Analysis on fractal spaces and walk dimension

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Introduction

Differentiation and integration have been the cornerstones of Analysis in Euclidean spaces since the days of Newton and Leibniz. Our purpose here is to discuss elements of Analysis on Ahlfors-regular *metric spaces*, in particular, on fractals.

Following Lebesgue, integration amounts to measure theory, and the latter on such spaces is determined by the Hausdorff dimension α and the Hausdorff measure $\mathcal{H}_\alpha$.

The *differential calculus* on metric spaces/fractals (if there is one) is determined by one more parameter β that is called the *walk dimension*. In $\mathbb{R}^n$ one does not see it because $\beta = 2$ independently of n. However, on fractal spaces one has typically $\beta > 2$.

Originally the walk dimension was introduced in connection with diffusion processes on fractal spaces where it determines the scaling *time* $\simeq$ *space*$^\beta$ for this process. The generator of this process is an analogue of the Laplace operator that gives rise to differential calculus on the underlying space.

In this talk we show how the notion of the walk dimension can be defined in any regular metric space independently of a diffusion process, using instead a critical exponent of the family of Besov function spaces. Hence, the walk dimension can be regarded as the second important invariant of a regular metric space, after the Hausdorff dimension, which can be used for classification of such spaces.

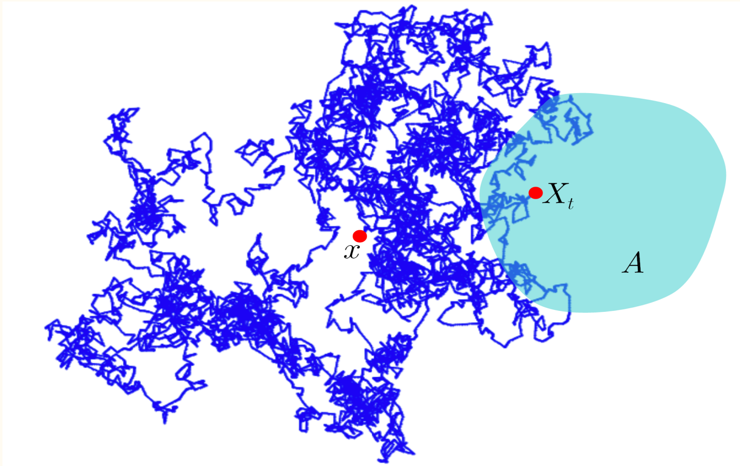
Brownian motion in $\mathbb{R}^n$

Let $\{X_t\}_{t\geq 0}$ be the classical Brownian motion in $\mathbb{R}^n$, whose transition density is given by the Gauss-Weierstrass function

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right),$$

that is, for any initial point $x \in \mathbb{R}^n$, any Borel set $A \subset \mathbb{R}^n$, and any $t > 0$,

$$\mathbb{P}_x(X_t \in A) = \int_A p_t(x,y)dy.$$



The function $p_t(x,y)$ is also referred to as *the heat kernel* since it is the fundamental solution of the heat equation

$$\partial_t u = \Delta u$$

where

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

is the Laplace operator.

Assume for simplicity that the initial point is 0. Let τ_R be the *first exit time* of X_t from the ball B_R of radius R centered at 0, that is,

$$\tau_R = \inf\left\{t > 0 : |X_t| > R\right\}.$$

The expected value $\mathbb{E}\tau_R$ is called the *mean exit time* of X_t from the ball B_R.

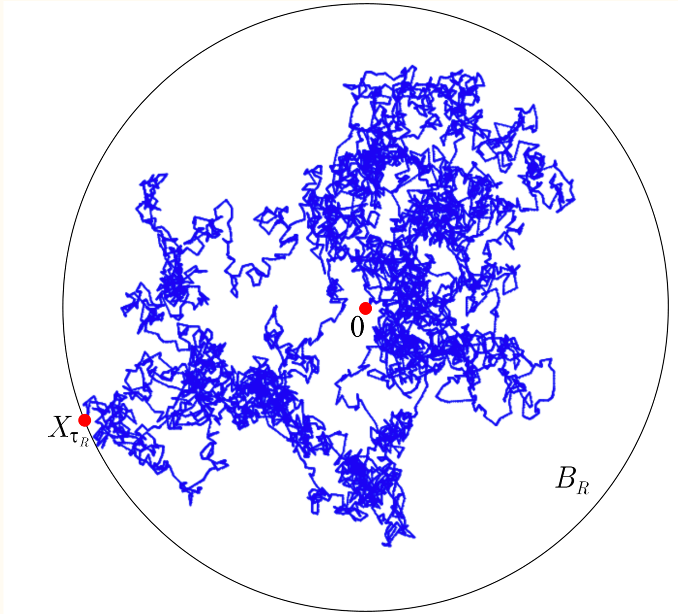
It is possible to prove that

$$\mathbb{E}\tau_R = cR^2$$

where $c = \frac{1}{2n}$.

Hence, Brownian motion needs time cR^2 to cover the distance R from the origin:

$$\textit{time} \simeq \textit{space}^2.$$



The exponent 2 here is called the *walk dimension* of Brownian motion.

Brownian motion on manifolds

Let M be a complete non-compact Riemannian manifold of dimension n and Δ be the Laplace-Beltrami operator on M. The associated heat equation

$$\partial_t u = \Delta u$$

has the minimal positive fundamental solution $p_t(x,y)$ (where $t>0$, $x,y\in M$). It serves as the transition density of a diffusion process $\{X_t\}$ on M that is also called Brownian motion.

For any open set $\Omega \subset M$, denote by τ_Ω the first exit time of X_t from Ω, that is,

$$\tau_\Omega = \inf\left\{t>0 : X_t \notin \Omega\right\}.$$

One can ask to estimate the mean exit time $\mathbb{E}_x \tau_{B(x,R)}$ where $B(x,R)$ denotes the geodesic ball of radius R centered at $x \in M$.

Assume that the heat kernel on M admits the following Gaussian estimates:

$$p_t(x,y) \asymp \frac{C}{t^{n/2}} \exp\left(-\frac{d(x,y)^2}{ct}\right), \tag{1}$$

where $d(x,y)$ is the geodesic distance, c, C are positive constants, and the sign $\asymp$ means both $\leq$ and $\geq$ but with possibly different values of c and C. Then one can that, for any ball $B(x,R)$, $\mathbb{E}_x \tau_{B(x,R)} \simeq R^2$. Hence, the walk dimension in this case is also equal to 2.

This class of manifolds satisfying (1) is characterized by the following two conditions:

(i) the volume estimate: for any geodesic ball $B(x,r)$,

$$\mu\left(B(x,r)\right) \simeq r^n,$$

where μ is the Riemannian volume;

(ii) the Poincaré inequality: for any $f \in C^1(B(x,r))$

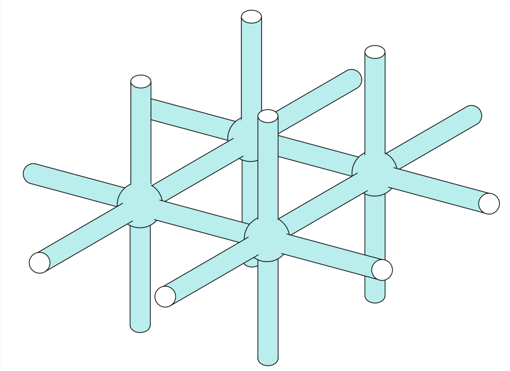
$$\int_{B(x,r)} |\nabla f|^2 \, d\mu \geq \frac{c}{r^2} \int_{B(x,\varepsilon r)} \left(f - \overline{f}\right)^2 d\mu,$$

where $c > 0$ and $\varepsilon \in (0,1)$ are constants, and $\overline{f} = \fint_{B(x,\varepsilon r)} f \, d\mu$.

For example, the Poincaré inequality is satisfied if $Ricci_M \geq 0$, and in this case (1) follows from the Li-Yau theorem.

Another case when (i) and (ii) are satisfied is when M covers a compact with a nilpotent deck transformation group G.

On this picture M is a *jungle gym* with $G = \mathbb{Z}^3$.



Regular metric spaces and fractals

Let (M, d) be a metric space and μ be a Borel measure on M. We say that M is *α-regular* if, for any metric ball $B(x, r) := \{y \in M : d(x, y) < r\}$ of radius $r < r_0$,

$$\mu\left(B\left(x, r\right)\right) \simeq r^{\alpha}, \tag{2}$$

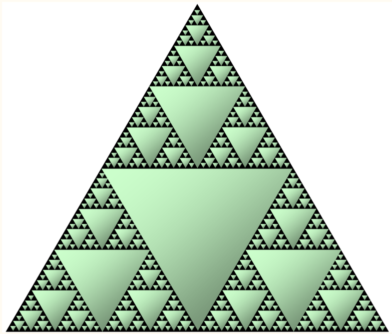
where $\alpha > 0$. It follows from (2) that α is the Hausdorff dimension of M and $\mu \simeq \mathcal{H}_{\alpha}$. Equivalently, we can define $\alpha = \dim_H M$, set $\mu = \mathcal{H}_{\alpha}$ and assume (2).

Hence, in some sense, α determines the *integral calculus* on M.

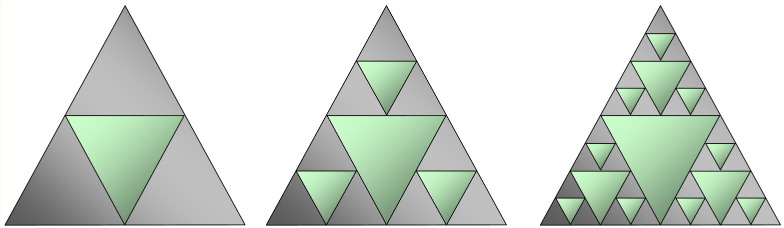
The α-regular spaces with fractional α are usually called *fractals*. The fractals first appeared in mathematics as curious examples of sets serving as counterexamples to illustrate various theorems.

A well-known fractal is the *Cantor set*, which however is disconnected.

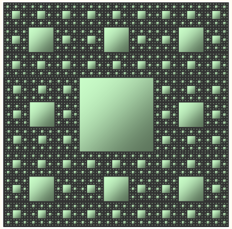
Here is a connected fractal set – the *Sierpinski gasket*:

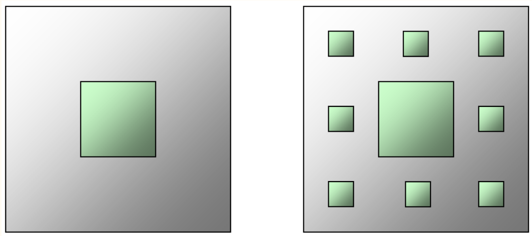


Here are three steps of construction of SG: $\alpha = \frac{\log 3}{\log 2} \approx 1.58.$

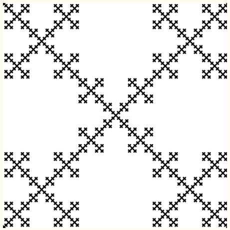


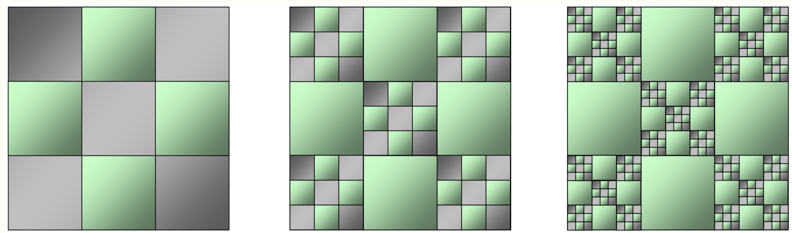
Sierpinski carpet and two steps of construction of SC: $\alpha = \frac{\log 8}{\log 3} \approx 1.89.$





Vicsek snowflake and three steps of construction of VS: $\alpha = \frac{\log 5}{\log 3} \approx 1.46.$





Dirichlet forms and their generators

On certain metric measure spaces, including fractals, it is possible to construct a *Laplace-type* operator, by means of the theory of Dirichlet forms (Beurling–Deny and Fukushima).

Definition. A *Dirichlet form* in $L^2(M,\mu)$ is a pair $(\mathcal{E},\mathcal{F})$ where $\mathcal{F}$ is dense subspace of $L^2(M,\mu)$ and $\mathcal{E}$ is a symmetric bilinear form on $\mathcal{F}$ with the following properties:

- It is *positive definite*, that is, $\mathcal{E}(f,f) \geq 0$ for all $f \in \mathcal{F}$.
- It is *closed*, that is, $\mathcal{F}$ is complete with respect to the norm
$$\int_M f^2 d\mu + \mathcal{E}(f,f).$$
- It is *Markovian*, that is, if $f \in \mathcal{F}$ then $\widetilde{f} := \min(f_+, 1) \in \mathcal{F}$ and $\mathcal{E}(\widetilde{f},\widetilde{f}) \leq \mathcal{E}(f,f)$.

Any Dirichlet form has the generator: a positive definite self-adjoint operator $\mathcal{L}$ in $L^2(M,\mu)$ with domain $\operatorname{dom}(\mathcal{L}) \subset \mathcal{F}$ such that

$$(\mathcal{L}f, g) = \mathcal{E}(f,g) \quad \text{for all } f \in \operatorname{dom}(\mathcal{L}) \text{ and } g \in \mathcal{F}.$$

For example, the classical Dirichlet integral in $\mathbb{R}^n$

$$\mathcal{E}(f,f)=\int_{\mathbb{R}^n}|\nabla f|^2\,dx \tag{3}$$

is the quadratic part of a Dirichlet form $(\mathcal{E},\mathcal{F})$ with domain $\mathcal{F}=W_2^1(\mathbb{R}^n)$. Its generator is $\mathcal{L}=-\Delta$ with $\mathrm{dom}\,(\mathcal{L})=W_2^2(\mathbb{R}^n)$.

Another example of a Dirichlet form in $\mathbb{R}^n$ arises from the quadratic form

$$\mathcal{E}(f,f)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(f(x)-f(y))^2}{|x-y|^{n+s}}dxdy, \tag{4}$$

where $s\in(0,2)$, with the domain $\mathcal{F}=B_{2,2}^{s/2}(\mathbb{R}^n)$. Its generator is $\mathcal{L}=(-\Delta)^{s/2}$.

The generator $\mathcal{L}$ of any Dirichlet form determines the *heat semigroup* $\{e^{-t\mathcal{L}}\}_{t\geq 0}$ in $L^2(M,\mu)$.

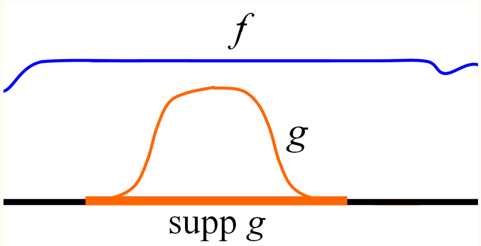
If the operator $e^{-t\mathcal{L}}$ for $t>0$ is an integral operator:

$$e^{-t\mathcal{L}}f(x)=\int_M p_t(x,y)f(y)d\mu(y)\ \text{ for all } f\in L^2,$$

then its integral kernel $p_t(x,y)$ (that is necessarily non-negative) is called the *heat kernel* of $\mathcal{L}$ or of $(\mathcal{E},\mathcal{F})$.

A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *strongly local* if $\mathcal{E}(f, g) = 0$ whenever

$$f = \text{const in a neighborhood of } \operatorname{supp} g.$$



For example, the Dirichlet form (3) is strongly local, while the Dirichlet form (4) is non-local.

The local Dirichlet form (3) with the generator $\mathcal{L} = -\Delta$ has the heat kernel

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x - y|^2}{4t}\right). \tag{5}$$

The non-local Dirichlet form (4) with the generator $\mathcal{L} = (-\Delta)^{s/2}$ has the heat kernel that admits the following estimate:

$$p_t(x, y) \simeq \frac{1}{t^{n/s}} \left(1 + \frac{|x - y|}{t^{1/s}}\right)^{-(n+s)}. \tag{6}$$

In the special case $s = 1$ the heat kernel of $(-\Delta)^{1/2}$ coincides with the Cauchy distribution with the scale parameter t:

$$p_t(x, y) = \frac{c_n t}{\left(t^2 + |x-y|^2\right)^{\frac{n+1}{2}}} = \frac{c_n}{t^n}\left(1 + \frac{|x-y|^2}{t^2}\right)^{-\frac{n+1}{2}}.$$

A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *regular* if $\mathcal{F} \cap C_0(M)$ is dense both in $\mathcal{F}$ and $C_0(M)$. For example, the both Dirichlet forms (3) and (4) are regular.

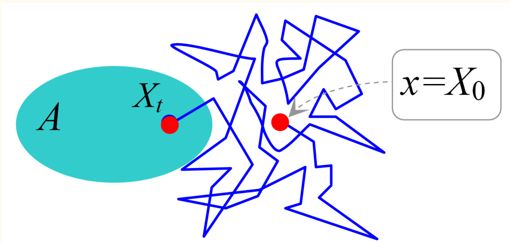
Any regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ determines a Markov processes $\{X_t\}_{t\geq 0}$ on M with the transition semigroup $e^{-t\mathcal{L}}$, which means that

$$\mathbb{E}_x f(X_t) = e^{-t\mathcal{L}} f(x) \quad \text{for all } f \in C_0(M) \text{ and } t \geq 0.$$

If the heat kernel of $(\mathcal{E}, \mathcal{F})$ exists then it serves as the transition density of $\{X_t\}$:

$$\mathbb{P}_x(X_t \in A) = \int_A p_t(x, y)\, d\mu(y),$$

for any Borel set $A \subset M$ and $t > 0$.



If $(\mathcal{E}, \mathcal{F})$ is local then $\{X_t\}$ is a diffusion process (=with continuous trajectories), while otherwise the trajectories of the process $\{X_t\}$ contain jumps.

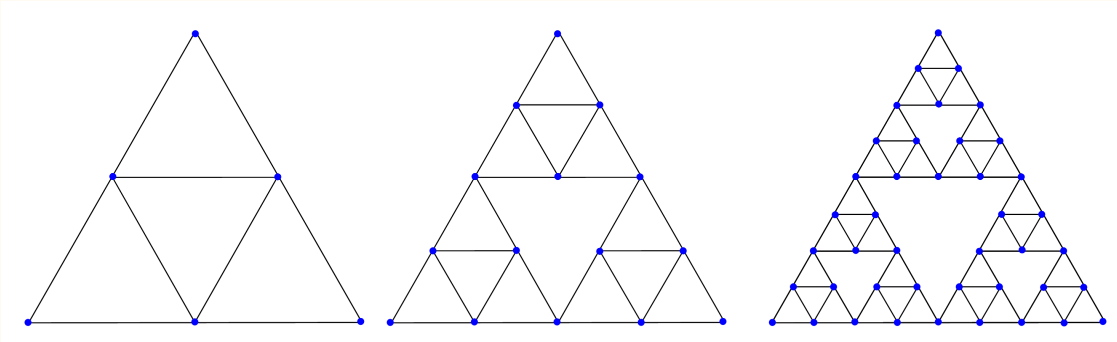
For example, the local Dirichlet form (3) with the generator $\mathcal{L} = -\Delta$ determines Brownian motion in $\mathbb{R}^n$ with the transition density (5), while the non-local Dirichlet form (4) with the generator $\mathcal{L} = (-\Delta)^{s/2}$ determines a symmetric stable Levy process in $\mathbb{R}^n$ of the index s with the transition density (6).

If a metric measure space M possesses a strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ then its generator $\mathcal{L}$ can be regarded as an analogue of the Laplace operator; hence, it determines in some sense a *differential calculus* on M.

Strongly local Dirichlet forms on fractals

Nontrivial strongly local regular Dirichlet forms have been successfully constructed on large families of fractals, in particular, on SG by Barlow–Perkins '88, Goldstein '87 and Kusuoka '87, on SC by Barlow–Bass '89 and Kusuoka–Zhou '92, on p.c.f. fractals (including VS) by Kigami '93.

Each of these fractals can be regarded as limit of a sequence of approximating graphs Γ_n.



Approximating graphs $\Gamma_1, \Gamma_2, \Gamma_3$ for SG

Define on each Γ_n a Dirichlet form $\mathcal{E}_n$ by

$$\mathcal{E}_n(f,f) = \sum_{x\sim y} (f(x) - f(y))^2$$

(where $x \sim y$ denotes neighboring vertices on Γ_n), and then consider a scaled limit

$$\mathcal{E}(f,f) = \lim_{n\to\infty} R_n \mathcal{E}_n(f,f) \tag{7}$$

with an appropriate renormalizing sequence $\{R_n\}$.

The main difficulty is to ensure the existence of $\{R_n\}$ such that this limit exists in $(0,\infty)$ for a dense in L^2 family of functions f.

For p.c.f. fractals one chooses $R_n = \rho^n$ where, for example, $\rho = \frac{5}{3}$ for SG and $\rho = 3$ for VS, and the limit in (7) exists due to monotonicity.

For SC the situation is much harder. Initially a strongly local Dirichlet form on SC was constructed by Barlow and Bass '89 in a different way by using a probabilistic approach. After a work of Barlow, Bass, Kumagai and Teplyaev '10 it became possible to claim that the limit (7) exists for a certain sequence $\{R_n\}$ such that $R_n \simeq \rho^n$, where the exact value of ρ is still unknown. Numerical computation indicates that $\rho \approx 1.25$.

Other methods of constructing a strongly local Dirichlet form on SC were proposed by Kusuoka and Zhou '92 and AG and M.Yang '19.

Walk dimension

In all the above examples of fractals, the strongly local Dirichlet form possesses the heat kernel that satisfies the following *sub-Gaussian* estimate:

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{8}$$

(where $C, c > 0$), for all $x, y \in M$ and $t \in (0, t_0)$ (Barlow–Perkins '88, Barlow–Bass '92).

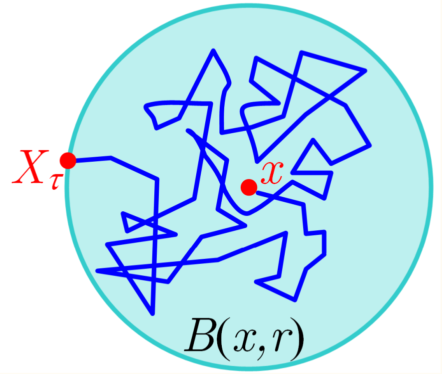
Here α as above is the Hausdorff dimension of the underlying metric space (M, d) while β is a new parameter.

For any open set $\Omega \subset M$, denote by τ_Ω the first exit time of diffusion X_t from Ω:

$$\tau_\Omega = \inf\{t > 0 : X_t \notin \Omega\}.$$

It is known that if (8) holds then, for any ball $B(x,r)$ with $r < r_0$,

$$\mathbb{E}_x \tau_{B(x,r)} \simeq r^\beta.$$

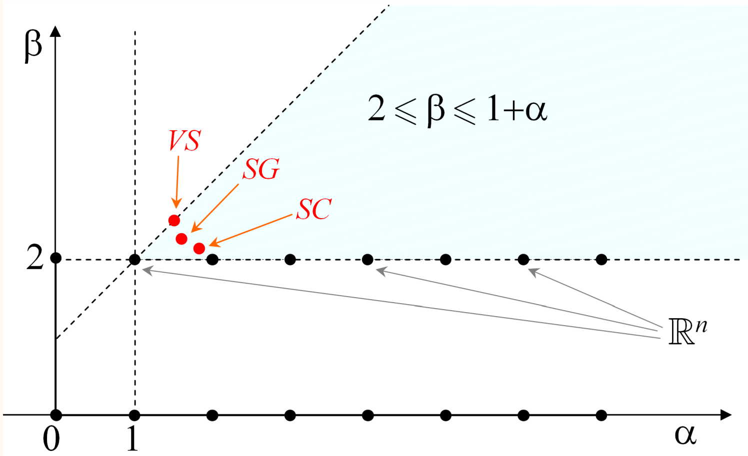


That is, β is the walk dimension of the diffusion process $\{X_t\}$.

Hence, we have the fractal scaling $time{=}distance^{\beta}$ that is different from the Euclidean one $time{=}distance^{2}$.

The walk dimension can be regarded as a numerical characteristic of the differential calculus on M that is determined by the generator $\mathcal{L}$.

It is known that always $\beta \geq 2$. Moreover, for any pair of reals $\alpha \geq 1$ and $\beta \in [2, \alpha + 1]$, there exists a *geodesic* metric measure space with a heat kernel satisfying (8) (Barlow '04).



Hence, we obtain a large family of regular metric measure spaces that are characterized by a pair (α, β), where α is responsible for integration while β is responsible for differentiation.

The Euclidean space $\mathbb{R}^n$ belongs to this family with $\alpha = n$ and $\beta = 2$ (in the case $\beta = 2$ the estimate (8) becomes Gaussian).

On fractals the values of β is determined by the scaling parameter ρ. It is known that:

- on SG: $\beta = \frac{\log 5}{\log 2} \approx 2.32$ (and $\alpha = \frac{\log 3}{\log 2} \approx 1.58$)
- on VS: $\beta = \frac{\log 15}{\log 3} \approx 2.46$ (and $\alpha = \frac{\log 5}{\log 3} \approx 1.46$)
- on SC: $\beta = \frac{\log(8\rho)}{\log 3} \approx 2.10$ (and $\alpha = \frac{\log 8}{\log 3} \approx 1.89$).

Besov spaces characterization of β

Given an α-regular metric measure space (M,d,μ), it is possible to define a family $B^{\sigma}_{p,q}$ of Besov spaces, where $p,q \in [1,\infty]$, $\sigma > 0$. Here we need only the following special cases: for any $\sigma > 0$ the space $B^{\sigma}_{2,2}$ consists of functions $f \in L^2(M,\mu)$ such that

$$\|f\|^2_{\dot{B}^{\sigma}_{2,2}} := \int\int_{M\times M} \frac{|f(x)-f(y)|^2}{d(x,y)^{\alpha+2\sigma}} d\mu(x)d\mu(y) < \infty, \tag{9}$$

and $B^{\sigma}_{2,\infty}$ consists of functions $f \in L^2(M,\mu)$ such that

$$\|f\|^2_{\dot{B}^{\sigma}_{2,\infty}} := \sup_{0<r<r_0} \frac{1}{r^{\alpha+2\sigma}} \int\int_{\{d(x,y)<r\}} |f(x)-f(y)|^2 \, d\mu(x)d\mu(y) < \infty.$$

It is easy to see that the space $B^{\sigma}_{2,2}$ shrinks as σ increases. Define

$$\boxed{\sigma^* = \sup\{\sigma > 0 : B^{\sigma}_{2,2} \text{ is dense in } L^2\}}. \tag{10}$$

If $\sigma < 1$ then $B^{\sigma}_{2,2}$ contains all Lipschitz functions with compact support. Hence, $\sigma^* \geq 1$. In $\mathbb{R}^n$, if $\sigma > 1$ then $B^{\sigma}_{2,2} = \{0\}$ so that $\sigma^* = 1$. On most fractal spaces $\sigma^* > 1$.

Theorem 1 (AG, Jiaxin Hu, Ka-Sing Lau) *Let $(\mathcal{E},\mathcal{F})$ be a strongly local Dirichlet form on (M,d,μ) such that its heat kernel exists and satisfies the sub-Gaussian estimate*

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{11}$$

with some α and β. Then the following is true:

(a) the space M is α-regular (consequently, $\alpha = \dim_H M$ and $\mu \simeq \mathcal{H}_a$);

(b) $\beta = 2\sigma^*$ (consequently, $\beta \geq 2$);

(c) $\mathcal{F} = B_{2,\infty}^{\sigma^*}$ *and* $\mathcal{E}(f,f) \simeq \|f\|^2_{\dot{B}^{\sigma^*}_{2,\infty}}$.

Corollary 2 *Both α and β in (11) are the invariants of the metric structure (M,d) alone.*

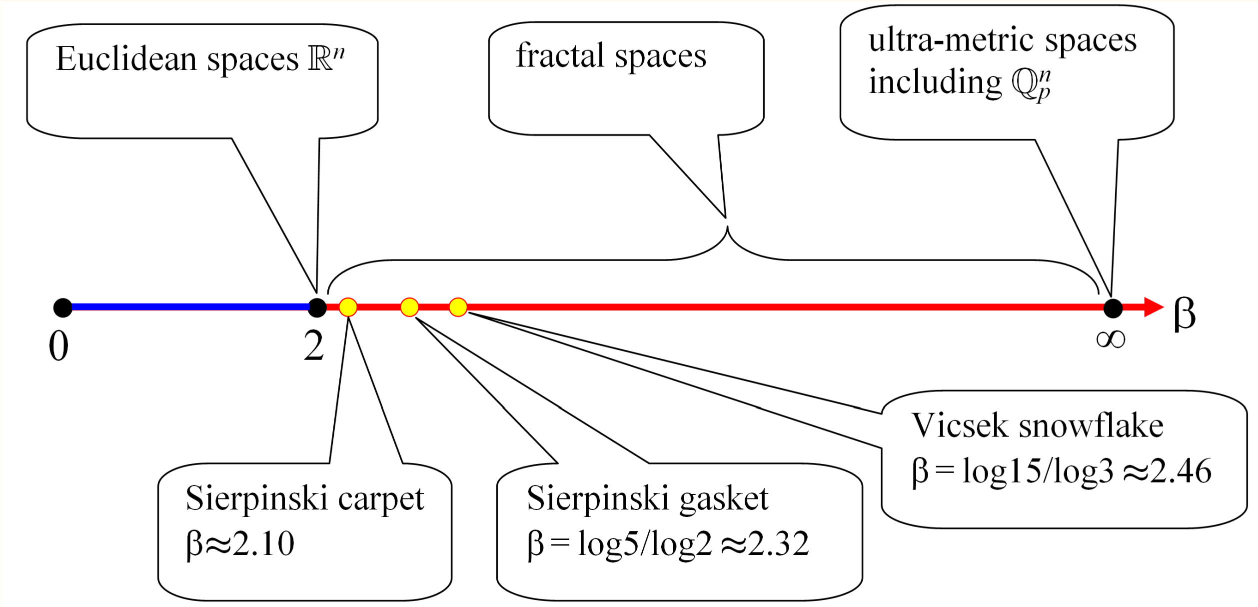
Indeed, σ^* is defined by using metric d and measure μ, while in this case $\mu \simeq \mathcal{H}_\alpha$ is also determined by d. Therefore, σ^* and β are also invariants of the metric space (M,d).

Note that σ^* is well defined by (10) for any α-regular metric space using $\mu = \mathcal{H}_\alpha$. In the view of Theorem 1, we *redefine* now the notion of the walk dimension by setting

$$\boxed{\beta := 2\sigma^*}. \tag{12}$$

Hence, β is the second invariant of a regular metric space after the Hausdorff dimension α.

Here is a classification of regular metric spaces according to their walk dimension.



Note that according to (10) and (12), the walk dimension β may take the value ∞, which is attained on *ultra-metric* spaces.

Ultra-metric spaces

A metric space (M, d) is called *ultra-metric* if it satisfies a stronger triangle inequality

$$d(x, y) \leq \max\left(d\left(x, z\right), d\left(y, z\right)\right) \quad \text{for all } x, y, z \in M. \tag{13}$$

For example, the field $\mathbb{Q}_p$ of p-adic numbers is an ultra-metric space with respect to the p-adic distance $d(x, y) = |x - y|_p$. Recall that, for a rational $z \in \mathbb{Q}$, the p-adic norm is defined by $|z|_p = p^{-k}$ provided $z = p^k \frac{a}{b}$ where the integers a, b are not divisible by p.
It is easy to see that

$$|z_1 + z_2|_p \leq \max\left(|z_1|_p, |z_2|_p\right),$$

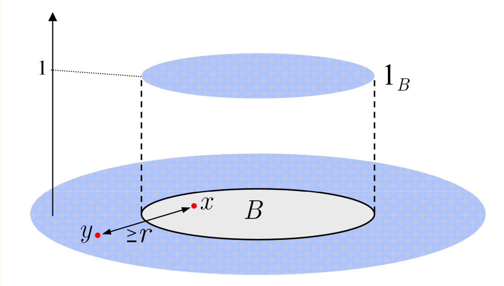
which implies (13) for all rational x, y, z. Since $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic norm, it follows that (13) holds for all $x, y, z \in \mathbb{Q}_p$.
Consequently, also $\mathbb{Q}_p^n$ is an ultra-metric space with the max-distance:

$$d(x, y) = \max_{1 \leq k \leq n}\left(|x_k - y_k|_p\right),$$

where x_k's are the p-adic components of $x \in \mathbb{Q}_p^n$. It is easy to prove that the Haar measure μ on $\mathbb{Q}_p^n$ is n-regular.

Let (M,d) be an α-regular ultra-metric space. The ultra-metric property (13) implies that, for any ball B of radius r, any point $x \in B$ is a center of B.

In particular, $d(x,y) \geq r$ for any $y \notin B$.



It follows that:

- the indicator function $\mathbf{1}_B$ is continuous;
- $\mathbf{1}_B \in B^{\sigma}_{2,2}$ for any $\sigma > 0$.

Consequently, $B^{\sigma}_{2,2}$ is dense in L^2 for any $\sigma > 0$, whence $\sigma^* = \infty$ and $\beta = \infty$.

Note that any ultra-metric space is totally disconnected and, hence, cannot carry a non-trivial diffusion. However, it carries a lot of jump processes.

Theorem 3 (A.Bendikov, AG, Eryan Hu, Jiaxin Hu, '21) *For any* $\sigma > 0$, *the Besov semi-norm* (9) *determines a regular Dirichlet form with the domain* $B^{\sigma}_{2,2}$, *its heat kernel satisfies a* stable-like *estimate*

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/s}} \left(1 + \frac{d(x,y)}{t^{1/s}}\right)^{-(\alpha+s)} \tag{14}$$

with the index $s = 2\sigma$, *and the walk dimension of the associated jump process is equal to* s.

An approach to construction of local Dirichlet forms

An open question. *Let (M, d, μ) be an α-regular metric measure space (or even self-similar). Assume $\sigma^* < \infty$. Does there exist a strongly local (regular) Dirichlet form in M? Does it have a heat kernel satisfying the sub-Gaussian estimate* (11) *with $\beta = 2\sigma^*$?*

Which additional conditions may be required?

Here is a possible approach to construction of such a Dirichlet form based on the family of Besov spaces. For any $\sigma < \sigma^*$ we need to define in $B^{\sigma}_{2,2}$ a quadratic form $\mathcal{E}_\sigma(f, f)$ with the following properties:

(i)
$$\mathcal{E}_\sigma(f, f) \simeq \|f\|^2_{\dot{B}^{\sigma}_{2,2}} = \iint\limits_{M \times M} \frac{|f(x) - f(y)|^2}{d(x, y)^{\alpha + 2\sigma}} d\mu(x) d\mu(y);$$

(ii) there should exist in some sense the limit

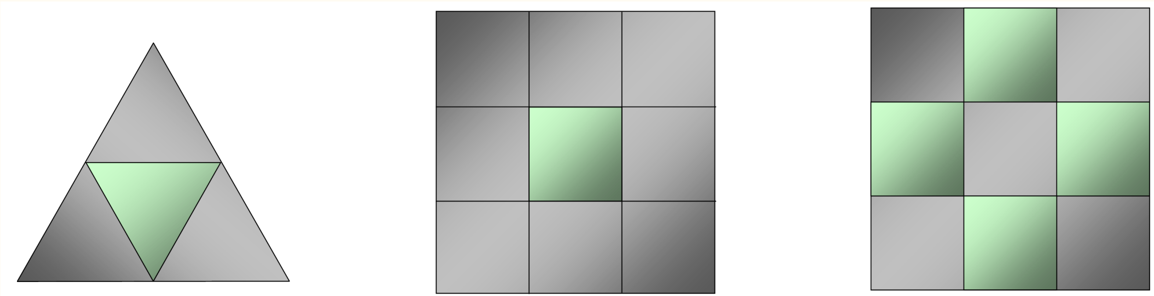
$$\lim_{\sigma \to \sigma^*} (\sigma^* - \sigma)\, \mathcal{E}_\sigma;$$

(iii) this limit should determine a strongly local regular Dirichlet form on M.

In $\mathbb{R}^n$ this method works with $\mathcal{E}_\sigma(f, f) = \|f\|^2_{\dot{B}^{\sigma}_{2,2}}$ and yields the Dirichlet integral. For SG and SC this method was realized by AG and Meng Yang '18 and '19.

An open question. *How to determine the walk dimension (equivalently, σ^*), even for self-similar sets?*

Each self-similar set is determined by the first step in its construction:



SG SC VS

It is well known how to compute the Hausdorff dimension: $\alpha = \frac{\log A}{\log B}$ where A is the number of remaining cells after the first step, and B is the contraction ratio.

An open question. *How to compute the walk dimension β using the first step in the fractal construction? This must be some graph invariant.*

The exact value of β remains open for the Sierpinski carpet.

Self-similar heat kernels

Let M be an α-regular metric space.

Theorem 4 (AG–Takashi Kumagai) *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on M. Assume that its heat kernel exists and satisfies the following estimate:*

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \Phi\left(c \frac{d(x,y)}{t^{1/\beta}}\right),$$

where $\beta > 0$ and Φ is a positive function on $[0,\infty)$. Then the following dichotomy holds :

- *either the Dirichlet form $\mathcal{E}$ is strongly local, $\beta \geq 2$ and $\Phi(s) \asymp C \exp(-cs^{\frac{\beta}{\beta-1}})$.*
- *or the Dirichlet form $\mathcal{E}$ is non-local and $\Phi(s) \simeq (1+s)^{-(\alpha+\beta)}$.*

That is, in the first case $p_t(x,y)$ satisfies the sub-Gaussian estimate

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{15}$$

while in the second case we obtain a *stable-like estimate*

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{16}$$