Hierarchical Schrödinger operators with singular potentials

Alexander Bendikov Alexander Grigor'yan Stanislav Molchanov

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Abstract

We consider the operator $H = L+V$ that is a perturbation of the Taibleson-Vladimirov operator $L = \mathfrak{D}^\alpha$ by a potential $V(x) = b\,\|x\|^{-\alpha}$, where $\alpha > 0$ and $b \geq b_*$, and prove that the operator H is closable and its minimal closure is a non-negative definite self-adjoint operator (where the critical value b_* depends on α). While the operator H is non-negative definite, the potential $V(x)$ may well take negative values as $b_* < 0$ for all $0 < \alpha < 1$. The equation $Hu = v$ admits a Green function $g_H(x, y)$, that is the integral kernel of the operator H^{-1}. We obtain sharp lower and upper bounds on the ratio of the Green functions $g_H(x, y)$ and $g_L(x, y)$.

Contents

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1 Introduction

The spectral theory of nested fractals similar to the Sierpinski gasket, i.e. the spectral theory of the corresponding Laplacians, is well understood. It has several important features: Cantor-like structure of the essential spectrum and, as result, the large number of spectral gaps, presence of infinite number of eigenvalues each of which has infinite multiplicity and compactly supported eigenstates, non-regularly varying heat kernels which contain oscillating in $\log t$ scale terms etc, see P. J. Grabner and W. Woess [21], G. Derfel and P. J. Grabner [16] and A. Bendikov, W. Cygan and W. Woess [8].

The spectral properties mentioned above occur in the very precise form for the Taibleson-Vladimirov Laplacian $\mathfrak{D}^\alpha$, the operator of fractional derivative of order α. This operator can be introduced in several different forms, say, as $L^2(\mathbb{Q}_p)$-multiplier where $\mathbb{Q}_p$ is the ring of p-adic numbers, see works of M. H. Taibleson [40], V. S. Vladimirov, I. V. Volovich, E. I. Zelenov [41]), [42], [43] and A. N. Kochubey [26].

The operator $\mathfrak{D}^\alpha$ is unitary equivalent to a hypersingular integral operator L acting in $L^2(0,+\infty)$,

$$Lf(x) = \int_0^\infty (f(x) - f(y))\, J(x,y)dy, \tag{1.1}$$

the kernel $J(x,y)$ will be specified in this section. We refer to the articles A. Bendikov [3], A. Bendikov and P. Krupski [6], and S. V. Kosyrev [27]. See also related articles S. Albeverio and W. Karwowski [2], A. Bendikov, A. Grigor'yan, S. A. Molchanov, G. P. Samorodnitsky and W. Woess [4], [5], [7], F. J. Dyson [17], S. A. Molchanov [34], [35], M. Del Muto and A. Figà-Talamanca [15], J. J. Rodríges-Vega and W. A. Zúñiga-Galindo [39], W. A. Zúñiga-Galindo [44].

Let us briefly outline the construction of the operator L. The equivalence $\mathfrak{D}^\alpha \simeq L$ will follow from the fact that $\mathfrak{D}^\alpha$ and L are essentially self-adjoint operators having pure point spectrums which consist of eigenvalues of infinite multiplicity with 0 as unique limit point, as subsets of $[0,+\infty)$ their spectrums coincide. For detailed exposition we refer the interested reader to the article A. Bendikov [3].

The ultrametric space Let us fix an integer $p \geq 2$ and consider the family of partitions $\{\Pi_r : r \in \mathbb{Z}\}$ of the set $X = [0,+\infty)$ such that each Π_r consists of all p-adic intervals $I = [kp^r, (k+1)p^r)$. We call r the rank of the partition Π_r (respectively, the rank of the interval $I \in \Pi_r$). Each interval of rank r is the union of p disjoint intervals of rank $(r-1)$. Each point $x \in X$ belongs to a certain interval $I_r(x)$ of rank r, and the intersection of all intervals $I_r(x)$, $r \in \mathbb{Z}$, is $\{x\}$.

The hierarchical distance $d(x,y)$ is defined as zero if $x = y$ and as the length $l(I)$ of the minimal p-adic interval I which contains x and y. Since any two points $x \neq y$ belong to a certain p-adic interval, $d(x,y) < \infty$. Clearly $d(x,y) = 0$ if and only if $x = y$, and $d(x,y) = d(y,x)$. Moreover, for arbitrary x, y and z holds the *ultrametric inequality* (which is stronger than the triangle inequality)

$$d(x,y) \leq \max\{d(x,z), d(z,y)\}. \tag{1.2}$$

The ultrametric space (X,d) is *complete, separable, non-compact* and *proper* metric space. In (X,d) the set of all open balls is countable and coincides with the set of all p-adic intervals. In particular, any two balls either do not intersect or one is a subset of another. Thus (X,d) is a totally disconnected topological space.

The Borel σ-algebra generated by the ultrametric balls coincides with the Borel σ-algebra generated by the Euclidean balls.

As it follows from a general theorem that is due to M. Del Muto and A. Figà-Talamanca [15, Section 2], the non-compact ultrametric space (X,d) is isometrically isomorphic to the ring of p-adic numbers $\mathbb{Q}_p$ equipped with its canonical ultrametric $\|x-y\|_p$. See Theorem 2.2 in Section 2.1 and discussion after this theorem about possible identification of a homogeneous ultrametric space with a totally disconnected locally compact Abelian group.

The hierarchical Laplacian. Let $\mathcal{D}$ be the set of all compactly supported locally constant functions. Let $\kappa \in]0,1[$ be a fixed parameter. *The hierarchical Laplacian* L is defined as a sum of Laplacians (i.e. minus Markov generators) L_r of pure jump processes

$$(Lf)(x)=\sum_{r=-\infty}^{+\infty}\underbrace{(1-\kappa)\kappa^{r-1}\left(f(x)-\frac{1}{l(I_r(x))}\int\limits_{I_r(x)} f dl\right)}_{(L_rf)(x)}. \tag{1.3}$$

The series in (1.3) diverges in general but it is finite and belongs to all spaces $L^p(0,\infty)$, $p\geq 1$, for any function $f\in\mathcal{D}$.

As each "elementary" Laplacian L_r can be written in the form

$$L_rf(x)=\int\limits_0^\infty \left(f(x)-f(y)\right)J_r(x,y)dy,$$

$$J_r(x,y)dy=\underbrace{(1-\kappa)\kappa^{r-1}}_{\lambda_r(x)}\cdot\underbrace{\mathbf{1}_{I_r(x)}(y)/l(I_r(x))dy}_{\mathcal{U}_r(x,dy)},$$

the operator L coincides with a hypersingular integral operator

$$Lf(x)=\int\limits_0^\infty \left(f(x)-f(y)\right)J(x,y)dy,$$

$$J(x,y)=\frac{\kappa^{-1}-1}{1-\kappa p^{-1}}\cdot\frac{1}{d(x,y)^{1+\alpha}},\ \ \alpha=-\frac{\log\kappa}{\log p}.$$

The operator L admits a complete system of compactly supported eigenfunctions. Indeed, let I be a p-adic interval of rank r, and $I_1,I_2,...,I_p$ be its p-adic subintervals of rank $r-1$. Let us consider p functions

$$\psi_{I_i}=\frac{1_{I_i}}{l(I_i)}-\frac{1_I}{l(I)}.$$

Each function ψ_{I_i} belongs to $\mathcal{D}$ and satisfies the equation

$$L\psi_{I_i}=\kappa^{r-1}\psi_{I_i}.$$

A Markov process $\{X(t),P_x\}$ with state space $\mathcal{X}$ is called a pure jump process if, starting from any point $x\in\mathcal{X}$, it has all sample paths constant except for isolated jumps, and right-continuous.

The basic data which defines the process are (i) a function $0<\lambda(x)<\infty$, and (ii) a Markov kernel $\mathcal{U}(x,dy)$ satisfying $\mathcal{U}(x,\{x\})=0$. Its Lalacian (i.e.minus Markov generator) has the form

$$Lf(x)=\int_{\mathcal{X}}\left(f(x)-f(y)\right)\lambda(x)\mathcal{U}(x,dy).$$

Intuitively a particle starting from x remains there for an exponentialy distributed time with parameter $\lambda(x)$ at which time it "jumps" to a new position x' according to distribution $\mathcal{U}(x,\cdot)$ etc.

When I runs over the set of all p-adic intervals the set of eigenfunctions ψ_{I_i} forms a complete system in $L^2(0,\infty)$. In particular, L is essentially self-adjoint operator having a pure point spectrum

$$Spec(L) = \{0\} \cup \{\kappa^r : r \in \mathbb{Z}\}.$$

Each eigenvalue κ^r has infinite multiplicity. In particular, the spectrum of L coincides with its essential part. It follows that writing $\kappa = p^{-\alpha}$ the operator L can be identified with the Taibleson-Vladimirov operator $\mathfrak{D}^\alpha$, the operator of fractional derivative of order α acting in $L^2(\mathbb{Q}_p)$,

$$\mathfrak{D}^\alpha \psi(x) = -\frac{1}{\Gamma_p(-\alpha)} \int_{\mathbb{Q}_p} \frac{\psi(x)-\psi(y)}{\|x-y\|_p^{1+\alpha}} dm(y).$$

Towards the general theory. There are already several publications on the spectrum of the hierarchical Laplacian acting on a general ultrametric measure space (X,d,m), see S. Albeverio and W. Karwowski [2], M. Aisenman and S. A. Molchanov [1], [35], [34], A. Bendikov, A. Grigor'yan, P. Krupski, S.A. Molchanov, Ch. Pittet and W. Woess resp. [4], [5], [6], [7]. Accordingly, the hierarchical Schrödinger-type operator, the subject of the present work, was studied in F. J. Dyson [18], S. A. Molchanov, B. Vainberg [35], [36], [37], A. Bovier [13], E. Kritchevski [30], [31], [32] (the hierarchical lattice of Dyson) and in V. S. Vladimirov, I. V. Volovich, E. I. Zelenov and A. N. Kochuvey resp. [43], [42], [26] (the field of p-adic numbers).

By the general theory developed in A. Bendikov, A. Grigor'yan, P. Krupski, Ch. Pittet and W. Woess resp. [4], [5] and [6], any hierarchical Laplacian L acts in $L^2(X,m)$, is essentially self-adjoint non-negative definite operator. It can be represented as a hypersingular integral operator

$$Lf(x) = \int_X (f(x)-f(y))J(x,y)dm(y). \tag{1.4}$$

Respectively, the quadratic form $Q_L(u,u) := (L^{1/2}u, L^{1/2}u)_{L^2(X,m)}$ is a regular Dirichlet form having representation

$$Q_L(u,u) = \frac{1}{2} \int_{X\times X} (f(x)-f(y))^2 J(x,y)dm(x)dm(y). \tag{1.5}$$

The operator L has a pure point spectrum, its Markovian semigroup $(e^{-tL})_{t>0}$ admits with respect to m a continuous transition density $p(t,x,y)$. In terms of certain (intrinsically related to L) ultrametric $d_*(x,y)$ the functions $J(x,y)$ and $p(t,x,y)$ can be represented in the form

$$J(x,y) = \int_0^{1/d_*(x,y)} N(x,\tau)d\tau, \tag{1.6}$$

$$p(t,x,y) = t \int_0^{1/d_*(x,y)} N(x,\tau)\exp(-t\tau)d\tau. \tag{1.7}$$

The function $N(x,\tau)$, the so called *Spectral function*, will be specified in the next section.

Outline. Let us describe the main body of the paper. In Section 2 we introduce the notion of homogeneous hierarchical Laplacian L and list its basic properties e.g. the spectrum of the operator L is pure point, all eigenvalues of L have infinite multiplicity and compactly supported eigenfunctions, the heat kernel $p(t,x,y)$ exists and is a continuous function having certain asymptotic properties etc. For the basic facts related to the ultrametric analysis of heat kernels listed here we refer to A. Bendikov, A. Grigor'yan, P. Krupski, Ch. Pittet and W. Woess [4], [5], [6].

As a special example we consider the case $X=\mathbb{Q}_p$, the ring of p-adic numbers endowed with its standard ultrametric $d(x,y)=\|x-y\|_p$ and the normed Haar measure m. The hierarchical Laplacian L in our example coincides with the Taibleson-Vladimirov operator $\mathfrak{D}^\alpha$, the operator of fractional derivative of order α, see V. S. Vladimirov, I. V. Volovich, E. I. Zelenov and A. N. Kochubey resp. [41], [43], and [26]. The most complete source for the basic definitions and facts related to the p-adic analysis is N. Koblitz [25] and M. H. Taibleson [40].

In the next section we consider the Schrödinger type operator $H=\mathfrak{D}^\alpha+V$ with potential $V\in L^1_{loc}$ having local singularity, e.g. $V(x)=b\,\|x\|_p^{-\alpha}$, $0<\alpha<1$. The main aim here is to prove that under certain conditions on V the quadratic form

$$Q(u,u):=Q_{\mathfrak{D}^\alpha}(u,u)+Q_V(u,u)$$

where

$$Q_{\mathfrak{D}^\alpha}(u,u)=\int\left|\mathfrak{D}^{\alpha/2}u\right|^2dm,\ Q_V(u,u)=\int|u|^2\,Vdm$$

is semibounded and whence defines a self-adjoint operator H. Under certain conditions on V we will prove that $\mathcal{D}$, the set of locally constant compactly supported functions, is indeed a form core for $Q(u,u)$.

We also prove several results about the negative part of the spectrum of H. For instance, if $V\in L^p$ for some $p>1/\alpha$, then the operator H has essential spectrum equals to the spectrum of $\mathfrak{D}^\alpha$. In particular, if H has any negative spectrum, then it consists of a sequence of negative eigenvalues of finite multiplicity. If this sequence is infinite then it converges to zero.

In the concluding section we consider the operator $H=\mathfrak{D}^\alpha+b\,\|x\|_p^{-\alpha}$ assuming that $0<\alpha<1$ and $b\geq b_*$, the critical value which will be specified later. We will prove that the equation $Hu=v$ admits a fundamental solution $g_H(x,y)$ (the Green function of the operator H). The function $g_H(x,y)$ is continuous and takes finite values off the diagonal. Let $g_{\mathfrak{D}^\alpha}(x,y)$ be the Green function of the operator $\mathfrak{D}^\alpha$. The main result of this section is the following statement: for any $b\geq b_*$ there exists $\frac{\alpha-1}{2}\leq\beta<\alpha$ such that

$$\frac{g_H(x,y)}{g_{\mathfrak{D}^\alpha}(x,y)}\asymp\left(\frac{\|x\|_p}{\|y\|_p}\wedge\frac{\|y\|_p}{\|x\|_p}\right)^\beta,$$

where the sign $\asymp$ means that the ratio of the left- and right hand sides is bounded from below and above by positive constants. This result must be compared with the Green function estimates for Schrödinger operators on complete Riemannian manifolds, see A. Grigor'yan [22].

2 Preliminaries

2.1 Homogeneous ultrametric space

Let (X,d) be a locally compact and separable ultrametric space. Recall that a metric d is called a *ultrametric* if it satisfies the ultrametric inequality

$$d(x,y) \leq \max\{d(x,z), d(z,y)\}, \tag{2.1}$$

that is stronger than the usual triangle inequality. The basic consequence of the ultrametric property is that each open ball is a closed set. Moreover, each point x of a ball B can be regarded as its center, any two balls A and B either do not intersect or one is a subset of another etc. In particular, the ultrametric space (X,d) is totally disconnected, see A. Bendikov and P. Krupski [6] and references therein. In this paper we assume that the ultrametric space (X,d) is *proper*, that is, each closed ball is a compact set.

To any ultrametric space (X,d) one can associate in a standard fashion a tree $\mathcal{T}$. The vertices of the tree are metric balls. $\mathcal{T}$ is an ultrametric space if distance between two balls u and v is defined as diameter $\operatorname{diam}(u \curlywedge v)$ of the minimal ball $u \curlywedge v$ which contains both u and v. The boundary $\partial\mathcal{T}$ can be identified with the one-point compactification $X \cup \{\varpi\}$ of X.

We refer the reader to the article A. Bendikov and P. Krupski [6] for a treatment of the association between an ultrametric space and the tree of its metric balls.

Definition 2.1. An ultrametric measure space (X,d,m) is called *homogeneous* if the group of its isometries acts transitively and preserves the measure.

The following remarkable result is due to M. Del Muto and A. Figà-Talamanca [15, Section 2].

Theorem 2.2. *Any homogeneous ultrametric measure space (X,d,m) can be identified with certain locally compact Abelian group equipped with a translation invariant ultrametric and the Haar measure.*

For example, the set $X = [0,+\infty[$ equipped with the ultrametric structure generated by p-adic intervals and normed Lebesgues measure is a non-compact homogeneous ultrametric measure space. Its tree of balls clearly coincides with the tree of balls associated with the ring of p-adic numbers $\mathbb{Q}_p$, whence these two homogeneous ultrametric measure spaces are isometrically isomorphic.

The identification in Theorem 2.2 is not unique. One possible way to define such identification is to choose the sequence $a = \{a_n\}$ of forward degrees associated with the tree of balls $\mathcal{T}$ (cf. Fig. 1). This sequence is two-sided if X is non-compact and perfect (has no isolated points), it is one-sided if X is compact and perfect, or if X is discrete. In the 1st case we identify X with Ω_a, the ring of a-adic numbers, in the 2nd case with $\Delta_a \subset \Omega_a$, the ring of a-adic integers, and in the 3rd case with the discrete group $[\Omega_a : \Delta_a]$. Notice, that $\Omega_a = \mathbb{Q}_p$, the ring of p-adic numbers (resp. $\Delta_a = \mathbb{Z}_p$, the ring of p-adic integers) whenever all numbers a_n are the same and equal p. The group $[\mathbb{Q}_p : \mathbb{Z}_p]$ is just infinite product of cyclic groups Z_p of order p.

We refer the reader to the monograph E. Hewitt and K. A. Ross [23] for the comprehensive treatment of special groups Ω_a, Δ_a and $[\Omega_a : \Delta_a]$.

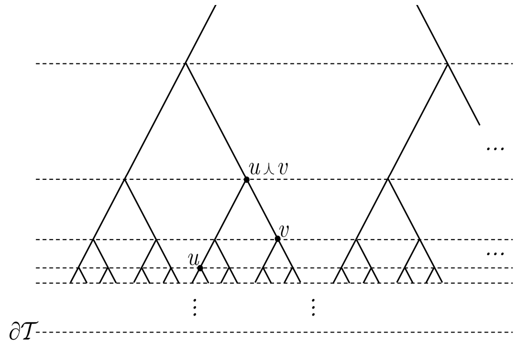

Figure 1: Tree of balls with forward degrees $a_n = 2$

2.2 Homogeneous hierarchical Laplacian

Let (X, d, m) be a non-compact homogeneous ultrametric measure space. Let $\mathcal{B}$ be the set of all open balls, $\mathcal{B}(x) \subset \mathcal{B}$ the set of balls centred at x, and $C : \mathcal{B} \to (0, \infty)$ a function satisfying the following conditions:

(i) $C(A) = C(B)$ for any two balls A and B of the same diameter,

(ii) $\lambda(B) := \sum\limits_{T \in \mathcal{B}:\ B \subseteq T} C(T) < \infty$ for all $B \in \mathcal{B}$,

(iii) $\sup_{B \in \mathcal{B}(x)} \lambda(B) = \infty$ for any non-isolated x.

The class of functions $C(B)$ satisfying these conditions is reach enough, e.g. one can choose

$$C(B) = (1/m(B))^{\alpha} - (1/m(B'))^{\alpha}$$

for any two closest neighboring balls $B \subset B'$. In this case $\lambda(B) = (1/m(B))^{\alpha}$.

The homogeneous hierarchical Laplacian L is defined (pointwise) as

$$Lf(x) := \sum_{B \in \mathcal{B}(x)} C(B) \left(f(x) - \frac{1}{m(B)} \int_B f dm \right). \tag{2.2}$$

In general, the series in (2.2) diverges but for $f \in \mathcal{D}$, the set of all locally constant compactly supported functions, it converges in Banach spaces $L^p(X, m)$, $1 \leq p < \infty$, and in $C_{\infty}(X)$.

Let us choose any two closest neighboring balls $B \subset B'$ and set

$$f_B = \frac{\mathbf{1}_B}{m(B)} - \frac{\mathbf{1}_{B'}}{m(B')}. \tag{2.3}$$

Then clearly $f_B \in \mathcal{D}$ and one can check that

$$Lf_B(x) = \lambda(B') f_B(x). \tag{2.4}$$

As the couple $B \subset B'$ runs over all nearest neighboring balls in $\mathcal{B}$ the system $\{f_B : B \in \mathcal{B}\}$ is complete. In particular, we conclude that $L : \mathcal{D} \to L^2(X, m)$ is an essentially self-adjoint operator.

The intrinsic ultrametric $d_*(x, y)$ associated with L is defined as follows

$$d_*(x, y) := \begin{cases} 0 & \text{when} \quad x = y \\ 1/\lambda(x \curlywedge y) & \text{when} \quad x \neq y \end{cases}, \tag{2.5}$$

where $x \curlywedge y$ is the minimal ball containing both x and y. In particular, for any non-singleton ball B we have

$$\lambda(B) = \frac{1}{\operatorname{diam}_*(B)}. \tag{2.6}$$

The spectral function $\tau \to N(\tau)$, see equation (1.6), is defined as the left-continuous step-function having jumps at the points $\lambda(B)$, and taking values

$$N(\lambda(B)) = 1/m(B).$$

The volume function $V(r)$ is defined by setting $V(r) = m(B)$ where the ball B has d_*-radius r. It is easy to see that

$$N(\tau) = 1/V(1/\tau). \tag{2.7}$$

The Markovian semigroup $P_t = e^{-tL}, t > 0$, admits a continuous density $p(t,x,y)$ w.r.t. m, we call it *the heat kernel.* The function $p(t,x,y)$ can be represented in the form (1.7).

For $\lambda > 0$ the Markovian resolvent $G_\lambda = (\lambda + L)^{-1}$ admits a continuous strictly positive integral kernel $g(\lambda,x,y)$. The operator G_λ is well defined for $\lambda = 0$ (i.e. the Markovian semigroup $(P_t)_{t>0}$ is transient) if and only if for some (equivalently, for all) $x \in X$ the reciprocal to the volume function $\tau \to 1/V(\tau)$ is integrable at ∞. The integral kernel $g(x,y) := g(0,x,y)$, called also the Green function, is of the form

$$g(x,y) = \int_r^{+\infty} \frac{d\tau}{V(\tau)}, \ r = d_*(x,y). \tag{2.8}$$

Under certain Tauberian conditions it takes the form

$$g(x,y) \asymp \frac{r}{V(r)}, \ r = d_*(x,y). \tag{2.9}$$

2.3 Subordination

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing homeomorphism. For any two nearest neighboring balls $B \subset B'$ we define

$$C(B) = \Phi\left(1/m(B)\right) - \Phi\left(1/m(B')\right). \tag{2.10}$$

Let L_Φ be the corresponding to $C(B)$ hierarchical Laplacian. The following properties hold true:

(i) $\lambda(B) = \Phi\left(1/m(B)\right)$. In particular, the Laplacians L_Φ and L_{Id} are related by the equation $L_\Phi = \Phi(L_{Id})$.

(ii) $d_*(x,y) = 1/\Phi\left(1/m(x \curlywedge y)\right)$.

(iii) $V(r) \leq 1/\Phi^{-1}(1/r)$.

(iv) $V(r) \asymp 1/\Phi^{-1}(1/r)$ whenever both Φ and Φ^{-1} are doubling and the inequality $m(B') \leq Cm(B)$ holds for some $C > 0$ and all nearest neighboring balls $B \subset B'$. In particular, in this case we have

$$p_\Phi(t,x,y) \asymp t \cdot \min\left\{\frac{1}{t}\Phi^{-1}\left(\frac{1}{t}\right), \frac{1}{m(x \curlywedge y)}\Phi\left(\frac{1}{m(x \curlywedge y)}\right)\right\}.$$

In the case $\Phi(\tau)$ is a Bernstein function the association $L_\Phi = \Phi(L_{Id})$ has been studied in the well-known Bochner's subordination theory W. Feller [19].

2.4 Multipliers

As a special case of the general construction consider $X = \mathbb{Q}_p$, the ring of p-adic numbers equipped with its standard ultrametric $\mathrm{d}(x,y) = \|x-y\|_p$. Notice that the ultrametric spaces $(\mathbb{Q}_p, \mathrm{d})$ and $([0,\infty), d)$ with non-Euclidean d, as explained in the introduction, are isometrically isomorphic (the isometry can be established via identification of their trees of metric balls).

Let $\mathcal{F} : f \to \widehat{f}$ be the Fourier transform of the function f. It is known, see M. H. Taibleson [40], V. S. Vladimirov, A. N. Kochubei [43], [26], that $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ is a bijection.

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing homeomorphism. The self-adjoint operator $\Phi(\mathfrak{D})$ we define as multiplier, that is,

$$\widehat{\Phi(\mathfrak{D})f}(\xi) = \Phi(\|\xi\|_p)\widehat{f}(\xi), \quad \xi \in \mathbb{Q}_p. \tag{2.11}$$

By A. Bendikov, A. Grigor'yan, Ch. Pittet and W. Woess [5, Theorem 3.1], $\Phi(\mathfrak{D})$ is a homogeneous hierarchical Laplacian. The eigenvalues $\lambda(B)$ of the operator $\Phi(\mathfrak{D})$ are numbers

$$\lambda(B) = \Phi\left(\frac{p}{m(B)}\right) = \Phi\left(\frac{p}{\mathrm{diam}(B)}\right). \tag{2.12}$$

Let $p_\Phi(t,x,y)$ be the heat kernel associated with the operator $\Phi(\mathfrak{D})$. Assuming that both Φ and Φ^{-1} are doubling we get the following relationship

$$p_\Phi(t,x,y) \asymp t \cdot \min\left\{\frac{1}{t}\Phi^{-1}\left(\frac{1}{t}\right), \frac{1}{\|x-y\|_p}\Phi\left(\frac{1}{\|x-y\|_p}\right)\right\}. \tag{2.13}$$

The Taibleson-Vladimirov operator $\mathfrak{D}^\alpha$ introduced in M. H. Taibleson [40] and V. S. Vladimirov [43] is the multiplier corresponding to the function $\Phi(\tau) = \tau^\alpha$. On the set $\mathcal{D}$ it can be represented in the form

$$\mathfrak{D}^\alpha \psi(x) = -\frac{1}{\Gamma_p(-\alpha)} \int_{\mathbb{Q}_p} \frac{\psi(x) - \psi(y)}{\|x-y\|_p^{1+\alpha}} dm(y), \tag{2.14}$$

where $\Gamma_p(z) = (1-p^{z-1})(1-p^{-z})^{-1}$ is the p-adic Gamma-function. The function $z \to \Gamma_p(z)$ is meromorphic in the complex plane $\mathbb{C}$ and satisfies the functional equation $\Gamma_p(z)\Gamma_p(1-z) = 1$.

By what we said above the heat kernel $p_\alpha(t,x,y)$, the transition density of the Markovian semigroup $(e^{-t\mathfrak{D}^\alpha})_{t>0}$, can be estimated as follows

$$p_\alpha(t,x,y) \asymp \frac{t}{(t^{1/\alpha} + \|x-y\|_p)^{1+\alpha}}, \tag{2.15}$$

In particular, the Markov semigroup $(e^{-t\mathfrak{D}^\alpha})_{t>0}$ is transient if and only if $\alpha < 1$. In the transient case the Green function $g_\alpha(x,y)$ can be computed explicitly

$$g_\alpha(x,y) = \frac{1}{\Gamma_p(\alpha)} \frac{1}{\|x-y\|_p^{1-\alpha}}. \tag{2.16}$$

For all facts listed above we refer the reader to A. Bendikov, A. Grigor'yan, P. Krupski, Ch. Pittet and W. Woess [4], [5] and [6].

2.5 The symbol of the hierarchical Laplacian

Identifying X with a locally compact Abelian group we can regard $-L$ as an isotropic Lévy generator. By (1.4), the operator L on $\mathcal{D}$ takes the form

$$Lf(x) = \int_X (f(x) - f(y))J(x-y)dm(y), \tag{2.17}$$

or equivalently, in terms of the Fourier transform,

$$\widehat{Lf}(\theta) = \widehat{L}(\theta) \cdot \widehat{f}(\theta),\ \theta \in \widehat{X}, \tag{2.18}$$

where $\widehat{X}$ is the dual Abelian group (e.g. $\widehat{\mathbb{Q}_p}$ can be identified with $\mathbb{Q}_p$) and

$$\widehat{L}(\theta) = \int_X [1 - \mathrm{Re}\,\langle h, \theta\rangle] J(h)dm(h). \tag{2.19}$$

The function $\widehat{L}(\theta) \geq 0$, the symbol of the Lévy generator $-L$, is a continuous *negative definite function* Ch. Berg and G. Forst [11]. In particular, the function $\sqrt{\widehat{L}(\theta)}$ is subadditive. By the subordination property A. Bendikov, A. Grigor'yan, Ch. Pittet and W. Woess [5, Theorem 3.1], the function $\widehat{L}(\theta)^2$ is the symbol of symmetric Lévy generator $-L^2$, so the function $\widehat{L}(\theta) = \sqrt{\widehat{L}(\theta)^2}$ is subadditive as well, i.e. it satisfies the triangle inequality

$$\widehat{L}(\theta_1 + \theta_2) \leq \widehat{L}(\theta_1) + \widehat{L}(\theta_2). \tag{2.20}$$

Since $-L$ is an isotropic Lévy generator [5, Sec. 5.2] , a stronger property holds true

Theorem 2.3. *The function $\widehat{L}(\theta)$ satisfies the ultrametric inequality*

$$\widehat{L}(\theta_1 + \theta_2) \leq \max\{\widehat{L}(\theta_1), \widehat{L}(\theta_2)\}. \tag{2.21}$$

Proof. In order to simplify notation we assume that $X = \mathbb{Q}_p$, the ring of p-adic numbers. Let $B \subset B'$ be two nearest neighboring balls centred at the neutral element. Notice that both B and B' are compact subgroups of the group $\mathbb{Q}_p$, say $B = p^{-k}\mathbb{Z}_p$ and $B' = p^{-k-1}\mathbb{Z}_p$. Applying the Fourier transform to the both sides of equation (2.4) we get

$$\widehat{L}(\theta)\widehat{f_B}(\theta) = \lambda(B')\widehat{f_B}(\theta). \tag{2.22}$$

The measure $\omega_B = (\mathbf{1}_B m)/m(B)$ is the normalized Haar measure of the compact subgroup B, similarly for $\omega_{B'}$. Since for any locally compact Abelian group, the Fourier transform of the normalized Haar measure of any compact subgroup A is the indicator of its annihilator group $A^\perp$, and in our particular case $B^\perp = p^k\mathbb{Z}_p$ and $(B')^\perp = p^{k+1}\mathbb{Z}_p$, we obtain

$$\widehat{f_B}(\theta) = \mathbf{1}_{B^\perp}(\theta) - \mathbf{1}_{(B')^\perp}(\theta) = \mathbf{1}_{\partial B^\perp}(\theta), \tag{2.23}$$

where $\partial B^\perp$ is the sphere $B^\perp \setminus (B')^\perp$. Equations (2.23) and (2.4) imply that the function $\widehat{L}(\theta)$ takes constant value $\lambda(B')$ on the sphere $\partial B^\perp$, i.e. $\widehat{L}(\theta) = \psi(\|\theta\|_p)$ for some function $\psi(\tau)$ such that $\psi(0) = 0$ and $\psi(+\infty) = +\infty$. Since $C \subset D$ implies $\lambda(C) > \lambda(D)$, the function $\psi(\tau)$ can be chosen to be continuous and increasing, so $\widehat{L}(\theta) = \psi(\|\theta\|_p)$ satisfies the ultrametric inequality (2.21) as claimed. ∎

3 Schrödinger-type operators

Let (X, d, m) be a homogeneous ultrametric measure space and L a homogeneous hierarchical Laplacian on it. In this section we embark on the study of Schrödinger-type operators

$$Hf(x) = Lf(x) + V(x)f(x).$$

Our goal is to find conditions such that one can associate with the equation above a self-adjoint operator H acting in $L^2(X, m)$.

3.1 Locally bounded potentials

If we assume that the potential V is a locally bounded function then

$$(Hu)(x) := (Lu)(x) + V(x)u(x)$$

is a well defined symmetric operator $H : \mathcal{D} \to L^2(X, m)$. For the proof of the following theorem we refer to the paper A. Bendikov, A. Grigor'yan and S. A. Molchanov [9, Theorem 3.1]

Theorem 3.1. *Assume that V is a locally bounded function, then*

1. The operator H is essentially self-adjoint.

2. If $V(x) \to +\infty$ as $x \to \varpi$, then the self-adjoint operator H has a compact resolvent. (Thus, its spectrum is discrete).

3. If $V(x) \to 0$ as $x \to \varpi$, then the essential spectrum of H coincides with the spectrum of L. (Thus, the spectrum of H is pure point and the negative part of the spectrum consists of isolated eigenvalues of finite multiplicity).

Remark 3.2. For the classical Schrödinger operator $H = -\Delta + V$ defined on the set of compactly supported smooth functions the statement similar to the statement 1 of Theorem 3.1 is known as the Sears's theorem: H is essentially self-adjoint if the potential V admits a low bound

$$V(x) \geq -Q(|x|),$$

where $Q(r) > 0$ is a continuous non-decreasing function such that

$$\int_0^\infty Q(r)^{-1/2} dr = \infty,$$

and H may fail to be essentially self-adjoint otherwise, see F. A. Beresin and M. A. Shubin [10, Chapter II, Theorem 1.1 and Example 1.1].

3.2 Potentials with local singularities

If we are interested in potentials with local singularities, such as $V(x) = b\,\|x\|_p^{-\beta}$, $b \in \mathbb{R}$, then certain local conditions on the potential V are necessary in order to prove that the quadratic form

$$Q(u, u) := Q_L(u, u) + Q_V(u, u), \tag{3.1}$$

defined on the set

$$dom(Q) := dom(Q_L) \cap dom(Q_V)$$

is a densely defined bounded below closed quadratic form, whence it is associated to a bounded below self-adjoint operator H, see E. B. Davies [14, Section 4.4]. It is customary to write $H = L+V$, but it must be remembered that this is a quadratic form sum and not an operator sum as in the previous subsection.

Theorem 3.3. *If* $0 \le V \in L^1_{loc}(X,m)$, *then the quadratic form* (3.1) *is a regular Dirichlet form. In particular, it is the form of a non-negative self-adjoint operator* H,

$$Q(u,u) = (H^{1/2}u, H^{1/2}u)$$

and the set $\mathcal{D}$ *is a core for* Q.

Proof. The set $\mathcal{D}$ belongs to both $dom(Q_L)$ and $dom(Q_V)$ hence Q is densely defined. Set $V_\tau = V \wedge \tau$ and define on the set $dom(Q_L)$ the form

$$Q^\tau(u,u) = Q_L(u,u) + Q_{V_\tau}(u,u).$$

Since V_τ is bounded the form Q^τ is closed. In particular, the function $u \to Q^\tau(u,u)$ is lower semicontinuous. It follows that the function $u \to Q(u,u) = \sup\{Q^\tau(u,u) : \tau > 0\}$ is lower semicontinuous as well. Hence by [14, Theorem 4.4.2] the form Q is closed, and thus it is the form of a non-negative definite self-adjoint operator H. Clearly the form Q is Markovian, i.e. the normal contraction operates on $(Q, \mathcal{F})$ where $\mathcal{F} = dom(Q)$. Thus Q is a Dirichlet form. Let us show that $\mathcal{D}$ is a core for Q, i.e. that Q is a regular Dirichlet form, see M. Fukushima [20]. **Step 1** For $u \in dom(Q)$ we set $u_n = ((-n) \vee u) \wedge n$, then $u_n \in dom(Q)$ and $Q(u-u_n, u-u_n) \to 0$, see M. Fukushima [20, Theorem 1.4.2]. Therefore the set of bounded functions in $dom(Q)$ is a core for Q. **Step 2** Let B be a ball centred at the neutral element. Let $u \in dom(Q_L)$ be bounded and $u_B = 1_B \cdot u$. The function 1_B is in $\mathcal{D} \subset dom(Q_L)$, whence applying M. Fukushima [20, Theorem 1.4.2] we get: $u_B \in dom(Q_L)$ and

$$\sqrt{Q_L(u_B,u_B)} \le \sqrt{Q_L(u,u)} + \|u\|_\infty \cdot \sqrt{Q_L(1_B,1_B)}.$$

The following auxiliary result is of its own interest: Let $B' \supset B$ be the closest neighboring balls and $\lambda(B')$ the eigenvalue of L corresponding to the ball B', see equations (2.3) and (2.4), then

$$\frac{1}{2}m(B)\lambda(B') < Q_L(1_B,1_B) < 2m(B)\lambda(B'). \tag{3.2}$$

Indeed, to prove inequality (3.2) we write

$$\frac{1_B}{m(B)} = \sum_{T\in\mathcal{B}:\ B\subseteq T} f_T$$

where the series converges in $L^2(X,m)$. As $\lambda(C) < \lambda(D)$ for any two balls $C \supset D$ the series below converges in $L^2(X,m)$ (and also in uniform metric) and

$$L1_B = m(B)\sum_{T\in\mathcal{B}:\ B\subseteq T} Lf_T = m(B)\sum_{T\in\mathcal{B}:\ B\subseteq T} \lambda(T')f_T.$$

Hence $1_B \in dom(L)$ and the following equation holds true

$$\begin{aligned} Q_L(1_B,1_B) = (L1_B,1_B) &= m(B)^2 \sum_{T\in\mathcal{B}:\ B\subseteq T} \lambda(T')\,\|f_T\|^2 \\ &= m(B)^2 \sum_{T\in\mathcal{B}:\ B\subseteq T} \lambda(T')\left(\frac{1}{m(T)} - \frac{1}{m(T')}\right). \end{aligned}$$

In turn, the above identity yield the desired inequalities

$$Q_L(1_B,1_B) > m(B)\lambda(B')\left(1 - \frac{m(B)}{m(B')}\right) \ge \frac{1}{2}m(B)\lambda(B')$$

and

$$Q_L(1_B,1_B) < m(B)\lambda(B') \sum_{T\in\mathcal{B}:\ B\subseteq T} \frac{m(B)}{m(T)} \leq 2m(B)\lambda(B')$$

as it was claimed. In particular, if we assume that

$$\lim_{B\nearrow X} m(B)\lambda(B') = 0 \tag{3.3}$$

(as it happens in the case of the operator $L = \mathfrak{D}^{\alpha},\ \ \alpha > 1$) then

$$\limsup_{B\nearrow X} Q_L(u_B,u_B) \leq Q_L(u,u). \tag{3.4}$$

On the other hand, since the form Q_L is closed the function $u \to Q_L(u,u)$ is lower semicontinuous whence we get

$$\liminf_{B\nearrow X} Q_L(u_B,u_B) \geq Q_L(u,u). \tag{3.5}$$

Thus, assuming that (3.3) holds, we obtain

$$\lim_{B\nearrow X} Q_L(u_B,u_B) = Q_L(u,u). \tag{3.6}$$

By the Lebesgue's convergence theorem,

$$\lim_{B\nearrow X} Q_V(u_B,u_B) = Q_V(u,u). \tag{3.7}$$

Hence applying (3.6) and (3.7) we get

$$\lim_{B\nearrow X} Q(u_B,u_B) = Q(u,u). \tag{3.8}$$

Step 3 Let $(R_\lambda)_{\lambda>0}$ be the Markov resolvent corresponding to Q. Let $Q_1(f,g) := Q(f,g) + (f,g)$. Then for any function $v \in L^2(X,m)$ by the Lebesgue's convergence theorem

$$Q_1(u_B, R_1 v) = (u_B, v) \to (u,v) = Q_1(u, R_1 v)$$

Since $R_1(L^2(X,m))$ is a dense set in the Hilbert space $(\mathcal{F}, \|\cdot\|_*)$, where $\mathcal{F} = dom(Q)$ and $\|u\|_* = \sqrt{Q_1(u,u)}$, the sequence u_B converges weakly in $(\mathcal{F}, \|\cdot\|_*)$ to u, i.e.

$$Q_1(u_B,w) \to Q_1(u,w), \quad \forall w \in dom(Q). \tag{3.9}$$

Using equations (3.8) and (3.9) we obtain:

$$\begin{aligned}\lim_{B\nearrow X} Q_1(u-u_B, u-u_B) &= \lim_{B\nearrow X} \left(Q_1(u,u) - 2Q_1(u_B,u) + Q_1(u_B,u_B)\right)\\ &= Q_1(u,u) - 2\lim_{B\nearrow X} Q_1(u_B,u) + \lim_{B\nearrow X} Q_1(u_B,u_B)\\ &= Q_1(u,u) - 2Q_1(u,u) + Q_1(u,u) = 0.\end{aligned}$$

Thus, if condition (3.8) holds, the set of bounded functions with compact support in $dom(Q)$ is a core for Q as desired. **Step 4** In order to prove property (3.8) without any limitation on the spectrum of L we are forced to apply the Fourier transform argument and the metric properties of the symbol $\widehat{L}(\theta)$ of the operator L. To simplify notation we assume that $X = \mathbb{Q}_p$ so that $\widehat{X} = \mathbb{Q}_p$. Any ball B centred at the neutral element is a compact subgroup of X.

Since the Fourier transform of the normalized Haar measure of a compact subgroup is the indicator of its annihilator group, we obtain

$$\begin{aligned}Q_L(u_B,u_B) &= \int_{\widehat{X}} \widehat{L}(\theta)\,|\widehat{u_B}(\theta)|^2\,d\widehat{m}(\theta)\\ &= \int_{\widehat{X}} \widehat{L}(\theta)\,|\widehat{u}*\widehat{m}_{B^\perp}(\theta)|^2\,d\widehat{m}(\theta),\end{aligned}$$

where $B^\perp$ is the annihilator group of the compact subgroup $B\subset X$ and $\widehat{m}_{B^\perp}$ is the normed Haar measure of $B^\perp$. Having this in mind and using the inequality

$$|\widehat{u}*\widehat{m}_{B^\perp}|^2 \le |\widehat{u}|^2 * \widehat{m}_{B^\perp}$$

we get

$$\begin{aligned}Q_L(u_B,u_B) &\le \int_{\widehat{X}} \widehat{L}(\theta)\left(|\widehat{u}|^2*\widehat{m}_{B^\perp}\right)(\theta)d\widehat{m}(\theta)\\ &= \int_{\widehat{X}} \widehat{L}(\theta)\left(\int_{B^\perp}|\widehat{u}(\theta+\zeta)|^2\,d\widehat{m}_{B^\perp}(\zeta)\right)d\widehat{m}(\theta)\\ &= \int_{B^\perp}\left(\int_{\widehat{X}}\widehat{L}(\theta+\zeta)\,|\widehat{u}(\theta)|^2\,d\widehat{m}(\theta)\right)d\widehat{m}_{B^\perp}(\zeta).\end{aligned}$$

By Theorem 2.3 $\widehat{L}(\theta)=\psi(\|\theta\|_p)$, where $\psi(\tau)$ is an increasing continuous function. It follows that $\theta\to\widehat{L}(\theta)$ satisfies the ultrametric inequality (2.21), and therefore

$$\begin{aligned}Q_L(u_B,u_B) &\le \int_{B^\perp}\left(\int_{\widehat{X}}\max\left\{\widehat{L}(\theta),\widehat{L}(\zeta)\right\}|\widehat{u}(\theta)|^2\,d\widehat{m}(\theta)\right)d\widehat{m}_{B^\perp}(\zeta)\\ &\le \int_{B^\perp}\left(\int_{\widehat{X}}\left(\widehat{L}(\theta)+\widehat{L}(\zeta)\right)|\widehat{u}(\theta)|^2\,d\widehat{m}(\theta)\right)d\widehat{m}_{B^\perp}(\zeta).\end{aligned}$$

As $\widehat{m}_{B^\perp}(1)=1$ all the above yield the following inequality

$$Q_L(u_B,u_B)\le Q_L(u,u)+\left(\int_{B^\perp}\widehat{L}(\zeta)d\widehat{m}_{B^\perp}(\zeta)\right)\|u\|^2.$$

When $B\nearrow X$ the measure $\widehat{m}_{B^\perp}$ converges weakly to the Dirac measure concentrated at the neutral element. As $\widehat{L}(0)=0$ we finally get

$$\limsup_{B\nearrow X} Q_L(u_B,u_B)\le Q_L(u,u). \tag{3.10}$$

Evidently (3.10), (3.5) and (3.7) yield the equation

$$\lim_{B\nearrow X} Q(u_B,u_B)=Q(u,u)$$

which holds without any restriction on the spectrum of the operator L. Thus, as in Step 3, we come to conclusion that the set of bounded compactly supported functions in $dom(Q)$ is a core for Q as desired. **Step 5** Let now $u\in dom(Q)$ be bounded and has a compact support. Let B be a ball centred at the neutral element of X (recall that B is a compact subgroup of X) and m_B be its normed Haar measure. We set $u^B=u*m_B$. The function u^B is locally constant and has a compact support, hence it belongs to $\mathcal{D}\subset dom(Q)$. We have $\widehat{u^B}=\widehat{u}\cdot 1_{B^\perp}$ whence

$$\lim_{B\to\{e\}}\left\|u-u^B\right\|_2^2=\lim_{B^\perp\to\widehat{X}}\int_{(B^\perp)^c}|\widehat{u}(\theta)|^2\,d\widehat{m}(\theta)=0, \tag{3.11}$$

similarly

$$\lim_{B\to\{e\}} Q_L(u-u^B,u-u^B) = \lim_{B^\perp\to\widehat{X}} \int\limits_{(B^\perp)^c} \widehat{L}(\theta)\left|\widehat{u}(\theta)\right|^2 d\widehat{m}(\theta) = 0. \tag{3.12}$$

There exists a compact set K which contains the support of every function $u-u^B$ provided $diam(B) \le 1$. Given $\varepsilon > 0$ there exists a decomposition $V|_K = V_1 + V_2$ such that $\|V_1\|_1 < \varepsilon$ and $V_2 \in L^\infty(X,m)$. It follows that

$$\begin{aligned} Q_V(u-u^B,u-u^B) &= \int\limits_K V\left|u-u^B\right|^2 dm \\ &= \int\limits_K V_1\left|u-u^B\right|^2 dm + \int\limits_K V_2\left|u-u^B\right|^2 dm \\ &\le 4\varepsilon \|u\|_\infty^2 + \|V_2\|_\infty \left\|u-u^B\right\|_2^2 \end{aligned}$$

whence

$$\limsup_{B\to\{e\}} Q_V(u-u^B,u-u^B) \le 4\varepsilon \|u\|_\infty^2 . \tag{3.13}$$

Clearly equations (3.11), (3.12) and (3.13) yield the desired result

$$\lim_{B\to\{e\}} Q_1(u-u^B,u-u^B) = 0,$$

i.e. $\mathcal{D}$ is indeed a core for $Q = Q_L + Q_V$. ■

Remark 3.4. It is clear that Theorem 3.3 can be extended for those V which are bounded below and in $L^1_{loc}(X,m)$ by simply adding a large enough positive constant. If, however, we are interested in V with negative local singularities, then stronger local conditions on V are necessary in order to be able to prove that the form Q is closed.

Definition 3.5. Let $p \ge 1$ be fixed. We say that a potential V lies in $L^p + L^\infty$ if one can write $V = V' + V''$where $V' \in L^p(X,m)$ and $V'' \in L^\infty(X,m)$. This decomposition is not unique, and, if it is possible at all, then one can arrange for $\|V'\|_p$ to be as small as one chooses.

Theorem 3.6. *Consider $X = \mathbb{Q}_p$ equipped with the Haar measure m. Let $L = \mathfrak{D}^\gamma$ acting in $L^2(\mathbb{Q}_p,m)$ and $Q = Q_L + Q_V$ be the quadratic form (3.1) where $V \in L^p + L^\infty$ for some $p > 1/\gamma$. Then:*

1. Q is a densely defined bounded below closed quadratic form. In particular, Q is associated with a bounded below self-adjoint operator H.

2. If $2 \le 1/\gamma < p$ then $dom(H) = dom(\mathfrak{D}^\gamma)$. The same is true if $1/\gamma < 2$ and $p = 2$.

Proof. The set $\mathcal{D}$ belongs to both $dom(Q_L)$ and $dom(Q_V)$ whence Q is densely defined. Given $\varepsilon > 0$ we may write $|V| = W + W'$ where $\|W\|_p < \varepsilon$ and $W' \in L^\infty(X,m)$. We claim that if $\varepsilon > 0$ is sufficiently small, then

$$\left\|W^{1/2}u\right\|_2^2 \le \frac{1}{2}Q_L(u,u) + c_0\|u\|_2^2 \tag{3.14}$$

for some constant $c_0 > 0$ and all $u \in dom(Q_L)$. Clearly inequality 3.14 yield that

$$\begin{aligned} \int |V|\,|u|^2\,dm &\le \left\|W^{1/2}u\right\|_2^2 + \|W'\|_\infty \|u\|_2^2 \\ &\le \frac{1}{2}Q_L(u,u) + c_1\|u\|_2^2 \end{aligned}$$

for some constant $c_1 > 0$ and all $u \in dom(Q_L)$. Thus for $c_2 > 2c_1$ we get

$$\frac{1}{2}\left\{Q_L(u,u) + c_2 \|u\|_2^2\right\} \leq Q(u,u) + c_2 \|u\|_2^2 \leq \frac{3}{2}\left\{Q_L(u,u) + c_2 \|u\|_2^2\right\}.$$

It follows that the quadratic form $u \to Q(u,u) + c_2 \|u\|_2^2$ is non-negative and closed whence it is associated with a non-negative self-adjoint operator, which is clearly equal to $H + c_2 I$. To prove the inequality 3.14 we need some auxiliary L^p-estimates which are of their own interest.

E1. If $0 < \alpha \leq 1/(2\gamma)$ and $2 \leq p < 2/(1-2\alpha\gamma)$, then $(\mathfrak{D}^\gamma + \mathrm{I})^{-\alpha}$ is a bounded linear operator from $L^2(X,m)$ to $L^p(X,m)$. If $\alpha > 1/(2\gamma)$, then $(\mathfrak{D}^\gamma + \mathrm{I})^{-\alpha}$ is a bounded linear operator from $L^2(X,m)$ to $L^\infty(X,m)$.

E2. If $0 < \alpha \leq 1/(2\gamma)$ and $\mathcal{W} \in L^q(X,m)$, then $\mathcal{A} := \mathcal{W} \cdot (\mathfrak{D}^\gamma + \lambda\mathrm{I})^{-\alpha}$ is a bounded linear operator on $L^2(X,m)$ provided $1/(\alpha\gamma) < q \leq \infty$. Moreover, there exists a constant $c > 0$ such that $\|\mathcal{A}\|_{L^2 \to L^2} \leq c\,\|\mathcal{W}\|_q$ for all such $\mathcal{W}$. The same bound holds in the case $\alpha > 1/(2\gamma)$ and $q = 2$. In both cases the operator $\mathcal{A}$ is a compact operator on L^2. Moreover, $\lim_{\lambda\to\infty} \|\mathcal{W} \cdot (\mathfrak{D}^\gamma + \lambda\mathrm{I})^{-\alpha}\|_{L^2 \to L^2} = 0$.

Proof of statement (E1). Assume first that $0 < \alpha \leq 1/(2\gamma)$. If we define the function $g(y) := (\|y\|_{\mathrm{p}}^\gamma + 1)^{-\alpha}$ and assume that $1/(\alpha\gamma) < s < \infty$ then

$$\|g\|_s^s = \int\limits_{\mathbb{Q}_\mathrm{p}} \frac{dm(y)}{(\|y\|_\mathrm{p}^\gamma + 1)^{\alpha s}} = \left(1 - \frac{1}{\mathrm{p}}\right) \sum_{\tau=-\infty}^{\infty} \frac{\mathrm{p}^\tau}{(\mathrm{p}^{\tau\gamma} + 1)^{\alpha s}} < \infty.$$

If $k = \widehat{(\mathfrak{D}^\gamma + I)^{-\alpha}} f$ and $f \in L^2$, then $k(y) = g(y)\widehat{f}(y)$. Putting $1/q = 1/s + 1/2$, $1/(\alpha\gamma) < s \leq \infty$, we deduce that $1 < q \leq 2$ and

$$\|k\|_q \leq \|g\|_s \left\|\widehat{f}\right\|_2 = c_1 \|f\|_2.$$

If $1/p + 1/q = 1$, then $2 \leq p < \infty$ and, as it follows from the Hausdorff-Young theorem,

$$\left\|(\mathfrak{D}^\gamma + I)^{-\alpha} f\right\|_p = \left\|\widehat{k}\right\|_p \leq \|k\|_q \leq c_1 \|f\|_2.$$

We have $1/p = 1 - 1/q = 1/2 - 1/s$ and $1/(\alpha\gamma) < s \leq \infty$, whence p increases from 2 to $2/(1-2\alpha\gamma)$ as s decreases from ∞ to $1/(\alpha\gamma)$. If $\alpha > 1/(2\gamma)$, then the function g defined above lies in L^2 and we deduce that

$$\|k\|_1 = \left\|g\widehat{f}\right\|_1 \leq \|g\|_2 \left\|\widehat{f}\right\|_2 = c_2 \|f\|_2$$

whence as above

$$\left\|(\mathfrak{D}^\gamma + I)^{-\alpha} f\right\|_\infty = \left\|\widehat{k}\right\|_\infty \leq \|k\|_1 \leq c_2 \|f\|_2$$

as desired. Proof of the statement (E2). For any fixed $\lambda > 0$ if $0 < \alpha \leq 1/(2\gamma)$, then

$$\left\|\mathcal{W} \cdot (\mathfrak{D}^\gamma + \lambda I)^{-\alpha} f\right\|_2 \leq \|\mathcal{W}\|_q \left\|(\mathfrak{D}^\gamma + \lambda I)^{-\alpha} f\right\|_p$$

provided $1/2 = 1/p + 1/q$. The condition $2 \leq p < 2/(1-2\alpha\gamma)$ is equivalent to $1/(\alpha\gamma) < q \leq \infty$. We apply the statement (E1) to get the desired conclusion. The case $\alpha > 1/(2\gamma)$ is similar,

$$\left\|\mathcal{W} \cdot (\mathfrak{D}^\gamma + \lambda I)^{-\alpha} f\right\|_2 \leq \|\mathcal{W}\|_2 \left\|(\mathfrak{D}^\gamma + \lambda I)^{-\alpha} f\right\|_\infty.$$

To prove compactness of the operator $\mathcal{A} = \mathcal{W} \cdot (\mathfrak{D}^\gamma + \lambda I)^{-\alpha}$ we choose a sequence $\mathcal{W}_n \in \mathcal{D}$ such that $\mathcal{W}_n \to \mathcal{W}$ in L^q. Let Φ_n be a strictly increasing function such that $\Phi_n(\tau) = \tau^\gamma$ for $0 \le \tau \le n$ and $\Phi_n(\tau) \asymp e^\tau$ as $\tau \to \infty$. If we set $\mathcal{A}_n = \mathcal{W}_n \cdot (\Phi_n(\mathfrak{D}) + \lambda I)^{-\alpha}$ then $\mathcal{A}_n \to \mathcal{A}$ in the operator norm. Since the set of compact operators is closed under norm limits, it is sufficient to prove that each $\mathcal{A}_n$ is a Hilbert-Schmidt operator. Each operator $\mathcal{A}_n$ is unitary equivalent to the integral operator $\widehat{\mathcal{A}_n} : \widehat{u} \to \widehat{\mathcal{A}_n u}$ which has the kernel

$$\widehat{\mathcal{A}_n}(\theta, \zeta) = \widehat{\mathcal{W}_n}(\theta - \zeta)(\Phi_n(\|\zeta\|) + \lambda)^{-\alpha} := \widehat{\mathcal{W}_n}(\theta - \zeta)\mathcal{G}(\zeta)$$

so that the Hilbert-Schmidt norm $\left\|\widehat{\mathcal{A}_n}\right\|$of the operator $\widehat{\mathcal{A}_n}$ is

$$\left\|\widehat{\mathcal{A}_n}\right\| = \|\mathcal{W}_n\|_2 \|\mathcal{G}\|_2 < \infty.$$

Thus the operator$\mathcal{A} = \mathcal{W} \cdot (\mathfrak{D}^\gamma + \lambda I)^{-\alpha}$ is compact and clearly its norm tend to zero as $\lambda \to \infty$. Let us turn to the proof of the claim 3.14. To prove the claim in the case $0 < \gamma \le 1$ and $p > 1/\gamma$ we write

$$\begin{aligned}
\left\|W^{1/2}u\right\|_2^2 &= \left\|W^{1/2} \cdot (\mathfrak{D}^\gamma + I)^{-1/2} \cdot (\mathfrak{D}^\gamma + I)^{1/2}u\right\|_2^2 \\
&\le \left\|W^{1/2} \cdot (\mathfrak{D}^\gamma + I)^{-1/2}\right\|_{L^2 \to L^2}^2 \left\|(\mathfrak{D}^\gamma + I)^{1/2}u\right\|_2^2 \\
&= \left\|W^{1/2} \cdot (\mathfrak{D}^\gamma + I)^{-1/2}\right\|_{L^2 \to L^2}^2 \left(Q_L(u,u) + \|u\|_2^2\right) \\
&\le c\left\|W^{1/2}\right\|_q^2 \left(Q_L(u,u) + \|u\|_2^2\right) \le \frac{1}{2}Q_L(u,u) + c_1\|u\|_2^2
\end{aligned}$$

provided $\varepsilon > 0$ is chosen small enough and $q = 2p > 2/\gamma$ as in the statement (E2) with $\alpha = 1/2$. The case $\gamma > 1$ is similar: The restriction $p > 1/\gamma$ becomes $p \ge 1$. We set $Y = \{|V| > \tau\}$ and $W = |V|\,1_Y$. By Markov inequality $m(Y) \le \tau^{-p}\|V\|_p^p < \infty$ whence $\|W\|_1 = o(1)$ as $\tau \to \infty$. In particular, $W^{1/2} \in L^2$ and $\left\|W^{1/2}\right\|_2 = o(1)$ as $\tau \to \infty$. Applying the second part of the statement (E2) with $\alpha = 1/2$ and $q = 2$ we come to the conclusion

$$\begin{aligned}
\left\|W^{1/2}u\right\|_2^2 &\le c\left\|W^{1/2}\right\|_2^2 \left(Q_L(u,u) + \|u\|_2^2\right) \\
&\le \frac{1}{2}Q_L(u,u) + c_1\|u\|_2^2,
\end{aligned}$$

as desired. To prove that $dom(H) = dom(\mathfrak{D}^\gamma)$ we first write $V = V' + V''$, where $V' \in L^p(X,m)$ and $V'' \in L^\infty(X,m)$. The statement (E2) yields that

$$\lim_{\lambda \to \infty} \left\|V' \cdot (\mathfrak{D}^\gamma + \lambda \mathrm{I})^{-1}\right\|_{L^2 \to L^2} = 0.$$

We also have

$$\left\|V'' \cdot (\mathfrak{D}^\gamma + \lambda \mathrm{I})^{-1}\right\|_{L^2 \to L^2} \le \left\|V''\right\|_\infty \left\|(\mathfrak{D}^\gamma + \lambda \mathrm{I})^{-1}\right\|_{L^2 \to L^2} = \lambda^{-1}\left\|V''\right\|_\infty$$

for all $\lambda > 0$, so

$$\lim_{t \to \infty} \left\|V \cdot (\mathfrak{D}^\gamma + \lambda \mathrm{I})^{-1}\right\|_{L^2 \to L^2} = 0.$$

For any $1 > \delta > 0$ small enough we conclude that if $\lambda > 0$ is chosen large enough then

$$\|Vf\|_2 \le \delta\|\mathfrak{D}^\gamma f\|_2 + \lambda\delta\|f\|_2$$

for all $f \in dom(\mathfrak{D}^\gamma)$. Thus V is a relatively bounded perturbation of $\mathfrak{D}^\gamma$with a relative bound $\delta < 1$ whence $dom(\mathfrak{D}^\gamma + V) = dom(\mathfrak{D}^\gamma)$ by an application of E. B. Davies [14, Theorem 1.4.2]. The proof is now completed. ■

Remark 3.7. The number $N := 2/\gamma$ is the so called *spectral dimension* related to the operator $\mathfrak{D}^{\gamma}$ (remember that the topological dimension of the state space $\mathbb{Q}_{\mathrm{p}}$ is zero). The value of the number N in our setting is similar to the value of the topological dimension in the classical potential theory, e.g. one can regard the estimates $(E.1)$ and $(E.2)$ as a version of the well known estimates for the operator $-\Delta$ in the Euclidean space $\mathbb{R}^N$, see E. B. Davies [14, Sec. 3.6]. One more example is the relation

$$p_{\gamma}(t,x,x) \asymp t^{-N/2} \tag{3.15}$$

which holds for the heat kernel $p_{\gamma}(t,x,y)$ of the operator $\mathfrak{D}^{\gamma}$, see equation (2.15) and A. Bendikov, W. Cygan and W. Woess [8] for a more advanced study of asymptotic relation (3.15).

3.3 The positive spectrum

We find criteria on the potential of a Schrödinger-type operator $H = L+V$ to have spectrum which is contained in the interval $[0,\infty)$. We assume that the quadratic form $Q = Q_L + Q_V$ is a densely defined bounded below closed quadratic form having $\mathcal{D}$ as a core, and thus H is a bounded below self-adjoint operator associated with Q. Remember that this happens if e.g. the potential V satisfy one of the hypotheses of Theorem 3.3 and Theorem 3.6) above. Notice however that even if $Spec(H)$ is contained in the interval $[0,\infty)$, the form Q is not a Dirichlet form unless $V \geq 0$.

In what follows we use the notion $\Gamma(u,v)$ for *the square of gradient* defined as follows: for all $u,v \in \mathcal{D}$ we set

$$\Gamma(u,v) := \frac{1}{2}\left\{uLv + vLu - L(uv)\right\}. \tag{3.16}$$

Let $J(x-y)$ be the jump kernel associated with the (non-local) hierarchical Laplacian L, see equations (2.17) and (2.19). It is straightforward to show that the following identities hold true:

$$\Gamma(u,v)(x) = \frac{1}{2}\int_X (u(y)-u(x))\,(v(y)-v(x))\,J(x-y)dm(y), \tag{3.17}$$

$$Q_L(u,v) = \int_X \Gamma(u,v)dm, \tag{3.18}$$

$$Q_L(uv,w) = \int_X v\Gamma(u,w)dm + \int_X u\Gamma(v,w)dm, \tag{3.19}$$

$$\begin{aligned}&\int_X v\Gamma(u^2,w)dm - 2\int_X vu\Gamma(u,w)dm \\ &= \frac{1}{2}\int_{X\times X} (u(y)-u(x))^2\,(w(y)-w(x))\,(v(y)-v(x))\,J(x-y)dm(x)dm(y).\end{aligned} \tag{3.20}$$

In particular, we have

$$\begin{aligned}&\int_X w\Gamma(u^2,w)dm - 2\int_X wu\Gamma(u,w)dm \\ &= \frac{1}{2}\int_{X\times X} (u(y)-u(x))^2\,(w(y)-w(x))^2\,J(x-y)dm(x)dm(y) \geq 0.\end{aligned} \tag{3.21}$$

The identities listed above can be extended to the set of all bounded functions u,v and w from $dom(Q_L)$. We refer to M. Fukushima [20, Sec. 5].

By the interpolation the operator $L : \mathcal{D} \to L^2(X,m)$ can be extended to each of the Banach spaces $C_{\infty}(X)$ and $L^q(X,m)$, $1 \leq q < \infty$, as minus Markov generator. To simplify our notation the extended operator we still denote by L denoting if required its domain.

Theorem 3.8. *Assume that there exists a function* $0 < f \in dom_{C_\infty(X)}(L)$ *such that the inequality*

$$V(x) \geq -\frac{Lf(x)}{f(x)}$$

holds m*-almost everywhere. Then there exists a self-adjoint operator* $H \geq 0$ *associated with the quadratic form* $Q = Q_L + Q_V$*, that is,* $Q(u,u) = (Hu,u)$*,* $\forall u \in dom(H) \subset dom(Q)$*. In particular,* $Spec(H) \subseteq [0,\infty)$*.*

Proof. Let us assume first that f is a locally constant function. Let us put $W_f := (-Lf)/f$ and let $\varphi \in \mathcal{D}$. If we put $\psi := \varphi/f \in \mathcal{D}$, then using equations (3.18)-(3.21) we get

$$\begin{aligned} Q(\varphi,\varphi) &= \int_X (\varphi L\varphi + V\varphi^2)dm \geq \int_X (\varphi L\varphi + W_f\varphi^2)dm \\ &= \int_X (\psi Lf - 2\Gamma(f,\psi) + fL\psi + W_f f\psi)f\psi dm. \end{aligned}$$

Since $Lf + W_f f = 0$, the right-hand side of the inequality from above (shortly RHS) can be written as

$$\begin{aligned} RHS &= \int_X (-2\psi f\Gamma(f,\psi) + f^2\psi L\psi)dm \\ &= \int_X -2\psi f\Gamma(f,\psi)dm + Q_L(f^2\psi,\psi). \end{aligned}$$

It follows that

$$\begin{aligned} Q(\varphi,\varphi) &\geq \int_X -2\psi f\Gamma(f,\psi)dm + Q_L(f^2\psi,\psi) \\ &= \int_X \{-2\psi f\Gamma(f,\psi) + f^2\Gamma(\psi,\psi) + \psi\Gamma(f^2,\psi)\}dm \end{aligned}$$

and thus finally, by (3.21),

$$\begin{aligned} Q(\varphi,\varphi) &\geq \int_X f^2\Gamma(\psi,\psi)dm + \int_X \{-2\psi f\Gamma(f,\psi) + \psi\Gamma(f^2,\psi)\}dm \\ &\geq \int_X f^2\Gamma(\psi,\psi)dm \geq 0. \end{aligned}$$

We have already shown that $Q(\varphi,\varphi) \geq 0$ for all $\varphi \in \mathcal{D}$. Since such functions φ form a core for Q, the result follows by an application of the variational formula

$$E = \inf\{Q(\varphi,\varphi) : \varphi \in \mathcal{D} \text{ and } \|\varphi\|_2 = 1\} \tag{3.22}$$

where E is the bottom of the spectrum of the operator H. In general one can choose a sequence of locally constant functions f_n such that $W_{f_n} \to W_f$ locally uniformly in X. For instance, one can choose a δ-sequence $\phi_n \in \mathcal{D}_+$ and set $f_n := f * \phi_n$. Then setting $\psi_n := \varphi/f_n$ we get

$$\begin{aligned} Q(\varphi,\varphi) &= \int_X (\varphi L\varphi + V\varphi^2)dm \geq \int_X (\varphi L\varphi + W_f\varphi^2)dm \\ &= \lim_{n\to\infty} \int_X (\varphi L\varphi + W_{f_n}\varphi^2)dm \geq \limsup_{n\to\infty} \int_X f_n^2\Gamma(\psi_n,\psi_n)dm \geq 0. \end{aligned}$$

The proof of the theorem is finished. ■

Corollary 3.9. *Consider the quadratic form $Q = Q_{\mathfrak{D}^\alpha} + Q_V$ with domain $\mathcal{D}$. Assume that $0 < \alpha < 1$ and that the following inequality*

$$V_-(x) \leq \left(\Gamma_p\left(\frac{1+\alpha}{2}\right)\right)^2 \|x\|_p^{-\alpha}$$

holds almost everywhere, then $Q(\varphi, \varphi) \geq 0$ for all $\varphi \in \mathcal{D}$.

Assume further that the non-negative definite quadratic form $(Q, \mathcal{D})$ is closable, let $\mathfrak{D}^\alpha + V$ be the non-negative definite operator associated with its minimal closed extension, then

$$Spec(\mathfrak{D}^\alpha + V) \subseteq [0, \infty).$$

Proof. Let us set $\mathfrak{u}_\beta(x) := \|x\|_p^\beta$. By V. S. Vladimirov [43, Sec. 8.1, Eq. (1.6)], the function $\mathfrak{u}_\beta$ defines a distribution (a generalized function) in $\mathcal{D}'$ which is holomorphic on β everywhere on the real line. The operator $\mathfrak{D}^\alpha : \psi \to \mathfrak{D}^\alpha \psi$ can be defined as convolution of distributions $\mathfrak{u}_{-\alpha-1}/\Gamma_p(-\alpha)$ and ψ, see V. S. Vladimirov [43, Sec. IX]. We claim that in the sense of distributions

$$\mathfrak{D}^\alpha \mathfrak{u}_\beta = \frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)} \mathfrak{u}_{\beta-\alpha}, \ \forall \beta \neq \alpha. \tag{3.23}$$

The case $\beta = 0$ is trivial. For $\beta \neq 0$ we apply the Fourier transform argument. Remind that the Fourier transform $f \to \widehat{f}$ is a linear isomorphism of $\mathcal{D}' \to \mathcal{D}'$. By virtue of the results of V. S. Vladimirov [43, Sec. VII.5], the equation

$$\widehat{\mathfrak{u}_{\gamma-1}}(\xi) = \Gamma_p(\gamma)\mathfrak{u}_{-\gamma}(\xi) \tag{3.24}$$

holds true for all $\gamma \neq 1$. Applying equation (3.24) we obtain

$$\begin{aligned}
\widehat{\mathfrak{D}^\alpha \mathfrak{u}_\beta}(\xi) &= \mathfrak{u}_\alpha(\xi)\widehat{\mathfrak{u}_\beta}(\xi) = \mathfrak{u}_\alpha(\xi)\widehat{\mathfrak{u}_{\beta+1-1}}(\xi) \\
&= \mathfrak{u}_\alpha(\xi)\Gamma_p(\beta+1)\mathfrak{u}_{-\beta-1}(\xi) = \Gamma_p(\beta+1)\mathfrak{u}_{-(1+\beta-\alpha)}(\xi) \\
&= \frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)}\Gamma_p(\beta+1-\alpha)\mathfrak{u}_{-(1+\beta-\alpha)}(\xi) \\
&= \frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)}\widehat{\mathfrak{u}_{(1+\beta-\alpha)-1}}(\xi) = \frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)}\widehat{\mathfrak{u}_{\beta-\alpha}}(\xi),
\end{aligned}$$

so by the uniqueness theorem the desired result follows. For $\phi \in \mathcal{D}_+$ and $\beta := (\alpha-1)/2$ we define the following function

$$W_\phi := \frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)} \frac{\mathfrak{u}_{\beta-\alpha} * \phi}{\mathfrak{u}_\beta * \phi} = \left(\Gamma_p\left(\frac{1+\alpha}{2}\right)\right)^2 \frac{\mathfrak{u}_{-\frac{1+\alpha}{2}} * \phi}{\mathfrak{u}_{-\frac{1-\alpha}{2}} * \phi}.$$

We claim that the function W_ϕ belongs to $C_\infty(X)$ and $W_\phi = \mathfrak{D}^\alpha f / f$ where

$$0 < f = \mathfrak{u}_{-\frac{1-\alpha}{2}} * \phi \in dom_{C_\infty(X)}(\mathfrak{D}^\alpha).$$

Indeed, $\|x\|_p > \|y\|_p$ implies that $\|x-y\|_p = \|x\|_p$ whence for any fixed ball B which is centred at the neutral element and contains the set $\{\phi > 0\}$ and for any x such that $\|x\|_p > diam(B)$ we have

$$0 < \mathfrak{u}_{-\frac{1\mp\alpha}{2}} * \phi(x) = \int_B \frac{\phi(y)}{\|x-y\|_p^{\frac{1\mp\alpha}{2}}} dm(y) = \frac{1}{\|x\|_p^{\frac{1\mp\alpha}{2}}} \int_B \phi dm. \tag{3.25}$$

It follows that the functions $\mathfrak{u}_{-\frac{1-\alpha}{2}} * \phi$, $\mathfrak{u}_{-\frac{1+\alpha}{2}} * \phi$ and W_ϕ belong to $C_\infty(X)$. In particular, applying equation (3.23) we get $f \in C_\infty(X)$ and

$$\mathfrak{D}^\alpha f = \left(\Gamma_p\left(\frac{1+\alpha}{2}\right)\right)^2 \mathfrak{u}_{-\frac{1+\alpha}{2}} * \phi := F \tag{3.26}$$

in the sense of distributions. So by V. S. Vladimirov [43, Sec. IX.3]

$$f(x)=\mathfrak{D}^{-\alpha}F(x)=\frac{1}{\Gamma_p(\alpha)}\int\frac{F(y)}{\|x-y\|_p^{1-\alpha}}dm(y). \tag{3.27}$$

Finally, equations (3.27) and (3.25) show that f (being the Green potential of the function $F\in C_\infty(X)$) belongs to $C_\infty(X)$-domain of the operator $\mathfrak{D}^\alpha$. In particular, equation (3.26) holds in the strong sense. Clearly $W_\phi=\mathfrak{D}^\alpha f/f$ and the proof of the claim is finished. Thus Theorem 3.8 is applicable and we conclude that

$$Q_{W_\phi}(\varphi,\varphi)\le Q_{\mathfrak{D}^\alpha}(\varphi,\varphi),\ \forall\varphi\in\mathcal{D}.$$

Let us choose a sequence $\{B_n:n=1,2,...\}$ of balls centred at the neutral element 0 such that $\cap_{n=1}^{\infty}B_n=\{0\}$ and set $\phi_n=1_{B_n}/m(B_n)$. Clearly $\phi_n*f\to f$ for any continuous function f, whence

$$W_{\phi_n}(x)\to W(x)=\left(\Gamma_p\left(\frac{1+\alpha}{2}\right)\right)^2\frac{\mathfrak{u}_{-\frac{1+\alpha}{2}}(x)}{\mathfrak{u}_{-\frac{1-\alpha}{2}}(x)}=\left(\Gamma_p\left(\frac{1+\alpha}{2}\right)\right)^2\|x\|_p^{-\alpha}.$$

Applying now Fatou lemma we conclude: for all $\varphi\in\mathcal{D}$, the following inequality holds (a $\mathbb{Q}_p$-version of the classical Hardy inequality in $\mathbb{R}^N$)

$$Q_W(\varphi,\varphi)\le Q_{\mathfrak{D}^\alpha}(\varphi,\varphi).$$

It follows that for all $\varphi\in\mathcal{D}$,

$$-Q_V(\varphi,\varphi)\le Q_{V_-}(\varphi,\varphi)\le Q_W(\varphi,\varphi)\le Q_{\mathfrak{D}^\alpha}(\varphi,\varphi),$$

or equivalently,

$$Q(\varphi,\varphi):=Q_{\mathfrak{D}^\alpha}(\varphi,\varphi)+Q_V(\varphi,\varphi)\ge 0.$$

The set $\mathcal{D}$ forms a core for $Q(\varphi,\varphi)$, for reasons which depend upon which assumption we make on V, and the proof is completed by an application of the variational formula (3.22). ■

Example 3.10. Consider the quadratic form $Q=Q_{\mathfrak{D}^\alpha}+Q_V$ with $V(x)=b\,\|x\|_p^{-\alpha}$, $0<\alpha<1$, $b\ge-\{\Gamma_p\left((1+\alpha)/2\right)\}^2$, and with $dom(Q)=\mathcal{D}$, the set of all locally constant functions having compact supports. In order to prove that the form Q is semibounded and closable Theorem 3.6 does not apply simply because $V\in L^p+L^\infty$ only if $p<1/\alpha$. Notice however that Theorem 3.6 applies in the case $Q=Q_{\mathfrak{D}^\alpha}+Q_{b\|\cdot\|_p^{-\beta}}$ where $0<\beta<\alpha<1$, $b\in\mathbb{R}^1$, because $V\in L^p+L^\infty$ for all $1/\alpha<p<1/\beta$.

In the interesting case $Q=Q_{\mathfrak{D}^\alpha}+Q_{b\|\cdot\|_p^{-\alpha}}$ we argue as follows: Let us consider the operator $H=\mathfrak{D}^\alpha+b\,\|x\|_p^{-\alpha}$ defined on the dense set $\mathcal{D}_0=\{\varphi\in\mathcal{D}:\varphi(0)=0\}$. Evidently the operator $H:\mathcal{D}_0\to L^2$ is symmetric and we can write

$$Q(\varphi,\varphi):=Q_{\mathfrak{D}^\alpha}(\varphi,\varphi)+Q_{b\|\cdot\|_p^{-\alpha}}(\varphi,\varphi)=(H\varphi,\varphi),\ \forall\varphi\in\mathcal{D}_0.$$

Since $b\ge-\{\Gamma_p\left((1+\alpha)/2\right)\}^2$, Corollary 3.9 applies and we conclude that $Q(\varphi,\varphi)\ge 0$, $\forall\varphi\in\mathcal{D}$. In particular, the operator H is a non-negative definite symmetric operator. As each non-negative definite symmetric operator is closable and its minimal closed extension $\overline{H}$ is a non-negative definite self-adjoint operator, the quadratic form Q with domain $\mathcal{D}_0$ is closable.

To prove that the non-negative quadratic form Q with domain $\mathcal{D}$ is closable it is enough to show that the indicator function 1_B of any ball B which contains the neutral element 0 can be approximated in Q-metric by a sequence of functions in $\mathcal{D}_0$. Let $B_n \downarrow \{0\}$ be a descending sequence of balls belonging to B such that any two balls $B_n \subset B_{n-i}$ are closest neighbors and $1_{B\setminus B_n} \in \mathcal{D}_0$ be the sequence of the indicators of sets $B\setminus B_n$. Let us show that

$$Q(1_B - 1_{B\setminus B_n}, 1_B - 1_{B\setminus B_n}) = Q(1_{B_n}, 1_{B_n}) \to 0,\ n \to \infty.$$

Clearly $Q_V(1_{B_n}, 1_{B_n}) \to 0$, so it is enough to show that $Q_{\mathfrak{D}^\alpha}(1_{B_n}, 1_{B_n}) \to 0$. Applying inequality (3.2) we get

$$\frac{1}{2}m(B_n)\lambda(B_{n-1}) < Q_{\mathfrak{D}^\alpha}(1_{B_n}, 1_{B_n}) < 2m(B_n)\lambda(B_{n-1}).$$

We can also assume that $m(B_n) = p^{-n}$ for all n large enough, then the eigenvalue $\lambda(B_{n-1}) = p^{\alpha(n-1)}$. Since $0 < \alpha < 1$ we get $m(B_n)\lambda(B_{n-1}) = p^{-(1-\alpha)(n-1)-1} \to 0$ as desired. Thus the form Q with $dom(Q) = \mathcal{D}$ is non-negative definite and closable and its minimal closure is associated with non-negative definite self-adjoint operator $\overline{H}$,

$$Q(u,u) = (\overline{H}^{1/2}u, \overline{H}^{1/2}u),\ \forall u \in dom(Q).$$

3.4 The negative spectrum

Next we discuss several results giving information about the negative part of the spectrum of the Schrödinger-type operator $H = L + V$. We consider $L = \mathfrak{D}^\gamma$ acting in $L^p(\mathbb{Q}_p, m)$ where m is the Haar measure.

Theorem 3.11. *Let $L = \mathfrak{D}^\gamma$ and let $V \in L^p(\mathbb{Q}_p, m)$ for some $p > 1/\gamma$. Then the following properties hold:*

1. The operator $H = L + V$ has essential spectrum equals to the spectrum of the operator L.

2. In particular, if H has any negative spectrum, then it consists of a sequence of negative eigenvalues of finite multiplicity. If this sequence is infinite then it converges to zero.

3. Suppose that there exists an open set $U \subset X$ on which V is negative. If E_λ is the bottom of the spectrum of the operator $H_\lambda = L + \lambda V$, then $E_\lambda \le 0$ for all $\lambda \ge 0$ and $\lim_{\lambda\to\infty} E_\lambda = -\infty$.

Proof. 1. By Theorem 3.6, if $c > 0$ is large enough then the operator $H + cI$ is non-negative and

$$\frac{1}{2}\left\|(L+cI)^{1/2}u\right\|_2 \le \left\|(H+cI)^{1/2}u\right\|_2 \le \frac{3}{2}\left\|(L+cI)^{1/2}u\right\|_2 \tag{3.28}$$

for all $u \in dom(Q_L)$. Let us define $\Delta := (L+cI)^{-1} - (H+cI)^{-1}$, then

$$\Delta = (L+cI)^{-1}V(H+cI)^{-1} = ABCDE$$

where $A = (L+cI)^{-1/2}$, $B = (L+cI)^{-1/2}\,|V|^{1/2}$, $C = sign(V)\cdot B^*$, $D = (L+cI)^{1/2}(H+cI)^{-1/2}$ and $E = (H+cI)^{-1/2}$. It is clear that A and E are bounded operators in $L^2(\mathbb{Q}_p, m)$, B^* and C are compact operators in $L^2(\mathbb{Q}_p, m)$, see statement (E2) in the proof of Theorem 3.6, and D is a bounded operator in $L^2(\mathbb{Q}_p, m)$ by equation (3.28). Thus, as a product of compact and bounded operators, the difference of two resolvents Δ is a compact operator on L^2. By the perturbation theory of linear operators, H and L have the same essential spectrum, see e.g. T. Kato [24]. Since $Spec_{ess}(L) = Spec(L) \subset [0,\infty[$, any negative point in the spectrum of H must be an isolated eigenvalue of finite multiplicity. Any limit of negative eigenvalues lies in the essential spectrum whence the only possible limit is zero. 2. That $E_\lambda \le 0$ for all

$\lambda \ge 0$ follows from the fact that $\{0\} \in Spec_{ess}(L)$ and that $Spec_{ess}(L+\lambda V) = Spec_{ess}(L)$. To prove the second statement we observe that $\mathcal{D}$ is a form core for $Q_L + Q_{\lambda V}$, whence

$$E_\lambda = \inf\{Q_L(u,u) + Q_{\lambda V}(u,u) : u \in \mathcal{D} \text{ and } \|u\|_2 = 1\}. \tag{3.29}$$

Let us choose $u \in \mathcal{D}$ having support in the set U, then as $\lambda \to \infty$ we get

$$\begin{aligned} E_\lambda &\le Q_L(u,u) + Q_{\lambda V}(u,u) \\ &= Q_L(u,u) - \lambda \int_U |V|\,|u|^2\, dm \to -\infty \end{aligned}$$

as it was claimed. ■

The following example shows that the crucial issue for the existence of negative eigenvalues in Theorem 3.11 for all $\lambda > 0$ is the rate at which the potential $V(x)$ converges to 0 as $\|x\|_{\mathrm{p}} \to \infty$.

Example 3.12. Let $0 < \alpha < 1$ and $H_\lambda = \mathfrak{D}^\alpha - \lambda V$ where

$$V(x) = (\|x\|_{\mathrm{p}} + 1)^{-\beta}$$

for some $0 < \beta < 1$ and $\lambda > 0$. We have:

1. If $\beta \ge \alpha$ then Theorem 3.11 and Corollary 3.9 are applicable and there exists a positive threshold for the existence of negative eigenvalues of H_λ.
2. If $\beta > \alpha$ then the number of negative eigenvalues of H_λ counted with their multiplicity can be estimated as follows

$$Neg(H_\lambda) \le c(\alpha,\beta)\lambda^{1/\alpha}.$$

Indeed, applying S. Molchanov and B. Vainberg [37, Theorem 2.1 and Remark 2.2], we obtain

$$\begin{aligned} Neg(H_\lambda) &\le c(\alpha) \int_{\mathbb{Q}_{\mathrm{p}}} (\lambda V)^{1/\alpha} dm \\ &= c(\alpha)\lambda^{1/\alpha} \int_{\mathbb{Q}_{\mathrm{p}}} \frac{dm(x)}{(\|x\|_{\mathrm{p}} + 1)^{\beta/\alpha}} = c(\alpha,\beta)\lambda^{1/\alpha}. \end{aligned}$$

3. If $0 < \beta < \alpha$ then the result is totally different.

Theorem 3.13. *In the notation of Example 3.12 assume that $0 < \beta < \alpha$, then H_λ has non-empty negative spectrum for all $\lambda > 0$.*

Proof. Let $f := \mathfrak{D}^{-\alpha} 1_B$ where B is a ball centred at the neutral element which we will specify later. The function f belongs to $\mathrm{dom}(\mathfrak{D}^\alpha)$ and calculations based on the spectral resolution formula and equation (2.12) show that

$$\begin{aligned} \mathfrak{D}^{-\alpha} 1_B / m(B) &= \mathfrak{D}^{-\alpha} \sum_{T:\, B \subseteq T} f_T = \sum_{T:\, B \subseteq T} \mathfrak{D}^{-\alpha} f_T \\ &= \sum_{T:\, B \subseteq T} \left(\frac{m(T')}{\mathrm{p}}\right)^\alpha f_T = \sum_{T:\, B \subseteq T} m(T)^\alpha \left(\frac{1_T}{m(T)} - \frac{1_{T'}}{m(T')}\right) \\ &= m(B)^{\alpha-1} \sum_{T:\, B \subseteq T} \left(\frac{m(T)}{m(B)}\right)^{\alpha-1} \left(1_T - \frac{1}{\mathrm{p}} 1_{T'}\right). \end{aligned}$$

In particular, $W := (\mathfrak{D}^\alpha f)/f$ is given by

$$W = \frac{1_B}{\mathfrak{D}^{-\alpha} 1_B} = \frac{\mathrm{p} - \mathrm{p}^\alpha}{\mathrm{p} - 1} \frac{1_B}{m(B)^\alpha} = \frac{\mathrm{p} - \mathrm{p}^\alpha}{\mathrm{p} - 1} \frac{1_B}{\mathrm{diam}(B)^\alpha}.$$

If $\lambda > 0$ and $0 < \beta < \alpha$, there exists a ball B such that $\mathrm{diam}(B)$ is large enough so that

$$W(x) < \frac{\lambda}{(\|x\|_{\mathrm{p}} + 1)^{\beta}} = \lambda V(x)$$

for all $x \in \mathbb{Q}_p$. Hence, as f belongs to $\mathrm{dom}(\mathfrak{D}^{\alpha})$, we obtain

$$\begin{aligned} Q_{H_\lambda}(f,f) &= Q_{\mathfrak{D}^\alpha}(f,f) - Q_{\lambda V}(f,f) < Q_{\mathfrak{D}^\alpha}(f,f) - Q_W(f,f) \\ &= (\mathfrak{D}^\alpha f, f) - (W \cdot f, f) = 0 \end{aligned}$$

and an application of the Rayleigh-Ritz formulae yields the desired result. ■

4 Green function estimates

In this section we consider the Schrödinger-type operator $H = L + V$ with L a homogeneous hierarchical Laplacian and show that under certain conditions the equation $Hu = v$ has unique solution u which can be represented in the form

$$u(x) = \int g_H(x,y)v(y)dm(y).$$

The kernel $g_H(x,y)$ is a continuous strictly positive function which is bounded outside the diagonal set Δ and $g_H|_\Delta = +\infty$. We call $g_H(x,y)$ the Green function defined by the operator $H = L + V$.

Our aim here is to compare the Green functions $g_H(x,y)$ and $g_L(x,y)$. We provide our calculations assuming that $L = \mathfrak{D}^\alpha$ and $V(x) = b\,\|x\|_p^{-\alpha}$ for $0 < \alpha < 1$ and $b \geq b_*$, where

$$b_* := -\{\Gamma_p\left((1+\alpha)/2\right)\}^2$$

is the critical value of the parameter b as will be explained in Theorem 4.1 below. We will prove, see Corollary 4.8 below, that for any $b \geq b_*$ there exists unique $\frac{\alpha-1}{2} \leq \beta < \alpha$ such that

$$\frac{g_H(x,y)}{g_{\mathfrak{D}^\alpha}(x,y)} \asymp \left(\frac{\|x\|_p}{\|y\|_p} \wedge \frac{\|y\|_p}{\|x\|_p}\right)^{\beta}.$$

4.1 Preliminary results

Recall that the p-adic Gamma-function is defined as

$$\Gamma_p(z) = (1 - p^{z-1})(1 - p^{-z})^{-1}.$$

The function $\Gamma_p(z)$ is meromorphic in the complex plane and satisfies the functional equation

$$\Gamma_p(z)\Gamma_p(1-z) = 1$$

(see [43, Sec.VIII.2]).

For a real β we regard the function $h(x) = \|x\|_p^{\beta}$ as a distribution in the spirit of [43]. For $\beta \neq \alpha$ equation (3.23) shows that (in the sense of distributions)

$$Lh(x) = \frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)}\,\|x\|_p^{\beta-\alpha}.$$

This relation must be compared with the Green function estimates for Schrödinger operators on Riemanian manifolds, see [22]

In particular, for $\beta > \alpha - 1$ and $0 < \alpha < 1$, the distributions $h(x)$ and $Lh(x)$ are regular (i.e. generated by locally integrable functions) and the function

$$V(x) := -\frac{Lh(x)}{h(x)} = -\frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)} \|x\|_p^{-\alpha} \tag{4.1}$$

belongs to $L^1_{loc}(\mathbb{Q}_p)$, so it defines a regular distribution as well.

Theorem 4.1. *Let the function $V(x)$ be defined by equation (4.1). Assume that $\alpha-1 < \beta < \alpha$ then the following statements hold true:*

1. For $0 < \beta < \alpha$ the function $V(x)$ is strictly positive and belongs to $L^1_{loc}(\mathbb{Q}_p)$. Moreover, for any $b > 0$ there exists $0 < \beta < \alpha$, the solution of the equation

$$-\frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)} = b, \tag{4.2}$$

such that $V(x) = b\,\|x\|_p^{-\alpha}$ for this value of β.

2. For $\alpha - 1 < \beta < 0$ the function $V(x)$ is strictly negative, and for all these values of β

$$V_-(x) = -V(x) \leq \left(\Gamma_p\left(\frac{1+\alpha}{2}\right)\right)^2 \|x\|_p^{-\alpha}.$$

Moreover, for $b_* := -\{\Gamma_p((1+\alpha)/2)\}^2$ and for any $0 > b \geq b_*$ there exist two values of β, solutions of the equation (4.2), $1 < \beta_1 \leq (\alpha-1)/2$ and $(\alpha-1)/2 \leq \beta_2 < 0$, such that $V(x) = b\,\|x\|_p^{-\alpha}$ for these values of β.

Proof. To prove the theorem we set $\vartheta = \beta + (1-\alpha)/2$ and write

$$-\frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)} = -\Gamma_p\left(\frac{1+\alpha}{2}+\vartheta\right)\Gamma_p\left(\frac{1+\alpha}{2}-\vartheta\right) =: C_\alpha(\vartheta).$$

The function $C_\alpha(\vartheta)$ is even, continuous and increasing on each interval $[0,(1+\alpha)/2[$ and $](1+\alpha)/2,+\infty[$. Using the very definition of the function $\Gamma_p(\xi)$ it is straightforward to show that the following properties hold true:

1. $C_\alpha(0) = -\{\Gamma_p((1+\alpha)/2)\}^2$, $C_\alpha((1-\alpha)/2) = 0$,
2. $C_\alpha((1+\alpha)/2 - 0) = +\infty$, $C_\alpha((1+\alpha)/2 + 0) = -\infty$,
3. $C_\alpha(+\infty) = -p^\alpha < C_\alpha(0)$.

Clearly properties (1)-(3) imply the result. The proof of the theorem is finished. ■

Let us define the linear space $\mathcal{D}_0 := \{u \in \mathcal{D} : u(0) = 0\}$. For V as above and $L = \mathfrak{D}^\alpha$ we define the linear operator $H = L + V$ with $dom(H) = \mathcal{D}_0$. Clearly the operator $H : \mathcal{D}_0 \to L^2(\mathbb{Q}_p)$ is symmetric and

$$Q(u,u) = (Hu,u),\ \forall u \in dom(H).$$

Theorem 4.2. *Assume that $\alpha - 1 < \beta < \alpha$. Then for L and V as above the operator $H = L + V$ is symmetric and non-negative definite. In particular, H is closable, its minimal closure $\overline{H}$ is the non-negative definite self-adjoint operator associated with the quadratic form $Q = Q_L + Q_V$, i.e.*

$$Q(u,u) = (\overline{H}u,u),\ \forall u \in dom(\overline{H}).$$

Proof. In the case $0 < \beta < \alpha$ we can apply the first statement of Theorem 4.1 and Theorem 3.3. In the case $\alpha - 1 < \beta < 0$ we can apply the second statement of Theorem 4.1, Example 3.10 and Corollary 3.9. The proof is finished. ■

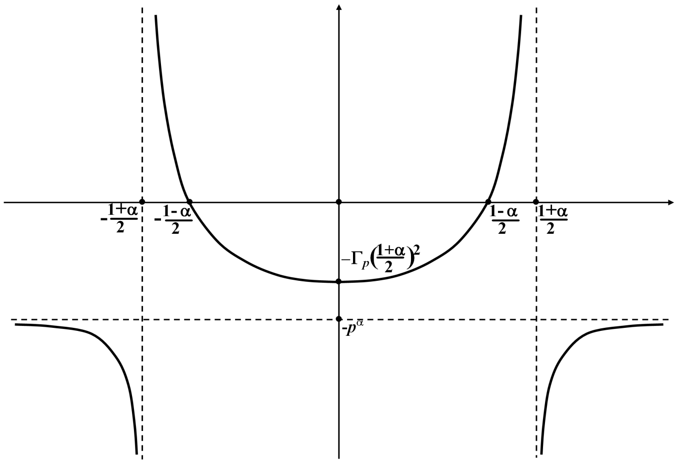

Figure 2: The function C_α

4.2 The time changed Dirichlet form

Let us choose in equation (4.1) the function $h(x) = \|x\|_p^\beta$ with β satisfying $(\alpha-1)/2 < \beta < \alpha$. If we set $V = (-Lh)/h$ and $H = L+V$ then the results of the previous subsection apply, so H is a non-negative definite symmetric operator acting in $L^2(X,m)$ (we keep notation $X = \mathbb{Q}_p$ equipped with the Haar measure m). Its minimal closure $\overline{H}$ is a non-negative self-adjoint operator associated with the quadratic form Q.

According to our choice $h^2 \in L^1_{loc}(X,m)$, so $h^2 \cdot m$ is a Radon measure. In particular, the operator

$$Ug = hg : L^2(X, h^2 \cdot m) \to L^2(X,m)$$

is an isometry. Consider the non-negative self-adjoint operator

$$\mathcal{H} = U^{-1} \circ \overline{H} \circ U : L^2(X, h^2 \cdot m) \to L^2(X, h^2 \cdot m)$$

and define the the following non-negative definite quadratic form

$$Q_{\mathcal{H}}(u,u) = \begin{cases} (\mathcal{H}^{1/2}u, \mathcal{H}^{1/2}u), & u \in dom(\mathcal{H}^{1/2}) \\ +\infty, & \text{otherwise} \end{cases} .$$

As $Q = Q_L + Q_V$ we get the equation

$$\begin{aligned} Q_{\mathcal{H}}(u,u) &= Q(hu,hu) = Q_L(hu,hu) + Q_V(hu,hu) \\ &= \frac{1}{2}\int_X \int_X (h(x)u(x) - h(y)u(y))^2 J(x,y)dm(y)dm(x) \\ &+ \int_X V(x)u^2(x)h^2(x)dm(x) \end{aligned}$$

where the kernel $J(x,y)$ is given by

$$J(x,y) = -\frac{1}{\Gamma_p(-\alpha)} \frac{1}{\|x-y\|_p^{1+\alpha}} . \tag{4.3}$$

Theorem 4.3. *Assume that $0 < \alpha < 1$ and $(\alpha-1)/2 < \beta < \alpha$. Then $\mathcal{D} \subset dom(Q_{\mathcal{H}})$ and the following equation holds*

$$Q_{\mathcal{H}}(u,u) = \frac{1}{2}\int\int (u(x) - u(y))^2 J(x,y)h(y)dm(y)h(x)dm(x). \tag{4.4}$$

In particular, $Q_{\mathcal{H}}$ is a densely defined, closed and Markovian quadratic form in $L^2(X, h^2 \cdot m)$. In other words, $Q_{\mathcal{H}}$ is a regular Dirichlet form relative to $L^2(X, h^2 \cdot m)$ having $\mathcal{D}$ as a core.

Proof. Let us prove that $\mathcal{D} \subset dom(Q_{\mathcal{H}})$. It is enough to show that $Q_{\mathcal{H}}(u,u)$ is finite for any u of the form $u = 1_B$, the indicator of any open ball in X. We have $Q_{\mathcal{H}}(u,u) = Q_L(hu, hu) + Q_V(hu, hu)$. Since $V(x) = b \|x\|_p^{-\alpha}$ and $\beta > (\alpha-1)/2$ we get

$$|Q_V(hu,hu)| = |b| \int_B \|x\|_p^{-\alpha+2\beta} \, dm(x) < \infty.$$

If $0 \notin B$, then clearly $hu \in \mathcal{D} \subset dom(L)$ and thus

$$Q_L(hu,hu) = (Lhu, hu) < \infty.$$

Assume now that $0 \in B$ and set $h_B := h\mathbf{1}_B$, then

$$\begin{aligned} Q_L(hu,hu) &= \frac{1}{2} \iint (h_B(x) - h_B(y))^2 \, J(x,y) dm(x) dm(y) \\ &= \iint\limits_{(x,y) \in B \times B: \, \|x\|_p < \|y\|_p} (h(x) - h(y))^2 \, J(x,y) dm(x) dm(y) \\ &\quad + \int_B h^2(x) dm(x) \int_{B^c} J(x,y) dm(y). \end{aligned}$$

The second term, call it II, is finite. Indeed, we have

$$II = \int_B h^2(x) dm(x) \int_{B^c} J(0,z) dm(z) < \infty.$$

Without loss of generality we may assume that $diam(B) = 1$. By the ultrametric inequality, $\|x\|_p < \|y\|_p$ implies that $\|x-y\|_p = \|y\|_p$, so the first term, call it I, can be estimated as follows:

$$\begin{aligned} I &= -\frac{1}{\Gamma_p(-\alpha)} \sum_{k=1}^{\infty} \sum_{l=1}^{k} \int\limits_{\|x\|_p = p^{-k}} dm(x) \int\limits_{\|y\|_p = p^{-k+l}} dm(y) \left(\|x\|_p^{\beta} - \|y\|_p^{\beta} \right)^2 \|y\|_p^{-(1+\alpha)} \\ &= -\frac{1}{\Gamma_p(-\alpha)} \left(1 - \frac{1}{p}\right)^2 \sum_{k=1}^{\infty} \sum_{l=1}^{k} p^{-k} p^{-k+l} p^{-(1+\alpha)(-k+l)} \left(p^{-k\beta} - p^{(-k+l)\beta} \right)^2 \\ &= -\frac{1}{\Gamma_p(-\alpha)} \left(1 - \frac{1}{p}\right)^2 \sum_{k=1}^{\infty} p^{-k(1-\alpha+2\beta)} \sum_{l=1}^{k} p^{-l\alpha} \left(1 - p^{l\beta}\right)^2. \end{aligned}$$

That the term I is finite for $(\alpha-1)/2 < \beta < \alpha$ follows by inspection. To prove equation (4.4) it is enough to check it for $u = \mathbf{1}_B$, the indicator of an open ball B. Let us first prove the following identity

$$Q_V(hu,hu) = -Q_L(hu,h). \tag{4.5}$$

It is enough to check the above identity assuming that B does not contain the neutral element. To certify this claim we act as in Example (3.10): if $0 \in B$ we choose a descending sequence of balls B_n in B which converges to $\{0\}$. Then by what we claim $Q_V(h1_{B \setminus B_n}, h1_{B \setminus B_n}) = -Q_L(h1_{B \setminus B_n}, h)$ and both sides of this equation converge to $Q_V(h1_B, h1_B)$ and $-Q_L(h1_B, h)$ respectively. Let us consider the distribution $f_\gamma(x) = \|x\|_p^{\gamma-1} / \Gamma_p(\gamma)$. According to V. S. Vladimirov [43, Section IX], $h(x) = \Gamma_p(\beta+1) f_{\beta+1}$ and

$$-Lh = f_{-\alpha} * \Gamma_p(\beta+1) f_{\beta+1} = \Gamma_p(\beta+1) f_{\beta-\alpha+1} \in L^1_{loc}.$$

As we assume that $0 \in B$ the functions $Vhu = (-Lh)u$ and $hu := h_B$ belong to $\mathcal{D}$ and

$$\begin{aligned}
Q_V(hu, hu) &= \int (-Lh) h_B dm = ((-Lh) * h_B)(0) \\
&= \Gamma_p(\beta+1)((f_{-\alpha} * f_{\beta+1}) * h_B)(0) \\
&= \Gamma_p(\beta+1)((f_{\beta+1} * (f_{-\alpha} * h_B))(0) = \int h(-Lh_B) dm \\
&= -\iint (h_B(x) - h_B(y)) h(x) J(x,y) dm(x) dm(y).
\end{aligned}$$

By symmetry $J(x,y) = J(y,x)$ we get

$$\begin{aligned}
Q_V(hu, hu) &= -\iint (h_B(y) - h_B(x)) h(y) J(x,y) dm(x) dm(y) \\
&= -\frac{1}{2} \iint (h_B(x) - h_B(y))(h(x) - h(y)) J(x,y) dm(x) dm(y)
\end{aligned}$$

or in other words $Q_V(hu, hu) = -Q_L(hu, h)$ as claimed. On the other hand, for $u = 1_B$ we have

$$Q_L(hu, hu) = \frac{1}{2} \iint (h_B(x) - h_B(y))^2 J(x,y) dm(x) dm(y). \tag{4.6}$$

Equations (4.5) and (4.6) yield the following equation

$$\begin{aligned}
Q_{\mathcal{H}}(u,u) = Q_L(hu, hu - h) = \frac{1}{2} \iint (h_B(x) - h_B(y))[(h_B(x) - h(x)) \\
- (h_B(y) - h(y))] J(x,y) dm(x) dm(y)
\end{aligned}$$

and thus

$$\begin{aligned}
Q_{\mathcal{H}}(u,u) = \frac{1}{2} \iint [h_B(x)(h_B(x) - h(x)) + h_B(y)(h_B(y) - h(y)) \\
- h_B(x)(h_B(y) - h(y)) - h_B(y)(h_B(x) - h(x))] J(x,y) dm(x) dm(y).
\end{aligned}$$

By symmetry we obtain

$$\begin{aligned}
Q_{\mathcal{H}}(u,u) &= \iint [h_B(x)(h_B(x) - h(x)) - h_B(x)(h_B(y) - h(y))] J(x,y) dm(x) dm(y) \\
&= \iint [(h_B(x)^2 - h(x) h_B(x)) - h_B(x)(h_B(y) - h(y))] J(x,y) dm(x) dm(y) \\
&= \iint h_B(x)(h(y) - h_B(y)) J(x,y) dm(x) dm(y).
\end{aligned}$$

Similarly, direct computations of the right hand side of equation (4.4) yield

$$\begin{aligned}
&\frac{1}{2} \int \int \left(1_B(x) - 1_B(y)\right)^2 J(x,y) h(y) dm(y) h(x) dm(x) \\
&= \frac{1}{2} \int \int \left(h_B(x) h(y) - 2 h_B(x) h_B(y) + h_B(y) h(x)\right) J(x,y) dm(y) dm(x) \\
&= \frac{1}{2} \int \int [h_B(x)(h(y) - h_B(y)) + h_B(y)(h(x) - h_B(x)] J(x,y) dm(y) dm(x) \\
&= \int \int h_B(x)(h(y) - h_B(y)) J(x,y) dm(y) dm(x).
\end{aligned}$$

This proves identity (4.4) together with the fact that the quadratic form $Q_{\mathcal{H}}(u,u)$ is a regular Dirichlet form in $L^2(X, h^2 \cdot m)$ as claimed. ■

Definition 4.4. A symmetric Markovian semigroup $(P_t)_{t>0}$ in $L^2(X,\mu)$ is called transient if its resolvent $(G_\lambda)_{\lambda>0}$ can be defined also for the value $\lambda = 0$ as a self-adjoint (possibly unbounded) operator $G_0 = \int_0^\infty P_t dt$ such that $\mathbf{1}_K \in dom(G_0)$ for every compact set $K \subset X$.

A Dirichlet form $Q(u,u)$ relative to $L^2(X,\mu)$ is called transient if the associated symmetric Markovian semigroup $(P_t)_{t>0}$ is transient.

One can show that the form $Q(u,u)$ is transient if and only if the following condition holds: for every compact set $K \subset X$ there exists a constant $C_K > 0$ such that

$$\int_X |u|\, d\mu \leq C_K \sqrt{Q(u,u)},\ \forall u \in dom(Q)\ .$$

Theorem 4.5. *In the setting of Theorem 4.3:*

1. *There exists a hierarchical Laplacian $\mathcal{L}$, related to the (non-homogeneous) ultrametric measure space $(X, h\cdot m)$, such that*

$$Q_{\mathcal{H}}(u,u) = Q_{\mathcal{L}}(u,u),\ \forall u \in L^2(X, h\cdot m)\cap L^2(X, h^2\cdot m).$$

2. *In particular, $\mathcal{D} \subset dom(Q_{\mathcal{L}})$ is a core for $Q_{\mathcal{L}}$ (i.e. $Q_{\mathcal{L}}(u,u)$ is a regular Dirichlet form relative to $L^2(X, h\cdot m)$).*

3. *The Dirichlet form $Q_{\mathcal{L}}$ is transient.*

Proof. Consider the function

$$J(B) := -\frac{1}{\Gamma_p(-\alpha)}\frac{1}{m(B)^{1+\alpha}},\ B\in\mathcal{B},$$

defined on the set $\mathcal{B}$ of all open balls. Since in the p-adic metric $m(B) = diam(B)$ for any ball B, we get

$$J(x,y) = J(x \curlywedge y)$$

where $x \curlywedge y$ is the minimal ball which contains x and y. Consider also the Radon measure $\widetilde{m} = h\cdot m$. We claim that the following properties hold true:

(i) $S\subset T \Longrightarrow J(S) > J(T)$ and $J(T)\to 0$ as $T\to X$.

(ii) $\widetilde{\lambda}(B) := \sum_{S:\ B\subseteq S} \widetilde{m}(S)\,(J(S)-J(S')) < \infty$ for any $B\in\mathcal{B}$.

(iii) $\widetilde{\lambda}(B)\to +\infty$ as $B\to\{x\}$ for any $x\in X$.

The property (i) is evident. To prove (ii) we write

$$\widetilde{\lambda}(B) = -\frac{1}{\Gamma_p(-\alpha)}\left(1-\frac{1}{p^{1+\alpha}}\right)\sum_{S:\ B\subseteq S}\frac{\widetilde{m}(S)}{m(S)^{1+\alpha}}$$

This condition of transience was first introduced by A. Beurling and J. Deny in the unreplacable paper A. Beurling and J. Deny [12]. It is slightly more restrictive than the definition of transience given in M. Fukushima [20, Section 1.5].

The following counterpart of Theorem 4.5 is in order: Let $X^{\mathcal{H}}$ and $X^{\mathcal{L}}$ be the Hunt processes associated with the Dirichlet forms $Q_{\mathcal{H}}$ and $Q_{\mathcal{L}}$ respectively. According to M. Fukushima [20, Theorem 5.5.2 and Example 5.5.1] their paths are related by the random time change $X^{\mathcal{H}}_t = X^{\mathcal{L}}_{\tau_t}$ where $\tau_t = \inf\{s>0 : A_t > t\}$ and $A_t = \int_0^t h(X^{\mathcal{H}}_s)ds$ is the positive continuous additive functional. It follows in particular, that the characteristic operators of Dynkin for these processes are related by the equation $(-\mathcal{H}u)(x) = (-\mathcal{L}u)(x)/h(x)$. This fact we are going to use in the next sections to solve the equation $Hu = v$.

$$= (p^\alpha - 1) \sum_{S:\ B \subseteq S} \frac{\widetilde{m}(S)}{m(S)^{1+\alpha}}.$$

Next, using the identity

$$\int f(\|x\|_p) dm(x) = \left(1 - \frac{1}{p}\right) \sum_{\gamma=-\infty}^{\infty} f(p^\gamma) p^\gamma,$$

we obtain that if $0 \in S$ then

$$\widetilde{m}(S) = \frac{p-1}{p - p^{-\beta}} m(S)^{1+\beta}, \tag{4.7}$$

so

$$\frac{\widetilde{m}(S)}{m(S)^{1+\alpha}} = \frac{p-1}{p-p^{-\beta}} \frac{1}{m(S)^{\alpha-\beta}}. \tag{4.8}$$

Clearly equality (4.8) implies (ii). On the other hand, for $B \in \mathcal{B}(x)$ small enough we have

$$\widetilde{\lambda}(B) \geq (p^\alpha - 1) \frac{\widetilde{m}(B)}{m(B)^{1+\alpha}} > (p^\alpha - 1) m(B)^{-\alpha} \min_{y \in B} \|y\|_p^\beta \tag{4.9}$$

and

$$\min_{y \in B} \|y\|_p^\beta = \begin{cases} \|x\|_p^\beta & \text{if} \quad x \neq 0 \\ (m(B))^\beta & \text{if} \quad x = 0 \end{cases}, \tag{4.10}$$

so (4.9) and (4.10) imply (iii). According to A. Bendikov [3, Section 2], properties $(i) - (iii)$ imply that the operator

$$\mathcal{L}u(x) = \int (u(x) - u(y)) J(x,y) d\widetilde{m}(y) \tag{4.11}$$

is a hierarchical Laplacian in $L^2(X, \widetilde{m})$. In particular, $\mathcal{D} \subset dom(\mathcal{L})$ and for $u \in \mathcal{D}$ we have

$$Q_{\mathcal{L}}(u,u) = \frac{1}{2} \int\int (u(x) - u(y))^2 J(x,y) d\widetilde{m}(y) d\widetilde{m}(x) = Q_{\mathcal{H}}(u,u).$$

That $\mathcal{D}$ is a core of $Q_{\mathcal{L}}$ follows from the fact that $\mathcal{L}$, as a hierarchical Laplacian, is essentially self-adjoint. Indeed, in this case $(Q_{\mathcal{L}}, dom(Q_{\mathcal{L}}))$ concedes with the minimal extension of $(Q_{\mathcal{L}}, \mathcal{D})$ which has $\mathcal{D}$ as a core. The proof of the fact that the Markovian semigroup $(e^{-t\mathcal{L}})_{t>0}$ is transient, i.e. that $\mathbf{1}_K$ belongs to $dom(G_0)$ for any compact set K, uses an ad hoc argument and we postpone it to the next section (Theorem 4.6). Let us show how to derive the Beurling-Deny condition of transience from the transience of the semigroup $(e^{-t\mathcal{L}})_{t>0}$. For any $u \in dom(Q_{\mathcal{L}})$ we have $|u| \in dom(Q_{\mathcal{L}})$ and $Q_{\mathcal{L}}(|u|,|u|) \leq Q_{\mathcal{L}}(u,u)$. Also $v := G_0 \mathbf{1}_K$ is in $dom(\mathcal{L})$ and $\mathcal{L}v = \mathbf{1}_K$ whence

$$\begin{aligned} \int_K |u| \, d\widetilde{m} &= Q_{\mathcal{L}}(|u|, v) \\ &\leq \sqrt{Q_{\mathcal{L}}(v,v)} \sqrt{Q_{\mathcal{L}}(u,u)}. \end{aligned}$$

Setting $C_K := \sqrt{Q_{\mathcal{L}}(v,v)}$ we get the desired result. The proof is finished. ■

4.3 The Green function $g_{\mathcal{L}}(x,y)$

In what follows we assume that $(\alpha-1)/2<\beta<\alpha$. The Markovian resolvent $G_\lambda=(\mathcal{L}+\lambda \mathrm{I})^{-1}$, $\lambda>0$, acts in Banach spaces $C_\infty(X)$ and $L^p(X,\widetilde{m})$, where $\widetilde{m}=h\cdot m$, as a bounded operator and admits the following representation

$$G_\lambda u(x)=\int g_{\mathcal{L}}(\lambda,x,y)u(y)d\widetilde{m}(y).$$

Here $g_{\mathcal{L}}(\lambda,x,y)$, the so called λ-Green function, is a continuous function taking finite values outside the diagonal set. As a function of λ it decreases, so the limit (finite or infinite)

$$g_{\mathcal{L}}(x,y):=\lim_{\lambda\to 0} g_{\mathcal{L}}(\lambda,x,y)$$

exists. The function $g_{\mathcal{L}}(x,y)$ is called the Green function of the operator $\mathcal{L}$.

Theorem 4.6. *The Green function $g_{\mathcal{L}}(x,y)$ is a continuous function taking finite values off the diagonal set (i.e. the Markovian semigroup $(e^{-t\mathcal{L}})_{t>0}$ is transient). Moreover, the following relationship holds:*

$$g_{\mathcal{L}}(x,y)\asymp \frac{\|x-y\|_p^{\alpha-1}}{\left(\|x\|_p\vee\|y\|_p\right)^{2\beta}}, \tag{4.12}$$

or equivalently

$$\frac{g_{\mathcal{L}}(x,y)}{g_L(x,y)}\asymp\left(\frac{1}{\|x\|_p}\wedge\frac{1}{\|y\|_p}\right)^{2\beta}. \tag{4.13}$$

Proof. Let us equip $X=\mathbb{Q}_p$ with the ultrametric $d(x,y)=p^{-\alpha}\|x-y\|_p^\alpha$, intrinsic for the hierarchical Laplacian L, and define the following variables

$$F(x,R)=\left(\int_R^\infty\left(\frac{1}{m(B_r(x))}\int_{B_r(x)}hdm\right)\frac{dr}{r^2}\right)^{-1}$$

and

$$\widetilde{d}(x,y)=F(x,d(x,y)). \tag{4.14}$$

Since for each fixed x the function $R\to F(x,R)$ is continuous, strictly increasing, 0 at 0 and ∞ at ∞, $\widetilde{d}(x,y)$ is an ultrametric on X. Let $\widetilde{B}_{\widetilde{R}}(x)$ be a $\widetilde{d}$-ball of radius $\widetilde{R}$ centred at x. Then $\widetilde{B}_{\widetilde{R}}(x)=B_R(x)$ whenever

$$\widetilde{R}=F(x,R).$$

Since L is a hierarchical Laplacian acting in $L^2(X,m)$ and $d(x,y)$ is its intrinsic ultrametric, we have (see [5, equation (3.11)])

$$\begin{aligned} J(x,y)&=\int_{d(x,y)}^\infty\frac{1}{m(B_R(x))}\frac{dR}{R^2} \\ &=\int_{\widetilde{d}(x,y)}^\infty\frac{1}{\widetilde{m}(\widetilde{B}_{\widetilde{R}}(x))}\frac{d\widetilde{R}}{\widetilde{R}^2}. \end{aligned}\tag{4.15}$$

It follows that $\widetilde{d}(x,y)$ is intrinsic ultrametric corresponding to the hierarchical Laplacian $\mathcal{L}$ and

$$\widetilde{V}(x,\widetilde{R}):=\widetilde{m}(\widetilde{B}_{\widetilde{R}}(x))$$

$$= \widetilde{m}(B_R(x)) = \int_{B_R(x)} h dm$$

is its volume-function. We claim that

$$\frac{\widetilde{m}(B_R(x))}{m(B_R(x))} \asymp \begin{cases} m(B_R(x))^{\beta} & \text{if} \quad d(0,x) \le R \\ h(x) & \text{if} \quad d(0,x) > R \end{cases}. \tag{4.16}$$

Indeed, if $d(0,x) \le R$ then $B_R(x) = B_R(0)$, so applying (4.7), we get

$$\begin{aligned}
\frac{\widetilde{m}(B_R(x))}{m(B_R(x))} &= \frac{1}{m(B_R(x))} \int_{B_R(x)} h dm \\
&= \frac{1}{m(B_R(0))} \int_{B_R(0)} h dm \\
&= \frac{p-1}{p-p^{-\beta}} m(B_R(0))^{\beta} = \frac{p-1}{p-p^{-\beta}} m(B_R(x))^{\beta}.
\end{aligned}$$

On the other hand, if $d(0,x) > R$ then from $y \in B_R(x)$ we get that $d(y,0) = d(x,0)$, so

$$\begin{aligned}
\frac{\widetilde{m}(B_R(x))}{m(B_R(x))} &= \frac{1}{m(B_R(x))} \int_{B_R(x)} h(y) dm(y) \\
&= \frac{1}{m(B_R(x))} \int_{B_R(x)} h(x) dm(y) = h(x).
\end{aligned}$$

Notice that asymptotic relationship (4.16) holds uniformly in x and R in the sense that the corresponding two sided inequality contains constants which do not depend on x and R. In turn, (4.16) implies the following (uniform) asymptotic relationship:

$$\widetilde{R} = F(x,R) \asymp \begin{cases} R/h(x) & \text{if} \quad R < d(0,x) \\ R^{\frac{\alpha-\beta}{\alpha}} & \text{if} \quad R \ge d(0,x) \end{cases} \tag{4.17}$$

Let us consider first the case $d(0,x) \le R$. We have

$$\begin{aligned}
\int_R^{\infty} \frac{\widetilde{m}(B_r(x))}{m(B_r(x))} \frac{dr}{r^2} &\asymp \int_R^{\infty} m(B_r(x))^{\beta} \frac{dr}{r^2} \\
&\asymp \int_R^{\infty} r^{-\left(2-\frac{\beta}{\alpha}\right)} dr \asymp R^{-\left(1-\frac{\beta}{\alpha}\right)},
\end{aligned}$$

so

$$\widetilde{R} := F(x,R) \asymp R^{1-\frac{\beta}{\alpha}}.$$

In the case $d(0,x) > R$ there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}
\int_R^{\infty} \frac{\widetilde{m}(B_r(x))}{m(B_r(x))} \frac{dr}{r^2} &= \int_R^{d(0,x)} \frac{\widetilde{m}(B_r(x))}{m(B_r(x))} \frac{dr}{r^2} + \int_{d(0,x)}^{\infty} \frac{\widetilde{m}(B_r(x))}{m(B_r(x))} \frac{dr}{r^2} \\
&= C_1 d(0,x)^{\frac{\beta}{\alpha}} \left(\frac{1}{R} - \frac{1}{d(0,x)} \right) + \frac{C_2}{d(0,x)^{1-\frac{\beta}{\alpha}}} \\
&\asymp \frac{d(0,x)^{\frac{\beta}{\alpha}}}{R} \asymp \frac{h(x)}{R},
\end{aligned}$$

so

$$\widetilde{R} := F(x,R) \asymp \frac{R}{h(x)}.$$

Furthermore, asymptotic relationships (4.16) and (4.17) yield the following (uniform) asymptotic relationship

$$\widetilde{V}(x,\widetilde{R}) = \widetilde{m}(B_R(x)) \tag{4.18}$$
$$\asymp \begin{cases} h(x)R^{\frac{1}{\alpha}} & \text{if} \quad R < d(0,x) \\ R^{\frac{1+\beta}{\alpha}} & \text{if} \quad R \geq d(0,x) \end{cases},$$

or equivalently, we get

$$\widetilde{V}(x,\widetilde{R}) \asymp \begin{cases} h(x)^{1+\frac{1}{\alpha}}\widetilde{R}^{\frac{1}{\alpha}} & \text{if} \quad \widetilde{R} < \widetilde{d}(0,x) \\ \widetilde{R}^{\frac{1+\beta}{\alpha-\beta}} & \text{if} \quad \widetilde{R} \geq \widetilde{d}(0,x) \end{cases}. \tag{4.19}$$

1. Let us consider the case $\|x-y\|_p = \|x\|_p \vee \|y\|_p$. Then clearly $d(x,y) = d(0,x) \vee d(0,y)$, and similar equation holds in $\widetilde{d}$ metric. If $R \geq d(0,x)$ then

$$\widetilde{R} := F(x,R) \asymp R^{1-\frac{\beta}{\alpha}} \tag{4.20}$$

and

$$\widetilde{V}(x,\widetilde{R}) \asymp R^{\frac{1+\beta}{\alpha}} \asymp \widetilde{R}^{\frac{1+\beta}{\alpha-\beta}}, \tag{4.21}$$

Equation (4.21) implies the following two results:

1. Since $\delta := \frac{1+\beta}{\alpha-\beta} > 1$, the function $\widetilde{R} \to 1/\widetilde{V}(x,\widetilde{R})$ is integrable at ∞ for any fixed x, so the Markovian semigroup $(e^{-t\mathcal{L}})_{t>0}$ (equivalently, the Dirichlet form $Q_{\mathcal{L}}$) is transient (see A. Bendikov, A. Grigor'yan, Ch. Pittet and W. Woess [5, Theorem 2.28]) as it has been stated in Theorem 4.5.

2. The fact that $\widetilde{V}(x,\widetilde{R}) \asymp \widetilde{R}^{\delta}$, $\delta > 1$, for $\widetilde{R} \geq \widetilde{d}(0,x)$, yield the following asymptotic relationship

$$g_{\mathcal{L}}(x,y) = \int\limits_{\widetilde{d}(x,y)}^{\infty} \frac{d\widetilde{R}}{\widetilde{V}(x,\widetilde{R})} \asymp \frac{\widetilde{d}(x,y)}{\widetilde{V}(x,\widetilde{d}(x,y))}, \tag{4.22}$$

or equivalently, see equations (4.20) and (4.21),

$$g_{\mathcal{L}}(x,y) \asymp \|x-y\|_p^{\alpha-1-2\beta} = \frac{\|x-y\|_p^{\alpha-1}}{\left(\|x\|_p \vee \|y\|_p\right)^{2\beta}} \tag{4.23}$$

provided $\|x\|_p \leq \|x-y\|_p$. Similarly, by symmetry, relationship (4.23) holds provided $\|y\|_p \leq \|x-y\|_p$. Thus finally, the assumption $\|x-y\|_p = \|x\|_p \vee \|y\|_p$ implies (4.23), as it was claimed. 2. Let us consider the case $\|x-y\|_p < \|x\|_p \vee \|y\|_p$. In this case we have: $\|x\|_p = \|y\|_p$ and $\|x-y\|_p < \|x\|_p$, similar relations hold in d and $\widetilde{d}$ metrics. Having this in mind we write

$$g_{\mathcal{L}}(x,y) = \int\limits_{\widetilde{d}(x,y)}^{\infty} \frac{d\widetilde{R}}{\widetilde{V}(x,\widetilde{R})} = \left(\int\limits_{\widetilde{d}(x,y)}^{\widetilde{d}(0,x)} + \int\limits_{\widetilde{d}(0,x)}^{\infty} \right) \frac{d\widetilde{R}}{\widetilde{V}(x,\widetilde{R})} = I + II.$$

Since $\widetilde{d}(0,x) \leq \widetilde{R}$ implies $\widetilde{V}(x,\widetilde{R}) \asymp \widetilde{R}^{\frac{1+\beta}{\alpha-\beta}}$, we get

$$II \asymp \frac{\widetilde{d}(0,x)}{\widetilde{V}(x,\widetilde{d}(0,x))} \asymp \frac{1}{\widetilde{d}(0,x)^{\frac{1-\alpha+2\beta}{\alpha-\beta}}}.$$

To estimate the first term we write

$$I=\int\limits_{\widetilde{d}(x,y)}^{\widetilde{d}(0,x)}\frac{d\widetilde{R}}{\widetilde{V}(x,\widetilde{R})}\asymp\frac{1}{h(x)^{1+\frac{1}{\alpha}}}\int\limits_{\widetilde{d}(x,y)}^{\widetilde{d}(0,x)}\frac{d\widetilde{R}}{\widetilde{R}^{\frac{1}{\alpha}}}$$

and

$$\begin{aligned}\frac{1}{h(x)^{1+\frac{1}{\alpha}}}\int\limits_{\widetilde{d}(x,y)}^{\widetilde{d}(0,x)}\frac{d\widetilde{R}}{\widetilde{R}^{\frac{1}{\alpha}}}&=\frac{1}{h(x)^{1+\frac{1}{\alpha}}}\left(\frac{1}{\widetilde{d}(x,y)^{\frac{1}{\alpha}-1}}-\frac{1}{\widetilde{d}(0,x)^{\frac{1}{\alpha}-1}}\right)\\&=\frac{\widetilde{d}(x,y)^{1-\frac{1}{\alpha}}}{h(x)^{1+\frac{1}{\alpha}}}\left(1-\left(\frac{\widetilde{d}(x,y)}{\widetilde{d}(0,x)}\right)^{\frac{1}{\alpha}-1}\right).\end{aligned}$$

Finally, since $\|x\|_p=\|y\|_p$ and $\|x-y\|_p<\|x\|_p$, we have

$$\begin{aligned}g_{\mathcal{L}}(x,y)&=I+II\\&\asymp\frac{\widetilde{d}(x,y)^{1-\frac{1}{\alpha}}}{h(x)^{1+\frac{1}{\alpha}}}\left(1-\left(\frac{\widetilde{d}(x,y)}{\widetilde{d}(0,x)}\right)^{\frac{1}{\alpha}-1}\right)+\frac{1}{\widetilde{d}(0,x)^{\frac{1-\alpha+2\beta}{\alpha-\beta}}}\\&=\frac{\widetilde{d}(x,y)^{1-\frac{1}{\alpha}}}{h(x)^{1+\frac{1}{\alpha}}}\left(\left(1-\left(\frac{\widetilde{d}(x,y)}{\widetilde{d}(0,x)}\right)^{\frac{1}{\alpha}-1}\right)+\frac{\widetilde{d}(x,y)^{\frac{1}{\alpha}-1}h(x)^{1+\frac{1}{\alpha}}}{\widetilde{d}(0,x)^{\frac{1-\alpha+2\beta}{\alpha-\beta}}}\right).\end{aligned}$$

According to (4.20) $h(x)\asymp\widetilde{d}(0,x)^{\frac{\beta}{\alpha-\beta}}$ whence

$$\frac{h(x)^{1+\frac{1}{\alpha}}}{\widetilde{d}(0,x)^{\frac{1-\alpha+2\beta}{\alpha-\beta}}}\asymp\frac{\widetilde{d}(0,x)^{\frac{\beta}{\alpha-\beta}(1+\frac{1}{\alpha})}}{\widetilde{d}(0,x)^{\frac{1-\alpha+2\beta}{\alpha-\beta}}}\asymp\frac{1}{\widetilde{d}(0,x)^{\frac{1}{\alpha}-1}}$$

and thus, using (4.17), we get

$$\begin{aligned}g_{\mathcal{L}}(x,y)&\asymp\frac{\widetilde{d}(x,y)^{1-\frac{1}{\alpha}}}{h(x)^{1+\frac{1}{\alpha}}}\asymp\left(\frac{d(x,y)}{h(x)}\right)^{1-\frac{1}{\alpha}}\frac{1}{h(x)^{1+\frac{1}{\alpha}}}\\&=\frac{d(x,y)^{1-\frac{1}{\alpha}}}{h(x)^2}\asymp\frac{\|x-y\|_p^{\alpha-1}}{\|x\|_p^{2\beta}}=\frac{\|x-y\|_p^{\alpha-1}}{\left(\|x\|_p\vee\|y\|_p\right)^{2\beta}}.\end{aligned}$$

The proof of the theorem is finished. ■

4.4 Solution of the equation $Hu=v$

Throughout this section we assume that $(\alpha-1)/2\leq\beta<\alpha$ and that b and β are related by equation (4.2). Then, by Theorem 4.1), the operator

$$H=\mathfrak{D}^{\alpha}+b\,\|x\|_p^{-\alpha}$$

is a self-adjoint and non-negative definite operator acting in $L^2(X,m)$.

Notice that b is an increasing continuous function of β which fulfill the whole range $[b_*,+\infty)$, where $b_*=-\{\Gamma_p\left((1+\alpha)/2\right)\}^2$. In particular, b fulfills the interval $[b_*,0)$ as β runs through the interval $[(\alpha-1)/2,0)$ and b fulfills $[0,+\infty)$ as β runs through the interval $[0,\alpha)$.

Theorem 4.7. *The equation $Hu = v$ has unique solution*

$$u(x) = \int_X g_H(x,y)v(y)dm(y),$$

where $g_H(x,y)$ is a continuous function given by

$$g_H(x,y) = h(x)g_{\mathcal{L}}(x,y)h(y).$$

We call $g_H(x,y)$ the Green function of the operator H, or the fundamental solution of the equation $Hu = v$.

Proof. We know that $\mathcal{L} : \mathcal{D} \to L^2(X, h \cdot m) \cap C_\infty(X)$. Let us show that $\mathcal{L} : \mathcal{D} \to L^q(X,m)$, $\forall 1 \le q \le \infty$. It is enough to check this property for $\psi = \mathbf{1}_B$, the indicator of an open ball B. In this case there exists a constant $C > 0$ such that as $x \to \infty$ the following asymptotic relationship holds:

$$\begin{aligned}\mathcal{L}\psi(x) &= -\int_B J(x,y)h(y)dm(y) \\ &= -\frac{1}{\Gamma_p(-\alpha)}\frac{1}{\|x\|_p^{1+\alpha}}\int_B hdm \asymp \frac{C}{\|x\|_p^{1+\alpha}}\end{aligned}$$

Clearly this relationship and the fact that $\mathcal{L}\psi(x)$ is bounded proofs the claim. In particular, $\mathcal{L}\psi \in L^2(X,m)$ and therefore $\frac{1}{h}\mathcal{L}\psi \in L^2(X, h^2 \cdot m)$ for any $\psi \in \mathcal{D}$. Having this in mind we do our computations for $\varphi, \psi \in \mathcal{D}$:

$$\begin{aligned}|Q_{\mathcal{H}}(\varphi,\psi)| &= |Q_{\mathcal{L}}(\varphi,\psi)| = |(\mathcal{L}\psi,\varphi)_{L^2(hm)}| \\ &= \left|\left(\frac{1}{h}\mathcal{L}\psi,\varphi\right)_{L^2(h^2m)}\right| \le \left\|\frac{1}{h}\mathcal{L}\psi\right\|_{L^2(h^2m)} \|\varphi\|_{L^2(h^2m)}.\end{aligned}$$

The above estimate means that $\varphi \to Q_{\mathcal{H}}(\varphi,\psi)$ is a bounded linear functional in $L^2(X, h^2 \cdot m)$ for any $\psi \in \mathcal{D}$. This fact, in turn, implies that $\mathcal{D} \subset dom(\mathcal{H})$ and

$$\mathcal{H}\psi = \frac{1}{h}\mathcal{L}\psi, \ \forall\psi \in \mathcal{D}. \tag{4.24}$$

Let us consider the equation $\mathcal{H}u = v$ for $v \in \mathcal{D}$. Since $\mathcal{D} \subset dom(\mathcal{H})$ we have

$$(\mathcal{H}u,\psi)_{L^2(X,h^2\cdot m)} = (u,\mathcal{H}\psi)_{L^2(X,h^2\cdot m)}, \ \forall\psi \in \mathcal{D}.$$

Applying equation (4.24) we get

$$(\mathcal{H}u,\psi)_{L^2(X,h^2\cdot m)} = \left(u, \frac{1}{h}\mathcal{L}\psi\right)_{L^2(X,h^2\cdot m)} = (u,\mathcal{L}\psi)_{L^2(X,h\cdot m)}.$$

On the other hand, we have

$$(\mathcal{H}u,\psi)_{L^2(X,h^2\cdot m)} = (v,\psi)_{L^2(X,h^2\cdot m)} = (hv,\psi)_{L^2(X,h\cdot m)}.$$

Our calculations from above show that for Hölder conjugated (p,q) we have

$$\left|(u,\mathcal{L}\psi)_{L^2(X,h\cdot m)}\right| = |(hv,\psi)_{L^2(X,h\cdot m)}| \le \|hv\|_{L^p(X,h\cdot m)} \|\psi\|_{L^q(X,h\cdot m)}.$$

It follows that if we choose $1 < p < \frac{1+\alpha}{1-\alpha}$, then $\psi \to (u, \mathcal{L}\psi)_{L^2(X,h\cdot m)}$ is a bounded linear functional in $L^q(X, h \cdot m)$ provided $q = \frac{p}{p-1}$, i.e. $\frac{1}{2}\left(1 + \frac{1}{\alpha}\right) < q < \infty$. As $(e^{-t\mathcal{L}})_{t>0}$ is a continuous symmetric Markovian semigroup an application of the Riesz-Thorin interpolation theorem shows that it can be extended to all $L^q(X, h \cdot m)$ as a continuous contraction semigroup. Let $\mathcal{L}_q$ be its minus infinitesimal generator, then $\mathcal{L}_q$ extends $\mathcal{L}$, and $\mathcal{L}_q^* = \mathcal{L}_p$. All the above shows that u must belong to the set $dom(\mathcal{L}_p)$ and $\mathcal{L}_p u = hv$. The equation $\mathcal{L}_p u = hv$ has unique solution

$$\begin{aligned} u(x) &= \int_X g_{\mathcal{L}}(x, y)(hv)(y)h(y)dm(y) \\ &= \int_X g_{\mathcal{L}}(x, y)v(y)h^2(y)dm(y). \end{aligned}$$

It follows that the operator $\mathcal{H}$ acting in $L^2(X, h^2 \cdot m)$ admits a Green function $g_{\mathcal{H}}(x, y)$ and that $g_{\mathcal{H}}(x, y)$ coincides with the function $g_{\mathcal{L}}(x, y)$, the Green function of the operator $\mathcal{L}$ acting in $L^2(X, h \cdot m)$:

$$g_{\mathcal{H}}(x, y) = g_{\mathcal{L}}(x, y). \tag{4.25}$$

Finally, let us consider the equation $Hu = v$. Since $H = U \circ \mathcal{H} \circ U^{-1}$, we get $\mathcal{H}(U^{-1}u) = U^{-1}v$. It follows that

$$(U^{-1}u)(x) = \int_X g_{\mathcal{H}}(x, y)(U^{-1}v)(y)h(y)^2 dm(y),$$

or equivalently

$$u(x) = \int_X h(x)g_{\mathcal{H}}(x, y)h(y)v(y)dm(y).$$

That means that equation $Hu = v$ admits a fundamental solution

$$\begin{aligned} g_H(x, y) &:= h(x)g_{\mathcal{H}}(x, y)h(y) \\ &= h(x)g_{\mathcal{L}}(x, y)h(y), \end{aligned}$$

thanks to (4.25). The proof of the theorem is finished. ■

Corollary 4.8. *The Green function $g_H(x, y)$ is a continuous function taking finite values off the diagonal set. Moreover, the following relationship holds:*

$$g_H(x, y) \asymp \frac{\|x\|_p^\beta \, \|x - y\|_p^{\alpha-1} \, \|y\|_p^\beta}{\left(\|x\|_p \vee \|y\|_p\right)^{2\beta}}, \tag{4.26}$$

or equivalently,

$$\frac{g_H(x, y)}{g_L(x, y)} \asymp \left(\frac{\|x\|_p}{\|y\|_p} \wedge \frac{\|y\|_p}{\|x\|_p}\right)^\beta.$$

Proof. Follows directly from Theorem 4.6 and Theorem 4.7. ■

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A. D. Bendikov
Institute of Mathematics, Wroclaw University, Wroclaw, Poland
E-mail address: bendikov@math.uni.wroc.pl

A. A. Grigor'yan
Department of Mathematics, University of Bielefeld, Bielefeld, Germany
E-mail address: grigor@math.uni-bielefeld.de

S. A. Molchanov
Department of Mathematics, University of North Carolina at Charlotte, U.S.A., and National Research University, Higher School of Economics, Russian Federation
E-mail address: smolchan@uncc.edu