Homologies of digraphs and the K"unneth formula

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Paths and boundary operator

Let $V$ be a finite set. For any $p \geq 0$, an *elementary $p$-path* is any sequence $i_0, ..., i_p$ of $p + 1$ vertices of $V$ that will be denoted by $i_0...i_p$ or by $e_{i_0...i_p}$. Fix a field $\mathbb{K}$. A *$p$-path* is any formal $\mathbb{K}$-linear combinations of elementary $p$-paths, that is, any $p$-path has a form

$$v = \sum_{i_0,i_1,...,i_p \in V} v^{i_0i_1...i_p} e_{i_0i_1...i_p}, \text{ where } v^{i_0i_1...i_p} \in \mathbb{K}.$$ 

Denote by $\Lambda_p = \Lambda_p(V)$ the $\mathbb{K}$-linear space of all $p$-paths. Set $\Lambda_{-1} = \{0\}$.

**Definition.** For any $p \geq 0$, the *boundary operator* $\partial : \Lambda_p \to \Lambda_{p-1}$ is a $\mathbb{K}$-linear operator that acts on elementary paths by

$$\partial e_{i_0...i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0...\hat{i}_q...i_p}, \quad (1)$$

where the hat $\hat{i}_q$ means omission of the index $i_q$.

For example, $e_{ij} \in \Lambda_1$, $e_{ijk} \in \Lambda_2$ and

$$\partial e_{ij} = e_j - e_i, \quad \partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}.$$
One can show that $\partial^2 v = 0$ for any $v \in \Lambda_p$ and $p \geq 1$. Hence, we obtain a chain complex $\Lambda_*(V)$:

$$0 \leftarrow \Lambda_0 \leftarrow \Lambda_1 \leftarrow ... \leftarrow \Lambda_{p-1} \leftarrow \Lambda_p \leftarrow ...$$

where arrows are given by the boundary operator $\partial$.

**Definition.** An elementary $p$-path $e_{i_0...i_p}$ is called *regular* if $i_k \neq i_{k+1}$ for all $k = 0,...,p-1$, and non-regular otherwise.

For example, $e_{iij}$ is non-regular, while $e_{iji}$ is regular provided $i \neq j$.

Consider the following subspace of $\Lambda_p$:

$$\mathcal{R}_p \equiv \mathcal{R}_p(V) := \text{span}_K \{e_{i_0...i_p} : i_0...i_p \text{ is regular}\},$$

whose elements are called *regular* $p$-paths. We would like to consider $\partial$ on the spaces $\mathcal{R}_p$. However, $\partial$ is not invariant on $\{\mathcal{R}_p\}$. For example, $e_{iij} \in \mathcal{R}_2$ for $i \neq j$ while $\partial e_{iij} = e_{ji} - e_{ii} + e_{ij}$ contains a non-regular component $e_{ii}$ and, hence, is not in $\mathcal{R}_1$.

To overcome this difficulty, consider the complementary subspace

$$N_p := \text{span}_K \{e_{i_0...i_p} : i_0...i_p \text{ is non-regular}\}.$$
One can show that $\partial N_p \subset N_{p-1}$ so that the boundary operator $\partial$ is well-defined on $\{N_p\}$ and hence, on the quotient spaces $\Lambda_p/N_p$. Since $\Lambda_p = R_p \oplus N_p$ and, hence, $R_p \cong \Lambda_p/N_p$, we can define a regular boundary operator $\partial : R_p \rightarrow R_{p-1}$ as pullback of $\partial : \Lambda_p/N_p \rightarrow \Lambda_{p-1}/N_{p-1}$.

For regular $\partial$, the formula (1)

$$\partial e_{i_0...i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0...\hat{i}_q...i_p}$$

should be read as follows: all non-regular paths in the right hand side are set to be 0.

For example, for non-regular $\partial : \Lambda_2 \rightarrow \Lambda_1$ we have $\partial e_{iji} = e_{ji} - e_{ii} + e_{ij}$ whereas for regular $\partial : R_2 \rightarrow R_1$ we have $\partial e_{iji} = e_{ji} + e_{ij}$ since $e_{ii}$ is set to be zero.

Denote by $R_\ast(V)$ the chain complex

$$0 \leftarrow R_0 \leftarrow R_1 \leftarrow ... \leftarrow R_{p-1} \leftarrow R_p \leftarrow ...$$

where all the arrows are given by regular operator $\partial$. Below we use always the regular boundary operator $\partial$. 

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Definition. A digraph (directed graph) is a pair $G = (V, E)$ of a set $V$ of vertices and a set $E \subset \{V \times V \setminus \text{diag}\}$ of (directed) edges. The fact that $(i, j) \in E$ is also denoted by $i \rightarrow j$.

Definition. Let $G = (V, E)$ be a digraph. An elementary $p$-path $i_0...i_p$ on $V$ is called allowed if $i_k \rightarrow i_{k+1}$ for any $k = 0, ..., p-1$, and non-allowed otherwise.

Consider the following linear space

$$A_p \equiv A_p (G) = \text{span}_\mathbb{K} \{e_{i_0...i_p} : i_0...i_p \text{ is allowed}\} . \quad (2)$$

Definition. The elements of $A_p$ are called allowed $p$-paths.

By construction $A_p \subset \mathcal{R}_p$ but spaces $A_p$ are in general not invariant for $\partial$. For example, let $e_{abc}$ be allowed, that is, $a \rightarrow b \rightarrow c$. Then $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab}$ is not allowed if $a \not\rightarrow c$. 
To fix this problem, consider the following subspace of $\mathcal{A}_p$

$$\Omega_p \equiv \Omega_p(G) := \{ v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1} \}.$$  \hfill (3)

**Definition.** The elements of $\Omega_p$ are called $\partial$-invariant $p$-paths.

**Claim.** If $v \in \Omega_p$ then $\partial v \in \Omega_{p-1}$.

Indeed, $v \in \Omega_p$ implies $\partial v \in \mathcal{A}_{p-1}$ and $\partial (\partial v) = 0 \in \mathcal{A}_{p-2}$, which implies that $\partial v \in \Omega_{p-1}$.

Hence, we obtain a chain complex $\Omega_* = \Omega_*(G) :

\begin{align*}
0 & \leftarrow \Omega_0 \leftarrow \Omega_1 \leftarrow \ldots \leftarrow \Omega_{p-1} \leftarrow \Omega_p \leftarrow \Omega_{p+1} \leftarrow \ldots
\end{align*}

Recall that by construction $\Omega_p \subset \mathcal{A}_p \subset \mathcal{R}_p$. Note also that

$$\Omega_0 = \mathcal{A}_0 = \mathcal{R}_0 = \text{span}_\mathbb{K} \{ e_i : i \in V \}, \quad \Omega_1 = \mathcal{A}_1 = \text{span}_\mathbb{K} \{ e_{ij} : (i, j) \in E \}.$$  

**Definition.** Define the **path homologies** of the digraph $G$ by

$$H_p(G, \mathbb{K}) = H_p(G) := H_p(\Omega_*(G)) = \ker \partial|_{\Omega_p} / \text{Im} \partial|_{\Omega_{p+1}}.$$  

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It is easy to show that $H_0(G) \cong \mathbb{K}$ if $G$ is connected, but all other $H_p(G)$ carry non-trivial information about $G$.

**Example.** Consider the *triangle* digraph

\[
\begin{array}{c}
2 \\
\downarrow \\
0 \rightarrow \bullet 1
\end{array}
\]

Then $e_{012} \in \Omega_2$ as $e_{012} \in A_2$ and $\partial e_{012} = e_{12} - e_{02} + e_{01} \in A_1$. In fact, $\Omega_2 = A_2 = \text{span} \{e_{012}\}$, $\Omega_p = A_p = \{0\}$ $\forall p \geq 3$, and $H_p = \{0\}$ $\forall p \geq 1$ (the only closed element in $\Omega_1$ is $e_{12} - e_{02} + e_{01}$, which is exact as it is the boundary of $e_{012}$; hence $H_1 = \{0\}$).

Consider the *square* digraph:

\[
\begin{array}{c}
2 \bullet \\
\downarrow \\
0 \rightarrow \bullet 1
\end{array} \quad \rightarrow \quad 
\begin{array}{c}
3 \\
\downarrow \\
1
\end{array}
\]

For this digraph $A_2 = \text{span} \{e_{013}, e_{023}\}$ but neither $e_{013}$ nor $e_{023}$ is $\partial$-invariant. However, the 2-path $v := e_{013} - e_{023}$ is $\partial$-invariant as

\[
\partial v = (e_{13} - e_{03} + e_{01}) - (e_{23} - e_{03} + e_{02}) = e_{13} + e_{01} - e_{23} - e_{02} \in A_1,
\]

In fact, $\Omega_2 = \text{span} \{v\}$, $\Omega_p = A_p = \{0\}$ $\forall p \geq 3$, and $H_p = \{0\}$ $\forall p \geq 1$. 
Consider one more example of a digraph $G$:

A computation shows that $H_1(G) = \{0\}$ and $H_p(G) = \{0\}$ for $p \geq 3$, whereas $\dim H_2(G) = 1$ and

$$H_2(G) = \text{span} \{e_{124} + e_{234} + e_{314} - (e_{125} + e_{235} + e_{315})\}.$$ 

It is interesting to observe that $G$ is a planar graph but nevertheless its second homology group is non-zero.
Cross product of paths

Given two finite sets $X, Y$, consider their Cartesian product $Z = X \times Y$.

**Definition.** A regular elementary path $z = z_0z_1...z_r$ on $Z$ is called *step-like* if, for any $k = 1, ..., r$, the vertices $z_{k-1}$ and $z_k$ have the same projections either on $X$ or on $Y$.

Any step-like $r$-path $z$ on $Z$ determines by projections regular elementary paths $x = x_0...x_p$ and $y = y_0...y_q$ on $X$ and $y = y_0...y_q$ on $Y$, where $p + q = r$. 

![Diagram of cross product of paths](image-url)
Every vertex \((x_i, y_j)\) of a step-like path \(z\) can be represented as a point \((i, j)\) of \(\mathbb{Z}^2\) so that the whole path \(z\) is represented by a \textit{staircase} \(S(z)\) in \(\mathbb{Z}^2\) connecting the points \((0, 0)\) and \((p, q)\).

\[
\begin{array}{c}
(0,q) \\
\hline
\hline
(0,0) \\
\hline
(p,0)
\end{array}
\]

\[
\begin{array}{c}
(0,q) \\
\hline
\hline
(0,0) \\
\hline
(p,0)
\end{array}
\]

\[
\text{S}(z)
\]

\[
\text{L}(z)
\]

\[
\text{(i,j)}
\]

\textbf{Definition.} Define the \textit{elevation} \(L(z)\) of the path \(z\) as the number of the cells in \(\mathbb{Z}_+^2\) below the staircase \(S(z)\).
By definition, any $p$-path $u$ on $X$ is given by $u = \sum_x u^x e_x$ where $x$ is any elementary $p$-paths on $X$ and $u^x \in \mathbb{K}$. Extend the summation to all elementary paths $x$ with arbitrary length, by setting $u^x = 0$ if the length of $x$ is not equal to $p$.

Definition. For any paths $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ with $p, q \geq 0$ define their cross product $u \times v$ as a path on $Z$ by the following rule: for any step-like elementary path $z$ on $Z$, the component $(u \times v)^z$ is defined by

$$(u \times v)^z = (-1)^{L(z)} u^x v^y,$$  \hspace{1cm} (4)$$

where $x$ and $y$ are the projections of $z$ onto $X$ and $Y$, while for the other paths $z$ set $(u \times v)^z = 0$. It follows that $u \times v \in \mathcal{R}_{p+q}(Z)$.

For any elementary regular $p$-path $x$ on $X$ and $q$-path $y$ on $Y$ with $p, q \geq 0$ denote by $\Pi_{x,y}$ the set of all step-like paths $z$ on $Z$ whose projections on $X$ and $Y$ are $x$ and $y$ respectively. It follows from (4) that, for all regular elementary paths $x, y$,

$$e_x \times e_y = \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} e_z.$$  \hspace{1cm} (5)
**Example.** Denote the vertices of $X$ by letters $a, b, c$ etc and the vertices of $Y$ by integers $0, 1, 2$, etc. The vertices of $Z = X \times Y$ will be denoted as $a0, b2, c1$, etc, as the fields on the chessboard. For example, we have
\[
e_a \times e_{01} = e_{a0a1}, \quad e_{ab} \times e_0 = e_{a0b0}
\]
\[
e_{ab} \times e_{01} = e_{a0b0b1} - e_{a0a1b1}
\]
\[
e_{abc} \times e_{01} = e_{a0b0c0c1} - e_{a0b0b1c1} + e_{a0a1b1c1}
\]
\[
e_{abc} \times e_{012} = e_{a0b0c0c1c2} - e_{a0b0b1c1c2} + e_{a0b0b1b2c2} + e_{a0a1b1c1c2} - e_{a0a1b1b2c2} + e_{a0a1a2b2c2}
\]

**Proposition 1** (Product rule for cross product) If $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ where $p, q \geq 0$, then
\[
\partial (u \times v) = (\partial u) \times v + (-1)^p u \times (\partial v).
\]
Cartesian product of digraphs

To simplify notation, we denote the set of vertices of a digraph by the same letter as the digraph itself.

**Definition.** Cartesian product $X \square Y$ of two digraphs $X, Y$ is a digraph $Z$ with the set of vertices $X \times Y = \{(x, y) : x \in X, y \in Y\}$ and with the set of edges as follows: for $x, x' \in X$ and $y, y' \in Y$,

$$(x, y) \to (x', y') \text{ if either } x \to x' \text{ and } y = y' \text{ or } x = x' \text{ and } y \to y'.$$

as is shown on the following diagram:

```
    y'•     ...     (x',y')     →     (x',y')     ...
      ↑     ↑     ↑     ↑
(   y•   ...     (x,y)     →     (x',y)     ...   )
     /     x     ..     →     ..     ...
```

Clearly, any regular elementary path on $Z = X \square Y$ is allowed if and only if it is step-like and its projections onto $X$ and $Y$ are allowed.
Proposition 2 Let $p, q \geq 0$ and $r = p + q$.

(a) If $u \in A_p (X)$ and $v \in A_q (Y)$ then $u \times v \in A_r (Z)$.

(b) If $u \in \Omega_p (X)$ and $v \in \Omega_q (Y)$ then $u \times v \in \Omega_r (Z)$. Moreover, the operation $u, v \mapsto u \times v$ extends to that for the homology classes $u \in H_p (X)$ and $v \in H_q (Y)$ so that $u \times v \in H_r (Z)$.

Proof. (a) It suffices to prove this for $u = e_x$ and $v = e_y$. By (5) $e_x \times e_y$ is a linear combination of $e_z$ with $z \in \Pi_{x,y}$. If $x$ and $y$ are allowed then any $z \in \Pi_{x,y}$ is allowed, which implies that $e_x \times e_y \in A_r (Z)$.

(b) We already know that $u \times v$ is allowed. Hence, it suffices to prove that $\partial (u \times v)$ is allowed, which follows from the product rule:

$$\partial (u \times v) = \partial u \times v + (-1)^p u \times \partial v$$

(7)
as the right hand side is allowed by (a). For the second claim it suffices to verify two properties. Firstly, if $u$ and $v$ are closed then $u \times v$ is closed, which is obvious from (7). Secondly, if one of $u$, $v$ is exact then also $u \times v$ is exact: indeed, if, for example, $u = \partial w$ then

$$\partial (w \times v) = \partial w \times v + (-1)^{p+1} w \times \partial v = u \times v$$

so that $u \times v$ is exact. □
Theorem 3 Let $X, Y$ be two finite digraphs and $Z = X \Box Y$. Then we have the following isomorphism of the chain complexes:

$$\Omega_* (Z) \cong \Omega_* (X) \otimes \Omega_* (Y),$$

which is given by the map $u \otimes v \mapsto u \times v$ with $u \in \Omega_* (X)$ and $v \in \Omega_* (Y)$.

The right hand side of (8) is the tensor product of the two chain complexes. More explicitly (8) means that, for any $r \geq 0$,

$$\Omega_r (Z) \cong \bigoplus_{\{p, q \geq 0: p+q=r\}} (\Omega_p (X) \otimes \Omega_q (Y)).$$

Isomorphism (8) and an abstract theorem of K"unneth yield

$$H_* (Z) \cong H_* (X) \otimes H_* (Y).$$

The latter is called the *K"unneth formula* for homologies. The K"unneth formula is known for simplicial and singular homologies of products. For Cartesian product of digraphs we have a stronger isomorphism (8), which can be referred to as the K"unneth formula for chain complexes. It has no analogue in algebraic topology.
Example. Consider the digraph $Z = X \boxtimes Y$, where

$$X = \begin{array}{c}
\bullet \quad b \\
\bullet \quad a \\
\bullet \quad c
\end{array}, \quad Y = \begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow
\end{array}.$$

For $r = 4$ we obtain from (9) that

$$\Omega_4 (Z) \cong \bigoplus_{\{p,q\geq 0: p+q=4\}} (\Omega_p (X) \otimes \Omega_q (Y)) = \Omega_2 (X) \otimes \Omega_2 (Y)$$

because on both digraphs $X, Y$ we have $\Omega_p = \{0\}$ for $p \geq 3$. 
We know that $\Omega_2 (X) = \text{span} (e_{abc})$ and $\Omega_2 (Y) = \text{span} (e_{013} - e_{023})$, whence it follows that $\Omega_4 (Z)$ is spanned by a single 4-path

$$e_{abc} \times (e_{013} - e_{023}) = e_{a0b0c01c3} - e_{a0b0b1c1c3} + e_{a0b0b1b3c3} + e_{a0a1b1c1c3} - e_{a0a1b1b3c3} + e_{a0a1a3b3c3} - e_{a0b0c02c3} + e_{a0b0b2c2c3} - e_{a0b0b2b3c3} - e_{a0a2b2c2c3} + e_{a0a2b2b3c3} - e_{a0a2a3b3c3}.$$

Similarly one can compute $\Omega_r (Z)$ for other values of $r$. For example,

$$\Omega_3 (Z) \cong \Omega_1 (X) \otimes \Omega_2 (Y) \bigoplus \Omega_2 (X) \otimes \Omega_1 (Y),$$

which implies $\dim \Omega_3 (Z) = 3 \cdot 1 + 1 \cdot 4 = 7$ and the generators of $\Omega_3 (Z)$ are

$$e_{ab} \times (e_{013} - e_{023}), \; e_{ac} \times (e_{013} - e_{023}), \; e_{bc} \times (e_{013} - e_{023})$$

$$e_{abc} \times e_{01}, \; e_{abc} \times e_{13}, \; e_{abc} \times e_{02}, \; e_{abc} \times e_{23}.$$

Since all the homology groups of $X, Y$ are trivial except for $H_0$, we obtain that the same is true for homologies of $Z$. 

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Example. Consider $Z = X \square Y$ where $X, Y$ are cyclic digraphs:

\[ X = \begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\leftarrow \\
\bullet \\
\end{array} \quad Y = \begin{array}{c}
\bullet \\
\uparrow \\
\downarrow \\
\leftarrow \\
\bullet \\
\end{array}. \]

Note that $X$ is not a triangle and $Y$ is not a square.

One can show that all homologies $H_p(X)$ and $H_q(Y)$ are trivial for $p, q \geq 2$ whereas

\[
H_1(X) = \text{span} \left( e_{ab} + e_{bc} + e_{ca} \right) \\
H_1(Y) = \text{span} \left( e_{01} + e_{12} + e_{23} + e_{30} \right).
\]

It follows from (10) that

\[
H_2(Z) \cong \bigoplus_{\{p, q \geq 0 : p + q = 2\}} \left( H_p(X) \otimes H_q(Y) \right) = H_1(X) \otimes H_1(Y),
\]

in particular, $\dim H_2(Z) = 1$. The generating element of $H_2(Z)$ is

\[
(e_{ab} + e_{bc} + e_{ca}) \times (e_{01} + e_{12} + e_{23} + e_{30}).
\]
For any digraph $X$, define the \textit{cylinder} over $X$ by

$$\text{Cyl } X := X \square Y \text{ with } Y = (\cdot \cdot \rightarrow \cdot 1).$$

Assuming that the vertices of $X$ are enumerated by $0, 1, ..., n - 1$, let us enumerate the vertices of Cyl $X$ by $0, 1, ..., 2n - 1$ as follows: the vertex $(i, 0)$ of Cyl $X$ receives the number $i$, while $(i, 1)$ receives $i + n$.

For any regular path $v$ on $X$, the \textit{lifted} path $\hat{v}$ on Cyl $X$ by $\hat{v} = v \times e_{01}$.

For example, if $v = e_{i_0...i_p}$ then

$$\hat{v} = e_{i_0...i_p} \times e_{01} = \sum_{k=0}^{p} (-1)^{p-k} e_{i_0...i_k(i_k+n)...(i_p+n)}. \quad (11)$$

Since $e_{01} \in \Omega_1 (Y)$, we see that if $v \in \Omega_p (X)$ then $\hat{v} \in \Omega_{p+1} (\text{Cyl } X)$. 

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**Example.** Let us define the digraph $\text{Cube}_n$ inductively: $\text{Cube}_0 = \{0\}$ and 

$$\text{Cube}_n = \text{Cyl Cube}_{n-1}.$$ 

For example, $\text{Cube}_1$ is

$$0 \rightarrow 1$$

$\text{Cube}_2$ is a square

$$2 \rightarrow 3$$

$$0 \rightarrow 1$$

and $\text{Cube}_3$ is shown here:
Since $\text{Cube}_n = \text{Cube}_{n-1} \times Y$, where $\Omega_q(Y)$ is non-trivial only for $q = 0, 1$, and $\Omega_n(\text{Cube}_{n-1}) = \{0\}$, we obtain from (9)

\[ \Omega_n(\text{Cube}_n) \cong \Omega_{n-1}(\text{Cube}_{n-1}) \otimes \Omega_1(Y). \]

Since $\Omega_1(Y)$ is generated by a single element $v_1 = e_{01}$, we obtain by induction that $\dim \Omega_n(\text{Cube}_n) = 1$. A generating element $v_n$ of $\Omega_n(\text{Cube}_n)$ can be computed inductively by

\[ v_n = v_{n-1} \times e_{01} = \hat{v}_{n-1}. \]

By (11) we obtain successively

\[ v_2 = \hat{v}_1 = e_{013} - e_{023}, \]
\[ v_3 = \hat{v}_2 = e_{0457} - e_{0157} + e_{0137} - e_{0467} + e_{0267} - e_{0237}, \]
\[ \ldots \]
\[ \ldots \]

In general, $v_n$ is an alternating sum of $n!$ elementary paths that correspond to partitioning of a solid $n$-cube into $n!$ simplexes.

By (10) all homology groups of $\text{Cube}_n$ are trivial except for $H_0$. 

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Sketch of proof of Theorem 3. The main difficulty is to show that each \( \partial \)-invariant path \( w \) on \( Z = X \square Y \) can be represented as a linear combination of the products \( u \times v \) where \( u \) is \( \partial \)-invariant on \( X \) and \( v \) is \( \partial \)-invariant on \( Y \).

For any \( r \geq 0 \) consider the space

\[
\tilde{\Omega}_r (Z) = \text{span} \{ u \times v : u \in \Omega_p (X), v \in \Omega_q (Y), p + q = r \}
\]

By Proposition 2 we have \( \tilde{\Omega}_r (Z) \subset \Omega_r (Z) \), but we have to prove the opposite inclusion. It suffices to prove that

\[
\dim \Omega_r (Z) \leq \dim \tilde{\Omega}_r (Z).
\]

In the next argument we take \( \mathbb{K} = \mathbb{R} \) (a general field \( \mathbb{K} \) requires a more complicated argument). Consider the space

\[
\tilde{A}_r (Z) = \text{span} \{ u \times v : u \in A_p (X), v \in A_q (Y), p + q = r \}.
\]

By Proposition 2 we have \( \tilde{A}_r (Z) \subset A_r (Z) \).

We prove separately, that any element from \( \Omega_r (Z) \) is a linear combination of \( e_x \times e_y \) with allowed \( x, y \), which implies

\[
\Omega_r (Z) \subset \tilde{A}_r (Z).
\] (12)
If digraphs $X, Y$ are such that $\Omega_p(X) = \mathcal{A}_p(X)$ and $\Omega_q(Y) = \mathcal{A}_q(Y)$ for all $p, q \geq 0$ then also $\tilde{\Omega}_r(Z) = \tilde{\mathcal{A}}_r(Z)$. Substitution into (12) yields $\Omega_r(Z) \subset \tilde{\Omega}_r(Z)$, which finishes the proof in this case. However, the main difficulty lies in the fact that in general $\Omega_p \not\subseteq \mathcal{A}_p$.

In the general case we use the inner product for regular paths $u, v$ on a digraph:

$$[u, v] = \sum_x u^x v^x,$$

for which we need $\mathbb{K} = \mathbb{R}$. We prove that if $u, u'$ are allowed paths on $X$ and $v, v'$ are allowed paths on $Y$ then

$$[u \times v, u' \times v'] = C [u, u'] [v, v'],$$

(13)

where $C$ is a constant depending on the lengths of the paths.

Define the following subspaces:

$\Omega_p^\perp(X)$ – the orthogonal complement of $\Omega_p(X)$ in $\mathcal{A}_p(X)$.

$\Omega_q^\perp(Y)$ – the orthogonal complement of $\Omega_q(Y)$ in $\mathcal{A}_q(Y)$.

$\Omega_r^\perp(Z)$ – the orthogonal complement of $\Omega_r(Z)$ in $\tilde{\mathcal{A}}_r(Z)$.
We use (13) in order to prove that, for $p + q = r$,

\begin{align*}
&u \in \Omega^\perp_p (X), \quad v \in A_q (Y) \Rightarrow u \times v \in \Omega^\perp_r (Z), \\
u \in A_p (X), \quad v \in \Omega^\perp_q (Y) \Rightarrow u \times v \in \Omega^\perp_r (Z),
\end{align*}

(14)

Since

$$A_p (X) = \Omega_p (X) \oplus \Omega^\perp_p (X),$$

any $u \in A_p (X)$ admits a decomposition $u = u_\Omega + u_\perp$ where $u_\Omega \in \Omega_p (X)$ and $u_\perp \in \Omega^\perp_p (X)$. Using also a similar decomposition $v = v_\Omega + v_\perp$ for $v \in A_q (Y)$, we obtain

$$u \times v = u_\Omega \times v_\Omega + u_\Omega \times v_\perp + u_\perp \times v_\Omega + u_\perp \times v_\perp.$$

where $u_\Omega \times v_\Omega \in \tilde{\Omega}_r (Z)$, while by (14) all other terms in the right hand side belong to $\Omega^\perp_r (Z)$. It follows that

$$u \times v \in \tilde{\Omega}_r (Z) + \Omega^\perp_r (Z).$$

Since $\tilde{A}_r (Z)$ is spanned by the products $u \times v$ where $u, v$ are allowed, we obtain that

$$\tilde{A}_r (Z) \subset \tilde{\Omega}_r (Z) + \Omega^\perp_r (Z).$$
Comparing with the decomposition

$$\tilde{\mathcal{A}}_r (Z) = \Omega_r (Z) \oplus \Omega_r^\perp (Z) ,$$

we obtain $\dim \Omega_r (Z) \leq \dim \tilde{\Omega}_r (Z)$, which was to be proved. ■