchical Laplacian. The eigenvalues $\lambda(B)$ of the operator $\Phi(\mathfrak{D})$ are of the form

$$\lambda(B) = \Phi\left(\frac{p}{m(B)}\right). \tag{2.11}$$

Let $p(t,x,y)$ be the heat kernel associated with the operator $\Phi(\mathfrak{D})$. Assume that both Φ and Φ^{-1} are doubling, then equation (2.9) applies. Since for any $x, y \in \mathbb{Q}_p$, $m(x \curlywedge y) = |x-y|_p$ we obtain

$$p(t,x,y) \asymp t \cdot \min\left\{\frac{1}{t}\Phi^{-1}\left(\frac{1}{t}\right), \frac{1}{|x-y|_p}\Phi\left(\frac{1}{|x-y|_p}\right)\right\}, \tag{2.12}$$

The Taibleson-Vladimirov operator $\mathfrak{D}^\alpha$ is L^2-multiplier, it can be written as a hyper-singular integral operator

$$\mathfrak{D}^\alpha f(x) = \frac{1}{\Gamma_p(-\alpha)}\int_{\mathbb{Q}_p} \frac{f(y)-f(x)}{|y-x|_p^{1+\alpha}} dm(y), \tag{2.13}$$

where $\Gamma_p(z) = (1-p^{z-1})(1-p^{-z})^{-1}$ is the p-adic Gamma-function, see [41, VIII.2].

The heat kernel $p_\alpha(t,x,y)$ of the operator $\mathfrak{D}^\alpha$ admits two-sided bounds

$$p_\alpha(t,x,y) \asymp \frac{t}{(t^{1/\alpha} + |x-y|_p)^{1+\alpha}}. \tag{2.14}$$

In particular, the Markov semigroup $(e^{-t\mathfrak{D}^\alpha})_{t>0}$ is transient if and only if $\alpha < 1$. In the transient case the Green function $\mathcal{R}_\alpha(x,y)$ is of the form

$$\mathcal{R}_\alpha(x,y) = \frac{1}{\Gamma_p(\alpha)} \frac{1}{|x-y|_p^{1-\alpha}}. \tag{2.15}$$

For all facts listed above we refer the reader to [3], [4] and [5].

3 Schrödinger type operators

3.1 Preliminary results

Let (X, d, m) be a homogeneous ultrametric measure space and L a homogeneous hierarchical Laplacian acting on (X, d, m). Identifying (X, d) with a locally compact Abelian group (say, $X = \mathbb{Q}_p$) one can regard $-L$ as a translation invariant isotropic Markov generator. By (1.4), the operator $L : \mathcal{D} \to L^2(X, m)$ is of the form

$$Lf(x) = \int\limits_{\mathbb{Q}_p} (f(x) - f(y))J(x - y)dm(y), \tag{3.16}$$

or equivalently, in terms of the Fourier transform $\mathcal{F} : g \to \widehat{g}$,

$$\widehat{Lf}(\theta) = \widehat{L}(\theta) \cdot \widehat{f}(\theta),\ \theta \in \mathbb{Q}_p, \tag{3.17}$$

where $\widehat{L}(\theta)$ is a *negative definite function* [9]. By the Lévy-Khinchin formula,

$$\widehat{L}(\theta) = \int\limits_{\mathbb{Q}_p} [1 - \operatorname{Re}\langle h, \theta\rangle] J(h)dm(h). \tag{3.18}$$

Let V be a real measurable function. Consider the Schrödinger type operator

$$Hu = Lu + V \cdot u,\ u \in \mathcal{D}. \tag{3.19}$$

Our goal is to show that under certain mild conditions on V one may associate a self-adjoint operator H with the equation (3.19).

Theorem 3.1 *Assume that V is locally bounded. Then the following is true:*

(i) The operator $H = L + V$ is essentially self-adjoint.[6]

(ii) Assume that $V(x) \to +\infty$ as $x \to \varpi$. Then the operator H has a compact resolvent. Consequently, the spectrum of H is discrete.

(iii) Assume that $V(x) \to 0$ as $x \to \varpi$. Then the essential spectrum of H coincides with the spectrum of L. Thus, the spectrum of H is pure point and the negative part of the spectrum consists of isolated eigenvalues of finite multiplicity.

Proof. (i) As the potential V is locally bounded $H : \mathcal{D} \to L^2(X, m)$ is a well-defined symmetric operator. Let us choose an open ball O which contains the neutral element and write equation (3.16) in the form

$$\begin{aligned} Lf(x) &= \left(\int\limits_{O} + \int\limits_{O^c} \right) [f(x) - f(x + y)]J(y)dm(y) \\ &= L_O f(x) + L_{O^c} f(x). \end{aligned}$$

We have $Hf = L_O f + L_{O^c} f + V f$, where the operator V is the operator of multiplication by the function $V(x)$. The operator $L_{O^c} f = J(O^c)(f - a * f)$, where

[6] Recall that, for the classical Schrödinger operator $H = -\Delta + V$ in $\mathbb{R}^n$, this statement is *not* true, unless V satisfies a certain lower bound, see [8, Chapter II, Theorem 1.1 and Example 1.1].

$a(y) = J(y)1_{O^c}(y)/J(O^c)$, is a bounded symmetric operator in $L^2(X,m)$ (as $f \to a * f$ is the operator of convolution with probability measure $a(y)dm(y)$) and thus does not influence self-adjointness. As L_O is minus Lévy generator it is essentially self-adjoint (one more way to make this conclusion is that the matrix of the operator L_O is diagonal in the basis $\{f_B\}$ of eigenfunctions of the operator L, see [22]).

For any ball B which belongs to the same horocycle $\mathcal{H}$ as O we denote $\mathfrak{H}_B$ the subspace of $L^2(X,m)$ which consists of all functions f having support in B. Since O is a subgroup of the Abelian group X and each ball $B \in \mathcal{H}$ is a coset (i.e. belongs to the quotient group $[X : O]$), we conclude that $\mathfrak{H}_B$ is an invariant subspace of the symmetric operator $H_O = L_O + V$. Moreover, by symmetry $\mathfrak{H}_B$ reduces H_O.

The ultrametric space X can be covered by a sequence of non-intersecting balls B_n (recall that due to the ultrametric property two balls of the same diameter either coincide or do not intersect). This leads to the orthogonal decomposition

$$L^2(X,m) = \bigoplus_n \mathfrak{H}_{B_n}$$

where each $\mathfrak{H}_{B_n}$ reduces H_O. The restriction of the essentially self-adjoint operator L_O to its invariant subspace $\mathfrak{H}_{B_n}$ is an essentially self-adjoint operator, while the restriction of the operator V is bounded. Thus H_O is essentially self-adjoint as orthogonal sum of essentially self-adjoint operators $H_{O,n}$, the restriction of H_O to $\mathfrak{H}_{B_n}$.

(ii) The proof is similar to the one for the Schrödinger operators given in [41, Theorem X.3]; the main tools are boundedness from below of the operator H and the Riesz-Rellich compactness criteria for subsets of $L^2(X,m)$.

(iii) Let us show that the operator V is $L-$*compact.* Then, by [18, Theorem IV.5.35], the essential spectrums of the operators H and L coincide. Recall that $L-$compactness means that if a sequence $\{u_n\}$ is such that both $\{u_n\}$ and $\{Lu_n\}$ are bounded then there exists a subsequence $\{u'_n\} \subset \{u_n\}$ such that the sequence $\{Vu'_n\}$ converges.

1. Denote $v_n = Lu_n + u_n$. By assumption the sequence $\{v_n\}$ is bounded and $u_n = \mathcal{R}_1 v_n = r_1 * v_n$. It follows that the quantity

$$\left(\int |u_n(x+h) - u_n(x)|^2 \, dm(x)\right)^{1/2} \leq \|v_n\|_{L^2} \int |r_1(z+h) - r_1(z)| \, dm(z)$$

tends to zero uniformly in n as h tends to the neutral element. Thus, the sequence $\{u_n\}$ consists of equicontinuous on the whole in $L^2(X,m)$ functions. The same is true for the sequence $\{Vu_n\}$. Indeed, for any ball B which contains the neutral element we write

$$\left(\int |V(x+h)u_n(x+h) - V(x)u_n(x)|^2 \, dm(x)\right)^{1/2} \leq I + II + III,$$

where

$$I = \|V\|_{L^\infty} \left(\int |u_n(x+h) - u_n(x)|^2 \, dm(x)\right)^{1/2},$$

$$II = \|u_n\|_{L^2} \left(\int_B |V(x+h) - V(x)|^2 \, dm(x)\right)^{1/2},$$

$$III = \|u_n\|_{L^2} \sup_{x \in B^c} |V(x+h) - V(x)| \, .$$

Clearly I, II and III tend to zero uniformly in n as h tends to the neutral element and $B \nearrow X$.

2. The sequence $\{Vu_n\}$ consists of functions with equicontinuous $L^2(X,m)$ integrals at infinity. Indeed, for any ball B which contains the neutral element we have

$$\int\limits_{B^c} |Vu_n(x)|^2 \, dm(x) \leq \|u_n\|_{L^2} \sup_{x \in B^c} |V(x)| \to 0$$

uniformly in n as $B \nearrow X$.

Thus, the sequence $\{Vu_n\}$ is bounded in $L^2(X,m)$, consists of equicontinuous on whole in $L^2(X,m)$ functions with equicontinuous $L^2(X,m)$ integrals at infinity. By the Riesz-Kolmogorov criterion of compactness in $L^2(X,m)$, the set $\{Vu_n\}$ is compact, whence it contains a convergent subsequence $\{Vu'_n\}$, as claimed. ■

In the case when the ultrametric measure space (X,d,m) is *countably infinite* the statement (ii) of Theorem 3.1 can be complemented as follows.

Theorem 3.2 *Assume that* (X,d,m) *is* countably infinite. *Then the following statements are equivalent:*

(i) *The operator* H *has a discrete spectrum.*

(ii) $|V(x)|$ *tend to infinity as* $x \to \varpi$.

Proof. $(ii) \Longrightarrow (i)$: Since X is discrete L is a bounded symmetric operator, let us set $d := \|L\|$. Suppose that $|V(x)|$ tend to infinity as $x \to \varpi$. Then for every given interval $I = [a,b]$ and its neighborhood $I' = [a-d-1, b+d+1]$ there exist a finite set A of points x such that $V(x) \in I'$. Let us choose $v \notin I'$ and define the operator $H' = L + V'$ where

$$V'(x) := \begin{cases} V(x) & \text{if} \quad x \notin A \\ v & \text{if} \quad x \in A \end{cases}.$$

The resolvent of the operator $V : u(x) \to V(x)u(x)$ is analytic inside of I' and, as a result, the resolvent of H' is analytic inside of I. Indeed, it is straightforward to show that

$$\left\| L(V' - \lambda \mathrm{I})^{-1} \right\| = \left\| (V' - \lambda \mathrm{I})^{-1} L \right\| \leq \frac{d}{d+1} < 1,$$

for any $\lambda \in I$. It follows that the operator

$$H' - \lambda \mathrm{I} = (V' - \lambda \mathrm{I}) \left(E + L(V' - \lambda \mathrm{I})^{-1} \right)$$

is invertible. This in turn implies that the operator H' has no spectrum inside the interval I. But the difference $H - H'$ is an operator of finite rank. Hence the operator H has (in the same interval I) not more than finite number of eigenvalues, see Lemma 3.12 below. Thus we have already proved that the spectrum of H is discrete.

$(i) \Longrightarrow (ii)$: Suppose that the operator H has a discrete spectrum. Then clearly the spectrum of H^2 is also discrete. Let $E_1 \leq E_2 \leq \cdots$ be the eigenvalues of H^2. Then by Courant's $\min - \max$ principle

$$E_n = \min_{\psi_1, \ldots, \psi_n} \max\{(\psi, H^2\psi) : \psi \in span(\psi_1, \ldots, \psi_n), \|\psi\| = 1\}. \tag{3.20}$$

Assume that $|V(x)|$ does not tend to $+\infty$ as $x \to \varpi$. Then there exists a sequence $\{x_n\} \subset X$ such that $|V(x_n)| \leq C$ for some $C > 0$ and all $n \geq 1$. It follows that

$$(\psi, H^2\psi) \leq 2(d^2 + C^2),\ \forall \psi \in span(\delta_{x_1}, \delta_{x_2}, \delta_{x_3}, ...), \|\psi\| = 1. \tag{3.21}$$

Equations (3.20) and (3.21) imply that the interval $[0, 2(d^2 + C^2)]$ contains at list one limit point of the sequence $\{E_n\}$, i.e. the essential spectrum of H^2 (equivalently of H) is not empty. This fact contradicts the discreetness of the spectrum of H^2 (or H). This proves the second part of the theorem. ■

The class $\mathcal{K}$ In the continuous case the situation is not so obvious. Let us consider a class class $\mathcal{K}$ of potentials V of the form

$$V = \sum_{B \in \mathcal{H}} \sigma(B) 1_B,$$

where $\mathcal{H}$ is any fixed horocycle in the tree of balls. For a potential $V \in \mathcal{K}$ let us select the following Hilbert subspaces of $L^2(X, m)$:

- $\mathcal{L}_+ = \mathrm{span}\{1_B : B \in \mathcal{H}\}$
- $\mathcal{L}_B = \mathrm{span}\{f_T : T \subsetneq B\}$
- $\mathcal{L}_- = L^2(X, m) \ominus \mathcal{L}_+ = \bigoplus_{B \in \mathcal{H}} \mathcal{L}_B$

The following three lemmas can be proved by inspection.

Lemma 3.3 *The linear spaces $\mathcal{L}_+$, $\mathcal{L}_B$ and $\mathcal{L}_-$ are invariant subspaces for both operators H and L. Let H_+, H_B and H_-(resp. L_+, L_B and L_-) be the restriction of the operator H (resp. L) to $\mathcal{L}_+$, $\mathcal{L}_B$ and $\mathcal{L}_-$ respectively. The following properties hold true:*

(i) $H = H_+ \oplus H_-$,

(ii) $H_B = L_B + \sigma(B)$,

(iii) $H_- = \bigoplus_{B \in \mathcal{H}} (L_B + \sigma(B))$.

Remind that $Lf_B = \lambda(B')f_B$ for any open ball B. As B converges to a singleton $\lambda(B') \to +\infty$ whence L_B has discrete spectrum. By the homogeneity property $Spec(L_A)$ is the same for all $A \in \mathcal{H}$. Let us set

- $\mathfrak{S}_{\mathcal{H}} := Spec(L_A)$,
- $\mathfrak{R}_V :=Range(V)$.

Lemma 3.4 *In the introduced notation*

$$Spec(H_-) = \overline{\mathfrak{S}_{\mathcal{H}} + \mathfrak{R}_V}.$$

In particular, the operator H_- has a pure point (not necessary discrete) spectrum.

Let us choose in each ball $B \in \mathcal{H}$ an element a_B and consider a discrete ultrametric space (X', m', d') with $X' = \{a_B : B \in \mathcal{H}\}$ induced by (X, m, d).

Lemma 3.5 *The operator L_+ can be identified with certain hierarchical Laplacian L' acting on (X', m', d'), respectively the operator H_+ can be identified with certain Schrödinger type operator $H' = L' + V'$ with potential $V' = \sum_{a \in X'} V(a)\delta_a$.*

Theorem 3.6 *For a potential $V \in \mathcal{K}$ the statements (i) and (ii) of Theorem 3.2 are related by the implication $(i) \Longrightarrow (ii)$. The inverse implication $(ii) \Longrightarrow (i)$ holds true if and only if the set $\mathfrak{S}_{\mathcal{H}} + \mathfrak{R}_V$ has no accumulating points.*

Proof. If we assume that $Spec(H)$ is discrete, then the operator H_+ (whence the operator H') has a discrete spectrum. Applying Theorem 3.2 we conclude that $|V(x)| \to +\infty$, i.e. $(i) \Longrightarrow (ii)$ as claimed.

If the sequence $\{\sigma(B) : B \in \mathcal{H}\}$ contains a subsequence $\sigma(B_k) \to -\infty$ then it may well happen that the set $Spec(H_-) = \overline{\mathfrak{S}_{\mathcal{H}} + \mathfrak{R}_V}$ will contain a number of accumulating points, i.e. $Spec(H)$ in this case is not discrete. In particular, $(ii) \Longrightarrow (i)$ if and only if the set $\mathfrak{S}_{\mathcal{H}} + \mathfrak{R}_V$ has no accumulating points. ■

3.2 Rank one perturbations

In this section we assume that the homogeneous ultrametric measure space (X, d, m) is *countably infinite*. In this case X can be identified with a countable Abelian group G equipped with an increasing sequence $\{G_n\}_{n \in \mathbb{N}}$ of finite subgroups such that $\cap G_n = \{0\}$ and $\cup G_n = G$. Each ball in G is a set of the form $g + G_n$ for some g and n.

As an example one can consider the group $G = \mathbb{Z}(p_1) \oplus \mathbb{Z}(p_2) \oplus \ldots$, the weak sum of cyclic groups, equipped with the sequence of its subgroups

$$G_n = \mathbb{Z}(p_1) \oplus \mathbb{Z}(p_2) \oplus \ldots \oplus \mathbb{Z}(p_n) \oplus \{0\} \oplus \ldots.$$

Let L be a homogeneous hierarchical Laplacian. We study spectral properties of the Schrödinger type operator $H = L + V$ with potential $V(x) = -\sigma\delta_a(x)$, $\sigma > 0$. Clearly H can be written in the form

$$Hf(x) = Lf(x) - \sigma(f, \delta_a)\delta_a(x),$$

that is, H can be regarded as a rank one perturbation of the operator L. In this connection let us recall an abstract form of the Simon-Wolff theorem [37, Theorems 2 and 2'] about pure point spectrum of rank one perturbations.

The Simon-Wolff criterion Let A be a self-adjoint operator with simple spectrum on a Hilbert space $\mathcal{H}$, and let φ be a cyclic vector for A, that is, $\{(A-\lambda)^{-1}\varphi \mid \operatorname{Im}\lambda > 0\}$ is a total set for $\mathcal{H}$. By the spectral theorem, $\mathcal{H}$ is unitary equivalent to $L^2(\mathbb{R}, \mu_0)$ in such a way that A is multiplication by x with cyclic vector $\varphi \equiv 1$. Here μ_0 is the spectral measure of φ for A. Let $H = A + \sigma(\varphi, \cdot)\varphi$ be a rank one perturbation of the operator A. Set

$$F(x) := \int (x-y)^{-2} d\mu_0(y) = \lim_{\epsilon \to 0} \left\| (A - (x + i\epsilon)\mathrm{I})^{-1}\varphi \right\|^2.$$

Theorem 3.7 *Fix an open interval* $]a,b[$. *The following are equivalent:*
(*i*) *For a.e.* σ, H *has only pure point spectrum in* $]a,b[$.
(*ii*) *For a.e.* $x \in]a,b[$, $F(x) < \infty$.

In general, if $\mathcal{H}_0$ is the closed subspace generated by vectors $\{(A-\lambda \mathrm{I})^{-1}\varphi \,|\mathrm{Im}\,\lambda > 0\}$, then its orthogonal complement $(\mathcal{H}_0)^{\perp}$ is an invariant space for H and $H = A$ on $(\mathcal{H}_0)^{\perp}$. Thus, the extension from the cyclic to general case is clear.

The function $\varphi = \delta_a$ is not a cyclic vector for L because the operator L has many compactly supported eigenfunctions ϕ having support outside of a. Indeed, for any such ϕ, for all $\lambda \in \mathbb{C}$ with $\mathrm{Im}\,\lambda > 0$ and for some eigenvalue λ_k we will have

$$((L-\lambda \mathrm{I})^{-1}\delta_a, \phi) = (\delta_a, (L-\overline{\lambda}\mathrm{I})^{-1}\phi) = (\delta_a, (\lambda_k - \overline{\lambda})^{-1}\phi) = 0.$$

We use the Krein type identity below to show that the spectrum of the operator $H = L - \sigma\delta_a$ is pure point for all σ. Let $\psi(x) = \mathcal{R}(\lambda, x, y)$ be the solution of the equation

$$L\psi(x) - \lambda\psi(x) = \delta_y(x).$$

Let $\psi_V(x) = \mathcal{R}_V(\lambda, x, y)$ be the solution of the equation

$$H\psi_V(x) - \lambda\psi_V(x) = \delta_y(x).$$

Notice that L and H are symmetric operators whence both $(x,y) \to \mathcal{R}(\lambda, x, y)$ and $(x,y) \to \mathcal{R}_V(\lambda, x, y)$ are symmetric functions.

Theorem 3.8 *In the notation introduced above*

$$\mathcal{R}_V(\lambda, x, y) = \mathcal{R}(\lambda, x, y) + \frac{\sigma\mathcal{R}(\lambda, x, a)\mathcal{R}(\lambda, a, y)}{1 - \sigma\mathcal{R}(\lambda, a, a)}, \tag{3.22}$$

$$\mathcal{R}_V(\lambda, a, y) = \frac{\mathcal{R}(\lambda, a, y)}{1 - \sigma\mathcal{R}(\lambda, a, a)} \tag{3.23}$$

and

$$\mathcal{R}_V(\lambda, a, a) = \frac{\mathcal{R}(\lambda, a, a)}{1 - \sigma\mathcal{R}(\lambda, a, a)}. \tag{3.24}$$

Proof. We have

$$\begin{aligned} L\psi_V(x) - \lambda\psi_V(x) &= \delta_y(x) + \sigma\delta_a(x)\psi_V(x) \\ &= \delta_y(x) + \sigma\delta_a(x)\psi_V(a). \end{aligned}$$

It follows that

$$\psi_V(x) = \mathcal{R}(\lambda, x, y) + \sigma\psi_V(a)\mathcal{R}(\lambda, x, a). \tag{3.25}$$

Setting $x = a$ in the above equation we obtain

$$\psi_V(a) = \mathcal{R}(\lambda, a, y) + \sigma\psi_V(a)\mathcal{R}(\lambda, a, a)$$

or

$$\psi_V(a)(1 - \sigma\mathcal{R}(\lambda, a, a)) = \mathcal{R}(\lambda, a, y).$$

Since $\psi_V(a) = \mathcal{R}_V(\lambda, a, y)$ we obtain equation (3.24). In turn, equations (3.24) and (3.25) imply (3.22) and (3.23). ■

Theorem 3.9 *The operator $H = L - \sigma\delta_a$ has a pure point spectrum which consists of at most one negative eigenvalue and countably many positive eigenvalues with accumulating point* 0.

The operator H has precisely one negative eigenvalue λ_-^σ if and only if $\sigma > 0$ and one of the following two conditions holds: (i) the semigroup $(e^{-tL})_{t>0}$ is recurrent, (ii) the semigroup $(e^{-tL})_{t>0}$ is transient and $\mathcal{R}(0, a, a) > 1/\sigma$. If it is the case, then Spec(H) consists of numbers

$$\lambda_-^\sigma < 0 < ... < \lambda_{k+1} < \lambda_k^\sigma < \lambda_k < ... < \lambda_2 < \lambda_1^\sigma < \lambda_1.$$

Otherwise Spec(H) consists of numbers

$$0 < ... < \lambda_{k+1} < \lambda_k^\sigma < \lambda_k < ... < \lambda_2 < \lambda_1^\sigma < \lambda_1.$$

If $\sigma < 0$, then Spec(H) consists of numbers

$$0 < ... < \lambda_{k+1} < \lambda_k^\sigma < \lambda_k < ... < \lambda_2 < \lambda_1^\sigma < \lambda_1 < \lambda_+^\sigma.$$

The eigenvalues λ_k are at the same time eigenvalues of the operator L. All λ_k have infinite multiplicity and compactly supported eigenfunctions, the eigenfunctions of the operator L, whose supports do not contain a.

The eigenvalue λ_k^σ (resp. λ_-^σ, λ_+^σ) is the unique solution of the equation

$$\mathcal{R}(\lambda, a, a) = 1/\sigma$$

in the interval $]\lambda_{k+1}, \lambda_k[$ (resp. $]-\infty, 0[$, $]\lambda_1, +\infty[$). Each λ_k^σ (resp. λ_-^σ, λ_+^σ) has multiplicity one and non-compactly supported eigenfunction $\psi_k(x) = \mathcal{R}(\lambda_k^\sigma, x, a)$ (resp. $\psi_-(x) = \mathcal{R}(\lambda_-^\sigma, x, a)$, $\psi_+(x) = \mathcal{R}(\lambda_+^\sigma, x, a)$).

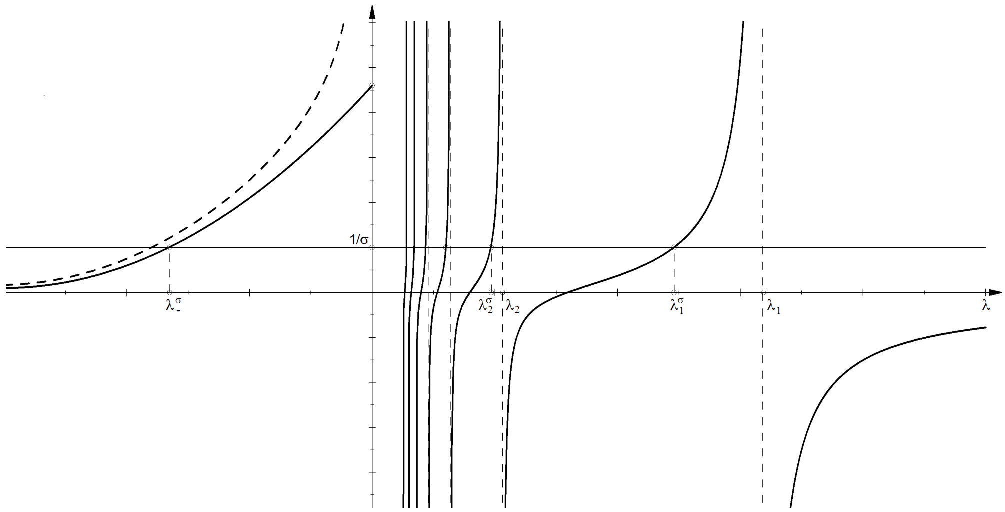

Figure 2: The roots $\{\lambda_*^\sigma\}$ of the equation $\mathcal{R}(\lambda, a, a) = 1/\sigma$. The dashed graph corresponds to a recurrent case, the solid graph – to the transient case.

Proof. Let $\Upsilon(X)$ be the tree of balls associated with the ultrametric space (X,d). Consider in $\Upsilon(X)$ the infinite geodesic path from a to ϖ : $\{a\}=B_0\subsetneq B_1\subsetneq ...\subsetneq B_k\subsetneq ...$. The series below converges uniformly and in L^2,

$$\delta_a=\left(\frac{1_{B_0}}{m(B_0)}-\frac{1_{B_1}}{m(B_1)}\right)+\left(\frac{1_{B_1}}{m(B_1)}-\frac{1_{B_2}}{m(B_2)}\right)+...=\sum_{k=0}^{\infty}f_{B_k}. \tag{3.26}$$

Notice that all f_{B_k} are eigenfunctions of the operator L, i.e. $Lf_{B_k}=\lambda(B_{k+1})f_{B_k}=\lambda_{k+1}f_{B_k}$. By definition $\mathcal{R}(\lambda,x,y)=(L-\lambda)^{-1}\delta_y(x)$ whence we obtain

$$\begin{aligned}\mathcal{R}(\lambda,a,a)&=\frac{1}{\lambda_1-\lambda}f_{B_0}(a)+\frac{1}{\lambda_2-\lambda}f_{B_1}(a)+...\\&=\frac{1}{\lambda_1-\lambda}\left(\frac{1}{m(B_0)}-\frac{1}{m(B_1)}\right)\\&+\frac{1}{\lambda_2-\lambda}\left(\frac{1}{m(B_1)}-\frac{1}{m(B_2)}\right)+...\ ,\end{aligned}$$

or in the final form

$$\mathcal{R}(\lambda,a,a)=\sum_{k=1}^{\infty}\frac{A_k}{\lambda_k-\lambda},\quad A_k=\left(\frac{1}{m(B_{k-1})}-\frac{1}{m(B_k)}\right). \tag{3.27}$$

Since $\lambda\to\mathcal{R}(\lambda,a,a)$ is an increasing function, the equation

$$1-\sigma\mathcal{R}(\lambda,a,a)=0,\quad \sigma\neq 0, \tag{3.28}$$

has precisely one solution λ_k^σ lying in each open interval $]\lambda_{k+1},\lambda_k[$,

$$\lambda_{k+1}<\lambda_k^\sigma<\lambda_k,\quad k=1,2,...\ .$$

Claim 1 All numbers λ_k^σ are eigenvalues of the operator H. Indeed, the function $\psi(x)=\mathcal{R}(\lambda,x,a)$ with $\lambda=\lambda_k^\sigma$ satisfies the equation

$$\begin{aligned}H\psi(x)-\lambda\psi(x)&=L\psi(x)-\lambda\psi(x)-\sigma\delta_a(x)\psi(x)\\&=L\psi(x)-\lambda\psi(x)-\sigma\delta_a(x)\psi(a)\\&=L\psi(x)-\lambda\psi(x)-\delta_a(x)=0.\end{aligned}$$

Claim 2 All numbers λ_k are eigenvalues of the operator H. Indeed, for any ball B which does not contain a but belongs to the horocycle $\mathcal{H}_{k-1}$ we have

$$Hf_B=Lf_B=\lambda_k f_B.$$

When $\sigma>0$ there may exist one more eigenvalue $\lambda_-^\sigma<0$, a solution of the equation (3.28). Indeed, $\lambda\to\mathcal{R}(\lambda,a,a)$ is an increasing function, continuous on the interval $]-\infty,0]$. Since $\mathcal{R}(\lambda,a,a)\to 0$ as $\lambda\to-\infty$ and $\mathcal{R}(\lambda,a,a)\to\mathcal{R}(0,a,a)\leq+\infty$ as $\lambda\to-0$, equation (3.28) has unique solution $\lambda=\lambda_-^\sigma<0$ in the cases (i) and (ii).The proof of the theorem is finished. ■

Example 3.10 *The Dyson's Laplacian. Consider the set* $X=\{0,1,2,...\}$ *equipped with the counting measure* m *and with the set of partitions* $\{\Pi_r : r=0,1,...\}$ *each of which consists of all rank* r *intervals* $I_r=\{x\in X : kp^r\leq x<(k+1)p^r\}$. *The set of partitions* $\{\Pi_r\}$ *generates the ultrametric structure on* X *and the hierarchical Laplacian*

$$\mathrm{D}^{\alpha}f(x)=\sum_{r=1}^{+\infty}(1-\kappa)\kappa^{r-1}\left(f(x)-\frac{1}{m(I_r(x))}\int\limits_{I_r(x)}fdm\right),\ \kappa=p^{-\alpha},$$

where the sum is taken over all rank r *intervals* $I_r(x)$ *containing* x.

The operator D^{α} *admits a complete system of compactly supported eigenfunctions. Indeed, let* I *be an interval of rank* r, *and* $I_1,I_2,...,I_p$ *be its subintervals of rank* $r-1$. *Let us consider* p *functions*

$$f_{I_i}=\frac{1_{I_i}}{m(I_i)}-\frac{1_I}{m(I)},\ i=1,2,...,p.$$

Each function f_{I_i} *belongs to the domain of the operator* D^{α} *and*

$$\mathrm{D}^{\alpha}f_{I_i}=\kappa^{r-1}f_{I_i}.$$

When I *runs over the set all* p*-adic intervals the set of eigenfunctions* f_{I_i} *forms a complete system in* $L^2(X,m)$. *In particular,* D^{α} *is essentially self-adjoint operator having pure point spectrum*

$$Spec(\mathrm{D}^{\alpha})=\{0\}\cup\{\kappa^{r-1}:r\in\mathbb{N}\}.$$

Clearly each eigenvalue $\lambda_r=\kappa^{r-1}$ *has infinite multiplicity. Let us compute the value* $\mathcal{R}(\lambda):=\mathcal{R}(\lambda,0,0)$ *of the resolvent kernel for* D^{α}. *By equation (3.27), we have*

$$\mathcal{R}(\lambda)=\sum_{k\geq 1}\frac{A_k}{\lambda_k-\lambda}=(p-1)\sum_{k\geq 1}\frac{1}{p^k(\lambda_k-\lambda)}.$$

In particular, $\mathcal{R}(0)=+\infty$ *if and only if* $\alpha\geq 1$, *otherwise*

$$\mathcal{R}(0)=\frac{p-1}{p}\sum_{k\geq 0}\frac{1}{p^{k(1-\alpha)}}=\frac{p-1}{p-p^{\alpha}}.$$

Consider the operator $H=\mathrm{D}^{\alpha}-\sigma\delta_0$, $\sigma>0$. *Let us compute the number* $Neg(H)$ *of negative eigenvalues of the operator* H *counted with their multiplicity. By Theorem 3.9, the operator* H *has at most one negative eigenvalue. It has exactly one negative eigenvalue if and only if either* $\alpha\geq 1$ *or* $0<\alpha<1$ *and* $\sigma>(p-p^{\alpha})(p-1)^{-1}$. *If we denote the set of pairs* (α,σ) *which satisfy the above conditions by* Ω *and by* $\Omega_0=\mathbb{R}^2_+\backslash\Omega$ *its complement, we obtain*

$$Neg(H)=\begin{cases}1 & \textit{if } (\alpha,\sigma)\in\Omega\\ 0 & \textit{if } (\alpha,\sigma)\in\Omega_0\end{cases}$$

which is shown on the picture below.

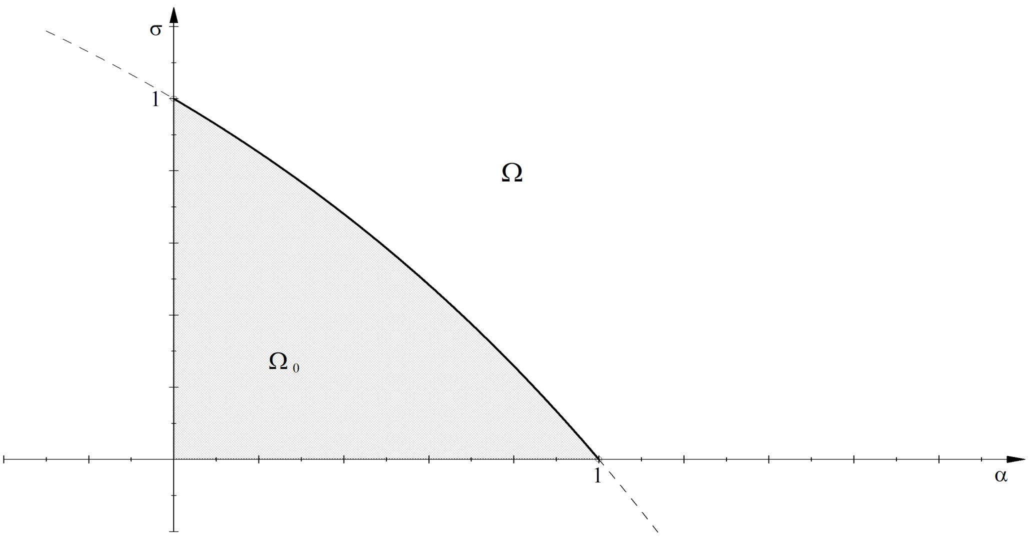

Figure 3: Sets Ω_0 and Ω

3.3 Finite rank perturbations

As in the previous section the ultrametric measure space (X,d,m) is *countably infinite and homogeneous. For convenience, we assume that* $m(B) = \mathrm{diam}(B)$ *for any non-singleton ball* B.

Let L be a homogeneous hierarchical Laplacian. We study spectral properties of the Schrödinger type operator $H = L + V$ with potential $V(x) = -\sum_{i=1}^{N} \sigma_i \delta_{a_i}(x)$, $\sigma_i > 0$. Clearly H can be written in the form

$$Hf(x) = Lf(x) - \sum_{i=1}^{N} \sigma_i (f, \delta_{a_i}) \delta_{a_i}(x),$$

that is, H can be regarded as rank N perturbation of the operator L. Throughout this section we use the following notation

- $\mathcal{R}(\lambda, x, y)$ is the solution of the equation $L\psi(x) - \lambda\psi(x) = \delta_y(x)$. We set $\mathcal{R}(\lambda, x, \overrightarrow{a}) := (\mathcal{R}(\lambda, x, a_i))_{i=1}^{N}$, and $\mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a}) := (\mathcal{R}(\lambda, a_j, a_i))_{i,j=1}^{N}$.
- $\mathcal{R}_V(\lambda, x, y)$ is the solution of the equation $H\psi(x) - \lambda\psi(x) = \delta_y(x)$. We set $\mathcal{R}_V(\lambda, x, \overrightarrow{a}) := (\mathcal{R}_V(\lambda, x, a_i))_{i=1}^{N}$, and $\mathcal{R}_V(\lambda, \overrightarrow{a}, \overrightarrow{a}) := (\mathcal{R}_V(\lambda, a_j, a_i))_{i,j=1}^{N}$.
- $\Sigma := \mathrm{diag}(\sigma_i : i = 1, ..., N)$.

Theorem 3.11 *The following properties hold true:*

1. *The set Spec(H) is pure point, its essential part Spec$_{ess}(H)$ coincides with the set Spec$(L) = \{0\} \cup \{\lambda_k\}$, its discrete part Spec$_d(H)$ in each open interval lying in the complement of Spec(L) consists of at most N distinct points, solutions of the equation*

$$\det(\Sigma^{-1} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a})) = 0. \tag{3.29}$$

2. *For each $k \in \mathbb{N}$ there exists $\delta > 1$ such that $\min_{i \neq j} d(a_i, a_j) > \delta$ implies that the operator H has precisely N distinct eigenvalues in each open interval $(\lambda_{s+1}, \lambda_s)$: $1 \leq s \leq k$. Moreover, there exists precisely N distinct negative eigenvalues of the operator H provided one of the following two conditions is satisfied:*

(2.1) *The semigroup $(e^{-tL})_{t>0}$ is recurrent.*

(2.2) *The semigroup $(e^{-tL})_{t>0}$ is transient and all $1/\sigma_i < \mathcal{R}(0, a, a)$.* [7]

The proof of the first part of Theorem 3.11 is based on the Weyl's theorem on the essential spectrum of compactly perturbed symmetric operators, see [18, Theorem IV.5.35], and on the following lemma.

Lemma 3.12 *Let A and B be two symmetric bounded operators and $H = A + B$. Assume that B is of rank N operator. Let (a, b) be an interval lying in the complement of the set $Spec(A)$. Then the set $Spec(H) \cap (a, b)$ consists of at most N distinct points.*

Proof. By the Weyl's essential spectrum theorem $Spec_{ess}(H)$ coincides with the set $Spec_{ess}(L) = \{0\} \cup \{\lambda_k\}$. Hence the set $Spec(H) \cap (a, b)$ may contain only finite number of eigenvalues each of which has finite multiplicity. Consider the case $N = 1$, that is, the operator B is of the form

$$Bf = \sigma_1(f, f_1)f_1.$$

Let $\lambda \in (a, b)$ and let f be a non-trivial solution of the equation $Hf - \lambda f = 0$. Then f can be written in the form

$$f = -\sigma_1(f, f_1)R_\lambda f_1 \tag{3.30}$$

where $R_\lambda = (A - \lambda)^{-1}$ is the resolvent operator. It follows that $(f, f_1) \neq 0$ and

$$(f, f_1) = -\sigma_1(f, f_1)(R_\lambda f_1, f_1),$$

or

$$\sigma_1(R_\lambda f_1, f_1) + 1 = 0. \tag{3.31}$$

The function $\phi(\lambda) = (R_\lambda f_1, f_1)$ is strictly increasing on the interval (a, b). Indeed, applying the resolvent identity we get

$$\frac{d\phi(\lambda)}{d\lambda} = (R_\lambda^2 f_1, f_1) = \|R_\lambda f_1\|^2 > 0.$$

It follows that equation (3.31) has at most one solution lying in the interval (a, b). Assume that equation (3.31) has a solution, denote it λ_*. Then (3.30) implies that the vector $f_* := R_{\lambda_*} f_1 / \|R_{\lambda_*} f_1\|$ satisfies the equation

$$Hf_* - \lambda_* f_* = 0.$$

Thus the operator H has at most one eigenvalue in the interval (a, b).

Without loss of generality we may provide the induction from $N = 1$ to $N = 2$. Thus assuming that the perturbation operator B is of the form

$$Bf = \sigma_1(f, f_1)f_1 + \sigma_2(f, f_2)f_2$$

[7] Thanks to the homogenuity assumption $\mathcal{R}(\lambda, a, a)$ does not depend on a

we set

$$A'f := Af + \sigma_1(f, f_1)f_1$$

and

$$Hf := A'f + \sigma_2(f, f_2)f_2.$$

Observe that the operator A' may have in the interval (a, b) at most one eigenvalue λ_*. The corresponding eigenspace is one-dimensional, call it $\langle f_* \rangle$, where $f_* := R_{\lambda_*}f_1 / \|R_{\lambda_*}f_1\|$. Let us consider two cases.

First case: Assume that $f_2 \perp f_*$. Then $Hf_* = A'f_* = \lambda_* f_*$, i.e. λ_* is an eigenvalue of the operator H. It follows that the orthogonal complement $\langle f_* \rangle^{\perp}$ is a joint invariant subspace of the operators H and A' and that these operators being restricted to $\langle f_* \rangle^{\perp}$, call them $H_\perp$ and $A'_\perp$, satisfy

$$H_\perp f = A'_\perp f + \sigma_2(f, f_2)f_2.$$

The operator $A'_\perp$ has no eigenvalues in the interval (a, b). Hence, by what we have already shown in the first part of the proof, the operator $H_\perp$ has at most one eigenvalue in the interval (a, b). It follows that the operator H has at most two eigenvalues in the interval (a, b).

Second case: Assume that f_2 and f_* are not orthogonal. Let $\mathcal{R}_\lambda := (H - \lambda \mathrm{I})^{-1}$ and $R'_\lambda := (A' - \lambda \mathrm{I})^{-1}$ be the resolvent operators. The following identity holds true

$$(\mathcal{R}_\lambda f, g) = (R'_\lambda f, g) - \frac{\sigma_2(R'_\lambda f, f_2)(R'_\lambda f_2, g)}{1 + \sigma_2(R'_\lambda f_2, f_2)} \tag{3.32}$$

for any f, g and $\lambda \neq \lambda_*$ lying in (a, b). Using the spectral resolution formula for the operator A', the fact that its spectral function E_λ in (a, b) has the only jump at $\lambda = \lambda_*$ and that the value of the jump ΔE_{λ_*} is the operator of orthogonal projection on the subspace $\langle f_* \rangle$ we get

$$(R'_\lambda f, f) = \frac{(f_*, f)^2}{\lambda - \lambda_*} + O_1(1) \tag{3.33}$$

and

$$(R'_\lambda f, f_2) = \frac{(f_*, f)(f_*, f_2)}{\lambda - \lambda_*} + O_2(1) \tag{3.34}$$

where $O_i(1)$ are analytic functions. Substituting asymptotic equations (3.33) and (3.34) in equation (3.32) we get analyticity of the function $\lambda \to (\mathcal{R}_\lambda f, f)$ at $\lambda = \lambda_*$. In particular, this shows that $\lambda = \lambda_*$ is not an eigenvalue of H.

On the other hand λ_* splits the interval (a, b) in two parts (a, λ_*) and (λ_*, b) each of which does not contain eigenvalues of the operator A'. Then, as we have already shown, each of these intervals contains at most one eigenvalue of the operator H. Since λ_* is not an eigenvalue of the operator H, the number of distinct eigenvalues of H in the interval (a, b) is at most two. The proof of the lemma is finished. ■

Proof of Theorem 3.11 (second part): Let $\lambda \in Spec_d(H)$ and let $\psi(x)$ be the corresponding eigenfunction, i.e.

$$H\psi(x) - \lambda\psi(x) = 0.$$

We have

$$L\psi(x) - \lambda\psi(x) = \sum_{i=1}^{N} \sigma_i \psi(a_i)\delta_{a_i}(x)$$

or applying to this equation the resolvent operator $(L-\lambda)^{-1}$ we get

$$\psi(x) = \sum_{i=1}^{N} \sigma_i \psi(a_i)\mathcal{R}(\lambda, x, a_i). \tag{3.35}$$

Taking consequently $x = a_1, a_2, ..., a_N$ in equation (3.35) we obtain a homogeneous system of N linear equations with N variables

$$\psi(a_j) = \sum_{i=1}^{N} \sigma_i \psi(a_i)\mathcal{R}(\lambda, a_j, a_i) \tag{3.36}$$

or in the vector form

$$\Psi = \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a})\Sigma\Psi, \tag{3.37}$$

where $\Psi = (\psi(a_i) : i = 1, ..., N)$. The system (3.37) has a non-trivial solution if and only if

$$\det(\Sigma^{-1} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a}) = 0. \tag{3.38}$$

Observe that the variable $z := \mathcal{R}(\lambda, a_i, a_i)$ does not depend on a_i, and its range is the whole interval $]-\infty, \infty[$ when λ takes values in each of open interval $]\lambda_{k+1}, \lambda_k[$. Equation (3.38) can be written as characteristic equation

$$\det(\mathfrak{A} - z\mathrm{I}) = 0 \tag{3.39}$$

where $\mathfrak{A} = (\mathfrak{a}_{ij})_{i,j=1}^{N}$ is symmetric $N \times N$ matrix with entries

$$\mathfrak{a}_{ij} = \begin{cases} 1/\sigma_i & \text{for} \quad i = j \\ -\mathcal{R}(\lambda, a_i, a_j) & \text{for} \quad i \neq j \end{cases}. \tag{3.40}$$

Let us compute $\mathcal{R}(\lambda, a_i, a_j)$. For any two neighboring balls $B \subset B'$ let us denote

$$A(B) = \frac{1}{m(B)} - \frac{1}{m(B')}.$$

Remember that we normalize m so that $m(B) = \mathrm{diam}(B)$ for any non-singleton ball B whence for such B,

$$A(B) = \frac{1}{\mathrm{diam}(B)} - \frac{1}{\mathrm{diam}(B')}. \tag{3.41}$$

Let $a_i \curlywedge a_j$ be the minimal ball which contains both a_i and a_j. Following the same line of reasons as in the proof of equation (3.27) we obtain

$$\mathcal{R}(\lambda, a_i, a_i) = \sum_{B:\ a_i \in B} \frac{A(B)}{\lambda(B) - \lambda}. \tag{3.42}$$

Similarly, for all $i \neq j$ we get

$$\mathcal{R}(\lambda, a_i, a_j) = -\frac{\mathrm{d}(a_i, a_j)^{-1}}{\lambda(a_i \curlywedge a_j) - \lambda} + \sum_{B:\ a_i \curlywedge a_j \subset B} \frac{A(B)}{\lambda(B) - \lambda}. \tag{3.43}$$

Let $\lambda > \lambda(a_i \curlywedge a_j)$. Equations (3.41), (3.43) and the fact $S \subset T \Rightarrow \lambda(S) > \lambda(T)$ imply that

$$\begin{aligned}
\mathcal{R}(\lambda, a_i, a_j) &= \frac{\mathrm{d}(a_i, a_j)^{-1}}{\lambda - \lambda(a_i \curlywedge a_j)} - \sum_{B:\, a_i \curlywedge a_j \subset B} \frac{A(B)}{\lambda - \lambda(B)} \\
&> \frac{\mathrm{d}(a_i, a_j)^{-1}}{\lambda - \lambda(a_i \curlywedge a_j)} - \frac{1}{\lambda - \lambda(a_i \curlywedge a_j)} \sum_{B:\, a_i \curlywedge a_j \subset B} A(B) \\
&= \frac{1}{\lambda - \lambda(a_i \curlywedge a_j)} \left(\frac{1}{\mathrm{d}(a_i, a_j)} - \frac{1}{\mathrm{diam}(a_i \curlywedge a_j)'} \right) > 0.
\end{aligned}$$

Hence for $\lambda > \lambda(a_i \curlywedge a_j)$ we obtain

$$0 < \mathcal{R}(\lambda, a_i, a_j) < \frac{\mathrm{d}(a_i, a_j)^{-1}}{\lambda - \lambda(a_i \curlywedge a_j)}. \tag{3.44}$$

Notice that $\lambda(B) \to 0$ as $\mathrm{diam}(B) \to \infty$. Let us fix k and let us consider $\lambda > \lambda_{k+1}$. Let us choose $\delta > 1$ such that if $\min_{i \neq j} d(a_i, a_j) \geq \delta$ then $\lambda(a_i \curlywedge a_j) < \lambda_k/2$. Then for all $i \neq j$ we get $\lambda - \lambda(a_i \curlywedge a_j) > \lambda_k/2$ and thus

$$|\mathcal{R}(\lambda, a_i, a_j)| < \frac{2}{\delta \lambda_k} := \frac{\varepsilon(\delta)}{N}. \tag{3.45}$$

Let us increase if necessary δ so that the intervals

$$\{s : |1/\sigma_i - s| \leq \varepsilon(\delta)\},\ i = 1, 2, ..., N,$$

do not intersect. By Gershgorin Circle Theorem the matrix $\mathfrak{A}$ admits N different eigenvalues $\mathfrak{a}_i$ each of which lies in the corresponding open interval

$$\{s : |1/\sigma_i - s| < \varepsilon(\delta)\},\ i = 1, 2, ..., N.$$

The eigenvalues $\mathfrak{a}_i, i = 1, 2, ..., N$, are analytic functions of λ in each open interval $(\lambda_{s+1}, \lambda_s)$, $1 \leq s \leq k$, see [34, Theorem XII.1]. Whence in each interval $(\lambda_{s+1}, \lambda_s)$ the number of different solutions of the equations $\mathfrak{a}_i = \mathcal{R}(\lambda, a_i, a_i)$ is at least N. By Lemma 3.12 the number of different solutions is at most N. Thus the number of different solutions is precisely N as claimed.

Theorem 3.13 *The set $Spec_d(H)$ coincides with the set of solutions of equation (3.29). Each eigenfunction $\psi_\lambda(x)$ corresponding to $\lambda \in Spec_d(H)$ can be represented as linear combination of functions $\mathcal{R}(\lambda, x, a_i)$, that is,*

$$\psi_\lambda(x) = \sum_{i=1}^{N} \zeta_i \mathcal{R}(\lambda, x, a_i).$$

Thus, support of ψ_λ is the whole space X whereas the eigenfunctions f_B corresponding to the eigenvalues $\lambda(B) \in Spec_{ess}(H)$ are compactly supported.

Proof. The proof is straightforward: we apply equations (3.35) and (3.36) to get the result, see the first part of the proof of Theorem 3.11 (second statement). ■

Theorem 3.14 *For $\lambda \notin Spec(H)$ the following identities hold true:*

$$\mathcal{R}_V(\lambda, x, y) = \mathcal{R}(\lambda, x, y) + \mathcal{R}(\lambda, x, \overrightarrow{a})(\Sigma^{-1} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a}))^{-1}\mathcal{R}(\lambda, \overrightarrow{a}, y),[8] \tag{3.46}$$

$$\Sigma\mathcal{R}_V(\lambda, \overrightarrow{a}, y) = (\Sigma^{-1} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a}))^{-1}\mathcal{R}(\lambda, \overrightarrow{a}, y) \tag{3.47}$$

and

$$\Sigma\mathcal{R}_V(\lambda, \overrightarrow{a}, \overrightarrow{a}) = (\Sigma^{-1} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a}))^{-1}\mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a}). \tag{3.48}$$

In particular, the operator $T(\lambda) := (H - \lambda\mathrm{I})^{-1} - (L - \lambda\mathrm{I})^{-1}$ is of finite rank N. Its operator norm can be estimated as follows

$$\|T(\lambda)\| \leq \left\|(\Sigma^{-1} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a}))^{-1}\right\| \left\|(L - \lambda\mathrm{I})^{-1}\right\|^2. \tag{3.49}$$

Proof. Recall that $Spec(H)$ coincides with the union of two sets: $Spec(L)$ and the set of those $\lambda \in \mathbb{R}$ for which $\det(\Sigma^{-1} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a})) = 0$. The proof of the theorem is similar to its one-dimensional version Theorem 3.8. Clearly we can write the following equation

$$\begin{aligned} L\mathcal{R}_V(\lambda, x, y) - \lambda\mathcal{R}_V(\lambda, x, y) &= \delta_y(x) + \sum_{i=1}^{N} \sigma_j \delta_{a_j}(x)\mathcal{R}_V(\lambda, x, y) \\ &= \delta_y(x) + \sum_{j=1}^{N} \sigma_j \mathcal{R}_V(\lambda, a_j, y)\delta_{a_j}(x), \end{aligned}$$

or equivalently

$$\mathcal{R}_V(\lambda, x, y) = \mathcal{R}(\lambda, x, y) + \sum_{j=1}^{N} \sigma_j \mathcal{R}_V(\lambda, a_j, y)\mathcal{R}(\lambda, x, a_j). \tag{3.50}$$

Substituting consequently $x = a_1, a_2, ..., a_N$ we obtain system of N linear equations with N variables

$$\mathcal{R}_V(\lambda, a_i, y) = \mathcal{R}(\lambda, a_i, y) + \sum_{j=1}^{N} \sigma_j \mathcal{R}(\lambda, a_i, a_j)\mathcal{R}_V(\lambda, a_j, y)$$

or in the vector form

$$(\mathrm{I} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a})\Sigma)\mathcal{R}_V(\lambda, \overrightarrow{a}, y) = \mathcal{R}(\lambda, \overrightarrow{a}, y). \tag{3.51}$$

Assuming that $\lambda \notin Spec(H)$, in particular $\det(\mathrm{I} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a})\Sigma) \neq 0$, we get

$$\mathcal{R}_V(\lambda, \overrightarrow{a}, y) = (\mathrm{I} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a})\Sigma)^{-1}\mathcal{R}(\lambda, \overrightarrow{a}, y) \tag{3.52}$$

Evidently equations (3.50) and (3.52) imply equations (3.46), (3.47) and (3.48).

[8]For a matrix A and vectors ξ and η we write $\xi A \eta := \sum_{i,j} a_{ij}\xi_i\eta_j$.

Equation $T(\lambda) = (H - \lambda \mathrm{I})^{-1}(L - H)(L - \lambda \mathrm{I})^{-1}$ applies that $T(\lambda))$ is of rank N. Finally, equation (3.49) follows from equation (3.46). Indeed, for $f \in L^2(X, m)$ we introduce (finite-dimensional) vectors $\mathcal{R}(\lambda, f, \overrightarrow{a}) := \sum_x f(x)\mathcal{R}(\lambda, x, \overrightarrow{a})$ and $\mathcal{R}(\lambda, \overrightarrow{a}, f) := \sum_y f(x)\mathcal{R}(\lambda, \overrightarrow{a}, y)$, then

$$\begin{aligned}(T(\lambda)f, f) &= \sum_{x,y} f(x)\mathcal{R}(\lambda, x, \overrightarrow{a})(\Sigma^{-1} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a}))^{-1}\mathcal{R}(\lambda, \overrightarrow{a}, y)f(y) \\ &= \mathcal{R}(\lambda, f, \overrightarrow{a})(\Sigma^{-1} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a}))^{-1}\mathcal{R}(\lambda, \overrightarrow{a}, f).\end{aligned}$$

By symmetry $\mathcal{R}(\lambda, \overrightarrow{a}, f) = \mathcal{R}(\lambda, f, \overrightarrow{a})$, whence

$$\begin{aligned}|(T(\lambda)f, f)| &\leq \left\|(\Sigma^{-1} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a}))^{-1}\right\| \left\|\mathcal{R}(\lambda, \overrightarrow{a}, f)\right\|^2 \\ &\leq \left\|(\Sigma^{-1} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a}))^{-1}\right\| \left\|(L - \lambda \mathrm{I})^{-1} f\right\|^2 \\ &\leq \left\|(\Sigma^{-1} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a}))^{-1}\right\| \left\|(L - \lambda \mathrm{I})^{-1}\right\|^2 \|f\|^2\end{aligned}$$

as desired. The proof of the theorem is finished. ■

3.4 Sparse potentials

We assume that the ultrametric measure space (X, d, m) is *countably infinite and homogeneous.* Our analysis of finite rank potentials $V = -\sum_{i=1}^{N} \sigma_i \delta_{a_i}$ indicates that in the case of increasing distances between locations $\{a_i\}$ of the *bumps* $V_i = -\sigma_i \delta_{a_i}$ their contributions to the spectrum of $H = L + V$ is close to the union of the contributions of the individual bumps V_i (each bump contributes one eigenvalue in each gap $(\lambda_{m+1}, \lambda_m)$ of the spectrum of the operator L).

The development of this idea leads to consideration of the class of *sparse potentials* $V = -\sum_{i=1}^{\infty} \sigma_i \delta_{a_i}$ where distances between locations $\{a_i : i = 1, 2, ...\}$ form a fast increasing sequence. In the classical theory this idea goes back to D. B. Pearson [32], see also S. Molchanov [29] and A. Kiselev, J. Last, and B. Simon [19].

Throughout this section we will assume that the sequence $\min_{i,j:\geq n,\ i\neq j} \mathrm{d}(a_i, a_j)$ tend to infinity with certain rate which will be specified later[9]. We will also assume that $\alpha < \sigma_i < \beta$ for all i and for some $\alpha, \beta > 0$. For $\lambda \notin Spec(L)$ we define the following infinite vectors and matrices:

- $\mathcal{R}(\lambda, x, \overrightarrow{a}) := (\mathcal{R}(\lambda, x, a_i) : i = 1, 2, ...)$.
- $\mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a}) := (\mathcal{R}(\lambda, a_i, a_j) : i, j = 1, 2, ...)$.
- $\Sigma := \mathrm{diag}(\sigma_i : i = 1, 2, ...)$, $\Sigma^{-1} := \mathrm{diag}(1/\sigma_i : i = 1, 2, ...)$.

Theorem 3.15 *The following properties hold true:*

(i) $\mathcal{R}(\lambda, x, \overrightarrow{a}) \in l^2$.

(ii) $\mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a})$, Σ *and* Σ^{-1} *act in* l^2 *as bounded symmetric operators.*

[9]We choose the ultrametric $\mathrm{d}(x, y)$ such that it coinsides with the measure $m(B)$ of the minimal ball B which contains both x and y, see e.g. (3.41).

(iii) *If the operator $\mathfrak{B}(\lambda) = \Sigma^{-1} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a})$ has a bounded inverse, then*

$$\mathcal{R}_V(\lambda, x, y) = \mathcal{R}(\lambda, x, y) + \mathcal{R}(\lambda, x, \overrightarrow{a})\mathfrak{B}(\lambda)^{-1}\mathcal{R}(\lambda, \overrightarrow{a}, y). \tag{3.53}$$

Proof. Let $\xi = (\xi_i) \in l^2$ has finite number non-zero coordinates. Define function $f = \sum \xi_i \delta_{a_i}$. Evidently $f \in L_2 = L_2(X, m)$ and $\|f\| = \|\xi\|$. Let $R_\lambda = (L - \lambda \mathrm{I})^{-1}$, $\lambda \notin Spec(L)$, be the resolvent. Then

$$\mathcal{R}(\lambda, x, \overrightarrow{a})\xi = \int \mathcal{R}(\lambda, x, y) f(y) dm(y) = R_\lambda f(x)$$

whence

$$|\mathcal{R}(\lambda, x, \overrightarrow{a})\xi| \leq \|R_\lambda\| \, \|f\| = \|R_\lambda\| \, \|\xi\|$$

which clearly proves (i).To prove (ii) we write

$$\begin{aligned} \xi\mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a})\xi &= \int\int f(x)\mathcal{R}(\lambda, x, y) f(y) dm(y) dm(x) \\ &= (f, R_\lambda f) \leq \|R_\lambda\| \, \|f\|^2 = \|R_\lambda\| \, \|\xi\|^2 \end{aligned}$$

which clearly proves boundedness of the symmetric operator $\mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a}) : l^2 \to l^2$. Since $\{\sigma_i\} \in (\alpha, \beta)$ for all i and some $\alpha, \beta > 0$, boundedness of the operators Σ and Σ^{-1} follows.

(iii) Assume that λ is such that the self-adjoint operator $\mathfrak{B}(\lambda)$ has a bounded inverse, then equation (3.53) follows from its finite dimensional version (3.46) by passage to limit. ∎

Theorem 3.16 $Spec(L) \subset Spec_{ess}(H)$.

Proof. Let V' be the sum of all but finite number of bumps V_i and $H' = L + V'$. By Weyl's essential spectrum theorem $Spec_{ess}(H) = Spec_{ess}(H')$. It follows that without loss of generality we may assume that the sequence of distances $\Delta_n = \min_{i,j:\geq n,\, i\neq j} \mathrm{d}(a_i, a_j)$ strictly increases to ∞. Having this in mind we can choose for any given τ from the range of the distance function an infinite sequence $\{B_n\}$ of disjoint balls of diameter τ such that $B_n \cap \{a_i\} = \varnothing$ for all n. Thanks to our choice we obtain

$$Hf_T = Lf_T = \lambda(T')f_T$$

for any ball $T \subset B_n$ and for all n. In particular, each $\lambda = \lambda(T)$, such that $T \subseteq B_n$ for some n, is an eigenvalue of the operator H having infinite multiplicity, whence it belongs to $Spec_{ess}(H)$. ∎

Theorem 3.17 *Let σ_* be a limit point of the sequence $\{\sigma_i\}$. Fix $m \in \mathbb{N}$ and let $\lambda_{*m} \in (\lambda_{m+1}, \lambda_m)$ be the unique solution of the equation*

$$\frac{1}{\sigma_*} = \mathcal{R}(\lambda, a, a).\ ^{10} \tag{3.54}$$

Then λ_{*m} belongs to the set $Spec_{ess}(H)$.

[10] Recall that the function $\lambda \to \mathcal{R}(\lambda, a, a)$ does not depend on a.

Before we embark on the proof of Theorem 3.17 let us state the Weyl's characterization of the essential spectrum $Spec_{ess}(A)$ of a self-adjoint operator A, see [42] and [35, Ch. IX, Sect. 2(133)].

Lemma 3.18 *A real number λ belongs to the set $Spec_{ess}(A)$ if and only if there exists a normed sequence $\{x_i\} \subset \mathrm{dom}(A)$ such that $x_i \to 0$ weakly and $Ax_i - \lambda x_i \to 0$ strongly.*

Proof of Theorem 3.17. To show that $\lambda_{*m} \in Spec_{ess}(H)$ we construct a λ_{*m}-sequence $\{f_{im}\}$ via Lemma 3.18. Let $\lambda_{im} \in (\lambda_{m+1}, \lambda_m)$ be the unique solution of the equation $1/\sigma_i = \mathcal{R}(\lambda, a_i, a_i)$. Let $\psi_{im}(x) = \mathcal{R}(\lambda_{im}, x, a_i)/\left\|\mathcal{R}(\lambda_{im}, \cdot, a_i)\right\|_2$ be the normed solution of the equation $H_i\psi = \lambda_{im}\psi$ where $H_i := L - \sigma_i\delta_{a_i}$ is a one-bump perturbation of L. Clearly $\lambda_{im} \to \lambda_{*m}$.

Passing if necessary to a subsequence of $\{\sigma_i\}$ we can assume that $\mathrm{d}(a_i, 0) \to \infty$ monotonically. Let us put $f_{im} := \psi_{im} \cdot 1_{B_i}$ where B_i is the maximal ball centred at a_i which does not contains a_{i-1} and a_{i+1}. Thanks to our choice $f_{im} \to 0$ weakly and

$$\|f_{im}\|_2^2 = \int_{B_i} |\psi_{im}|^2 \, dm \to 1.$$

Thus what is left is to show that $Hf_{im} - \lambda_{*m}f_{im} \to 0$ strongly. We have

$$\begin{aligned}\|Hf_{im} - \lambda_{*m}f_{im}\|_2 &\leq \|Hf_{im} - \lambda_{im}f_{im}\|_2 + \|f_{im}\|_2 \, |\lambda_{im} - \lambda_{*m}| \\ &\leq \|Hf_{im} - \lambda_{im}f_{im}\|_2 + |\lambda_{im} - \lambda_{*m}| \\ &= \|Hf_{im} - \lambda_{im}f_{im}\|_2 + o(1),\end{aligned}$$

$$\begin{aligned}\|Hf_{im} - \lambda_{im}f_{im}\|_2 &\leq \|H\psi_{im} - \lambda_{im}\psi_{im}\|_2 + \|(H - \lambda_{im}\mathrm{I})(f_{im} - \psi_{im})\|_2 \\ &\leq \|H\psi_{im} - \lambda_{im}\psi_{im}\|_2 + \|(H - \lambda_{im}\mathrm{I})\| \; \|(f_{im} - \psi_{im})\|_2 \\ &= \|H\psi_{im} - \lambda_{im}\psi_{im}\|_2 + o(1),\end{aligned}$$

$$\|H\psi_{im} - \lambda_{im}\psi_{im}\|_2 \leq \|H_i\psi_{im} - \lambda_{im}\psi_{im}\|_2 + \left\|\sum_{j\neq i} \sigma_j\delta_{a_j}\psi_{im}\right\|_2$$

and

$$\left\|\sum_{j\neq i} \sigma_j\delta_{a_j}\psi_{im}\right\|_2 = \sqrt{\sum_{j\neq i} \sigma_j^2|\psi_{im}(a_j)|^2} \leq \sup\{\sigma_j^2\}\sqrt{\int_{X\setminus B_i} |\psi_{im}|^2 dm}.$$

The right-hand side of this inequality tends to zero as $i \to \infty$ and we finally conclude that $\{f_{im}\}$ is the desired λ_{*m}-sequence in the sense of Lemma 3.18. The proof is finished.

Let us introduce the following notation

- Σ_* is the set of limit points of the sequence $\{\sigma_i\}$
- $1/\Sigma_* := \{1/\sigma_* : \sigma_* \in \Sigma_*\}$
- $\mathcal{R}^{-1}(1/\Sigma_*) := \{\lambda : \mathcal{R}(\lambda, a, a) \in 1/\Sigma_*\}$

Theorem 3.19 *Assume that the following condition holds*

$$\lim_{N\to\infty} \sup_{i\geq N} \sum_{j\geq N:\ j\neq i} \frac{1}{\mathrm{d}(a_i, a_j)} = 0, \tag{3.55}$$

then

$$Spec_{ess}(H) = Spec(L) \cup \mathcal{R}^{-1}(1/\Sigma_*). \tag{3.56}$$

Proof. That $Spec(L)$ and $\mathcal{R}^{-1}(1/\Sigma_*)$ are subsets of $Spec_{ess}(H)$ follows from Theorem 3.16 and Theorem 3.17. We are left to prove that

$$Spec_{ess}(H) \subset Spec(L) \cup \mathcal{R}^{-1}(1/\Sigma_*).$$

Let us fix $m \in \mathbb{N}$ and choose a closed interval $\mathcal{I}$ from the spectral gap $(\lambda_{m+1}, \lambda_m)$. We claim that

$$\mathcal{I} \cap Spec_{ess}(H) = \varnothing.$$

Indeed, since $\mathcal{R}(\lambda) := \mathcal{R}(\lambda, a, a)$ is strictly increasing and continuous in the interval $(\lambda_{m+1}, \lambda_m)$, closed sets $\mathcal{R}(\mathcal{I})$ and $1/\Sigma_*$ do not intersect. Hence there exists only a finite number of σ_i such that $1/\sigma_i \in \mathcal{R}(\mathcal{I})$. Let us choose N big enough so that the sets $\{1/\sigma_i : i > N\}$ and $\mathcal{R}(\mathcal{I})$ do not intersect. Let us write $H = H' + V'$ where V' is a finite number of bumps $-\sigma_i\delta_{a_i}$, $i \leq N$. By Weyl's essential spectrum theorem

$$Spec_{ess}(H) = Spec_{ess}(H').$$

Notice however that the sets $Spec_d(H)$ and $Spec_d(H')$, discrete parts of $Spec(H)$ and $Spec(H')$, may well be quite different. Observe that for the operators H and H' the sets of limit points, the function $\mathcal{R}$, the set of gaps etc are the same. Thus in all our further considerations we may *assume that* $\{1/\sigma_i\} \cap \mathcal{R}(\mathcal{I}) = \varnothing$.

Making this assumption consider now the operator $\mathfrak{B}(\lambda) = \Sigma^{-1} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a})$, $\lambda \in \mathcal{I}$. According to identity (3.53), if $\mathfrak{B}(\lambda)$ has a bounded inverse then $\lambda \notin Spec(H)$. Let us write

$$\mathfrak{B}(\lambda) = \Sigma^{-1} - \mathcal{R}(\lambda, \overrightarrow{a}, \overrightarrow{a}) := \left[\Sigma^{-1} - \mathcal{R}(\lambda)\mathrm{I}\right] - \widetilde{\mathcal{R}}(\lambda).$$

Since we assume that the closed bounded sets $\overline{\{1/\sigma_i\}}$ and $\mathcal{R}(\mathcal{I})$ do not intersect, the operator $\mathcal{A}(\lambda) := \Sigma^{-1} - \mathcal{R}(\lambda)\mathrm{I}$ has a bounded inverse $\mathcal{A}(\lambda)^{-1}$ for all $\lambda \in \mathcal{I}$. Clearly the norm $\left\|\mathcal{A}(\lambda)^{-1}\right\|$ can be estimated by the reciprocal of the distance between sets $\overline{\{1/\sigma_i\}}$ and $\mathcal{R}(\mathcal{I})$, denote it by C_1. Thus writing for $\lambda \in \mathcal{I}$ the identity

$$\mathfrak{B}(\lambda) = \mathcal{A}(\lambda)(\mathrm{I} - \mathcal{A}(\lambda)^{-1}\widetilde{\mathcal{R}}(\lambda)) \tag{3.57}$$

we get

$$\left\|\mathcal{A}(\lambda)^{-1}\widetilde{\mathcal{R}}(\lambda))\right\| \leq C_1 \left\|\widetilde{\mathcal{R}}(\lambda))\right\|. \tag{3.58}$$

Writing again H as $H' + V'$ where V'consists of a finite number, say N, of bumps and applying inequality (3.44) for the operator H' :

$$|\mathcal{R}(\lambda, a_i, a_j)| < \frac{1}{\mathrm{d}(a_i, a_j)} \frac{1}{\lambda - \lambda(a_i \curlywedge a_j)},\ i \neq j,\ i, j \geq N,$$

we will get, thanks to our assumption (3.55), the following inequality

$$\left\| \widetilde{\mathcal{R}}(\lambda)) \right\| \leq C_2 \sup_{i \geq N} \sum_{j:\, j \neq i, j \geq N} \frac{1}{\mathrm{d}(a_i, a_j)} < \frac{1}{2C_1} \tag{3.59}$$

for some constant $C_2 > 0$ which depends only on $\mathcal{I}$, and for N chosen big enough. Clearly inequalities (3.58) and (3.59) imply the fact that the operator $\mathrm{I} - \mathcal{A}(\lambda)^{-1}\widetilde{\mathcal{R}}(\lambda)$ has bounded inverse for all $\lambda \in \mathcal{I}$,

$$\left(\mathrm{I} - \mathcal{A}(\lambda)^{-1}\widetilde{\mathcal{R}}(\lambda)\right)^{-1} = \sum_{k \geq 0} \left(\mathcal{A}(\lambda)^{-1}\widetilde{\mathcal{R}}(\lambda)\right)^{k}.$$

This fact, in turn, implies that the operator $\mathfrak{B}(\lambda)$ given by equation (3.57) has bounded inverse for all $\lambda \in \mathcal{I}$ therefore $\mathcal{I} \cap Spec(H') = \varnothing$. In particular, since $Spec_{ess}(H') = Spec_{ess}(H)$ by Weyl's essential spectrum theorem, we finally get

$$\mathcal{I} \cap Spec_{ess}(H) = \varnothing$$

as desired. The proof is finished. ∎

Remark 3.20 *Theorem 3.19 does not contain information about sets $Spec_{ac}(H)$ and $Spec_{sc}(H)$, the absolutely continuous and singular continuous parts of $Spec(H)$. In the next section we will show that under more restrictive assumption $Spec_{ac}(H)$ and $Spec_{sc}(H)$ are indeed empty sets, that is, $Spec(H)$ is pure point. Moreover, the eigenfunctions of H decay exponentially in certain metric at infinity. This is the so called* localization property.

3.5 Localization

As in the previous section the ultrametric measure space (X, d, m) is *countably infinite and homogeneous.* We consider the operator $H = L + V$ where L, the deterministic part of H, is a hierarchical Laplacian and

$$V = -\sum_{a \in I} \sigma(a, \omega)\delta_a, \ \omega \in (\Omega, \mathcal{F}, P),$$

is a random potential defined by a family of locations $I = \{a_i\}$ and a family $\sigma(a_i, \omega)$ of i.i.d. random variables. Henceforth, we assume that the probability distribution of $\sigma(a_i, \omega)$ is absolutely continuous with respect to the Lebesgues measure and has a bounded density supported by a finite interval $[\alpha, \beta]$.

In the case when X is the Dyson lattice and $L = \mathrm{D}^{\alpha}$, the Dyson Laplacian (see Example 3.10), the perturbed operator

$$H = \mathrm{D}^{\alpha} - \sum_{a \in X} \sigma(a, \omega)\delta_a$$

has a pure point spectrum for $P-$a.s. ω. This statement (*the localization theorem*) appeared first in the paper of Molchanov [28] ($\sigma(a, \omega)$ is the Cauchy random variable) and later in a more general form in the papers of Kritchevski [26] and [25]. The proof of this statement is based on the self-similarity property of the operator H.

The localization theorem 3.23 below concerns the case where the family of locations I does not coincide with the whole space X, whence the operator H is not self-similar. The technique developed in [28], [26] and [25] does not apply here to prove Theorem 3.23.

Our approach is based on the different technique: the abstract form of the Aizenman-Molchanov criterion for pure point spectrum, the Krein type identity from the previous section, technique of fractional moments, decoupling lemma of Molchanov and Borel-Cantelli type arguments, see papers [1], [27].

The Aizenman-Molchanov Criterion Let $H = H_0 + V$ be a self-adjoint operator in $l^2(\Gamma)$ (Γ is a countable set of sites) with H_0 a bounded operator and $V = -\sum_{a\in\Gamma}\sigma(a,\omega)\delta_a$. Assume that the collection of random variables $\{\sigma(a,\omega) : a \in \Gamma\}$ has the property that for each site a the conditional probability distribution of $\sigma(a,\omega)$ (conditioned on the values of the potential at all other sites) is absolutely continuous with respect to the Lebesgues measure (in particular, this assumption holds if $\{\sigma(a,\omega) : a \in \Gamma\}$ are mutually independent random variables having absolutely continuous w.r.t. the Lebesgues measure l probability distributions).

Let $H = \int \lambda dE_\lambda$ be the spectral resolution of symmetric operator H. Let $G(\lambda, x, y)$ be the integral kernel of the operator $(H - \lambda\mathrm{I})^{-1}$. Then for any fixed x, τ and $\epsilon \neq 0$,

$$\sum_{y\in\Gamma} |G(\tau + i\epsilon, x, y)|^2 = \left\|(H - (\tau + i\epsilon)\mathrm{I})^{-1}\delta_x\right\|^2 = \int \frac{d(E_\lambda\delta_x, \delta_x)}{(\lambda - \tau)^2 + \epsilon^2} \tag{3.60}$$

As the left-hand side of equation (3.60) (as a function of ϵ) decreases on the interval $]0, +\infty[$, the limit (finite or infinite) in equation (3.60) exists and equals

$$\lim_{\epsilon\downarrow 0} \sum_{y\in\Gamma} |G(\tau + i\epsilon, x, y)|^2 = \int \frac{d(E_\lambda\delta_x, \delta_x)}{(\lambda - \tau)^2}.$$

Theorem 3.21 *If for any $x \in \Gamma$, and Lebesgues a.a. $\tau \in [a, b]$:*

$$\lim_{\epsilon\downarrow 0} \sum_{y\in\Gamma} |G(\tau + i\epsilon, x, y)|^2 < \infty, \tag{3.61}$$

for a.e. realizations of $\{\sigma(x,\cdot)\}$, then almost surely the operator H has only pure point spectrum in the interval $[a, b]$. Furthermore, if under condition (3.61), the integral kernel

$$G(\tau + i0, x, y) := \lim_{\epsilon\downarrow 0} G(\tau + i\epsilon, x, y)$$

(which exists a.e. τ) decays exponentially at infinity (in some metric $\rho(x, y)$ on Γ), then do the eigenfunctions $\varphi_\tau(y)$, for $\tau \in [a, b]$ [11]*.*

Proof. The first part of the statement follows from Simon-Wolff theorem 3.7. For completeness of exposition we comment on the proof. To prove the second part one needs an ad hoc argument and we refer to the cited above paper [1, Theorems 3.1 and 3.3 in Sec. 3]).

[11] An even more versatile version can be found in [1, Theorems 3.1 and 3.3 in Sec. 3].

Note that in the case of the Dyson-Vladimirov Laplacian D^α and $H = \mathrm{D}^\alpha - \sum \sigma_i(a_i,\omega)\delta_a$ one can use the metric $\rho(x,y) = \ln(1+\mathrm{d}(x,y))$ where $\mathrm{d}(x,y)$ is the ultrametric generated by p-adic intervals as in example 3.10. In this case the exponential decay of eigenfunctions in $\rho-$metric follows directly from two facts: (1) each eigenfunction $\varphi_\tau(y)$ of H can be represented as a linear combination of functions $\mathcal{R}(\tau,a_i,y)$, where $\mathcal{R}(\lambda,x,y)$ is the resolvent kernel of D^α, see Theorem 3.13, and (2) $\mathcal{R}(\lambda,x,y)$ has an exponential decay because the heat kernel $p(t,x,y)$ does, see equation (1.2).

By the spectral theory, one can represent $l^2(\Gamma)$ as the direct sum of three H-invariant subspaces:

$$l^2(\Gamma) = \mathcal{H}_{ac} \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{pp},$$

where $\mathcal{H}_{ac}$ (resp. $\mathcal{H}_{sc}$, $\mathcal{H}_{pp}$) is the set of all functions $f \in l^2(\Gamma)$ such that the spectral measure

$$\sigma^f(A) = \int 1_A(\lambda) d(E_\lambda f, f)$$

is absolutely continuous (resp. singular continuous, pure point) with respect to the Lebesgues measure. By Theorem 3.7, condition (3.61) implies that for any $x \in \Gamma$ the probability measure

$$\sigma^x(A) = \int 1_A(\lambda) d(E_\lambda \delta_x, \delta_x)$$

is pure point, that is, $\sigma^x(A) = \sigma^x(A \cap S_x)$ for any open set A and some at most countable set S_x. Set $S := \cup_{x\in\Gamma} S_x$, then for any $f \in l^2(\Gamma)$ and measurable set A,

$$\begin{aligned}\sigma^f(A) &= \int 1_A(\lambda) d(E_\lambda f, f) = \|1_A(H)f\|^2 \\ &= \sum_{x\in\Gamma} |f(x)|^2 \left|(1_A(H)f, \delta_x)\right|^2\end{aligned}$$

and, if A lies in the complement of S,

$$\begin{aligned}\left|(1_A(H)f, \delta_x)\right|^2 &\le |(1_A(H)f, f)|\, |(1_A(H)\delta_x, \delta_x)| \\ &= \|1_A(H)f\|^2 \sigma^x(A) = 0.\end{aligned}$$

Thus for any $f \in l^2(\Gamma)$ the spectral measure σ^f is pure point, that is, $f \in \mathcal{H}_{pp}$. That means that the operator H has a pure point spectrum. ■

Remark 3.22 *The function $z \to G(z,x,y)$, analytic in the domain $\mathbb{C}_+$, is represented by the Borel-Stieltjes transform of a signed measure of finite variation*

$$G(z,x,y) = \int \frac{d(E_\lambda \delta_x, \delta_y)}{\lambda - z}.$$

It follows that the limit $G(\tau + i0, x, y)$ exists and takes finite values for Lebesgues a.e. τ, see e.g. [36, Theorem 1.4]. Moreover, the limit $G(\tau + i0, x, y)$ exists even in a more restrictive sense, as the non-tangential limit, see [33, Ch. III, Sec. 2.2, 3.1 and 3.2]. We will apply this fact in the proof of Theorem 3.23 below.

The localization theorem Coming back to our setting, let $H = L + V$ where L is a hierarchical Laplacian and V a random potential of the form $V = -\sum_i \sigma_i(\omega)\delta_{a_i}$. Here $\sigma_i(\omega) := \sigma(a_i, \omega)$ are i.i.d. random variables corresponding to the set of locations $I = \{a_i\}$.

Let $\mathrm{d}(x, y)$ be the ultrametric which is chosen such that it coincides with the measure $m(B)$ of the minimal ball B containing both x and y.

Let $\mathcal{R}(\lambda, x, y)$ be the integral kernel of the operator $(L - \lambda \mathrm{I})^{-1}$, i.e. the solution of the equation $Lu - \lambda u = \delta_y$. The function $\lambda \to \mathcal{R}(\lambda, x, x)$ does not depend on x, we denote its value $\mathcal{R}(\lambda)$. This is strictly increasing continuous in each spectral gap function, we denote by $\mathcal{R}^{-1}(\nu)$ its inverse function.

Theorem 3.23 *The operator H has a pure point spectrum for $P-$a.s. ω provided for some (whence for all) $y \in X$ the sequence $\mathrm{d}(a_i, y)$ eventually increases, and for some small r (say, $0 < r < 1/3$):*

$$\lim_{M\to\infty} \sup_{i\geq M} \sum_{j\geq M:\ j\neq i} \frac{1}{\mathrm{d}(a_i, a_j)^r} = 0. \text{ [12]} \tag{3.62}$$

Proof. The set of limit points of the sequence $\{\sigma_i(\omega)\}$ coincides (for $P-$a.s. ω) with the whole interval $[\alpha, \beta]$. Hence, by Theorem 3.19, the closed set $Spec_{ess}(H)$ consists (for P-a.a. ω) of two parts: (1) the set $Spec(L)$ and (2) the collection of countably many disjoint closed intervals $\mathcal{I}_k = \mathcal{R}^{-1}([1/\beta, 1/\alpha]) \cap\ ]\lambda_{k+1}, \lambda_k, [$ and the interval $\mathcal{I}_- = \mathcal{R}^{-1}([1/\beta, 1/\alpha]) \cap\ ] - \infty, 0[$, i.e.

$$Spec_{ess}(H) = Spec(L) \cup \mathcal{I}_- \cup \mathcal{I}_1 \cup \mathcal{I}_2 ... \ .$$

Let $\mathcal{R}_V(\lambda, x, y)$ be the integral kernel of the operator $(H - \lambda \mathrm{I})^{-1}$, i.e. solution of the equation $Hu - \lambda u = \delta_y$. Due to the Aizenman-Molchanov criterion, the operator H has only pure point spectrum (for $P-$a.s. ω) provided *for each $y \in X$, for each interval* $\mathcal{I}_k$, and for *Lebesgues a.e. $\tau \in \mathcal{I}_k$* :

$$\lim_{\epsilon\to+0} \sum_x |\mathcal{R}_V(\tau + i\epsilon, x, y)|^2 < \infty, \tag{3.63}$$

for a.e. realization of $\{\sigma(y, \omega)\}$.We split the proof of equation (3.63) in seven steps.

Step I. When V has a finite rank, Theorem 3.11(i) and Theorem 3.14 imply that for each fixed ω the function $\mathcal{R}_V(\tau + i0, x, y) = (H - \tau \mathrm{I})^{-1}\delta_y(x)$ belongs to $L^2(X, m)$ for each $y \in X$ and for all but finitely many $\tau \in \mathcal{I}_k$ (which are eigenvalues of H).

In general, when the rank of V is infinite, we split V in two parts $V' = -\sigma_1\delta_{a_1}$ and $V'' = -\sum_{i>1} \sigma_i\delta_{a_i}$. Writing the set of locations as $\{a\} = \{a_1\} \cup \{a_i : i > 1\}$ we get similarly to equation (3.53): for λ in the domain $\mathbb{C}_+$,

$$\mathcal{R}_V(\lambda, x, y) = \mathcal{R}_{V''}(\lambda, x, y) + \mathcal{R}_{V''}(\lambda, x, a_1)\mathfrak{B}(\lambda)^{-1}\mathcal{R}_{V''}(\lambda, a_1, y),$$

where $\mathfrak{B}(\lambda) = 1/\sigma_1 - \mathcal{R}_{V''}(\lambda, a_1, a_1)$ is a non-constant analytic in the domain $\mathbb{C}_+$ function. It follows that

$$\begin{aligned}\|\mathcal{R}_V(\lambda, \cdot, y)\|_2 &\leq \|\mathcal{R}_{V''}(\lambda, \cdot, y)\|_2 \\ &\quad + |\mathfrak{B}(\lambda)|^{-1} \|\mathcal{R}_{V''}(\lambda, \cdot, a_1)\|_2 \, |\mathcal{R}_{V''}(\lambda, a_1, y)|.\end{aligned}$$

[12]Clearly this condition implies condition (3.55).

Hence the function $\mathcal{R}_V(\lambda, x, y)$ satisfies condition (3.63), i.e. $\|\mathcal{R}_V(\tau + i0, \cdot, y)\|_2$ is finite for all y and a.e. τ provided the function $\mathcal{R}_{V''}(\lambda, x, y)$ satisfies condition (3.63), i.e. $\|\mathcal{R}_{V''}(\tau + i0, \cdot, a)\|_2$ is finite for all a and a.e. τ, and also one more restriction on τ, it does not belong to the exceptional set

$$\Upsilon := \{s : \mathfrak{B}(s + i0) = 0\}.$$

The function $\mathfrak{B}(\lambda)$, analytic in the domain $\mathbb{C}_+$, admits non-tangential boundary values $\mathfrak{B}(s + i0)$ for a.e. s. By the Lusin-Privalov uniqueness theorem on boundary-values of analytic functions [33, Ch. IV, Sec. 2.5], see also [36, Theorem 1.5], the *Lebesgues measure* of the exceptional set Υ equals to zero. Thus, we come to the conclusion that condition (3.63) for the potential V can be reduced to the case of truncated potential V''.

Repeating this argument finitely many times we come to the final conclusion: in order to prove that (3.63) holds for V we can consider, if necessary, any *finitely truncated* potential V'' (the potential corresponding to the finitely truncated system of locations $\{a_i : i > k\}$) and to prove that (3.63) holds for V'' instead of V.

Step II. Writing for $\lambda \in \mathbb{C}_+$ equation $Hu - \lambda u = \delta_y$ in the form $Lu - \lambda u = \delta_y - Vu$ we obtain

$$\mathcal{R}_V(\lambda, x, y) = \mathcal{R}(\lambda, x, y) + \sum_{j=1}^{\infty} \sigma_j \mathcal{R}(\lambda, x, a_j) \mathcal{R}_V(\lambda, a_j, y). \tag{3.64}$$

Equation (3.64) shows that to estimate the function $y \to \|\mathcal{R}_V(\lambda, \cdot, y)\|_2$ it is enough to estimate the quantity $|\mathcal{R}_V(\lambda, a_j, y)|$ for $j = 1, 2, ...$ etc. Indeed, since $\|\mathcal{R}(\lambda, \cdot, y)\|_2$ does not depend on y, we get

$$\|\mathcal{R}_V(\lambda, \cdot, y)\|_2 \leq \|\mathcal{R}(\lambda, \cdot, y)\|_2 \left(1 + \beta \sum_{j=1}^{\infty} |\mathcal{R}_V(\lambda, a_j, y)|\right). \tag{3.65}$$

Choosing $x = a_i$, $i = 1, 2, ...$, in equation (3.64) and setting $\mathcal{R}(\lambda, a_i, a_i) = \mathcal{R}(\lambda)$ we obtain

$$\mathcal{R}_V(\lambda, a_i, y) = \frac{\mathcal{R}(\lambda, a_i, y)}{1 - \sigma_i \mathcal{R}(\lambda)} + \sum_{j:\, j \neq i} \frac{\sigma_j \mathcal{R}(\lambda, a_j, a_i) \mathcal{R}_V(\lambda, a_j, y)}{1 - \sigma_i \mathcal{R}(\lambda)}. \tag{3.66}$$

Step III. Applying in equation (3.66) the inequality

$$\left|\sum_{j=1}^{\infty} Z_j\right|^s \leq \sum_{j=1}^{\infty} |Z_j|^s, \;\; Z_j \in \mathbb{C}, \;\; 0 < s \leq 1,$$

we will get

$$\begin{aligned} |\mathcal{R}_V(\lambda, a_i, y)|^s &\leq \left|\frac{\mathcal{R}(\lambda, a_i, y)}{1 - \sigma_i \mathcal{R}(\lambda)}\right|^s + \sum_{j:\, j \neq i} \left|\frac{\sigma_j \mathcal{R}(\lambda, a_j, a_i)}{1 - \sigma_i \mathcal{R}(\lambda)}\right|^s |\mathcal{R}_V(\lambda, a_j, y)|^s \\ &\leq \left|\frac{\mathcal{R}(\lambda, a_i, y)}{1 - \sigma_i \mathcal{R}(\lambda)}\right|^s + \beta^s \sum_{j:\, j \neq i} \left|\frac{\mathcal{R}(\lambda, a_j, a_i)}{1 - \sigma_i \mathcal{R}(\lambda)}\right|^s |\mathcal{R}_V(\lambda, a_j, y)|^s. \end{aligned}$$

Taking the expectation over $\{\sigma_i\}$ we obtain the following inequality

$$\mathbb{E}\left|\mathcal{R}_V(\lambda, a_i, y)\right|^s \leq \mathbb{E}\left|\frac{1}{1-\sigma_i\mathcal{R}(\lambda)}\right|^s \left|\mathcal{R}(\lambda, a_j, y)\right|^s \qquad (3.67)$$
$$+\beta^s \sum_{j:\, j\neq i} \mathbb{E}\left|\frac{\mathcal{R}_V(\lambda, a_j, y)}{1-\sigma_i\mathcal{R}(\lambda)}\right|^s \left|\mathcal{R}(\lambda, a_j, a_i)\right|^s.$$

Step IV. Due to equation (3.22) the random variable $\mathcal{R}_V(\lambda, a_j, y)$ can be represented in the form

$$\mathcal{R}_V(\lambda, a_j, y) = \mathcal{R}_{V'}(\lambda, a_j, y) + \frac{\sigma_i\mathcal{R}_{V'}(\lambda, a_j, a_i)\mathcal{R}_{V'}(\lambda, a_i, y)}{1-\sigma_i\mathcal{R}_{V'}(\lambda, a_i, a_i)} := \frac{a\sigma_i + b}{c\sigma_i + d}$$

where the random variables $V' = -\sum_{k:\, k\neq i}\sigma_k(\omega)\delta_{a_k}$, a, b, c and d do not dependent on σ_i (but they of course depend on the truncated sequence $\{\sigma_k : k \neq i\}$). This observation and the following two general inequalities from Molchanov's lectures [27, Chapter II, Lemma 2.2]): There exist constants $c_0, c_1 > 0$ such that for all complex numbers a, b, c, d, σ'

$$\int_0^1 \frac{d\sigma}{|\sigma-\sigma'|^s} \leq \frac{c_0}{1-s}, \text{ for all } 0 < s < 1,$$

and

$$\int_0^1 \left|\frac{a\sigma+b}{c\sigma+d}\right|^s \frac{d\sigma}{|\sigma-\sigma'|^s} \leq c_1 \int_0^1 \left|\frac{a\sigma+b}{c\sigma+d}\right|^s d\sigma, \text{ for all } 0 < s < 1/2,$$

yield the following lemma, which is the fundamental point of our reasons.

Lemma 3.24 *(Decoupling lemma) There exist constants $C_0, C_0' > 0$ which depend on s, α, β and k such that the inequalities*

$$\mathbb{E}\left|\frac{1}{1-\sigma_i\mathcal{R}(\lambda)}\right|^s \leq C_0$$

and

$$\mathbb{E}\left|\frac{\mathcal{R}_V(\lambda, a_j, y)}{1-\sigma_i\mathcal{R}(\lambda)}\right|^s \leq C_0'\mathbb{E}\left|\mathcal{R}_V(\lambda, a_j, y)\right|^s$$

hold for all $0 < s < 1/2$ and all $\lambda \in \mathbb{C}_+$ such that $\operatorname{Re}\lambda \in \mathcal{I}_k$.

Step V. For any fixed $y \in X$ and λ as above let us denote $\psi_i := \mathbb{E}\left|\mathcal{R}_V(\lambda, a_i, y)\right|^s$. Applying Decoupling lemma to inequality (3.67) and setting $C_1 := \beta^s C_0'$ we get an infinite system of inequalities

$$\psi_i \leq C_0\left|\mathcal{R}(\lambda, a_i, y)\right|^s + C_1 \sum_{j:\, j\neq i} \left|\mathcal{R}(\lambda, a_j, a_i)\right|^s \psi_j.$$

In the vector form this system reads as follows

$$\psi_i \leq g_i + (\mathcal{A}\psi)_i\,, \; i = 1, 2, \ldots,$$

where $\psi = (\psi_i)$, $g = (g_i)$ has entries $g_i = C_0 \left|\mathcal{R}(\lambda, a_i, y)\right|^s$, and where $\mathcal{A}$ is an infinite matrix with non-negative entries $\mathfrak{a}_{ij} = C_1 \left|\mathcal{R}(\lambda, a_j, a_i)\right|^s$ if $i \neq j$ and 0 otherwise.

Iterating formally this infinite system of inequalities we get

$$\psi_i \leq g_i + (\mathcal{A}g)_i + \left(\mathcal{A}^2 g\right)_i + \left(\mathcal{A}^3 g\right)_i + ... \leq \left((I - \mathcal{A})^{-1} g\right)_i .$$

In particular, this would yield the following inequality (one of the fundamental points in the proof of (3.63)),

$$\|\psi\| \leq 2 \|g\| \tag{3.68}$$

given $\mathcal{A} : \mathcal{L} \to \mathcal{L}$ is a bounded linear operator acting in some Banach space $\mathcal{L}$ of sequences such that

$$\|\mathcal{A}\| \leq 1/2. \tag{3.69}$$

For instance, choosing $\mathcal{L} = \{\psi : \|\psi\| = \sum_i \mu_i |\psi_i| < \infty\}$ we obtain

$$\sum_i \mu_i \mathbb{E} \left|\mathcal{R}_V(\lambda, a_i, y)\right|^s \leq 2C_0 \sum_i \mu_i \left|\mathcal{R}(\lambda, a_i, y)\right|^s \tag{3.70}$$

given

$$\|\mathcal{A}\| = \sup_{\|\psi\|=1} \sum_i \mu_i \left|(\mathcal{A}\psi)_i\right| \leq 1/2. \tag{3.71}$$

For ψ such that $\|\psi\| = 1$ we have

$$\begin{aligned} \sum_i \mu_i \left|(\mathcal{A}\psi)_i\right| &\leq \sum_i \mu_i \sum_j \mathfrak{a}_{ij} |\psi_j| \\ &= \sum_j \mu_j |\psi_j| \left(\sum_i \mu_i \mathfrak{a}_{ij}\right) / \mu_j \leq \sup_j \left(\sum_i \mu_i \mathfrak{a}_{ij}\right) / \mu_j . \end{aligned}$$

In particular, inequality (3.71) holds whenever

$$\sup_j \left(\sum_{i:\ i \neq j} \mu_i \left|\mathcal{R}(\lambda, a_j, a_i)\right|^s\right) / \mu_j \leq \frac{1}{2C_1}. \tag{3.72}$$

Finally, 3.44 together with (3.72) allow us to conclude that (3.69) holds provided

$$\sup_j \left(\sum_{i:\ i \neq j} \mu_i \frac{1}{\mathrm{d}(a_j, a_i)^s}\right) / \mu_j \leq \frac{1}{2C_1 C_2}. \tag{3.73}$$

<u>Step VI.</u> For λ as above and $\varepsilon_j > 0$ which we will choose later consider events

$$A_j = \{|\mathcal{R}_V(\lambda, a_j, y)| > \varepsilon_j\}.$$

Applying Chebyshev inequality, we will get, for each $j = 1, 2, ...,$ the following inequality

$$P(A_j) \leq \frac{\mathbb{E} \left|\mathcal{R}_V(\lambda, a_j, y)\right|^s}{\varepsilon_j^s}. \tag{3.74}$$

Equations (3.74), (3.70) and (3.73), yield

$$\sum_j P(A_j) \le \sum_j \frac{\mathbb{E}\,|\mathcal{R}_V(\lambda, a_j, y)|^s}{\varepsilon_j^s}$$
$$\le 2C_0 \sum_j \frac{|\mathcal{R}(\lambda, a_j, y)|^s}{\varepsilon_j^s} \le 2C_0C_2 \sum_j \frac{1}{\mathrm{d}(a_j, y)^s \varepsilon_j^s}$$

provided ε_j are chosen such that

$$\sup_j \varepsilon_j^s \left(\sum_{i:\, i\neq j} \frac{1}{\varepsilon_i^s} \frac{1}{\mathrm{d}(a_j, a_i)^s} \right) \le \frac{1}{2C_1C_2}. \tag{3.75}$$

Let us choose $s = 1/2 - \delta$ and $\varepsilon_j = 1/\mathrm{d}(a_j, y)^r$. Then, truncating if necessary the potential V, i.e. passing to the potential $V'' = V - V'$ with V' of finite rank as explained in Step I, we can assume that the sequence ε_j is a strictly decreasing sequence. By the ultrametric inequality, we have $\mathrm{d}(a_i, a_j) = \mathrm{d}(a_i, y)$. Hence

$$\sup_j \varepsilon_j^s \left(\sum_{i:\, i\neq j} \frac{1}{\varepsilon_i^s} \frac{1}{\mathrm{d}(a_j, a_i)^s} \right) \le \sup_j \left(\sum_{i:\, i<j} \frac{1}{\mathrm{d}(a_j, a_i)^s} \right) + \sup_j \varepsilon_j^s \left(\sum_{i:\, i>j} \frac{\mathrm{d}(a_i, y)^{rs}}{\mathrm{d}(a_j, a_i)^s} \right)$$
$$\le \sup_j \left(\sum_{i:\, i<j} \frac{1}{\mathrm{d}(a_j, a_i)^s} \right) + \sup_j \varepsilon_j^s \left(\sum_{i:\, i>j} \frac{1}{\mathrm{d}(a_j, a_i)^{(1-r)s}} \right)$$
$$\le \sup_j \left(\sum_{i:\, i<j} \frac{1}{\mathrm{d}(a_j, a_i)^s} \right) + \sup_j \left(\sum_{i:\, i\neq j} \frac{1}{\mathrm{d}(a_j, a_i)^{(1-r)s}} \right).$$

Thus, truncating the potential V and then choosing $0 < \delta < 1/2 - r/(1-r)$ we obtain inequality (3.75). Moreover, thanks to our choice, the series $\sum_j \varepsilon_j$ converges. Hence of course converges the series $\sum_j P(A_j)$. Applying the Borel-Cantelli lemma we conclude: For $P-$a.s. ω there exists $j_0(\omega)$ such that

$$|\mathcal{R}_V(\lambda, a_j, y)| \le \varepsilon_j, \text{ for all } j \ge j_0(\omega), \tag{3.76}$$

holds for all $\lambda \in \mathbb{C}_+$ such that $\operatorname{Re}\lambda \in \mathcal{I}_k$.

Step VII. For λ as above, the function $\mathcal{R}(\lambda, x, y) = (L - \lambda \mathrm{I})^{-1}\delta_y(x)$ belongs to $L^2(x, m)$ and, by the homogeneity assumption, its norm $\|\mathcal{R}(\lambda, \cdot, y)\|_2$ does not depend on y. Having this in mind we write inequality (3.65) (for the truncated potential V'')

$$\|\mathcal{R}_{V''}(\lambda, \cdot, y)\|_2 \le \|\mathcal{R}(\lambda, \cdot, y)\|_2 \left(1 + \beta \sum_j |\mathcal{R}_{V''}(\lambda, a_j, y)| \right)$$
$$\le \|\mathcal{R}(\lambda, \cdot, y)\|_2 \left(1 + \beta \sum_{j\ge j_0(\omega)} \varepsilon_j + \beta \sum_{j<j_0(\omega)} |\mathcal{R}_{V''}(\lambda, a_j, y)| \right)$$

which clearly holds for all λ as above and for $P-$a.s. ω. As $\operatorname{Im}\lambda \downarrow 0$ we get finite limit for $P-$a.s. ω and for each $\lambda \in \mathcal{I}_k$ which does not belong to some exceptional set $\mathcal{I}_k(\omega) \subset \mathcal{I}_k$ of Lebesgues measure zero (the exceptional set appears because we pass to the boundary values of the Cauchy-Stieltjes integrals $\mathcal{R}_{V''}(\lambda, a_j, y)$, $j < j_0(\omega)$, as explained in Theorem 3.21). This is precisely what we claim in equation (3.66). The proof is finished. ■

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