Homologies of digraphs and Künneth formulas

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Abstract

We state and prove Künneth formulas for path homologies of Cartesian product and join of two digraphs.

Contents

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1 Introduction

The purpose of this paper is to prove Künneth formulas for homology groups of digraphs (=directed graphs). We use the notion of *path homology* of digraphs that was introduced in [9] and [10]. A digraph G is a pair (V, E) where V is a set of vertices and E the set of edges that is a subset of $V \times V \setminus \mathrm{diag}$. An n-*path* on G is a formal linear combination of sequences $v_0, ..., v_n$ of $n+1$ vertices. If in any sequence, involved in the linear combination, all pairs (v_i, v_{i+1}) are edges, then the n-path is called *allowed*.

There is a natural definition of the boundary ∂ of an n-path that is an $(n-1)$-path. However, the boundary of an allowed path does not have to be allowed. Those allowed paths whose boundaries are also allowed, are called ∂-*invariant*. The linear space of ∂-invariant n-paths is an element of the chain complex whose homologies are called the path homologies of the digraph.

An undirected graph can also be considered as a digraph by replacing each undirected edge by two oppositely directed edges. In this way we obtain path homologies of undirected graphs.

There has been a number of attempts to define the notions of homology and cohomology for graphs. At a trivial level, any graph can be regarded as an one-dimensional simplicial complex, so that its simplicial homologies are defined. However, all homology groups of order 2 and higher are trivial, which makes this approach uninteresting.

Another way to make a graph into a simplicial complex is to consider all its cliques (=complete subgraphs) as simplexes of the corresponding dimensions (cf. [5], [15]). Then higher dimensional homologies may be non-trivial, but in this approach the notion of graph looses its identity and becomes a particular case of the notion of a simplicial complex. Besides, some desirable functorial properties of homologies fail, for example, the Künneth formula is not true for Cartesian product of graphs.

Yet another approach to homologies of digraphs can be realized via Hochschild homology. Indeed, allowed paths on a digraph have a natural operation of product, which allows to define the notion of a *path algebra* of a digraph. The Hochschild homology of the path algebra is a natural object to consider. However, it was shown in [14] that Hochschild homologies of order ≥ 2 are trivial, which makes this approach less attractive.

More recently there have been a number of attempts to define singular homologies of graphs [2], [17]. In these theories one uses predefined "small" graphs as basic cell elements and defines singular chains using maps of the basic cells into the graph. However, simple examples show that the homology groups obtained in this way, depend essentially on the choice of the basic cells and, moreover, lack some the functorial properties of homologies. Not to say that computation of singular homologies for the graphs beyond the trivial ones is very hard.

The path homologies of digraphs that are dealt with in this paper have many advantages in comparison with other notions of graph homologies.

Firstly, the homologies of all dimensions could be non-trivial. Also, the chain complex may have a rich structure. It contains not only cliques but also binary hypercubes and many other interesting shapes. Besides, path homologies can be relatively easily computed, by definition for small graphs and by means of any conventional linear algebra computing package for larger graphs.

Secondly, there is an independently developed notion of homotopy of digraphs [10] (similar to homotopy theory on graphs [1], [3]) that is compatible with path homology. For example, homotopy equivalent digraphs have the same homology groups, and the first homology group is abelization of the fundamental group.

Thirdly, there is a dual cohomology theory with the coboundary operator that arises independently and naturally as an exterior derivative on the algebra of functions on the vertex set of the digraph. This approach to the cohomology of digraphs, that is based on the classification of Bourbaki [4] of exterior derivations on algebras, was introduced by Dimakis and Müller-Hoissen in [6], [7] and further developed in [13]. Here we do not touch cohomologies of digraph and refer the reader to [9] for details.

Finally, the notion of path homology has good functorial properties with respect to graph-theoretical operations. The main result of this paper goes exactly in this direction: we prove the Künneth formulas for the path homologies of the Cartesian product and of the join of two digraphs.

We feel that the notion of path homology of digraphs has a rich mathematical content and hope that it will become a useful tool in various areas of pure and applied mathematics. For example, this notion was employed in [12] to give a new elementary proof of a theorem of Gerstenhaber and Schack [8] that identifies simplicial homology as a Hochschild homology. A link between path homologies of digraphs and cubical homologies was revealed in [11]. On the other hand, it is conceivable that the notion of path homology can be used in practical applications such as coverage verification in sensor networks (cf. [16]), and many others.

Let us briefly describe the structure of the paper. In Section 2 we define the notion of the boundary operator, path homology and give some simple examples.

In Section 3 we introduce the operation *join* of two digraphs and state a Künneth formula for it (Theorem 3.3). Particular cases of join are operations of building a cone and suspension over a digraph, which behave homologically in the same way as those in the classical algebraic topology.

In Section 4 we introduce the notions of cross product of paths and Cartesian product of digraphs. We state a Künneth formula for Cartesian product (Theorem 4.7) and give some examples.

In Sections 5, 6 we prove both Theorems 3.3 and 4.7 in a unified way. Note that in the both cases of join and Cartesian product we prove not only Künneth formulas for homologies but also similar formulas for chain complexes that have no analog in the classical algebraic topology.

The main difficulty in the proof lies in distinction between the notions of allowed paths and ∂-invariant paths. This difficulty does not occur in the classical algebraic topology and in order to overcome it we have developed a new tool of homological algebra that is stated in Theorem 5.1.

2 Path homologies

2.1 Paths on finite sets

Let V be an arbitrary non-empty finite set whose elements will be called vertices. For any non-negative integer p, an *elementary p-path* on a set V is any sequence $\{i_k\}_{k=0}^{p}$ of $p+1$ vertices of V (the vertices in the path do not have to be distinct). For $p=-1$, an elementary p-path is an empty set $\emptyset$. The p-path $\{i_k\}_{k=0}^{p}$ will also be denoted simply by $i_0...i_p$, without delimiters between the vertices.

Fix a field $\mathbb{K}$ and consider a $\mathbb{K}$-linear space $\Lambda_p = \Lambda_p(V)$ that consists of all formal linear combinations of all elementary p-paths with the coefficients from $\mathbb{K}$. The elements of Λ_p are called *p-paths* on V. An elementary p-path $i_0...i_p$ as an element of Λ_p will be denoted by $e_{i_0...i_p}$. The empty set as an element of Λ_{-1} will be denoted by e.

By definition, the family $\left\{e_{i_0...i_p} : i_0, ..., i_p \in V\right\}$ is a basis in Λ_p. Hence, each p-path v has a unique representation in the form

$$v = \sum_{i_0,\dots,i_p \in V} v^{i_0\dots i_p} e_{i_0\dots i_p}, \tag{2.1}$$

where $v^{i_0\dots i_p} \in \mathbb{K}$ are the components of v. For example, Λ_0 consists of all linear combinations of elements e_i that are the vertices of V, Λ_1 consists of all linear combinations of the elements e_{ij} that are pairs of vertices, etc. Note that , Λ_{-1} consists of all multiples of e so that $\Lambda_{-1} \cong \mathbb{K}$.

Definition 2.1 For any $p \geq 0$, the *boundary operator* $\partial : \Lambda_p \to \Lambda_{p-1}$ is a $\mathbb{K}$-linear operator that acts on elementary paths by

$$\partial e_{i_0\dots i_p} = \sum_{q=0}^{p} (-1)^q \, e_{i_0\dots \widehat{i_q}\dots i_p}, \tag{2.2}$$

where the hat $\widehat{i_q}$ means omission of the index i_q.

For example, we have

$$\partial e_i = e, \quad \partial e_{ij} = e_j - e_i, \quad \partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}. \tag{2.3}$$

It follows that, for any $v \in \Lambda_p$,

$$(\partial v)^{j_0\dots j_{p-1}} = \sum_{k\in V} \sum_{q=0}^{p} (-1)^q \, v^{j_0\dots j_{q-1} k \, j_q \dots j_{p-1}}. \tag{2.4}$$

Set also $\Lambda_{-2} = \{0\}$ and define $\partial : \Lambda_{-1} \to \Lambda_{-2}$ to be zero.

Lemma 2.2 *We have $\partial^2 v = 0$ for any $v \in \Lambda_p$ with $p \geq 0$.*

Proof. For $p = 0$ this is trivial. For $p \geq 1$ we have by (2.2)

$$\begin{aligned}
\partial^2 e_{i_0\dots i_p} &= \sum_{q=0}^{p} (-1)^q \, \partial e_{i_0\dots \widehat{i_q}\dots i_p} \\
&= \sum_{q=0}^{p} (-1)^q \left(\sum_{r=0}^{q-1} (-1)^r \, e_{i_0\dots \widehat{i_r}\dots \widehat{i_q}\dots i_p} + \sum_{r=q+1}^{p} (-1)^{r-1} \, e_{i_0\dots \widehat{i_q}\dots \widehat{i_r}\dots i_p} \right) \\
&= \sum_{0\leq r<q\leq p} (-1)^{q+r} \, e_{i_0\dots \widehat{i_r}\dots \widehat{i_q}\dots i_p} - \sum_{0\leq q<r\leq p} (-1)^{q+r} \, e_{i_0\dots \widehat{i_q}\dots \widehat{i_r}\dots i_p}.
\end{aligned}$$

After switching q and r in the last sum we see that the two sums cancel out, whence $\partial^2 e_{i_0\dots i_p} = 0$. This implies $\partial^2 v = 0$ for all $v \in \Lambda_p$. ■

Definition 2.3 *For all $p, q \geq -1$ and for any two paths $u \in \Lambda_p$ and $v \in \Lambda_q$ define their* join *$uv \in \Lambda_{p+q+1}$ as follows:*

$$(uv)^{i_0\dots i_p j_0\dots j_q} = u^{i_0\dots i_p} v^{j_0\dots j_q}. \tag{2.5}$$

Clearly, join of paths is a bilinear operation that satisfies the associative law (but not commutative). It follows from (2.5) that

$$e_{i_0\dots i_p} e_{j_0\dots j_q} = e_{i_0\dots i_p j_0\dots j_q}. \tag{2.6}$$

Let us extend the definition of uv to the case when either $p = -2$ and $q \geq -1$ or $q = -2$ and $p \geq -1$, just by setting $uv = 0 \in \Lambda_{p+q+1}$.

Lemma 2.4 (Product rule for join) *For all $p, q \geq -1$ and $u \in \Lambda_p$, $v \in \Lambda_q$ we have*

$$\partial (uv) = (\partial u)v + (-1)^{p+1} u\partial v. \tag{2.7}$$

Proof. Assume first that $p, q \geq 0$. It suffices to prove (2.7) for $u = e_{i_0...i_p}$ and $v = e_{j_0...j_q}$. We have

$$\begin{aligned} \partial (uv) &= \partial e_{i_0...i_p j_0...j_q} = e_{i_1...i_p j_0...j_q} - e_{i_0 i_2...i_p j_0...j_q} + ... \\ &\quad + (-1)^{p+1} \left(e_{i_0...i_p j_1...j_q} - e_{i_0...i_p j_0 j_2...j_q} + ... \right) \\ &= \left(\partial e_{i_0...i_p} \right) e_{j_0...j_q} + (-1)^{p+1} e_{i_0...i_p} \partial e_{j_0...j_q}, \end{aligned}$$

whence (2.7) follows. If $p = -1$ then it suffices to prove (2.7) for $u = e$. In this case $uv = v$, $(\partial u) v = 0$, and $u\partial v = \partial v$ so that the both sides of (2.7) are equal to ∂v. The case $q = -1$ is similar. ∎

2.2 Regular paths

Definition 2.5 An elementary p-path $e_{i_0...i_p}$ on a set V is called *regular* if $i_k \neq i_{k+1}$ for all $k = 0, ..., p-1$, and *non-regular* otherwise.

For any $p \geq -1$, consider the following subspace of Λ_p

$$\mathcal{R}_p = \mathcal{R}_p (V) := \operatorname{span} \left\{ e_{i_0...i_p} : i_0...i_p \text{ is regular} \right\},$$

whose elements are called *regular* p-paths. We would like to consider the operator ∂ on the spaces $\mathcal{R}_p$. However, in the present form ∂ is not invariant on the spaces $\mathcal{R}_p$. For example, $e_{iji} \in \mathcal{R}_2$ for $i \neq j$ while $\partial e_{iji} = e_{ji} - e_{ii} + e_{ij}$ contains a non-regular component e_{ii}. The same applies to the notion of join of paths: the join of two regular path does not have to be regular, for example, $e_{ij} e_{ji} = e_{ijji}$.

To overcome this difficulty, consider the complementary subspace

$$\mathcal{I}_p = \mathcal{I}_p (V) := \operatorname{span} \left\{ e_{i_0...i_p} : i_0...i_p \text{ is non-regular} \right\}$$

and observe the following property of $\mathcal{I}_p$.

Lemma 2.6 *Let $u \in \mathcal{I}_p$. Then $\partial u \in \mathcal{I}_{p-1}$ and $uv \in \mathcal{I}_{p+q+1}$ for any $v \in \Lambda_q$.*

Proof. It suffices to prove both claims for $u = e_{i_0...i_p}$. Since this path is non-regular, there exists an index k such that $i_k = i_{k+1}$. Then we have

$$\begin{aligned} \partial e_{i_0...i_p} &= e_{i_1...i_p} - e_{i_0 i_2...i_p} + ... \\ &\quad + (-1)^k e_{i_0...i_{k-1} i_{k+1} i_{k+2}...i_p} + (-1)^{k+1} e_{i_0...i_{k-1} i_k i_{k+2}...i_p} \\ &\quad + ... + (-1)^p e_{i_0...i_{p-1}}. \end{aligned} \tag{2.8}$$

By $i_k = i_{k+1}$ the two terms in the middle line of (2.8) cancel out, whereas all other terms are non-regular, whence $\partial e_{i_0...i_p} \in \mathcal{I}_{p-1}$.

If $v = e_{j_0...j_q}$ then $uv = e_{i_0...i_p j_0...j_q}$ is obviously non-regular. ∎

It follows from Lemma 2.6 that the both operations ∂ and join are well defined on the quotients $\Lambda_p / \mathcal{I}_p$. On the other hand, it is clear that $\Lambda_p = \mathcal{R}_p \oplus \mathcal{I}_p$ and, hence, $\mathcal{R}_p \cong \Lambda_p / \mathcal{I}_p$, where each element of $\Lambda_p / \mathcal{I}_p$ has a unique representative in $\mathcal{R}_p$.

Definition 2.7 Define the *regular* operations ∂ and join on the spaces $\mathcal{R}_p$ as pullbacks of those from $\Lambda_p/\mathcal{I}_p$ using the natural linear isomorphism $\mathcal{R}_p \cong \Lambda_p/\mathcal{I}_p$. The previously defined operations ∂ and join on Λ_p will be then referred to as non-regular.

Of course, Lemmas 2.2 and 2.4 remain true for the regular operations. When applying the formulas (2.2), (2.4) for the regular boundary operator ∂ and (2.5), (2.6) for regular join, one should make the following adjustments:

(I) all the components $v^{i_0 \dots i_p}$ of $v \in \mathcal{R}_p$ for non-regular paths $i_0 \dots i_p$ are equal to 0 by definition;

(II) all non-regular elementary paths $e_{i_0 \dots i_p}$, should they arise as a result of an operation, are treated as zeros.

For example, for non-regular operator $\partial : \Lambda_2 \to \Lambda_1$ we have $\partial e_{iji} = e_{ji} - e_{ii} + e_{ij}$ whereas for the regular operator $\partial : \mathcal{R}_2 \to \mathcal{R}_1$ we have $\partial e_{iji} = e_{ji} + e_{ij}$ since e_{ii} is set to be zero. Similarly, for non-regular join we have $e_{ij}e_{ji} = e_{ijji}$ whereas for the regular join $e_{ij}e_{ji} = 0$.

Hence, we obtain the *regular chain complex* of the set V:

$$0 \leftarrow \mathbb{K} \leftarrow \mathcal{R}_0 \leftarrow \mathcal{R}_1 \leftarrow \dots \leftarrow \mathcal{R}_{p-1} \leftarrow \mathcal{R}_p \leftarrow \dots \tag{2.9}$$

where all the arrows are given by regular operator ∂. We will need also the *truncated regular chain complex*

$$0 \leftarrow \mathcal{R}_0 \leftarrow \mathcal{R}_1 \leftarrow \dots \leftarrow \mathcal{R}_{p-1} \leftarrow \mathcal{R}_p \leftarrow \dots \tag{2.10}$$

where we follow a different convention for $\mathcal{R}_{-1}$, by setting $\mathcal{R}_{-1} = 0$. In this case ∂v for $v \in \mathcal{R}_0$ is redefined by $\partial v = 0$. Note that for the latter definition the product rule (2.7) for join breaks down if $p = 0$ or $q = 0$. However, (2.10) will be useful when dealing with cross product of paths in Section 4.

In the rest of this paper ∂ means always the regular boundary operator acting on $\mathcal{R}_p$, although in two modifications: (2.9) and (2.10), depending on the context. More precisely, in Section 3 we use (2.9) whereas in Section 4 – the version (2.10).

2.3 Paths on digraphs

A digraph is a pair $G = (V, E)$ where V is an arbitrary set, called the set of vertices, and $E \subset V \times V \setminus \{\text{diag}\}$ is the set of directed edges. For vertices $a, b \in V$ the fact that $(a, b) \in E$ will be denoted by $a \to b$. Note that $a \to b$ excludes $a = b$. The set V will always be assume finite.

Let $n \geq -1$ be an integer.

Definition 2.8 An elementary n-path $i_0 \dots i_n \in \Lambda_n(V)$ is called *allowed* on (V, E) if $i_{k-1} \to i_k$ for all $k = 1, \dots, n$.

Denote by E_n the set of all allowed n-paths on (V, E). We have $E_{-1} = \{e\}$, $E_0 = V$ and $E_1 = E$.

For any integer $n \geq -1$ denote by $\mathcal{A}_n$ the $\mathbb{K}$-linear space that is spanned by all the paths from E_n, that is

$$\mathcal{A}_n = \mathcal{A}_n(G) = \operatorname{span}\{e_{i_0 \dots i_n} : i_0 \dots i_n \text{ is allowed}\}.$$

Set also $\mathcal{A}_{-2} = \{0\}$.

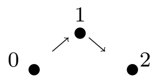
Definition 2.9 The elements of $\mathcal{A}_n(G)$ are called *allowed* n-paths.

By construction, $\mathcal{A}_n$ is a subspace of $\mathcal{R}_n$. For example, $\mathcal{A}_p = \Lambda_p = \mathcal{R}_p$ for $p \leq 0$, while $\mathcal{A}_1$ is spanned by all the edges from E and can be smaller than $\mathcal{R}_1$. Sometimes we will need also the space

$$\mathcal{N}_n = \mathcal{N}_n(G) = \operatorname{span}\{e_{i_0...i_n} : i_0...i_n \text{ is regular and non-allowed}\}. \tag{2.11}$$

Clearly, we have $\mathcal{R}_n = \mathcal{A}_n \oplus \mathcal{N}_n$.

We would like to restrict the regular boundary operator ∂ to the spaces $\mathcal{A}_n$. For some digraphs it can happen that $\partial \mathcal{A}_n \subset \mathcal{A}_{n-1}$, so that the restriction of ∂ to $\mathcal{A}_n$ is straightforward. However, in general $\partial \mathcal{A}_n$ does not have to be a subspace of $\mathcal{A}_{n-1}$. For example, this is the case for a digraph



where the 2-path e_{012} is allowed, while $\partial e_{012} = e_{12} - e_{02} + e_{01}$ is non-allowed because e_{02} is non-allowed.

For a general digraph $G = (V, E)$ and for any $n \geq -1$, consider the following subspaces of $\mathcal{A}_n$:

$$\Omega_n = \Omega_n(G) := \{v \in \mathcal{A}_n : \partial v \in \mathcal{A}_{n-1}\}. \tag{2.12}$$

Set also $\Omega_{-2} = \{0\}$. Note that $\Omega_n = \mathcal{A}_n$ for $n \leq 1$; in particular Ω_0 consists of all $\mathbb{K}$-linear combinations of the vertices and Ω_1 consists of all $\mathbb{R}$-linear combination of the edges, so that

$$\dim \Omega_0 = |V| \text{ and } \dim \Omega_1 = |E|.$$

For $n \geq 2$ the space Ω_n can be actually smaller that $\mathcal{A}_n$.

Claim. *We have $\partial \Omega_n \subset \Omega_{n-1}$ for all $n \geq -1$.*

Proof. Indeed, if $v \in \Omega_n$ then $\partial v \in \mathcal{A}_{n-1}$ and $\partial(\partial v) = \partial^2 v = 0 \in \mathcal{A}_{n-2}$ whence it follows that $\partial v \in \Omega_{n-1}$, which was to be proved. ■

Definition 2.10 The elements of $\Omega_n(G)$ are called ∂-*invariant* n-paths.

Thus, we obtain the chain complex of ∂-invariant paths:

$$0 \leftarrow \mathbb{K} \leftarrow \Omega_0 \leftarrow \Omega_1 \leftarrow ... \leftarrow \Omega_{n-1} \leftarrow \Omega_n \leftarrow \Omega_{n+1} \leftarrow ... \tag{2.13}$$

where all arrows are given by ∂, which is a subcomplex of (2.9). Consider also its *truncated* version

$$0 \leftarrow \Omega_0 \leftarrow \Omega_1 \leftarrow ... \leftarrow \Omega_{n-1} \leftarrow \Omega_n \leftarrow \Omega_{n+1} \leftarrow ... \tag{2.14}$$

that is a subcomplex of (2.10). In the case (2.13) we have $\Omega_{-1} = \mathcal{A}_{-1} \cong \mathbb{K}$, whereas in the case (2.14) we redefine $\Omega_{-1} = \{0\}$.

Definition 2.11 The homology groups of (2.14) are referred to as the *path homology groups* of the digraph G and are denoted by $H_n(G, \mathbb{K}), n \geq 0$, that is,

$$H_n(G, \mathbb{K}) = \ker \partial|_{\Omega_n} / \operatorname{Im} \partial|_{\Omega_{n+1}}.$$

The homology groups of (2.13) are called the *reduced path homology groups* of G and are denoted by $\widetilde{H}_n(G, \mathbb{K}), n \geq -1$.

We will also use short notations $H_n(G)$ and $\widetilde{H}_n(G)$ since the field $\mathbb{K}$ is usually fixed. Clearly, $\widetilde{H}_n(G) = H_n(G)$ for $n \geq 1$, $\widetilde{H}_0(G) \cong H_0(G)/\mathbb{K}$ and $\widetilde{H}_{-1}(G) = \{0\}$. Although we have assumed from the very beginning that $\mathbb{K}$ is a field, the notion of path homology can be defined in the same way when $\mathbb{K}$ is a commutative unital ring. However, the main theorems of the present paper are proved under the assumption that $\mathbb{K}$ is a field.

The notion of path homologies of digraphs is the main object of this paper. It is easy to prove that $\dim H_0(G)$ is equal to the number of (undirected) connected components of the G. In particular, for connected digraphs $\dim H_0(G) = 1$ and, hence, $\widetilde{H}_0(G) = \{0\}$.

Example 2.12 Consider a digraph on Fig. 1. A direct computation shows that $H_1(G) = \{0\}$ and $H_2(G) \cong \mathbb{K}$; moreover, $H_2(G)$ is generated by

$$e_{124} + e_{234} + e_{314} - (e_{125} + e_{235} + e_{315}),$$

which will be proved in Example 3.8. It is easy to see that G is a planar digraph but nevertheless its second homology group is non-trivial.

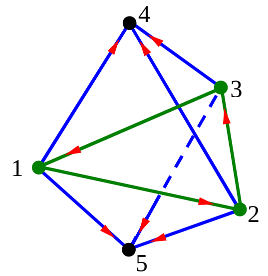


Figure 1: An example of a planar digraph G with non-trivial H_2

2.4 Cyclic digraphs

To give an example of computation of $\Omega_p(G)$ and $H_p(G)$, we consider here a class of cyclic digraphs. We say that a digraph $G = (V, E)$ is a *cycle* if it is connected (as an undirected graph), every vertex had the degree 2, and there are no double edges. We refer to G as an n-cycle if the number $|V|$ of its vertices is n. For an n-cycle we have $\dim \Omega_0(G) = \dim \Omega_1(G) = n$ and $\dim H_0(G) = 1$.

Consider two specific examples of cycles. Let us call by a *triangle* the following digraph

$$\begin{tikzcd} & \overset{2}{\bullet} & \\ \overset{0}{\bullet} \arrow[ur] \arrow[rr] & & \overset{1}{\bullet} \arrow[ul] \end{tikzcd} \tag{2.15}$$

Note that the triangle contains a 2-path $e_{012} \in \Omega_2$ as $e_{012} \in \mathcal{A}_2$ and

$$\partial e_{012} = e_{12} - e_{02} + e_{01} \in \mathcal{A}_1.$$

Let us called by a *square* the following digraph

$$\begin{tikzcd} {}^{2}\bullet \arrow[r] & \bullet_{3} \\ {}^{0}\bullet \arrow[u] \arrow[r] & \bullet_{1} \arrow[u] \end{tikzcd} \tag{2.16}$$

The square contains a 2-path $v := e_{013} - e_{023} \in \Omega_2$ because $v \in \mathcal{A}_2$ and

$$\begin{aligned} \partial v &= (e_{13} - e_{03} + e_{01}) - (e_{23} - e_{03} + e_{02}) \\ &= e_{13} + e_{01} - e_{23} - e_{02} \in \mathcal{A}_1, \end{aligned}$$

where the non-allowed path e_{03} cancels out.

Proposition 2.13 *Let G be a cycle. Then*

$$\Omega_p(G) = \{0\} \quad \textit{for all } p \geq 3 \quad \textit{and} \quad H_p(G) = \{0\} \ \textit{for all } p \geq 2.$$

If G is a triangle or a square then

$$\dim \Omega_2(G) = 1, \ \dim H_1(G) = 0$$

whereas otherwise

$$\dim \Omega_2(G) = 0, \quad \dim H_1(G) = 1.$$

Proof. Let G be a triangle (2.15). For $p \geq 3$ there are no allowed p-paths, in particular, $\Omega_p = \{0\}$ and $H_p = \{0\}$. Obviously, $\mathcal{A}_2$ is spanned by a single 2-path e_{012}, and this path is also ∂-invariant, so that $\Omega_2 = \operatorname{span}\{e_{012}\}$. Since $\partial e_{012} \neq 0$, we see that $\ker \partial|_{\Omega_2} = 0$ and, hence, $H_2 = \{0\}$. Clearly, $\Omega_1 = \operatorname{span}\{e_{01}, e_{02}, e_{12}\}$ and it is easy to see that $\ker \partial|_{\Omega_1}$ is spanned by $e_{12} - e_{02} + e_{01}$, which coincides with $\operatorname{Im} \partial|_{\Omega_2}$; hence, $H_1 = \{0\}$.

Let G be a square (2.16). As above, we obtain for $p \geq 3$ that $\Omega_p = \{0\}$ and $H_p = \{0\}$. The space $\mathcal{A}_2$ is now 2-dimensional:

$$\mathcal{A}_2 = \operatorname{span}\{e_{013}, e_{023}\}.$$

A 2-path $v = \alpha e_{013} + \beta e_{023} \in \mathcal{A}_2$ has the boundary

$$\partial v = \alpha (e_{13} - e_{03} + e_{01}) + \beta (e_{23} - e_{03} + e_{02})$$

that is allowed if and only if the terms e_{03} cancel out, that is, when $\alpha + \beta = 0$. Hence, Ω_2 is one-dimensional and

$$\Omega_2 = \operatorname{span}\{e_{013} - e_{023}\}.$$

As in the case of a triangle, we obtain $\ker \partial|_{\Omega_2} = 0$ and $H_2 = \{0\}$. Also, $\ker \partial|_{\Omega_1}$ is spanned by $e_{13} + e_{01} - e_{23} - e_{02}$, which coincides with $\operatorname{Im} \partial|_{\Omega_2}$; hence, $H_1 = \{0\}$.

Assume first that G is neither triangle nor square. Then G contains neither triangle nor square as subgraph. Let us show that $\Omega_p = \{0\}$ for any $p \geq 2$. Indeed, let v be a ∂-invariant p-path. Consider one of the elementary paths $e_{i_0 \ldots i_p}$ that enters v with non-zero coefficients. Then we have $i_0 \to i_1 \to i_2 \to \ldots$ but $i_0 \not\to i_2$ because otherwise $i_0 i_1 i_2$ would be a triangle. Note that $\partial e_{i_0 \ldots i_p}$ contains the term $e_{i_0 \widehat{i_1} i_2 \ldots i_p}$ that is not allowed. Hence, the latter term should cancel out with a similar term that comes from another elementary path $e_{i_0 i_1' i_2 \ldots i_n}$ path being also a part of v. But then we have $i_0 \to i_1' \to i_2$ so that the vertices i_0, i_1, i_1', i_2 form a square, which is impossible. Consequently, we have $H_p = \{0\}$ for $p \geq 2$.

Finally, let us compute $H_1 = \ker \partial|_{\Omega_1}$. Set $n = |V|$. By the definition of a cycle, the set of vertices of G can be identified with $\mathbb{Z}_n$ so that there is an edge between i and $i+1$ for any $i \in \mathbb{Z}_n$. Denote this edge by v_i, that is, v_i is either $e_{i(i+1)}$ or $e_{(i+1)i}$. Then we have

$$\partial v_i = \sigma_i (e_{i+1} - e_i),$$

where

$$\sigma_i = \begin{cases} 1, & \text{if } i \to (i+1) \\ -1, & \text{if } (i+1) \to i, \end{cases} \tag{2.17}$$

For any allowed (=∂-invariant) 1-path $v = \sum_i \alpha_i v_i$ we have

$$\partial v = \sum_i \alpha_i \sigma_i (e_{i+1} - e_i) = \sum_i (\alpha_{i-1}\sigma_{i-1} - \alpha_i \sigma_i) e_i,$$

which vanishes if and only if for all i

$$\alpha_{i-1}\sigma_{i-1} = \alpha_i \sigma_i. \tag{2.18}$$

Hence, $H_1 = \ker \partial|_{\Omega_1}$ is one-dimensional. The condition (2.18) is, in particular, satisfied if $\alpha_i = \sigma_i$ for all i, which implies that H_1 is spanned by $v = \sum_i \sigma_i v_i$. An example of such a path is shown on Fig. 2. ■

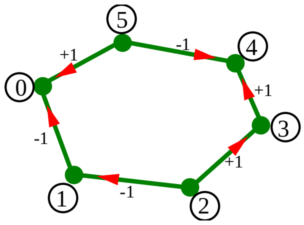


Figure 2: The 1-path $v = -e_{01} - e_{12} + e_{23} + e_{34} - e_{45} + e_{50}$ spans H_1.

3 Join of digraphs

To simplify notation, we denote the set of vertices of a digraph by the same letter as the digraph itself. In this section we always use the version (2.13) of the chain complex $\{\Omega_p\}_{p\geq -1}$ and, hence, the reduced homologies $\{\widetilde{H}_p\}_{p\geq -1}$.

3.1 Join of two digraphs

Definition 3.1 Let X, Y be two digraphs whose sets of vertices are disjoint. Consider the digraph Z with the set of vertices $X \sqcup Y$ and with the set of edges, that consists of all the edges of X and Y, as well as of all the edges of the form $x \to y$ for all $x \in X$ and $y \in Y$. The digraph Z is called the *join* of X and Y and is denoted by $X * Y$.

An example of a join of two digraphs is shown on Fig. 3(right). The operation $*$ on the digraphs is obviously non-commutative but associative.

Since X and Y are subgraphs of $Z = X * Y$, every (regular) path on X or Y can be considered as a (regular) path on Z. In particular, we can consider the operation join uv of regular paths u on X and v on Y, so that the result uv is a regular path on Z (see Fig. 3(left)). Clearly, if $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ then $uv \in \mathcal{R}_{p+q+1}(Z)$.

It is clear from construction that an elementary path e_z on Z is allowed if and only if it has the form $e_x e_y$ where e_x is an elementary allowed path on X and e_y is that on Y. Moreover, x and y in the representation $e_z = e_x e_y$ are uniquely defined.

Proposition 3.2 *Let $p, q \geq -1$ and $r = p + q + 1$.*

(a) If $u \in \mathcal{A}_p(X)$ and $v \in \mathcal{A}_q(Y)$ then $uv \in \mathcal{A}_r(Z)$. If $u \in \mathcal{A}_p(X)$ and $v \in \mathcal{N}_q(Y)$ then $uv \in \mathcal{N}_r(Z)$, and the same is true for $u \in \mathcal{N}_p(X)$ and $v \in \mathcal{A}_p(Y)$ (where $\mathcal{N}_*$ is defined in (2.11)).

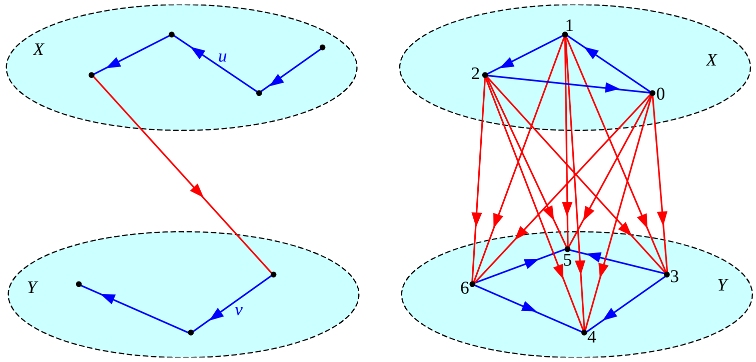


Figure 3: Join of two paths (left) and join of two digraphs (right)

(b) If $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $uv \in \Omega_r(Z)$. Moreover, the operation $u, v \mapsto uv$ extends to that for the homology classes $u \in \widetilde{H}_p(X)$ and $v \in \widetilde{H}_q(Y)$ so that $uv \in \widetilde{H}_r(Z)$.

Proof. (a) For $u = e_x$, $v = e_y$ the both claims are obvious, for general u, v they follow by linearity.

(b) If $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then by (a) the path uv is allowed. Since ∂u and ∂v are allowed, by (a) also $(\partial u)v$ and $u(\partial v)$ are allowed, whence $\partial(uv)$ is allowed by the product rule (2.7). It follows that uv is ∂-invariant.

If u, v are representatives of homology classes, that is, closed paths, then by (2.7) the join uv is also closed, so that uv represents a homology class of Z. We are left to verify that the class of uv depends only on the classes of u and v. For that it suffices to prove that if either u or v is exact then so is uv. Indeed, if $u = \partial w$ then

$$\partial(wv) = (\partial w)v + (-1)^p w(\partial v) = uv$$

so that uv is exact. ■

One of our main results is the following theorem. Here we denote by Ω_* the full chain complex (2.13) and deal with its homologies $\widetilde{H}_*$. Set also, for any $p \geq 0$,

$$\Omega'_p := \Omega_{p-1}$$

so that Ω'_* is the same chain complex as Ω_* but with the shifted index p.

Theorem 3.3 *Let X, Y be two finite digraphs and $Z = X * Y$. Then we have the following isomorphism of the chain complexes:*

$$\Omega_*(Z) \cong \Omega'_*(X) \otimes \Omega_*(Y), \tag{3.1}$$

which is given by the map $u \otimes v \mapsto uv$ with $u \in \Omega'_*(X)$ and $v \in \Omega_*(Y)$. In particular, for any $r \geq -1$,

$$\Omega_r(Z) \cong \bigoplus_{\{p,q \geq -1 : p+q=r-1\}} (\Omega_p(X) \otimes \Omega_q(Y)). \tag{3.2}$$

Consequently, we have, for any $r \geq 0$,

$$\widetilde{H}_r(Z) \cong \bigoplus_{\{p,q \geq 0 : p+q=r-1\}} \left(\widetilde{H}_p(X) \otimes \widetilde{H}_q(Y)\right) \tag{3.3}$$

(a Künneth formula for join).

The proof will be given in Section 6. It follows from (3.2) that $\Omega_r(Z)$ has a basis

$$\bigsqcup_{\{p,q\geq -1: p+q=r-1\}} \{u_i^{(p)} v_j^{(q)}\}_{i,j}, \tag{3.4}$$

where $\{u_i^{(p)}\}$ is a basis in $\Omega_p(X)$ and $\{v_j^{(q)}\}$ is a basis in $\Omega_q(Y)$; in particular,

$$\dim \Omega_r(Z) = \sum_{\{p,q\geq -1: p+q=r-1\}} \dim \Omega_p(X) \dim \Omega_q(Y).$$

In the same way one expresses a basis in $\widetilde{H}_r(Z)$ via the basis in $\widetilde{H}_p(X)$ and $\widetilde{H}_q(Y)$.

Example 3.4 Consider the digraph $Z = X * Y$ as on Fig. 3(right). In this case we have by Proposition 2.13 that all the homology groups $\widetilde{H}_p(X)$ and $\widetilde{H}_q(Y)$ are trivial except for

$$\begin{aligned} \widetilde{H}_1(X) &= \operatorname{span}\{e_{01} + e_{12} + e_{20}\}, \\ \widetilde{H}_1(Y) &= \operatorname{span}\{e_{35} - e_{65} + e_{64} - e_{34}\}. \end{aligned}$$

It follows from (3.3) that all $\widetilde{H}_r(Z)$ are trivial except for $\widetilde{H}_3(Z)$, and the latter is generated by a single element

$$e_{0135} - e_{0165} + e_{0164} - e_{0134} + e_{1235} - e_{1265} + e_{1264} - e_{1234} + e_{2035} - e_{2065} + e_{2064} - e_{2034}$$

that is the join of the generators of $\widetilde{H}_1(X)$ and $\widetilde{H}_1(Y)$.

Given a digraph X, the digraph $\operatorname{Cone} X$ is obtained from X by adding one more vertex a and all the edges of the form $x \to a$ for all $x \in X$. The vertex a is called the cone vertex. Clearly, we have $\operatorname{Cone} X = X * Y$ where the digraph Y consists of a single vertex a. Observe that $\Omega_{-1}(Y) \cong \mathbb{K}$, $\Omega_0(Y) = \operatorname{span}\{e_a\}$, $\Omega_q(Y) = \{0\}$ for $q \geq 1$ and $\widetilde{H}_q(Y) = \{0\}$ for all $q \geq -1$. Hence, applying Theorem 3.3 with $Y = \{a\}$, we obtain the following statement.

Proposition 3.5 *For any digraph X and for any $r \geq 0$, we have*

$$\Omega_r(\operatorname{Cone} X) \cong \Omega_r(X) \oplus \Omega_{r-1}(X), \tag{3.5}$$

where the isomorphism is given by the map $u, v \mapsto u + ve_a$, where $u \in \Omega_r(X)$, $v \in \Omega_{r-1}(X)$ and a is the cone vertex. Furthermore, all the reduced homologies $\widetilde{H}_r(\operatorname{Cone} X)$ are trivial.

Example 3.6 Let us define for any $n \geq 0$ a *simplex-digraph* Sm_n as follows: its set of vertices is $\{0, 1, ..., n\}$ and the edges are $i \to j$ for all $i < j$. For example, we have

$$\mathrm{Sm}_1 = {}^0\bullet \ \to \ \bullet^1, \qquad \mathrm{Sm}_2 = \begin{array}{ccc} & \overset{2}{\bullet} & \\ \nearrow & & \nwarrow \\ {}_0\bullet & \to & \bullet_1 \end{array},$$

and Sm_3 is shown on Fig. 4.

Clearly, we have $\mathrm{Sm}_n = \operatorname{Cone} \mathrm{Sm}_{n-1}$. Since $\Omega_0(\mathrm{Sm}_0) = \operatorname{span}\{e_0\}$ and $\Omega_n(\mathrm{Sm}_{n-1}) = \{0\}$, we obtain by induction from (3.5) that $\Omega_n(\mathrm{Sm}_n) = \operatorname{span}\{e_{01...n}\}$. Of course, all the reduced homologies of Sm_n are trivial.

A *suspension* over a digraph X is a digraph $\operatorname{Sus} X$ that is obtained from X by adding two vertices a, b and all the edges $x \to a$ and $x \to b$ for all $x \in X$. The vertices a, b are called the suspension vertices. Clearly, we have $\operatorname{Sus} X = X * Y$ where Y is a digraph that consists of two vertices a, b and no edges. Observe that $\Omega_{-1}(Y) \cong \mathbb{K}$, $\Omega_0(Y) = \operatorname{span}\{e_a, e_b\}$, $\Omega_q(Y) = \{0\}$ for $q \geq 1$, $\widetilde{H}_0(Y) = \operatorname{span}(e_b - e_a)$ and $\widetilde{H}_q(Y) = \{0\}$ for $q \geq 1$. Applying Theorem 3.3 with $Y = \{a, b\}$, we obtain the following statement.

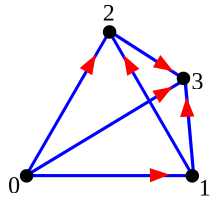


Figure 4: A simplex-digraph Sm_3

Proposition 3.7 *For any digraph X and for any $r \geq 0$, we have*

$$\Omega_r(\operatorname{Sus} X) \cong \Omega_r(X) \oplus \Omega_{r-1}(X) \oplus \Omega_{r-1}(X), \tag{3.6}$$

and the isomorphism is given by the map $u, v, w \mapsto u + ve_a + we_b$, where $u \in \Omega_r(X)$, $v, w \in \Omega_{r-1}(X)$ and a, b are the suspension vertices. Furthermore, we have

$$\widetilde{H}_r(\operatorname{Sus} X) \cong \widetilde{H}_{r-1}(X), \tag{3.7}$$

and the isomorphism is given by the map $u \mapsto u(e_a - e_b)$, where $u \in \widetilde{H}_{r-1}(X)$.

Example 3.8 Let S be any cycle that is neither triangle nor square, so that by Proposition 2.13 $\dim H_1(S) = 1$. We regards S as an analog of a circle. Define S_n inductively by $S_1 = S$ and $S_{n+1} = \operatorname{Sus} S_n$, so that S_n can be regarded as an analog of the n-dimensional sphere. Proposition 3.7 implies by induction that $\dim H_n(S_n) = \dim H_1(S) = 1$, which gives an example of a non-trivial H_n with an arbitrary n. In the same way one shows that $H_p(S_n) = \{0\}$ for all $p \geq 1, p \neq n$.

Let v be an 1-path on S that spans $H_1(S)$ (see the proof of Proposition 2.13). Denoting by a_n, b_n the suspension vertices of $S_{n+1} = \operatorname{Sus} S_n$, we obtain by induction that $H_n(S_n)$ is generated by the path

$$v(e_{a_1} - e_{b_1})(e_{a_2} - e_{b_2}) \ldots (e_{a_{n-1}} - e_{b_{n-1}}).$$

For example, the digraph G on Fig. 1 is an S_2 based on 3-cycle S with the vertices $1, 2, 3$. Since by Proposition 2.13 $v = e_{12} + e_{23} + e_{31}$, we obtain that $H_2(G)$ is generated by

$$v(e_4 - e_5) = (e_{124} + e_{234} + e_{314}) - (e_{125} + e_{235} + e_{315}).$$

Another example of an S_2 is shown on Fig. 5 in two ways.

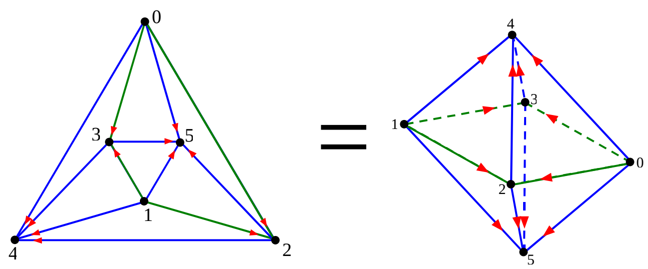


Figure 5: An octahedron digraph

Indeed, denoting by S the 4-cycle with vertices $\{0, 1, 2, 3\}$, we see that the digraph G on Fig. 5 is $\operatorname{Sus} S = S_2$. Hence, we obtain that $H_2(G)$ is generated by

$$e_{024} - e_{025} - e_{034} + e_{035} - e_{124} + e_{125} + e_{134} - e_{135}.$$

3.2 Some auxiliary results

Given a digraph G, set $E_* (G) = \bigcup_{p \geq -1} E_p (G)$, that is, $E_* (G)$ is the set of all allowed elementary p-paths on G with $p \geq -1$. Denote by $\mathcal{A}_* (G)$ the union of all $\mathcal{A}_p (G)$ with $p \geq -1$ and define in $\mathcal{A}_* (G)$ the $\mathbb{K}$-scalar product as follows: for all $u, v \in \mathcal{A}_* (G)$

$$[u, v] := \sum_{x \in E_*(X)} u^x v^x.$$

In particular, if $u \in \mathcal{A}_p$ and $v \in \mathcal{A}_{p'}$ with $p \neq p'$ then clearly $[u, v] = 0$.

As before, let $Z = X * Y$ for two digraphs X, Y. Let us prove some simple properties of join of paths that we will need later for the proof of Theorem 3.3.

Lemma 3.9 *Any path $w \in \mathcal{A}_* (Z)$ admits a representation*

$$w = \sum_{x \in E_*(X), y \in E_*(Y)} c^{xy} e_x e_y \tag{3.8}$$

where $c^{xy} \in \mathbb{K}$.

Proof. By construction of join, any allowed elementary path on Z has the form $e_x e_y$, where $x \in E_* (X)$ and $y \in E_* (Y)$. Hence, any path $w \in \mathcal{A}_* (Z)$ has the form (3.8). ■

Lemma 3.10 *If $u \in \mathcal{A}_p (X), \varphi \in \mathcal{A}_{p'} (X)$ and $v \in \mathcal{A}_q (Y), \psi \in \mathcal{A}_{q'} (Y)$ with $p, q, p', q' \geq -1$ then*

$$[uv, \varphi\psi] = [u, \varphi] [v, \psi] . \tag{3.9}$$

Proof. Set $r = p + q + 1$. If $p' + q' + 1 \neq r$ then then both sides of (4.22) vanish. Assuming $p' + q' + 1 = r$ we obtain

$$\begin{aligned} [uv, \varphi\psi] &= \sum_{z \in E_r(Z)} (uv)^z (\varphi\psi)^z \\ &= \sum_{x \in E_p(X), y \in Eq(Y)} u^x v^y \varphi^x \psi^y. \end{aligned}$$

If $p' \neq p$ then $\varphi^x = 0$ and then both sides of (4.22) vanish. If $p' = p$ and, hence, $q' = q$, then we obtain

$$[uv, \varphi\psi] = \sum_{x \in E_p(X)} u^x \varphi^x \sum_{y \in E_q(Y)} v^y \psi^y = [u, \varphi] [v, \psi] ,$$

which finishes the proof. ■

4 Cartesian product

In this section we use the truncated chain complexes $\{\mathcal{R}_p\}_{p \geq 0}$ and $\{\Omega_p\}_{p \geq 0}$ (cf. (2.10) and (2.14)) and the homologies $\{H_p\}_{p \geq 0}$ of the latter.

4.1 Step-like paths

Given two finite sets X, Y, consider their Cartesian product $Z = X \times Y$. Fix $r \geq 0$ and let $z = z_0 z_1 ... z_r$ be a regular elementary r-path on Z, where $z_k = (x_k, y_k)$ with $x_k \in X$ and $y_k \in Y$. We say that the path z is *step-like* if, for any $k = 1, ..., r$, either $x_{k-1} = x_k$ or $y_{k-1} = y_k$. In fact, exactly one of these conditions holds as z is regular.

Any step-like path z on Z determines regular elementary paths x on X and y on Y by projection. More precisely, x is obtained from z by taking the sequence of all X-components of the vertices of z and then by collapsing in it any subsequence of repeated vertices to one vertex. The same rule applies to y. By construction, the projections x and y are *regular* elementary paths on X and Y, respectively. If the projections of z are $x = x_0 ... x_p$ and $y = y_0 ... y_q$ then $p + q = r$ (cf. Fig. 6(left)).

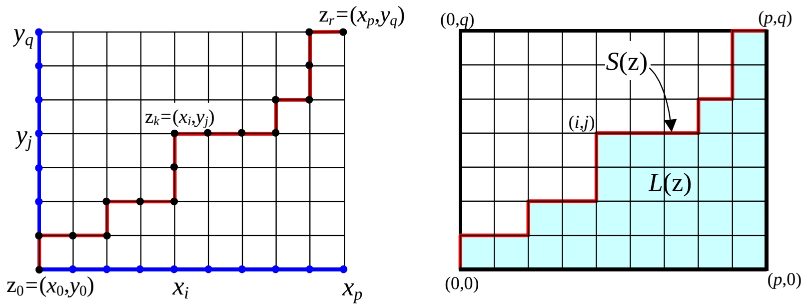


Figure 6: Left: a step-like path z and its projections x and y. Right: a staircase $S(z)$ and its elevation $L(z)$ (here $L(z) = 30$).

Every vertex (x_i, y_j) of a step-like path z can be represented as a point (i, j) of $\mathbb{Z}^2$ so that the whole path z is represented by a *staircase* $S(z)$ in $\mathbb{Z}^2$ connecting the points $(0, 0)$ and (p, q).

Definition 4.1 Define the *elevation* $L(z)$ of the path z as the number of the cells in $\mathbb{Z}_+^2$ below the staircase $S(z)$ (cf. the shaded area on Fig. 6(right)).

By definition, any p-path u on X is given by

$$u = \sum_x u^x e_x$$

where the summation is taken over all elementary p-paths x on X and $u^x \in \mathbb{K}$ are the components of u. It will be convenient to extend the summation here to all elementary paths x with arbitrary length, by setting $u^x = 0$ if the length of x is not equal to p.

Definition 4.2 For any paths $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ with $p, q \geq 0$ define their *cross product* $u \times v$ as a path on Z by the following rule: for any step-like elementary path z on Z, the component $(u \times v)^z$ is defined by

$$(u \times v)^z = (-1)^{L(z)} u^x v^y, \tag{4.1}$$

where x and y are the projections of z onto X and Y, while for the rest paths z set $(u \times v)^z = 0$. Hence, we have $u \times v \in \mathcal{R}_{p+q}(Z)$.

In the context of this section we use the truncated regular chain complex (2.10), that is, the convention that $\mathcal{R}_{-1} = \{0\}$. Let us extend the definition of the cross product $u \times v$ to the case when either $p = -1$ and $q \geq 0$ or $p \geq 0$ and $q = -1$ simply by setting $u \times v = 0 \in \mathcal{R}_{p+q}$.

For any regular elementary p-path x on X and q-path y on Y with $p, q \geq 0$ denote by $\Pi_{x,y}$ the set of all step-like paths z on Z whose projections on X and Y are x and y respectively. Clearly, we have $|\Pi_{x,y}| = \binom{p+q}{p}$. It follows from (4.1) that, for all regular elementary paths x, y,

$$e_x \times e_y = \sum_z (e_x \times e_y)^z \, e_z = \sum_z (-1)^{L(z)} \, (e_x)^{x'} \, (e_y)^{y'} \, e_z,$$

where x' and y' are projections of z, whence

$$e_x \times e_y = \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} \, e_z. \tag{4.2}$$

It is not difficult to see that cross product is associative.

Example 4.3 Let us denote the vertices of X by letters a, b, c etc. and the vertices of Y by integers $0, 1, 2$, etc. Then the vertices of $Z = X \times Y$ will be denoted as chessboard fields, for example, $a0, b1$ etc. Here are some examples of cross products:

1. $e_a \times e_{01} = e_{a0a1}, \quad e_{ab} \times e_0 = e_{a0b0}$
2. $e_{ab} \times e_{01} = e_{a0b0b1} - e_{a0a1b1}$
3. $e_{abc} \times e_{01} = e_{a0b0c0c1} - e_{a0b0b1c1} + e_{a0a1b1c1}$
4. $e_{abc} \times e_{012} = e_{a0b0c0c1c2} - e_{a0b0b1c1c2} + e_{a0b0b1b2c2} + e_{a0a1b1c1c2} - e_{a0a1b1b2c2} + e_{a0a1a2b2c2}$ (cf. Fig. 7).

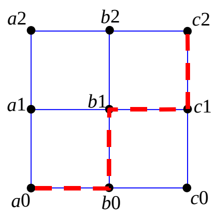


Figure 7: The staircase $a0b0b1c1c2$ has elevation 1. Hence, $e_{a0b0b1c1c2}$ enters the product $e_{abc} \times e_{012}$ with the negative sign.

Proposition 4.4 (Product rule for cross product) *If $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ where $p, q \geq 0$, then*

$$\partial(u \times v) = (\partial u) \times v + (-1)^p \, u \times (\partial v). \tag{4.3}$$

Proof. It suffices to prove (4.3) for the case $u = e_x$ and $v = e_y$ where $x = x_0 ... x_p$ and $y = y_0 ... y_q$ are regular elementary p-path on X and q-path on Y, respectively. Set $r = p + q$ so that $e_x \times e_y \in \mathcal{R}_r(Z)$.

If $p = q = 0$ then all the terms in (4.3) vanish. Assume $p = 0$ and $q \geq 1$. Then $\Pi_{x,y}$ contains the only element $z = z_0 ... z_q$ where $z_i = (x_0, y_i)$. Since $L(z) = 0$, we obtain by (4.2) that

$$e_x \times e_y = e_{z_0 ... z_q}$$

By (2.2) obtain

$$\partial\left(e_x \times e_y\right) = \partial e_{z_0 \ldots z_q} = e_x \times \partial e_{y_0 \ldots y_q},$$

which is equivalent to (4.3), because $\partial u = 0$. In the same way (4.3) is proved if $q = 0$ and $p \geq 1$.

Consider now the main case $p, q \geq 1$. We have by (4.2) and (2.2)

$$\partial\left(e_x \times e_y\right) = \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} \partial e_z = \sum_{z \in \Pi_{x,y}} \sum_{k=0}^{r} (-1)^{L(z)+k} e_{z_{(k)}}, \tag{4.4}$$

where we use a shortcut

$$z_{(k)} = z_0 \ldots \widehat{z_k} \ldots z_r = z_0 \ldots z_{k-1} z_{k+1} \ldots z_r.$$

Switching the order of the sums, rewrite (4.4) in the form

$$\partial\left(e_x \times e_y\right) = \sum_{k=0}^{r} \sum_{z \in \Pi_{x,y}} (-1)^{L(z)+k} e_{z_{(k)}}. \tag{4.5}$$

Given an index $k = 0, \ldots, r$ and a path $z \in \Pi_{x,y}$, consider the following four logically possible cases how horizontal and vertical couples combine around z_k:

$$(H): \begin{tikzcd} \overset{z_{k-1}}{\bullet} \arrow[r] & \overset{z_k}{\bullet} \arrow[r] & \overset{z_{k+1}}{\bullet} \end{tikzcd} \qquad (V): \begin{tikzcd} \overset{z_{k+1}}{\bullet} \\ \overset{z_k}{\bullet} \arrow[u] \\ \overset{z_{k-1}}{\bullet} \arrow[u] \end{tikzcd}$$

$$(R): \begin{tikzcd} & \overset{z_{k+1}}{\bullet} \\ \overset{z_{k-1}}{\bullet} \arrow[r] & \overset{z_k}{\bullet} \arrow[u] \end{tikzcd} \qquad (L): \begin{tikzcd} \overset{z_k}{\bullet} \arrow[r] & \overset{z_{k+1}}{\bullet} \\ \overset{z_{k-1}}{\bullet} \arrow[u] & \end{tikzcd}$$

Here (H) stands for a horizontal position, (V) for vertical, (R) for right and (L) for left. If $k = 0$ or $k = r$ then z_{k-1} or z_{k+1} should be ignored, so that one has only two distinct positions (H) and (V).

If $z \in \Pi_{x,y}$ and z_k stands in (R) or (L) then consider a path $z' \in \Pi_{x,y}$ such that $z'_i = z_i$ for all $i \neq k$, whereas z'_k stands in the opposite position (L) or (R), respectively, as on the diagrams:

$$\begin{tikzcd} \overset{z'_k}{\bullet} \arrow[r] & \overset{z_{k+1}}{\bullet} \\ \overset{z_{k-1}}{\bullet} \arrow[u] \arrow[r] & \overset{z_k}{\bullet} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \overset{z_k}{\bullet} \arrow[r] & \overset{z_{k+1}}{\bullet} \\ \overset{z_{k-1}}{\bullet} \arrow[u] \arrow[r] & \overset{z'_k}{\bullet} \arrow[u] \end{tikzcd}$$

Clearly, we have $L(z') = L(z) \pm 1$ which implies that the terms $e_{z_{(k)}}$ and $e_{z'_{(k)}}$ in (4.5) cancel out.

Denote by $\Pi^k_{x,y}$ the set of paths $z \in \Pi_{x.y}$ such that z_k stands in position (V) and by $\Pi_{x,y}{}^k$ the set of paths $z \in \Pi_{x,y}$ such that z_k stands in position (H). By the above observation, we can restrict the summation in (4.5) to those pairs k, z where z_k is either in vertical or horizontal position, that is,

$$\partial\left(e_x \times e_y\right) = \sum_{k=0}^{r} \sum_{z \in \Pi^k_{x,y} \sqcup \Pi_{x,y}{}^k} (-1)^{L(z)+k} e_{z_{(k)}}. \tag{4.6}$$

Let us now compute the first term in the right hand side of (4.3):

$$(\partial e_x) \times e_y = \sum_{l=0}^{p} (-1)^l e_x \times e_y = \sum_{l=0}^{p} \sum_{w \in \Pi_{x_{(l)},y}} (-1)^{L(w)+l} e_w. \tag{4.7}$$

Fix some $l = 0, ..., p$ and $w \in \Pi_{x_{(l)},y}$. Since the projection of w on X is $x_{(l)} = x_0...x_{l-1}x_{l+1}...x_p$, there exists a unique index k such that w_{k-1} projects onto x_{l-1} and w_k projects onto x_{l+1}. Then w_{k-1} and w_k have a common projection onto Y, say y_m (cf. Fig. 8).

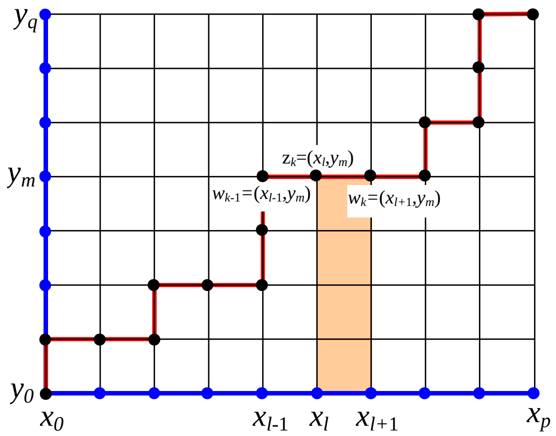


Figure 8: Step-like paths w and z. The shaded area represents the difference $L(z) - L(w)$.

Define a path $z \in \Pi_{x,y}^{k}$ by setting

$$z_i = \begin{cases} w_i & \text{for } i \le k-1, \\ (x_l, y_m) & \text{for } i = k, \\ w_{i-1} & \text{for } i \ge k+1. \end{cases} \tag{4.8}$$

By construction we have $z_{(k)} = w$. It also follows from the construction that

$$L(z) = L(w) + m.$$

Since $k = l + m$, we obtain that

$$L(z) + k = L(w) + l + 2m.$$

We see that each pair l, w where $l = 0, ..., p$ and $w \in \Pi_{x_{(l)},y}$ gives rise to a pair k, z where $k = 0, ..., r$, $z \in \Pi_{x,y}^{k}$, and

$$(-1)^{L(z)+k} e_{z_{(k)}} = (-1)^{L(w)+l} e_w.$$

By reversing this argument, we obtain that each such pair k, z gives back l, w so that this correspondence between k, z and l, w is bijective. Hence, we conclude that

$$(\partial e_x) \times e_y = \sum_{l=0}^{p} \sum_{w \in \Pi_{x_{(l)},y}} (-1)^{L(w)+l} e_w = \sum_{k=0}^{r} \sum_{z \in \Pi_{x,y}^{k}} (-1)^{L(z)+k} e_{z_{(k)}}. \tag{4.9}$$

The second term in the right hand side of (4.3) is computed similarly:

$$(-1)^p\, e_x \times \partial e_y = \sum_{m=0}^{q} (-1)^{m+p}\, e_x \times e_{y_{(m)}} = \sum_{m=0}^{q} \sum_{w\in\Pi_{x,y_{(m)}}} (-1)^{L(w)+m+p}\, e_w.$$

Each pair m, w here gives rise to a pair k, z where $k = 0, ..., r$ and $z \in \Pi^k_{x,y}$ in the following way: choose k such that w_{k-1} projects onto y_{m-1} and w_k projects onto y_{m+1}. Then w_{k-1} and w_k have a common projection onto X, say x_l.

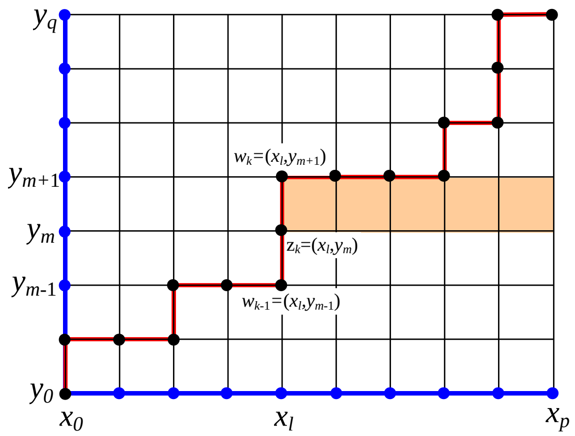


Figure 9: Paths w and z. The shaded area represents $L(z) - L(w)$.

Define the path $z \in \Pi^k_{x,y}$ as in (4.8) (cf. Fig. 9). Then we have $w = z_{(k)}$ and

$$L(z) = L(w) + p - l.$$

Since $k = l + m$, we obtain

$$L(z) + k = L(w) + p + m$$

and

$$(-1)^p\, e_x \times \partial e_y = \sum_{m=0}^{q} \sum_{w\in\Pi_{x,y_{(m)}}} (-1)^{L(w)+m+p}\, e_w = \sum_{k=0}^{r} \sum_{z\in\Pi^k_{x,y}} (-1)^{L(z)+k}\, e_{z_{(k)}}.$$

Combining this with (4.6) and (4.9), we obtain (4.3). ■

4.2 Cartesian product of digraphs

Definition 4.5 Given two digraphs X and Y, consider the digraph Z with the set of vertices $X \times Y$ and with the set of edges defined by the following rule: for $x, x' \in X$ and $y, y' \in Y$,

$$(x, y) \to (x', y') \text{ if either } x \to x' \text{ and } y = y' \text{ or } x = x' \text{ and } y \to y'.$$

The digraph Z is called the *Cartesian product* of X and Y and is denoted by $X\Box Y$.

A fragment of the graph Z is shown here:

$$\begin{array}{ccccccc}
y'\bullet & \dots & \overset{(x,y')}{\bullet} & \longrightarrow & \overset{(x',y')}{\bullet} & \dots \\
\uparrow & & \uparrow & & \uparrow & \\
y\bullet & \dots & \overset{(x,y)}{\bullet} & \longrightarrow & \overset{(x',y)}{\bullet} & \dots \\
& & & & & \\
{}^{Y}\diagup_{X} & \dots & \underset{x}{\bullet} & \longrightarrow & \underset{x'}{\bullet} & \dots
\end{array}$$

It is not difficult to see that the Cartesian product of digraphs is associative.

Clearly, any regular elementary path on $Z = X \Box Y$ is allowed if and only if it is step-like and its projections onto X and Y are allowed.

Proposition 4.6 *Let $p, q \geq 0$ and $r = p + q$.*

(a) If $u \in \mathcal{A}_p(X)$ and $v \in \mathcal{A}_q(Y)$ then $u \times v \in \mathcal{A}_r(Z)$. If $u \in \mathcal{N}_p(X)$ and $v \in \mathcal{A}_q(Y)$ then $u \times v \in \mathcal{N}_r(Z)$, and the same is true if $u \in \mathcal{A}_p(X)$ and $v \in \mathcal{N}_q(Y)$.

(b) If $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $u \times v \in \Omega_r(Z)$. Moreover, the operation $u, v \mapsto u \times v$ extends to that for the homology classes $u \in H_p(X)$ and $v \in H_q(Y)$ so that $u \times v \in H_r(Z)$.

Proof. (a) It suffices to prove the both claims for $u = e_x$ and $v = e_y$. By (4.2) $e_x \times e_y$ is a linear combination of e_z with $z \in \Pi_{x,y}$. If x and y are allowed then any $z \in \Pi_{x,y}$ is allowed, which implies that $e_x \times e_y \in \mathcal{A}_r(Z)$.

Let x be non-allowed. Since the projection of $z \in \Pi_{x,y}$ onto X is x, it follows that z is non-allowed. Hence, $e_x \times e_y$ is a linear combination of non-allowed paths, that is $e_x \times e_y \in \mathcal{N}_r(Z)$.

(b) The proof is based on the product rule (4.3) and goes the same way as the proof of Proposition 3.2(b). ■

The next theorem gives a complete description of ∂-invariant paths on Z. We denote by $\Omega_* = \{\Omega_p\}_{p\geq 0}$ the truncated chain complex (2.14) and by $H_* = \{H_p\}_{p\geq 0}$ its homologies.

Theorem 4.7 *Let X, Y be two finite digraphs and $Z = X \Box Y$. Then we have the following isomorphism of the chain complexes:*

$$\Omega_*(Z) \cong \Omega_*(X) \otimes \Omega_*(Y), \tag{4.10}$$

which is given by the map $u \otimes v \mapsto u \times v$ with $u \in \Omega_*(X)$ and $v \in \Omega_*(Y)$. In particular, for any $r \geq 0$

$$\Omega_r(Z) \cong \bigoplus_{\{p,q\geq 0: p+q=r\}} (\Omega_p(X) \otimes \Omega_q(Y)). \tag{4.11}$$

Consequently, we have

$$H_*(Z) \cong H_*(X) \otimes H_*(Y), \tag{4.12}$$

that is, for any $r \geq 0$,

$$H_r(Z) \cong \bigoplus_{\{p,q\geq 0: p+q=r\}} (H_p(X) \otimes H_q(Y)) \tag{4.13}$$

(a Künneth formula for product).

Example 4.8 Consider the digraph $Z = X \Box Y$ (shown on Fig. 10), where

$$X = \begin{array}{ccc} & \overset{b}{\bullet} & \\ \nearrow & & \searrow \\ {}^{a}\bullet & \rightarrow & \bullet^{c} \end{array} \quad \text{and} \quad Y = \begin{array}{ccc} {}^{2}\bullet & \longrightarrow & \bullet_{3} \\ \uparrow & & \uparrow \\ {}_{0}\bullet & \longrightarrow & \bullet_{1} \end{array}.$$

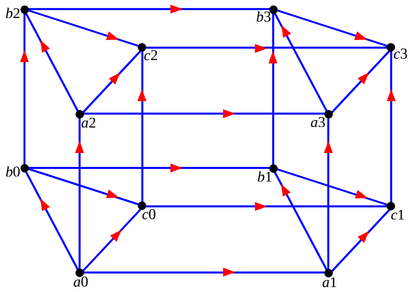


Figure 10: Cartesian product of a triangle and a square

For $r = 4$ we obtain from (4.11) that

$$\Omega_4(Z) \cong \Omega_2(X) \otimes \Omega_2(Y)$$

because on both digraphs X, Y we have $\Omega_p = \{0\}$ for $p \geq 3$. By the proof of Proposition 2.13, $\Omega_2(X) = \mathrm{span}(e_{abc})$ and $\Omega_2(Y) = \mathrm{span}(e_{013} - e_{023})$, whence it follows that $\Omega_4(Z)$ is spanned by a single 4-path

$$\begin{aligned}
& e_{abc} \times (e_{013} - e_{023}) = e_{a0b0c0c1c3} - e_{a0b0b1c1c3} + e_{a0b0b1b3c3} \\
& + e_{a0a1b1c1c3} - e_{a0a1b1b3c3} + e_{a0a1a3b3c3} \\
& - e_{a0b0c0c2c3} + e_{a0b0b2c2c3} - e_{a0b0b2b3c3} \\
& - e_{a0a2b2c2c3} + e_{a0a2b2b3c3} - e_{a0a2a3b3c3}.
\end{aligned}$$

Similarly one can compute $\Omega_r(Z)$ for other values of r. For example,

$$\Omega_3(Z) \cong \Omega_1(X) \otimes \Omega_2(Y) \bigoplus \Omega_2(X) \otimes \Omega_1(Y),$$

which implies $\dim \Omega_3(Z) = 3 \cdot 1 + 1 \cdot 4 = 7$ and the generators of $\Omega_3(Z)$ are

$$\begin{aligned}
& e_{ab} \times (e_{013} - e_{023}), \ e_{ac} \times (e_{013} - e_{023}), \ e_{bc} \times (e_{013} - e_{023}) \\
& e_{abc} \times e_{01}, \ e_{abc} \times e_{13}, \ e_{abc} \times e_{02}, \ e_{abc} \times e_{23}
\end{aligned}$$

Since all the homology groups of X, Y are trivial except for H_0, we obtain that the same is true for homologies of Z.

Example 4.9 Consider $Z = X \Box Y$ where X, Y are cyclic digraphs:

$$X = \begin{array}{ccc} & \overset{b}{\bullet} & \\ \nearrow & & \searrow \\ {}_a\bullet & \leftarrow & \bullet_c \end{array}, \quad Y = \begin{array}{ccc} {}^1\bullet & \longrightarrow & \bullet_2 \\ \uparrow & & \downarrow \\ {}^0\bullet & \longleftarrow & \bullet_3 \end{array}.$$

By Proposition 2.13 all the homologies $H_p(X)$ and $H_q(Y)$ are trivial for $p, q \geq 2$ whereas

$$\begin{aligned}
H_1(X) &= \mathrm{span}(e_{ab} + e_{bc} + e_{ca}) \\
H_1(Y) &= \mathrm{span}(e_{01} + e_{12} + e_{23} + e_{30}).
\end{aligned}$$

It follows from (4.12) that

$$H_2(Z) \cong \bigoplus_{\{p,q \geq 0: p+q=2\}} (H_p(X) \otimes H_q(Y)) = H_1(X) \otimes H_1(Y),$$

in particular, $\dim H_2(Z) = 1$. The generating element of $H_2(Z)$ is

$$(e_{ab} + e_{bc} + e_{ca}) \times (e_{01} + e_{12} + e_{23} + e_{30}).$$

For any digraph X, define the *cylinder* over X by

$$\operatorname{Cyl} X := X \Box Y \text{ with } Y = \left({}^0\bullet \to \bullet^1 \right).$$

Assuming that the vertices of X are enumerated by $0, 1, ..., n-1$, let us enumerate the vertices of $\operatorname{Cyl} X$ by $0, 1, ..., 2n-1$ using the following rule: $(x, 0)$ is assigned the number x, while $(x, 1)$ is assigned $x + n$.

Define the operation of *lifting* paths from X to $\operatorname{Cyl} X$ as follows: for any regular path v on X, the lifted path $\widehat{v}$ is defined by $\widehat{v} = v \times e_{01}$. For example, if $v = e_{i_0 \dots i_p}$ then

$$\widehat{v} = e_{i_0 \dots i_p} \times e_{01} = (-1)^p \sum_{k=0}^{p} (-1)^k e_{i_0 \dots i_k (i_k + n) \dots (i_p + n)}. \tag{4.14}$$

Since $e_{01} \in \Omega_1(Y)$, we see that if $v \in \Omega_p(X)$ then $\widehat{v} \in \Omega_{p+1}(\operatorname{Cyl} X)$.

Example 4.10 Let us define *n-cube* Cube_n inductively as follows: $\operatorname{Cube}_0 = \{0\}$ and

$$\operatorname{Cube}_n = \operatorname{Cyl} \operatorname{Cube}_{n-1}.$$

For example, Cube_1 is

$${}^0\bullet \to \bullet^1$$

Cube_2 is a square

$$\begin{array}{ccc} {}^2\bullet & \longrightarrow & \bullet^3 \\ \uparrow & & \uparrow \\ {}^0\bullet & \longrightarrow & \bullet^1 \end{array}$$

and Cube_3 is shown on Fig. 11.

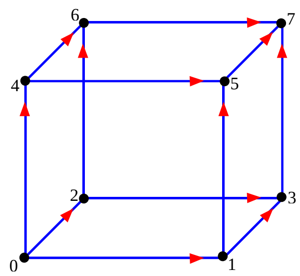


Figure 11: A 3-cube

Since $\operatorname{Cube}_n = \operatorname{Cube}_{n-1} \times Y$, where $\Omega_q(Y)$ is non-trivial only for $q = 0, 1$, and $\Omega_n(\operatorname{Cube}_{n-1}) = \{0\}$, we obtain from (4.11)

$$\Omega_n(\operatorname{Cube}_n) \cong \Omega_{n-1}(\operatorname{Cube}_{n-1}) \otimes \Omega_1(Y).$$

Since $\Omega_1(Y)$ is generated by a single element $v_1 = e_{01}$, we obtain by induction that $\dim \Omega_n(\operatorname{Cube}_n) = 1$. A generating element v_n of $\Omega_n(\operatorname{Cube}_n)$ can be computed inductively by

$$v_n = v_{n-1} \times e_{01} = \widehat{v_{n-1}}.$$

By (4.14) we obtain successively

$$\begin{aligned} v_2 &= \widehat{v_1} = e_{013} - e_{023}, \\ v_3 &= \widehat{v_2} = e_{0457} - e_{0157} + e_{0137} - e_{0467} + e_{0267} - e_{0237}, \end{aligned}$$

etc. In general, v_n is an alternating sum of $n!$ elementary paths that correspond to partitioning of a solid n-cube into $n!$ simplexes.

By (4.13) all homology groups of Cube_n are trivial except for H_0.

4.3 Some auxiliary results

For any digraph G denote by $E_*(G)$ the union of all $E_p(G)$ with $p \geq 0$, where $E_p(G)$ is the set of all allowed elementary p-paths. Note that the notation E_* here is different from that in Section 3.2 where $p = -1$ was also allowed.

Denote by $\mathcal{A}_*(G)$ the union of all $\mathcal{A}_p(G)$ with $p \geq 0$ and by $\Omega_*(G)$ the similar union of all $\Omega_p(G)$ with $p \geq 0$. We will use the $\mathbb{K}$-scalar product in $\mathcal{A}_*(G)$ defined by for all $u, v \in \mathcal{A}_*(G)$ by

$$[u, v] := \sum_{x \in E_*(X)} u^x v^x.$$

As in Section 4.2, we work with two digraphs X, Y and their Cartesian product $Z = X \Box Y$. Here we prove some auxiliary results.

Lemma 4.11 *The family of paths $\{e_x \times e_y\}$ where $x \in E_*(X)$ and $y \in E_*(Y)$ is linearly independent.*

Proof. Set for some $c^{xy} \in \mathbb{K}$

$$w = \sum_{x \in E_*(X), y \in E_*(Y)} c^{xy} e_x \times e_y$$

and prove that $w = 0$ implies that $c^{xy} = 0$ for all couple x, y as in the summation. Fix such a couple x', y' and choose one $z \in \Pi_{x',y'}$. By (4.1) we have

$$(e_x \times e_y)^z = \begin{cases} (-1)^{L(z)}, & \text{if } x = x' \text{ and } y = y', \\ 0, & \text{otherwise,} \end{cases}$$

which implies that $w^z = (-1)^{L(z)} c^{x'y'}$. Hence, $w = 0 \Rightarrow c^{x'y'} = 0$, which was to be proved. ■

Lemma 4.12 *Any path $w \in \Omega_*(Z)$ admits a representation*

$$w = \sum_{x \in E_*(X),\ y \in E_*(Y)} c^{xy} (e_x \times e_y) \tag{4.15}$$

with some coefficients $c^{xy} \in \mathbb{K}$ (note that c^{xy} are uniquely defined by Lemma 4.11).

Proof. Fix $w \in \Omega_r(Z)$ for some $r \geq 0$. For any $x \in E_*(X)$ and $y \in E_*(Y)$ choose some $z \in \Pi_{x,y}$ and set

$$c^{xy} = (-1)^{L(z)} w^z. \tag{4.16}$$

Let us first show that the value of c^{xy} in (4.16) is independent of the choice of $z \in \Pi_{x,y}$. Set $z = i_0 ... i_r$. Let k be an index such that one of the couples $i_{k-1} i_k$, $i_k i_{k+1}$ is vertical and the other is horizontal. If $i_{k-1} = (a, b)$ and $i_{k+1} = (a', b')$ where $a, a' \in X$ and $b, b' \in Y$, then i_k is either (a', b) or (a, b'). Denote the other of these two vertices by i'_k, as, for example, on the diagram:

$$
\begin{array}{cccccc}
\vdots & & & & \vdots & \\
b' \bullet & & \overset{i'_k}{\bullet} & \longrightarrow & \overset{i_{k+1}}{\bullet} & \cdots \\
\uparrow & & \uparrow & & \uparrow & \\
b \bullet & \cdots & \overset{i_{k-1}}{\bullet} & \longrightarrow & \overset{i_k}{\bullet} & \\
\vdots & & \vdots & & & \\
\| & & & & & \\
y & & & & & \\
 & x= \cdots & \underset{a}{\bullet} & \longrightarrow & \underset{a'}{\bullet} & \cdots
\end{array}
$$

Replacing in the path $z = i_0 ... i_r$ the vertex i_k by i'_k, we obtain the path $z' = i_0 ... i_{k-1} i'_k i_{k+1} ... i_r$ that clearly belongs to $\Pi_{x,y}$ and, hence, is allowed. Since the $(r-1)$-path $i_0 ... i_{k-1} i_{k+1} ... i_r$ is regular but non-allowed (as it is not step-like), while ∂w is allowed, we have

$$(\partial w)^{i_0 ... i_{k-1} i_{k+1} ... i_r} = 0. \tag{4.17}$$

On the other hand, we have by (2.4)

$$
\begin{aligned}
(\partial w)^{i_0 ... i_{k-1} i_{k+1} ... i_r} &= \sum_{j \in Z} \left(\sum_{m=0}^{k-1} (-1)^m \, w^{i_0 ... i_{m-1} j i_m ... i_{k-1} i_{k+1} ... i_r} \right. && (4.18) \\
&\quad + (-1)^k \, w^{i_0 ... i_{k-1} j i_{k+1} ... i_r} && (4.19) \\
&\quad \left. + \sum_{m=k+2}^{r+1} (-1)^{m-1} \, w^{i_0 ... i_{k-1} i_{k+1} ... i_{m-1} j i_m ... i_r} \right). && (4.20)
\end{aligned}
$$

All the components of w in the sums (4.18) and (4.20) vanish since they correspond to non-allowed paths, while w is allowed. The path $i_0 ... i_{k-1} j i_{k+1} ... i_r$ in the term (4.19) is also non-allowed unless $j = i_k$ or $j = i'_k$ (note that i_k and i'_k are uniquely determined by i_{k-1} and i_{k+1}). Hence, the only non-zero terms in (4.18)-(4.20) are $w^{i_0 ... i_{k-1} i_k i_{k+1} ... i_r} = w^z$ and $w^{i_0 ... i_{k-1} i'_k i_{k+1} ... i_r} = w^{z'}$. Combining (4.17) and (4.18)-(4.20), we obtain

$$0 = w^z + w^{z'}.$$

Since $L(z') = L(z) \pm 1$, it follows that

$$(-1)^{L(z')} \, w^{z'} = (-1)^{L(z)} \, w^z. \tag{4.21}$$

The transformation $z \mapsto z'$ described above, allows us to obtain from a given $z \in \Pi_{x,y}$ in a finite number of steps any other path in $\Pi_{x,y}$. Since the quantity $(-1)^{L(z)} \, w^z$ does not change under this transformation, it follows that it does not depend on a particular choice of $z \in \Pi_{x,y}$, which was claimed. Hence, the coefficients c^{xy} are well-defined by (4.16).

Finally, let us show that the identity (4.15) holds with the coefficients c^{xy} from (4.16). By (4.2) we have

$$e_x \times e_y = \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} \, e_z.$$

Using (4.16) we obtain

$$\begin{aligned}\sum_{x\in E_*(X),\ y\in E_*(Y)} c^{xy}\left(e_x\times e_y\right) &= \sum_{x\in E_*(X),\ y\in E_*(Y)} c^{xy}\sum_{z\in\Pi_{x,y}}(-1)^{L(z)}e_z\\ &= \sum_{x\in E_*(X),\ y\in E_*(Y)}\sum_{z\in\Pi_{x,y}} w^z e_z\\ &= \sum_{z\in E_*(Z)} w^z e_z = w,\end{aligned}$$

which finishes the proof. ∎

Lemma 4.13 *If $u\in\mathcal{A}_p(X),\varphi\in\mathcal{A}_{p'}(X)$ and $v\in\mathcal{A}_q(Y),\psi\in\mathcal{A}_{q'}(Y)$, where $p,q\geq 0$, then*

$$[u\times v,\varphi\times\psi]=\binom{p+q}{p}[u,\varphi][v,\psi]. \tag{4.22}$$

Proof. Set $r=p+q$. If $p'+q'\neq r$ then the both sides of (4.22) vanish. Assume further that $p'+q'=r$ so that $w:=\varphi\times\psi\in\mathcal{A}_r(Z)$. We have then

$$\begin{aligned}[u\times v,w] &= \sum_{z\in E_r(Z)}(u\times v)^z w^z\\ &= \sum_{z\in E_r(Z)}(-1)^{L(z)}u^x v^y w^z \quad (x,y \text{ are projections of } z)\\ &= \sum_{x\in E_p(X)}\sum_{y\in E_q(Y)}\sum_{z\in\Pi_{x,y}}(-1)^{L(z)}u^x v^y w^z. \end{aligned}\tag{4.23}$$

By definition of cross product, we have

$$w^z=(-1)^{L(z)}\varphi^x\psi^y,$$

where x and y are the projections of z. In (4.23) we have $x\in E_p(X)$ and $y\in E_q(Y)$. Therefore, if $p'\neq p$ then $\varphi^x=0$, $w^z=0$ and the sum in (4.23) vanishes. In this case, the both sides of (4.22) are equal to zero again.

In the main case $p'=p$ and, hence, $q'=q$, we obtain

$$\begin{aligned}[u\times v,\varphi\times\psi] &= \sum_{x\in E_*(X)}\sum_{y\in E_*(Y)}\sum_{z\in\Pi_{x,y}}(-1)^{L(z)}u^x v^y(-1)^{L(z)}\varphi^x\psi^y\\ &= \sum_{x\in E_*(X)}\sum_{y\in E_*(Y)}\left|\Pi_{x,y}\right|u^x\varphi^x v^y\psi^y\\ &= \binom{p+q}{p}[u,\varphi][v,\psi]\end{aligned}$$

where we have used $\left|\Pi_{x,y}\right|=\binom{p+q}{p}$. ∎

5 ∂-Invariant elements on abstract products

The main part of the proof of Theorems 3.3 and 4.7 is contained in Theorem 5.1 below. This theorem will be stated in terms of an abstract chain complex. Fix some integer f and let $\{\mathcal{R}_p\}_{p\geq f}$ be a sequence of finite dimensional $\mathbb{K}$-linear spaces, that together with a boundary operator $\partial:\mathcal{R}_p\to\mathcal{R}_{p-1}$ forms a chain complex

$$0\leftarrow\mathcal{R}_f\leftarrow\mathcal{R}_{f+1}\leftarrow\ldots\leftarrow\mathcal{R}_{p-1}\leftarrow\mathcal{R}_p\leftarrow\ldots$$

Fix a basis $\{e_x\}$ in $\mathcal{R}_p$ where the index x varies in a finite set R_p. For any $u \in \mathcal{R}_p$ we have then a unique expansion $u = \sum_{x\in R_p} u^x e_x$ where $u^x \in \mathbb{K}$. We extend the notation u^x to any $x \in R_* = \bigcup_{p\geq f} R_p$ by setting $u^x = 0$ if $x \in R_q$ with $q \neq p$. Define in $\mathcal{R}_* = \bigcup_{p\geq f} \mathcal{R}_p$ a $\mathbb{K}$-scalar product by

$$[u, v] = \sum_{x\in R_*} u^x v^x.$$

Fix a subset E_p of R_p and denote by $\mathcal{A}_p$ the subspace of $\mathcal{R}_p$ spanned by $\{e_x\}$ with $x \in E_p$. The elements of $\mathcal{A}_* = \bigcup_{p\geq f} \mathcal{A}_p$ are called allowed. Define also

$$\mathcal{N}_p = \operatorname{span}\{e_x : x \in R_p \setminus E_p\}$$

so that $\mathcal{R}_p = \mathcal{A}_p \oplus \mathcal{N}_p$. The elements of $\mathcal{N}_*$ are called non-allowed.

Define for any $p \geq f$ a subspace Ω_p of $\mathcal{A}_p$ by

$$\Omega_p = \{u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1}\}.$$

Obviously, we have $\partial\Omega_p \subset \Omega_{p-1}$ so that $\{\Omega_p\}_{p\geq f}$ is a chain complex. The elements of Ω_p are called ∂-invariant.

Of course, the above definitions represent abstract version of the construction of ∂-invariant paths. If we start with a digraph G then R_p is the set of regular elementary p-paths on G and E_p – the set of allowed elementary p-paths. The value of f is a flag that distinguishes the full chain complex (2.13) from the truncated one (2.14). Therefore, $f = -1$ in Theorem 3.3 and $f = 0$ in Theorem 4.7.

Assume now that we have three sets of the structure $\{\mathcal{R}_*, \mathcal{A}_*, \Omega_*\}$, which we will distinguish by adding to these notations $(X), (Y), (Z)$. Of course, in application X, Y, Z will be digraphs where $Z = X * Y$ or $Z = X\Box Y$, but abstractly X, Y, Z are nothing else but indices to distinguish the three structures. This convention makes the notation in the abstract setting identical to those for the digraph setting. Therefore, the reader who is interested only in digraphs may safely assume that X, Y, Z are digraphs.

Assume that there exists a $\mathbb{K}$-bilinear operation

$$u \in \mathcal{R}_*(X), v \in \mathcal{R}_*(Y) \mapsto u \cdot v \in \mathcal{R}_*(Z)$$

with the following properties (everywhere $p, q \geq f$).

1. If $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ then $u \cdot v \in \mathcal{R}_r(Z)$ with $r = p + q - f$.
2. If $u \in \mathcal{A}_*(X)$ and $v \in \mathcal{A}_*(Y)$ then $u \cdot v \in \mathcal{A}_*(Z)$.
3. If $u \in \mathcal{A}_*(X)$ and $v \in \mathcal{N}_*(Y)$ then $u \cdot v \in \mathcal{N}_*(Z)$; the same is true if $u \in \mathcal{N}_*(X)$ and $v \in \mathcal{A}_*(Y)$.
4. The sequence $\{e_x \cdot e_y\}_{x\in E_*(X), y\in E_*(Y)}$ is linearly independent in $\mathcal{A}_*(Z)$.
5. If $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ then

$$\partial(u \cdot v) = \alpha\partial u \cdot v + \beta u \cdot \partial v, \tag{5.1}$$

where α and β are non-zero scalars depending on p, q. In the case $p = f$ we understand $\partial u \cdot v$ as zero, and the same convention applies to $u \cdot \partial v$ in the case $q = f$.

6. If $u\in\mathcal{A}_p(X),\varphi\in\mathcal{A}_{p'}(X),v\in\mathcal{A}_q(Y),\psi\in\mathcal{A}_{q'}(Y)$ then
$$[u\cdot v,\varphi\cdot\psi]=\gamma\,[u,\varphi]\,[v,\psi]\,, \tag{5.2}$$
where γ is a scalar depending on p,p',q,q'.
7. For any $w\in\Omega_*(Z)$ there is a representation
$$w=\sum_{x\in E_*(X),\ y\in E_*(Y)}c^{xy}e_x\cdot e_y \tag{5.3}$$
where $c^{xy}\in\mathbb{K}$ (c^{xy} are unique by 4).

Note that for the both operations of join (with $f=-1$) and cross product (with $f=0$) all the above properties 1-7 are satisfied as they were proved in the previous sections:

- for join: in Lemma 2.4, Proposition 3.2, Lemmas 3.9, 3.10;
- for cross product: in Propositions 4.4, 4.6, Lemmas 4.11, 4.12, 4.13.

The next theorem is our main technical result.

Theorem 5.1 *Under the above hypotheses, any element $w\in\Omega_*(Z)$ admits a representation in the form*
$$w=\sum_{i=1}^{k}u_i\cdot v_i \tag{5.4}$$
for some finite k, where $u_i\in\Omega_*(X)$ and $v_i\in\Omega_*(Y)$.

For the proof we need some lemmas.

Lemma 5.2 *If $u\in\Omega_*(X)$ and $v\in\Omega_*(Y)$ then $u\cdot v\in\Omega_*(Z)$.*

Proof. Indeed, we know by hypothesis 2 that $u\cdot v\in\mathcal{A}_*(Z)$. Since ∂u and ∂v are allowed, we see that the right hand side of (5.1) is allowed, whence $\partial(u\cdot v)\in\mathcal{A}_*(Z)$ and, hence, $u\cdot v\in\Omega_*(Z)$. ■

Lemma 5.3 *Any $w\in\Omega_*(Z)$ admits a representation*
$$w=\sum_{y\in E_*(Y)}u^y\cdot e_y, \tag{5.5}$$
where $u^y\in\Omega_*(X)$. Similarly, there is a representation
$$w=\sum_{x\in E_*(X)}e_x\cdot v^x,$$
where $v^x\in\Omega_*(Y)$.

Proof. It follows from (5.3) that

$$w=\sum_{y\in E_*(Y)} u^y\cdot e_y, \tag{5.6}$$

where

$$u^y=\sum_{x\in E_*(X)} c^{xy}e_x\in\mathcal{A}_*(X).$$

It is obvious that u^y are uniquely determined as so are the coefficients c^{xy}. Let us show that, in fact, $u^y\in\Omega_*(X)$. Since the operator $\partial:\mathcal{R}_*(Y)\to\mathcal{R}_*(Y)$ is linear, for any $y\in R_*(Y)$ there is an expansion

$$\partial e_y=\sum_{y'\in R_*(Y)}\delta_y^{y'}e_{y'},$$

where $\delta_y^{y'}\in\mathbb{K}$. Using (5.6) and by the product rule (5.1) we obtain (here α_y,β_y are scalars):

$$\begin{aligned}
\partial w &= \sum_{y\in E_*(Y)}\left(\alpha_y\partial u^y\cdot e_y+\beta_y u^y\cdot\partial e_y\right)\\
&= \sum_{y\in E_*(Y)}\alpha_y\partial u^y\cdot e_y+\sum_{y\in E_*(Y)}\sum_{y'\in R_*(Y)}\beta_y\delta_y^{y'}u^y\cdot e_{y'}\\
&= \sum_{y\in E_*(Y)}\alpha_y\partial u^y\cdot e_y+\sum_{y\in R_*(Y)}\sum_{y'\in E_*(Y)}\beta_{y'}\delta_{y'}^{y}u^{y'}\cdot e_y\\
&= \sum_{y\in E_*(Y)}\left(\alpha_y\partial u^y+\sum_{y'\in E_*(Y)}\beta_{y'}\delta_{y'}^{y}u^{y'}\right)\cdot e_y \qquad (5.7)\\
&\quad+\sum_{y\in R_*(Y)\setminus E_*(Y)}\left(\sum_{y'\in E_*(Y)}\beta_{y'}\delta_{y'}^{y}u^{y'}\right)\cdot e_y. \qquad (5.8)
\end{aligned}$$

Note that the whole term in (5.7) belongs to $\mathcal{A}_*(Z)$ by hypothesis 2, while (5.8) consists of products of elements of $\mathcal{A}_*(X)$ and $\mathcal{N}_*(Y)$, which by hypothesis 3 lie in $\mathcal{N}_*(Z)$. Since $\partial w\in\mathcal{A}_*(Z)$, it follows that the whole term (5.8) vanishes. On the other hand, since $\partial w\in\Omega_*(Z)$, we have a representation

$$\partial w=\sum_{y\in E_*(Y)}\widetilde{u}^y\cdot e_y\ ,$$

where $\widetilde{u}^y\in\mathcal{A}_*(X)$. Comparison with (5.7) yields

$$\widetilde{u}^y=\alpha_y\partial u^y+\sum_{y'\in E_*(Y)}\beta_{y'}\delta_{y'}^{y}u^{y'}.$$

Since $u^{y'}\in\mathcal{A}_*(X)$, it follows that $\partial u^y\in\mathcal{A}_*(X)$, which proves that $u^y\in\Omega_*(X)$.

The second claim is proved similarly. ■

If $u,v\in\mathcal{A}_p$ and $[u,v]=0$ then we write $u\perp v$. More generally, we write $u\perp V$ where V is a subspace of $\mathcal{A}_p$ if $u\perp v$ for all $v\in V$.

Lemma 5.4 *Fix $p,q\geq f$ and set $r=p+q-f$. If $u\in\mathcal{A}_p(X)$ and $u\perp\Omega_p(X)$ then $(u\cdot v)\perp\Omega_r(Z)$ for all $v\in\mathcal{A}_q(Y)$. Similarly, if $v\in\mathcal{A}_q(Y)$ and $v\perp\Omega_q(Y)$ then $(u\cdot v)\perp\Omega_r(Z)$ for any $u\in\mathcal{A}_p(X)$.*

Proof. To prove the first claim, we need to show that, for any $w \in \Omega_r(Z)$,

$$[u \cdot v, w] = 0. \tag{5.9}$$

By Lemma 5.3, w is a sum of the terms $\varphi \cdot \psi$ where $\varphi \in \Omega_*(X)$ and $\psi \in \mathcal{A}_*(Y)$, so that it suffices to prove (5.9) for $w = \varphi \cdot \psi$. Observe that $[u, \varphi] = 0$. Indeed, if $\varphi \in \Omega_p(X)$ then this follows from $u \perp \Omega_p(X)$. If $\varphi \in \Omega_{p'}(X)$ with $p' \neq p$ then $[u, \varphi] = 0$ holds trivially. Finally, we conclude $[u \cdot v, \varphi \cdot \psi] = 0$ by (5.2). The second claim is proved similarly. ■

Proof of Theorem 5.1. Given two subspaces $U \subset \mathcal{A}_p(X)$ and $V \subset \mathcal{A}_q(Y)$, denote by $U \cdot V$ the subspace of $\mathcal{A}_{p+q-f}(Z)$ that is spanned by all the products $u \cdot v$ with $u \in U$ and $v \in V$. For any $r \geq f$ set

$$\widetilde{\Omega}_r(Z) = \sum_{p+q-f=r} \Omega_p(X) \cdot \Omega_q(Y), \tag{5.10}$$

that is, $\widetilde{\Omega}_r(Z)$ is the subspace of $\mathcal{A}_r(Z)$ that is spanned by all the elements of the form $u \cdot v$ where $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ with some p, q such that $p + q - f = r$. By Lemma 5.2 we have $u \cdot v \in \Omega_r(Z)$ for all such u, v, whence it follows that

$$\widetilde{\Omega}_r(Z) \subset \Omega_r(Z).$$

The existence of the representation (5.4) is equivalent to the opposite inclusion, that is, to the identity of the two spaces. For that, it suffices to prove that

$$\dim \Omega_r(Z) \leq \dim \widetilde{\Omega}_r(Z). \tag{5.11}$$

Consider also the space

$$\widetilde{\mathcal{A}}_r(Z) = \sum_{p+q-f=r} \mathcal{A}_p(X) \cdot \mathcal{A}_q(Y). \tag{5.12}$$

By the properties of the operation "$\cdot$" we have

$$\widetilde{\mathcal{A}}_r(Z) \subset \mathcal{A}_r(Z).$$

By (5.3) any element from $\Omega_r(Z)$ is a linear combination of $e_x \cdot e_y$ with allowed x, y, which implies

$$\Omega_r(Z) \subset \widetilde{\mathcal{A}}_r(Z). \tag{5.13}$$

If $\Omega_p(X) = \mathcal{A}_p(X)$ and $\Omega_q(Y) = \mathcal{A}_q(Y)$ for all $p, q \geq f$ then we obtain from (5.10) and (5.12) that $\widetilde{\Omega}_r(Z) = \widetilde{\mathcal{A}}_r(Z)$. Clearly, (5.13) implies $\Omega_r(Z) \subset \widetilde{\Omega}_r(Z)$, which finishes the proof in this case. However, the main difficulty in the present proof lies in the fact that in general $\Omega_p \subsetneqq \mathcal{A}_p$.

In the general case consider the spaces:

- $\Omega_p^{\perp}(X)$ – the orthogonal complement of $\Omega_p(X)$ in $\mathcal{A}_p(X)$, that is,

$$\Omega_p^{\perp}(X) = \{u \in \mathcal{A}_p(X) : [u, v] = 0 \text{ for all } v \in \Omega_p(X)\}. \tag{5.14}$$

- $\Omega_q^{\perp}(Y)$ – the orthogonal complement of $\Omega_q(Y)$ in $\mathcal{A}_q(Y)$.

- $\Omega_r^{\perp}(Z)$ – the orthogonal complement of $\Omega_r(Z)$ in $\widetilde{\mathcal{A}}_r(Z)$.

Lemma 5.4 implies that

$$
\begin{array}{ll}
u \in \Omega_p^{\perp}(X), \quad v \in \mathcal{A}_q(Y) & \Rightarrow u \cdot v \in \Omega_r^{\perp}(Z), \\
u \in \mathcal{A}_p(X), \quad v \in \Omega_q^{\perp}(Y) & \Rightarrow u \cdot v \in \Omega_r^{\perp}(Z),
\end{array}
\tag{5.15}
$$

where $r = p + q - f$. Indeed, we have by (5.12) $u \cdot v \in \widetilde{\mathcal{A}}_r(Z)$, whereas by Lemma 5.4 $(u \cdot v) \perp \Omega_r(Z)$.

Consider first a simple case when the field $\mathbb{K}$ is $\mathbb{R}$. In this case $[\cdot, \cdot]$ is a proper inner product in $\mathcal{A}_p$ and $\Omega_p^{\perp}$ is a proper orthogonal complement of Ω_p in $\mathcal{A}_p$; in particular, we have[1]

$$
\mathcal{A}_p(X) = \Omega_p(X) \oplus \Omega_p^{\perp}(X) \tag{5.16}
$$

(cf. Fig. 12)

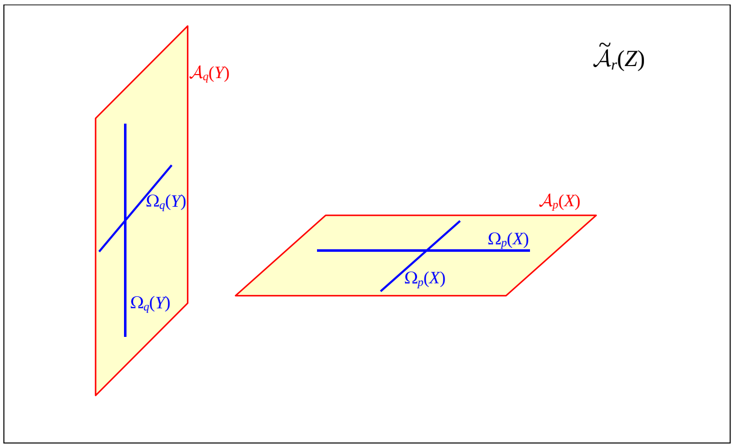


Figure 12: Decomposition of the spaces $\mathcal{A}_p(X)$ and $\mathcal{A}_q(Y)$

For each $u \in \mathcal{A}_p(X)$ consider a decomposition

$$
u = u_{\Omega} + u_{\perp} \tag{5.17}
$$

where $u_{\Omega} \in \Omega_p(X)$ and $u_{\perp} \in \Omega_p^{\perp}(X)$, and a similar decomposition $v = v_{\Omega} + v_{\perp}$ for $v \in \mathcal{A}_q(Y)$. Then we have

$$
u \cdot v = u_{\Omega} \cdot v_{\Omega} + u_{\Omega} \cdot v_{\perp} + u_{\perp} \cdot v_{\Omega} + u_{\perp} \cdot v_{\perp}.
$$

Here we have $u_{\Omega} \cdot v_{\Omega} \in \widetilde{\Omega}_r(Z)$, while by (5.15) all other terms in the right hand side belong to $\Omega_r^{\perp}(Z)$. It follows that

$$
u \cdot v \in \widetilde{\Omega}_r(Z) + \Omega_r^{\perp}(Z).
$$

Since $\widetilde{\mathcal{A}}_r(Z)$ is spanned by the products $u \cdot v$ where u, v are allowed, we obtain that

$$
\widetilde{\mathcal{A}}_r(Z) \subset \widetilde{\Omega}_r(Z) + \Omega_r^{\perp}(Z).
$$

Comparing with the decomposition

$$
\widetilde{\mathcal{A}}_r(Z) = \Omega_r(Z) \oplus \Omega_r^{\perp}(Z),
$$

we obtain (5.11).

[1] For a general field $\mathbb{K}$, the decomposition (5.16) is not true, because Ω_p and $\Omega_p^{\perp}$ may have a non-trivial intersection.

Consider now the most general case of an arbitrary field $\mathbb{K}$. Let us introduce the following notation:

$$\begin{aligned}a_p &= \dim \mathcal{A}_p(X), \ a_q = \dim \mathcal{A}_q(Y), \ a_r = \dim \widetilde{\mathcal{A}}_r(Z), \\ \omega_p &= \dim \Omega_p(X), \ \omega_q = \dim \Omega_q(Y), \ \omega_r = \dim \Omega_r(Z),\end{aligned}$$

and observe that, by the rank-nullity theorem,

$$\dim \Omega_p^{\perp}(X) = a_p - \omega_p, \ \dim \Omega_q^{\perp}(Y) = a_q - \omega_q, \ \dim \Omega_r^{\perp}(Z) = a_r - \omega_r \tag{5.18}$$

(cf. Lemma 6.1 below). Let us also observe that

$$a_r = \sum_{p+q-f=r} a_p a_q. \tag{5.19}$$

Indeed, by (5.12) $\widetilde{\mathcal{A}}_r(Z)$ is spanned by all the products $e_x \cdot e_y$, where $x \in E_p(X)$, $y \in E_q(Y)$ and $p+q-f=r$. The number of such products $e_x \cdot e_y$ is equal to the right hand side of (5.19), so that the identity (5.19) follows from the linear independence of the family $\{e_x \cdot e_y\}$. The latter implies also

$$\dim (\mathcal{A}_p(X) \cdot \mathcal{A}_q(Y)) = a_p a_q, \tag{5.20}$$

whence it follows that the sum in (5.12) is direct, that is,

$$\widetilde{\mathcal{A}}_r(Z) = \bigoplus_{p+q-f=r} (\mathcal{A}_p(X) \cdot \mathcal{A}_q(Y)). \tag{5.21}$$

Let us show that, for arbitrary subspaces $U \subset \mathcal{A}_p(X)$ and $V \subset \mathcal{A}_q(Y)$,

$$\dim (U \cdot V) = \dim U \dim V. \tag{5.22}$$

Indeed, let $u_1, u_2, ..u_k$ be a basis in U and $v_1, ...v_l$ be a basis in V. Then $U \cdot V$ is spanned by all the products $u_i \cdot v_j$, so that

$$\dim (U \cdot V) \leq kl. \tag{5.23}$$

Let us complement the basis $\{u_i\}$ to a basis in $\mathcal{A}_p(X)$ by adding additional paths $u_1', ..., u_{k'}'$, and similarly complement $\{v_j\}$ to a basis in $\mathcal{A}_q(Y)$ by adding $v_1', ..., v_{l'}'$. Set $U' = \operatorname{span}\{u_i'\}$ and $V' = \operatorname{span}\{v_j'\}$, so that

$$\mathcal{A}_p(X) = U \oplus U' \quad \text{and} \quad \mathcal{A}_q(Y) = V \oplus V'. \tag{5.24}$$

Then

$$\mathcal{A}_p(X) \cdot \mathcal{A}_q(Y) = \left(U + U'\right) \cdot \left(V + V'\right) = U \cdot V + U \cdot V' + U' \cdot V + U' \cdot V', \tag{5.25}$$

whence by (5.20) and (5.23)

$$\begin{aligned}a_p a_q &\leq \dim (U \cdot V) + \dim \left(U \cdot V'\right) + \dim \left(U' \cdot V\right) + \dim \left(U' \cdot V'\right) \\ &\leq kl + kl' + k'l + k'l'.\end{aligned} \tag{5.26}$$

However, the right hand side here is equal to $(k+k')(l+l') = a_p a_q$, which implies that all the inequalities in (5.26) are equalities, in particular, $\dim (U \cdot V) = kl$, which proves (5.22).

It follows from this argument (or directly from (5.22)) that the sum at the right hand side of (5.25) is direct and that

$$U \cdot \mathcal{A}_q(Y) = U \cdot \left(V \oplus V'\right) = (U \cdot V) \oplus \left(U \cdot V'\right)$$

and

$$A_p(X)\cdot V=\left(U\oplus U'\right)\cdot V=(U\cdot V)\oplus\left(U'\cdot V\right).$$

Consequently, we obtain

$$\left(U\cdot\mathcal{A}_q(Y)\right)\cap\left(A_p(X)\cdot V\right)=U\cdot V. \tag{5.27}$$

By (5.15) we have

$$\Omega_p^{\perp}(X)\cdot\mathcal{A}_q(Y)\subset\Omega_r^{\perp}(Z)$$

and

$$\mathcal{A}_p(X)\cdot\Omega_q^{\perp}(Y)\subset\Omega_r^{\perp}(Z),$$

whence it follows that

$$\sum_{p+q-f=r}\left[\left(\Omega_p^{\perp}(X)\cdot\mathcal{A}_q(Y)\right)+\left(\mathcal{A}_p(X)\cdot\Omega_q^{\perp}(Y)\right)\right]\subset\Omega_r^{\perp}(Z). \tag{5.28}$$

Note that the space in the square brackets under the summation sign is a subspace of $\mathcal{A}_p(X)\cdot\mathcal{A}_q(Y)$. It follows from (5.21) that the sum in (5.28) is direct, hence

$$\sum_{p+q-f=r}\dim\left[\left(\Omega_p^{\perp}(X)\cdot\mathcal{A}_q(Y)\right)+\left(\mathcal{A}_p(X)\cdot\Omega_q^{\perp}(Y)\right)\right]\leq\dim\Omega_r^{\perp}(Z). \tag{5.29}$$

The intersection of $\Omega_p^{\perp}(X)\cdot\mathcal{A}_q(Y)$ and $\mathcal{A}_p(X)\cdot\Omega_q^{\perp}(Y)$ is $\Omega_p^{\perp}(X)\cdot\Omega_q^{\perp}(Y)$ (cf. (5.27)), which implies that

$$\begin{aligned}
&\dim\left[\left(\Omega_p^{\perp}(X)\cdot\mathcal{A}_q(Y)\right)+\left(\mathcal{A}_p(X)\cdot\Omega_q^{\perp}(Y)\right)\right]\\
=&\dim\left(\Omega_p^{\perp}(X)\cdot\mathcal{A}_q(Y)\right)+\dim\left(\mathcal{A}_p(X)\cdot\Omega_q^{\perp}(Y)\right)\\
&-\dim\left(\Omega_p^{\perp}(X)\cdot\Omega_q^{\perp}(Y)\right)\\
=&\left(a_p-\omega_p\right)a_q+a_p\left(a_q-\omega_q\right)-\left(a_p-\omega_p\right)\left(a_q-\omega_q\right)\\
=&a_pa_q-\omega_p\omega_q.
\end{aligned}$$

Hence, (5.29) yields

$$\sum_{p+q-f=r}\left(a_pa_q-\omega_p\omega_q\right)\leq a_r-\omega_r.$$

Finally, we are left to observe that, by (5.10), (5.21) and (5.22),

$$\dim\widetilde{\Omega}_r(Z)=\sum_{p+q-f=r}\omega_p\omega_q,$$

whence (5.11) follows. ∎

6 Proof of Theorems 3.3 and 4.7

In the next proof Ω_* denotes the truncated chain complex (2.14), that is, $\Omega_*=\{\Omega_p\}_{p\geq 0}$.

Proof of Theorem 4.7. By definition, the chain complex $\Omega_*(X)\otimes\Omega_*(Y)$ consists of the spaces

$$\Omega_r(X,Y)=\bigoplus_{\{p,q\geq 0:p+q=r\}}\left(\Omega_p(X)\otimes\Omega_q(Y)\right)$$

for all $r \geq 0$ and the boundary operator $\partial : \Omega_r(X, Y) \to \Omega_{r-1}(X, Y)$ is given by

$$\partial(u \otimes v) = (\partial u) \otimes v + (-1)^p \, u \otimes v \tag{6.1}$$

for all $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ (set also $\Omega_{-1}(X, Y) = \{0\}$).

Denoting by $H_*(X, Y)$ the homologies of $\Omega_*(X, Y)$, we have by the abstract Künneth theorem that

$$H_*(X, Y) \cong H_*(X) \otimes H_*(Y).$$

Therefore, all the statements of Theorem 4.7 will be proved if we show the isomorphism of the chain complexes

$$\Omega_*(X, Y) \cong \Omega_*(Z).$$

Consider for any $r \geq 0$ the space

$$\mathcal{A}_r(X, Y) = \bigoplus_{\{p,q \geq 0 : p+q=r\}} (\mathcal{A}_p(X) \otimes \mathcal{A}_q(Y))$$

and define a linear map

$$\Phi : \mathcal{A}_r(X, Y) \to \mathcal{A}_r(Z),$$

by

$$\Phi(e_x \otimes e_y) = e_x \times e_y$$

for all $x \in E_p(X)$ and $y \in E_q(Y)$ with $p + q = r$, and then extend Φ to full $\mathcal{A}_r(X, Y)$ by linearity. Note that $e_x \times e_y \in \mathcal{A}_r(Z)$ by Proposition 4.6. By Lemma 4.11 the family $\{e_x \times e_y\}$ is linearly independent, which implies that the map Φ is injective.

For $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ we obtain by Proposition 4.6

$$\Phi(u \otimes v) = u \times v \in \Omega_r(Z),$$

which implies that the restriction of Φ to $\Omega_r(X, Y)$ is a monomorphism

$$\Phi : \Omega_r(X, Y) \to \Omega_r(Z). \tag{6.2}$$

By Theorem 5.1, any $w \in \Omega_r(Z)$ admits a representation

$$w = \sum_i u_i \times v_i,$$

where $u_i \in \Omega_p(X)$ and $v_i \in \Omega_q(Y)$ with $p + q = r$. It follows that $w = \Phi\left(\sum_i u_i \otimes v_i\right)$, that is, the map (6.2) is surjective. Hence, Φ is isomorphism, which was to be proved.

It remains to observe that Φ commutes with ∂, which immediately follows from the identical form of the product rules (6.1) and (4.3). ■

In the next proof Ω_* denotes the full chain complex (2.13), that is, $\Omega_* = \{\Omega_p\}_{p \geq -1}$. Recall that $\Omega'_* = \{\Omega'_p\}_{p \geq 0}$ where $\Omega'_p = \Omega_{p-1}$.

Proof of Theorem 3.3. The chain complex $\Omega'_*(X) \otimes \Omega_*(Y)$ consists of the spaces

$$\begin{aligned} \Omega_r(X, Y) &= \bigoplus_{\{p' \geq 0, q \geq -1 : p'+q=r\}} \left(\Omega'_{p'}(X) \otimes \Omega_q(Y)\right) \\ &= \bigoplus_{\{p,q \geq -1 : p+q=r-1\}} (\Omega_p(X) \otimes \Omega_q(Y)) \end{aligned}$$

for all $r \geq -1$, and the boundary operator $\partial : \Omega_r(X,Y) \to \Omega_{r-1}(X,Y)$ is given by

$$\partial(u \otimes v) = (\partial u) \otimes v + (-1)^{p+1} u \otimes v \tag{6.3}$$

for all $u \in \Omega_p(X) = \Omega'_{p+1}(X)$ and $v \in \Omega_q(Y)$ (set also $\Omega_{-2}(X,Y) = \{0\}$). The isomorphism (3.1) is equivalent to

$$\Omega_*(Z) \cong \Omega_*(X,Y). \tag{6.4}$$

Let us first show how (6.4) implies (3.2) and (3.3). Indeed, (6.4) and the Künneth theorem yield

$$H_*(\Omega_*(Z)) \cong H_*(\Omega_*(X,Y)) \cong H_*\left(\Omega'_*(X)\right) \otimes H_*(\Omega_*(Y)).$$

More explicitly this means that, for any $r \geq -1$,

$$\begin{aligned} H_r(\Omega_*(Z)) &\cong \bigoplus_{\{p' \geq 0, q \geq -1 : p'+q=r\}} \left(H_{p'}\left(\Omega'_*(X)\right) \otimes H_q(\Omega_*(Y))\right) \\ &= \bigoplus_{\{p,q \geq -1 : p+q=r-1\}} \left(H_p(\Omega_*(X)) \otimes H_q(\Omega_*(Y))\right). \end{aligned}$$

Since the homology group $H_{-1}(\Omega_*)$ is always trivial, the condition $p,q \geq -1$ can be replaced here by $p,q \geq 0$. Finally, observing that $H_p(\Omega_*(X)) = \widetilde{H}_p(X)$, $H_q(\Omega_*(Y)) = \widetilde{H}_q(Y)$ and $H_r(\Omega_*(Z)) = \widetilde{H}_r(Z)$ are the reduced homologies, we obtain (3.3).

Now we concentrate on the proof of (6.4). Consider for any $r \geq -1$ the space

$$\mathcal{A}_r(X,Y) = \bigoplus_{\{p,q \geq -1 : p+q=r-1\}} \left(\mathcal{A}_p(X) \otimes \mathcal{A}_q(Y)\right)$$

and define a linear map

$$\Phi : \mathcal{A}_r(X,Y) \to \mathcal{A}_r(Z)$$

by

$$\Phi(e_x \otimes e_y) = e_{xy}, \tag{6.5}$$

for all $x \in E_p(X)$ and $y \in E_q(Y)$ with $p+q = r-1$. Note that e_{xy} is the join of e_x and e_y on $Z = X * Z$. Since the family $\{e_{xy}\}$ of paths is linearly independent, we see that Φ is a injective.

For $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ we obtain by Proposition 3.2

$$\Phi(u \otimes v) = uv \in \Omega_r(Z),$$

which implies that the restriction of Φ to $\Omega_r(X,Y)$ is a monomorphism

$$\Phi : \Omega_r(X,Y) \to \Omega_r(Z). \tag{6.6}$$

By Theorem 5.1, each $w \in \Omega_r(Z)$ admits a representation $w = \sum_i u_i v_i$, where $u_i \in \Omega_p(X)$ and $v_i \in \Omega_q(Y)$ with $p+q = r-1$. It follows that $w = \Phi\left(\sum_i u_i \otimes v_i\right)$, that is, the map (6.6) is surjective. Hence, Φ is isomorphism, which was to be proved.

It remains to observe that Φ commutes with ∂, which immediately follows from the identical form of the product rules (6.3) and (2.7). ■

In conclusion, let us prove a statement justifying (5.18) for an arbitrary field $\mathbb{K}$. The subtlety is that this identity does not hold for arbitrary bilinear form $[u,v]$ and requires an additional condition that replaces the positive definiteness in the case of $\mathbb{K} = \mathbb{R}$.

Lemma 6.1 *Let $\mathcal{A}$ be a vector space over $\mathbb{K}$ of dimension n. Fix a basis $\{e_1, ..., e_n\}$ in $\mathcal{A}$ and define on $\mathcal{A}$ a bilinear form $[u, v] = \sum_{i=1}^{n} u_i v_i$, where $u = \sum_{i=1}^{n} u_i e_i$ and $v = \sum_{i=1}^{n} v_i e_i$ are arbitrary elements of $\mathcal{A}$. Then, for any subspace Ω of $\mathcal{A}$,*

$$\dim \Omega^{\perp} = n - \dim \Omega,$$

where $\Omega^{\perp} = \{v \in \mathcal{A} : [u, v] = 0 \ \forall u \in \Omega\}$.

Proof. Set $\dim \Omega = k$ and fix a basis $\{u_1, ..., u_k\}$ in Ω. Set $u_j = \sum_{i=1}^{n} u_{ij} e_i$ so that the matrix (u_{ij}) has the rank k over $\mathbb{K}$. The condition $[u, v] = 0$ for $v = \sum_{i=1}^{n} v_i e_i$ amounts to k linear equations $\sum_{i=1}^{n} u_{ij} v_i = 0$ for n unknown v_i. By the rank-nullity theorem, the dimension of the space of solutions v of this linear system is equal to $n - k$, which was to be proved. ∎

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