Analysis and Geometry on Graphs
Part 2. Differential forms on digraphs

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1 Differential forms on a finite set

Let V be a non-empty finite set. Denote by $\Lambda^0 = \Lambda^0(V)$ the linear space of all $\mathbb{K}$-valued functions on V, where $\mathbb{K}$ is a fixed scalar field, say $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. More generally, for any integer $p \geq 0$, denote by $\Lambda^p = \Lambda^p(V)$ the linear space of all $\mathbb{K}$-valued functions on $V^{p+1} = \underbrace{V \times ... \times V}_{p+1}$. Clearly,

$\dim \Lambda^p = |V|^{p+1}$.

Definition. Elements of Λ^p are referred to as *p-forms* on V.

The value of a p-form ω at a point $(i_0, i_1, ..., i_p) \in V^{p+1}$ will be denoted by $\omega_{i_0 i_1 \dots i_p}$. In particular, the value of a function $f \in \Lambda^0(V)$ at $i \in V$ will be denoted by f_i.

Denote by $e^{j_0 \cdots j_p}$ a p-form that takes value $1 \in \mathbb{K}$ at the point $(j_0, j_1, ..., j_p)$ and 0 at all other points. For example, e^j is a function on V that is equal to 1 at j and 0 away from j. Let us refer to $e^{j_0 \cdots j_p}$ as an *elementary* p-form. Clearly, the family $\{e^{j_0 \cdots j_p}\}$ of all elementary p-forms is a basis in the linear space Λ^p and, for any $\omega \in \Lambda^p$,

$$\omega = \sum_{j_0, \dots, j_p \in V} \omega_{j_0 \dots j_p} e^{j_0 \cdots j_p}.$$

1.1 Exterior derivative

Definition. Define *the exterior derivative* $d : \Lambda^p \to \Lambda^{p+1}$ by

$$\boxed{(d\omega)_{i_0 \dots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \, \omega_{i_0 \dots \widehat{i_q} \dots i_{p+1}}}, \tag{1.1}$$

for any $\omega \in \Lambda^p$, where the hat $\widehat{i_q}$ means omission of the index i_q.

For example, for a function $f \in \Lambda^0$ we have

$$(df)_{ij} = f_j - f_i,$$

for 1-form ω

$$(d\omega)_{ijk} = \omega_{jk} - \omega_{ik} + \omega_{ij},$$

for a 2-form ω

$$(d\omega)_{ijkl} = \omega_{jkl} - \omega_{ikl} + \omega_{ijl} - \omega_{ijk}.$$

It follows from (1.1) that

$$
\begin{aligned}
\left(de^{j_0\dots j_p}\right)_{i_0\dots i_{p+1}} &= \sum_{q=0}^{p+1} (-1)^q \, e^{j_0\dots j_p}_{i_0\dots\widehat{i_q}\dots i_{p+1}} \\
&= \sum_{q=0}^{p+1} (-1)^q \, e^{j_0\dots j_{q-1} i_q j_q \dots j_p}_{i_0\dots i_q\dots i_{p+1}} \\
&= \sum_{q=0}^{p+1} (-1)^q \sum_i e^{j_0\dots j_{q-1} i j_q \dots j_p}_{i_0\dots i_{p+1}}
\end{aligned}
$$

whence

$$
\boxed{de^{j_0\dots j_p} = \sum_i \sum_{q=0}^{p+1} (-1)^q \, e^{j_0 j_1\dots j_{q-1} i j_q \dots j_p}.} \tag{1.2}
$$

For example,

$$
de^j = \sum_i \left(e^{ij} - e^{ji}\right),
$$

$$
de^{jk} = \sum_i \left(e^{ijk} - e^{jik} + e^{jki}\right).
$$

Lemma 1.1 *For any* $p \geq 0$ *and all* $\omega \in \Lambda^p$,

$$d^2\omega = 0. \tag{1.3}$$

Proof. We have

$$\begin{aligned}
\left(d^2\omega\right)_{i_0\ldots i_{p+2}} &= \sum_{q=0}^{p+2} (-1)^q \, (d\omega)_{i_0\ldots\widehat{i_q}\ldots i_{p+2}} \\
&= \sum_{q=0}^{p+2} (-1)^q \left(\sum_{r=0}^{q-1} (-1)^r \, \omega_{i_0\ldots\widehat{i_r}\ldots\widehat{i_q}\ldots i_{p+2}} + \sum_{r=q+1}^{p+2} (-1)^{r-1} \, \omega_{i_0\ldots\widehat{i_q}\ldots\widehat{i_r}\ldots i_{p+2}} \right) \\
&= \sum_{0\leq r<q\leq p+2} (-1)^{q+r} \, \omega_{i_0\ldots\widehat{i_r}\ldots\widehat{i_q}\ldots i_{p+2}} \\
&\quad - \sum_{0\leq q<r\leq p+2} (-1)^{q+r} \, \omega_{i_0\ldots\widehat{i_q}\ldots\widehat{i_r}\ldots i_{p+2}}.
\end{aligned}$$

After switching q and r in the second sum we see that it is equal to the first one, whence $d^2\omega = 0$ follows. ■

1.2 Concatenation and product rule

Definition. For forms $\varphi \in \Lambda^p$ and $\psi \in \Lambda^q$ denote by $\varphi\psi$ a form from Λ^{p+q} that is defined by

$$\boxed{(\varphi\psi)_{i_0\dots i_{p+q}} = \varphi_{i_0\dots i_p}\psi_{i_p i_{p+1}\dots i_{p+q}}.} \tag{1.4}$$

The form $\varphi\psi$ is called the *concatenation* of φ and ψ.

Clearly, $\varphi\psi$ is a bilinear operation with respect to φ, ψ. For example, if φ is a function, that is, $p = 0$, then $\varphi\psi \in \Lambda^q$ and

$$(\varphi\psi)_{i_0\dots i_q} = \varphi_{i_0}\psi_{i_0\dots i_q}.$$

Also $\psi\varphi \in \Lambda^q$ and

$$(\psi\varphi)_{i_0\dots i_q} = \psi_{i_0\dots i_q}\varphi_{i_q}.$$

For the elementary forms $e^{i_0\dots i_p}$ and $e^{j_0\dots j_q}$ we have

$$e^{i_0\dots i_p}e^{j_0\dots j_q} = \begin{cases} 0, & i_p \neq j_0, \\ e^{i_0\dots i_p j_1\dots i_q}, & i_p = j_0. \end{cases}$$

Lemma 1.2 *For all* $\varphi \in \Lambda^p$ *and* $\psi \in \Lambda^q$, *we have*

$$d(\varphi\psi) = (d\varphi)\psi + (-1)^p \varphi d\psi. \tag{1.5}$$

Proof. Denoting $\omega = \varphi\psi$, we have

$$\begin{aligned}
(d\omega)_{i_0 \ldots i_{p+q+1}} &= \sum_{r=0}^{p+q+1} (-1)^r \omega_{i_0 \ldots \widehat{i_r} \ldots i_{p+q+1}} \\
&= \sum_{r=0}^{p} (-1)^r \omega_{i_0 \ldots \widehat{i_r} \ldots i_{p+1} \ldots i_{p+q+1}} + \sum_{r=p+1}^{p+q+1} (-1)^r \omega_{i_0 \ldots i_p \ldots \widehat{i_r} \ldots i_{p+q+1}} \\
&= \sum_{r=0}^{p} (-1)^r \varphi_{i_0 \ldots \widehat{i_r} \ldots i_{p+1}} \psi_{i_{p+1} \ldots i_{p+q+1}} + \sum_{r=p+1}^{p+q+1} (-1)^r \varphi_{i_0 \ldots i_p} \psi_{i_p \ldots \widehat{i_r} \ldots i_{p+q+1}}.
\end{aligned}$$

Noticing that

$$(d\varphi)_{i_0 \ldots i_{p+1}} = \sum_{r=0}^{p+1} (-1)^r \varphi_{i_0 \ldots \widehat{i_r} \ldots i_{p+1}}$$

and

$$(d\psi)_{i_p \ldots i_{p+q+1}} = \sum_{r=p}^{p+q+1} (-1)^{r-p} \psi_{i_p \ldots \widehat{i_r} \ldots i_{p+q+1}},$$

we obtain

$$\begin{aligned}(d\omega)_{i_0\ldots i_{p+q+1}} &= \left[(d\varphi)_{i_0\ldots i_{p+1}} - (-1)^{p+1}\varphi_{i_0\ldots i_p}\right]\psi_{i_{p+1}\ldots i_{p+q+1}} \\ &\quad + (-1)^p\varphi_{i_0\ldots i_p}\left[(d\psi)_{i_p\ldots i_{p+q+1}} - \psi_{i_{p+1}\ldots i_{p+q+1}}\right] \\ &= ((d\varphi)\,\psi)_{i_0\ldots i_{p+q+1}} + (-1)^p(\varphi d\psi)_{i_0\ldots i_{p+q+1}}\end{aligned}$$

which was to be proved. ■

1.3 Spaces of paths and Stokes's formula

An *elementary p-path* is any (ordered) sequence $i_0, ..., i_p$ of $p+1$ vertices of V that will be denoted simply by $i_0...i_p$ or by $e_{i_0...i_p}$. We use the notation $e_{i_0...i_p}$ when we consider the elementary path as an element of a linear space $\Lambda_p = \Lambda_p(V)$ that consists of all formal linear combination of all elementary p-paths. The elements of Λ_p are called *p-paths*. Each p-path has a form

$$v = \sum_{i_0 i_1 \ldots i_p} v^{i_0 i_1 \ldots i_p} e_{i_0 i_1 \ldots i_p},$$

with arbitrary scalars $v^{i_0 i_1 \cdots i_p}$, that are called the coefficients of v.

We have a natural pairing of p-forms and p-paths as follows:

$$(\omega, v) := \sum_{i_0 \ldots i_p} \omega_{i_0 \ldots i_p} v^{i_0 \ldots i_p}$$

for all $\omega \in \Lambda^p$ and $v \in \Lambda_p$. It follows that the spaces Λ^p and Λ_p are dual.

Definition. Define the *boundary operator* $\partial : \Lambda_{p+1} \to \Lambda_p$ by

$$\boxed{(\partial v)^{i_0 \ldots i_p} = \sum_k \sum_{q=0}^{p+1} (-1)^q \, v^{i_0 \ldots i_{q-1} k i_q \ldots i_p}} \tag{1.6}$$

where the index k is inserted so that it is preceded by q indices.

This definition is valid for $p \geq 0$. Sometimes we need also the operator $\partial : \Lambda_0 \to \Lambda_{-1}$ where we set $\Lambda_{-1} = \{\emptyset\}$, so that Λ_{-1} can be understood as a 0-dimensional linear space. Then by definition $\partial v = \emptyset$ for all $v \in \Lambda_0$.

If v is an 1-path, then ∂v is given by

$$(\partial v)^i = \sum_k \left(v^{ki} - v^{ik}\right).$$

If v is a 2-path then

$$(\partial v)^{ij} = \sum_k \left(v^{kij} - v^{ikj} + v^{ijk}\right).$$

It follows from (1.6) that

$$\begin{aligned}
\left(\partial e_{j_0 \ldots j_{p+1}}\right)^{i_0 \ldots i_p} &= \sum_k \sum_{q=0}^{p+1} (-1)^q \, e_{j_0 \ldots j_{p+1}}^{i_0 \ldots i_{q-1} k i_q \ldots i_p} \\
&= \sum_{q=0}^{p+1} \sum_k (-1)^q \, e_{j_0 \ldots j_{p+1}}^{i_0 \ldots i_{q-1} k i_q \ldots i_p} \\
&= \sum_{q=0}^{p+1} (-1)^q \, e_{j_0 \ldots j_{q-1} j_{q+1} \ldots j_{p+1}}^{i_0 \ldots i_{q-1} i_q \ldots i_p}
\end{aligned}$$

whence

$$\boxed{\partial e_{j_0 \ldots j_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \, e_{j_0 \ldots \widehat{j_q} \ldots j_{p+1}}}. \tag{1.7}$$

For example, $\partial e_{ij} = e_j - e_i$ and $\partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}$.

Lemma 1.3 *For any p-form ω and any $(p+1)$-path v the following identity holds*

$$(d\omega, v) = (\omega, \partial v) .$$

Hence, the operators $d : \Lambda^p \to \Lambda^{p+1}$ and $\partial : \Lambda_{p+1} \to \Lambda_p$ are dual.

Proof. It suffices to prove this for $v = e_{i_0 \ldots i_{p+1}}$. We have

$$(d\omega, v) = (d\omega)_{i_0 \ldots i_{p+1}} = (d\omega)_{i_0 \ldots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \, \omega_{i_0 \ldots \widehat{i_q} \ldots i_{p+1}}$$

while

$$(\omega, \partial v) = \left(\omega, \sum_{q=0}^{p+1} (-1)^q \, e_{i_0 \ldots \widehat{i_q} \ldots i_{p+1}} \right) = \sum_{q=0}^{p+1} (-1)^q \, \omega_{i_0 \ldots \widehat{i_q} \ldots i_{p+1}},$$

whence the identity of the two expressions follows. ■

Corollary 1.4 *For any $v \in \Lambda_p$, we have $\partial^2 v = 0$.*

1.4 Product of paths

For any two paths $u \in \Lambda_p$ and $v \in \Lambda_q$ define the product $uv \in \Lambda_{p+q+1}$ as follows:

$$\boxed{(uv)^{i_0...i_p j_0...j_q} = u^{i_0...i_p} v^{j_0...j_q}.} \tag{1.8}$$

For example, if $u = e_{i_0...i_p}$ and $v = e_{j_0...j_q}$, then

$$e_{i_0...i_p} e_{j_0...j_q} = e_{i_0...i_p j_0...j_q}. \tag{1.9}$$

This definition is valid for all $p, q \geq 0$.

To state a product rule for $\partial(uv)$ we need also the notion of a product also for $p = -1$ or $q = -1$. For that consider instead of $\Lambda_{-1} = \{\emptyset\}$ a modified space $\widetilde{\Lambda}_{-1} \equiv \mathbb{K}$ so that any $u \in \widetilde{\Lambda}_{-1}$ is just a scalar. Then (1.8) can be used again to define the product uv for $u \in \widetilde{\Lambda}_{-1}$ (or $v \in \widetilde{\Lambda}_{-1}$) because the right hand side of (1.8) amounts to multiplying by the scalar u (resp. v). That is, if $p = -1$ then uv is just the multiple of v with the coefficient u.

We need then a modified version of ∂ when acting from Λ_0 to $\widetilde{\Lambda}_{-1}$.Define the operator $\widetilde{\partial} : \Lambda_p \to \Lambda_{p-1}$ as follows. If $p > 0$ then $\widetilde{\partial} \equiv \partial$, and for $p = 0$ define $\widetilde{\partial} : \Lambda_0 \to \mathbb{K}$ by setting $\widetilde{\partial} e_i = 1$ and extending to all $v \in \Lambda_0$ by linearity. In other word, for $v \in \Lambda_0$ we have $\widetilde{\partial} v = (1, v)$. This definition of $\widetilde{\partial}$ is the same as the one used in the extended chain complex (2.20). It is easy to see that $\widetilde{\partial}^2 = 0$.

Lemma 1.5 *For any paths $u \in \Lambda_p$ and $v \in \Lambda_q$ with $p, q \geq 0$, we have*

$$\partial (uv) = (\widetilde{\partial} u) v + (-1)^{p+1} u \widetilde{\partial} v. \tag{1.10}$$

Proof. By bilinearity it suffices to prove (1.10) for $u = e_{i_0 \ldots i_p}$ and $v = e_{j_0 \ldots j_q}$. Consider first the case $p = q = 0$. Then $u = e_i$, $v = e_j$ and $\widetilde{\partial} u = \widetilde{\partial} v = 1$ and

$$\partial (uv) = \partial (e_{ij}) = e_j - e_i = (\widetilde{\partial} u) v - u (\widetilde{\partial} v),$$

which proves (1.10) in this case.

If $p = 0$ and $q \geq 1$ then $u = e_i$ and $v = e_{j_0 \ldots j_q}$, whence

$$\begin{aligned} \partial (uv) &= \partial e_{i j_0 \ldots j_q} = e_{j_0 \ldots j_q} - e_{i j_1 j_2 \ldots j_q} + e_{i j_0 j_2 \ldots j_q} - \ldots \\ &= v - e_i (\partial v) = (\widetilde{\partial} u) v - u (\widetilde{\partial} v) \end{aligned}$$

which proves (1.10) in this case.

If $p\geq 1$ and $q=0$ then $u=u_{i_0...i_p}$, $v=e_j$, whence

$$\begin{aligned}\partial\left(uv\right) &= \partial e_{i_0...i_qj}=e_{i_1...i_qj}-e_{i_0i_2...i_qj}-...+\left(-1\right)^{p+1}e_{i_0...i_p}\\ &= \left(\partial u\right)e_j+\left(-1\right)^{p+1}u=(\widetilde{\partial}u)v+\left(-1\right)^{p+1}u\widetilde{\partial}v.\end{aligned}$$

Finally, if $p\geq 1$ and $q\geq 1$ then

$$\begin{aligned}\partial\left(uv\right) &= \partial e_{i_0...i_pj_0...j_q}=e_{i_1...i_pj_0...j_q}-e_{i_0i_2...i_pj_0...j_q}+...\\ &\quad+\left(-1\right)^{p+1}\left(e_{i_0...i_pj_1...j_q}-e_{i_0...i_pj_0j_2...j_q}+...\right)\\ &= \left(\partial u\right)v+\left(-1\right)^{p+1}u\left(\partial v\right),\end{aligned}$$

which finishes the proof. ∎

1.5 Regular forms

We say that a path $i_0...i_p$ is *regular* if $i_k\neq i_{k+1}$ for all $k=0,...,p-1$, and irregular otherwise. Consider the following subspace of Λ^p:

$$\begin{aligned}\mathcal{R}^p &= \operatorname{span}\left\{e^{i_0...i_p}:i_0...i_p\text{ is regular}\right\}\\ &= \left\{\omega\in\Lambda^p:\omega_{i_0...i_p}=0\text{ if }i_0...i_p\text{ is irregular}\right\}.\end{aligned}$$

The elements of $\mathcal{R}^p$ are called regular p-forms. For example, $\omega \in \mathcal{R}^1$ if $\omega_{ii} \equiv 0$ and $\omega \in \mathcal{R}^2$ if $\omega_{iij} \equiv \omega_{jii} \equiv 0$. The condition $f \in \mathcal{R}^0$ has no additional restriction so that $\mathcal{R}^0 = \Lambda^0$.

The operations of exterior derivative and concatenation can be restricted to regular forms.

Lemma 1.6 *If $\omega \in \mathcal{R}^p$ then $d\omega \in \mathcal{R}^{p+1}$. If $\varphi \in \mathcal{R}^p$ and $\psi \in \mathcal{R}^q$ then $\varphi\psi \in \mathcal{R}^{p+q}$.*

Proof. Let $\omega \in \mathcal{R}^p$. To prove that $d\omega \in \mathcal{R}^{p+1}$, we must show that

$$(d\omega)_{i_0 \ldots i_{p+1}} = 0 \tag{1.11}$$

whenever $i_0 \ldots i_{p+1}$ is irregular, say $i_k = i_{k+1}$. We have by (1.1)

$$(d\omega)_{i_0 \ldots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \, \omega_{i_0 \ldots \widehat{i_q} \ldots i_{p+1}}.$$

If $q \neq k, k+1$ then both i_k, i_{k+1} are present in $\omega_{i_0 \ldots \widehat{i_q} \ldots i_{p+1}}$ which makes this term equal to 0 since ω is regular. In the remaining two cases $q = k$

and $q = k + 1$ the term $\omega_{i_0...\widehat{i_q}...i_{p+1}}$ has the same values (because the sequences $i_0...\widehat{i_q}...i_{p+1}$ are the same) but the signs $(-1)^q$ are opposite. Hence, they cancel out, which proves (1.11).

Let us prove that $\varphi\psi$ is regular provided so are φ and ψ. By (1.4), we have

$$(\varphi\psi)_{i_0...i_{p+q}} = \varphi_{i_0...i_p}\psi_{i_p...i_{p+q}}.$$

If the sequence $i_0...i_{p+q}$ is irregular, say $i_k = i_{k+1}$ then the both indices i_k, i_{k+1} are present either in the sequence $i_0...i_p$ or in $i_p...i_{p+q}$, which implies that one of the terms $\varphi_{i_0...i_p}$, $\psi_{i_p...i_{p+q}}$ vanishes and, hence, $(\varphi\psi)_{i_0...i_{p+q}} = 0$. ■

1.6 Regular paths

We say that an elementary p-path $e_{i_0...i_p}$ is regular (or irregular) if the path $i_0...i_p$ is regular (resp. irregular). We would like to define the boundary operator ∂ on the subspace of Λ_p spanned by regular elementary paths. Just restriction of ∂ to the subspace does not work as ∂ is not invariant on this subspace, so that we have to consider a quotient space instead.

Let I_p be the subspace of Λ_p that is spanned by irregular $e_{i_0 \ldots i_p}$. Consider the quotient spaces

$$\mathcal{R}_p := \Lambda_p / I_p.$$

The elements of $\mathcal{R}_p$ are the equivalence classes $v \bmod I_p$ where $v \in \Lambda_p$, and they are called *regularized* p-paths. The next lemma shows that the boundary operator ∂, the product and the pairing are well-defined for regularized paths.

Lemma 1.7 (a) *If* $v_1, v_2 \in \Lambda_p$ *and* $v_1 = v_2 \bmod I_p$ *then* $\partial v_1 = \partial v_2 \bmod I_{p-1}$.

(b) *If* $\omega \in \mathcal{R}^p$, $v_1, v_2 \in \Lambda_p$ *and* $v_1 = v_2 \bmod I_p$ *then* $(\omega, v_1) = (\omega, v_2)$.

(b) *Let* $u_1, u_2 \in \Lambda_p$ *and* $v_1, v_2 \in \Lambda_q$. *If* $u_1 = u_2 \bmod I_p$ *and* $v_1 = v_2 \bmod I_q$ *then* $u_1 v_1 = u_2 v_2 \bmod I_{p+q+1}$.

Proof. (a) It suffices to prove that if $v = 0 \bmod I_p$ then $\partial v = 0 \bmod I_{p-1}$. Since v is a linear combination of irregular paths, it suffices to prove that $\partial e_{i_0 \ldots i_p}$ is irregular provided $e_{i_0 \ldots i_p}$ is irregular. If $e_{i_0 \ldots i_p}$ is irregular then there exists an index k such that $i_k = i_{k+1}$. Then we have

$$\begin{aligned}\partial e_{i_0\ldots i_p} &= e_{i_1\ldots i_p} - e_{i_0 i_2\ldots i_p} + \ldots \\ &\quad + (-1)^k e_{i_0\ldots i_{k-1} i_{k+1} i_{k+2}\ldots i_p} + (-1)^{k+1} e_{i_0\ldots i_{k-1} i_k i_{k+2}\ldots i_p} \\ &\quad + \ldots + (-1)^p e_{i_0\ldots i_{p-1}}.\end{aligned} \tag{1.12}$$

By $i_k = i_{k+1}$ the two terms in the middle line of (1.12) cancel out, whereas all other terms are irregular, whence, $\partial e_{i_0\ldots i_p} \in I_{p-1}$.

(b) Indeed, $v_1 - v_2 \in I_p$ is a linear combination of irregular paths $e_{i_0\ldots i_p}$. Since $\left(\omega, e_{i_0\ldots i_p}\right) = 0$ for irregular paths, it follows that $(\omega, v_1 - v_2) = 0$ and $(\omega, v_1) = (\omega, v_2)$.

(c) Observe first that if $u \in \Lambda_p$, $v \in \Lambda_q$ then $uv = 0 \bmod I_{p+q+1}$ provided $u = 0 \bmod I_p$ or $v = 0 \bmod I_q$. Indeed, if for example $u = 0 \bmod I_p$ then u is a linear combination of irregular paths $e_{i_0\ldots i_p}$, and the product of an irregular path with any path is irregular. Since

$$u_1 v_1 - u_2 v_2 = (u_1 - u_2)\, v_1 + u_2\,(v_1 - v_2)$$

and $u_1 - u_2 = 0 \bmod I_p$, $v_1 - v_2 = 0 \bmod I_q$, we conclude that

$$u_1 v_1 = u_2 v_2 \bmod I_{p+q+1}.$$

■

It follows from Lemma 1.3 that, for all $\omega \in \mathcal{R}^p$ and $v \in \mathcal{R}_{p+1}$,

$$(d\omega, v) = (\omega, \partial v) . \tag{1.13}$$

By Lemma 1.5, we obtain that, for all $u \in \mathcal{R}_p$ and $v \in \mathcal{R}_q$,

$$\partial (uv) = (\partial u) v + (-1)^{p+1} u \partial v. \tag{1.14}$$

Clearly, $\mathcal{R}_p$ is linearly isomorphic to the space of regular paths:

$$\operatorname{span} \left\{ e_{i_0 \ldots i_p} : i_0 \ldots i_p \text{ is regular} \right\} .$$

For simplicity of notation, we will identify $\mathcal{R}_p$ with this space, by setting all irregular p-paths to be equal to 0. Hence, when applying the formulas for ∂ and for the product in the spaces $\mathcal{R}_p$, one should make the following adjustments: all elementary irregular paths $e_{i_0 \ldots i_p}$ are equal to zero, and the components $v^{i_0 \ldots i_p}$ for irregular paths $i_0 \ldots i_p$ vanish by definition. In particular, the formula (1.6) for the component $(\partial v)^{i_0 \ldots i_p}$ is valid only for regular $i_0 \ldots i_p$, whereas the formula (1.7) for $\partial e_{j_0 \ldots j_{p+1}}$ remains valid for all $j_0 \ldots j_{p+1}$.

Let V' be a subset of V. Clearly, every elementary regular p-path $e_{i_0 \ldots i_p}$ on V' is also a regular p-path on V, so that we have a natural inclusion

$$\mathcal{R}_p(V') \subset \mathcal{R}_p(V). \tag{1.15}$$

By (1.7), $\partial e_{i_0 \ldots i_p}$ has the same expression in the both spaces $\mathcal{R}_p(V')$, $\mathcal{R}_p(V)$ so that ∂ commutes with the inclusion (1.15).

Note for comparison that for p-forms the inclusion $\mathcal{R}^p(V') \subset \mathcal{R}^p(V)$ is also valid, but the operator d *does not* commute with it. For example, in the formula

$$de^i = \sum_j \left(e^{ji} - e^{ij}\right)$$

the summation index j on the right hand side runs over all vertices, and the result depends on the set of vertices.

2 Elements of homological algebra

2.1 Cochain complexes

A *cochain complex* X is a sequence

$$0 \rightarrow X^0 \xrightarrow{d} X^1 \xrightarrow{d} \dots \xrightarrow{d} X^{p-1} \xrightarrow{d} X^p \xrightarrow{d} \dots \tag{2.1}$$

of vector spaces $\{X^p\}_{p=0}^{\infty}$ over a field $\mathbb{K}$ and linear mappings $d : X^p \rightarrow X^{p+1}$ with the property that $d^2 = 0$ at each level. The latter means that $\operatorname{Im} d|_{X^{p-1}} \subset \ker d|_{X^p}$ that allows to define the de Rham cohomologies of the complex X by

$$H^p(X) = \ker d|_{X^p} / \operatorname{Im} d|_{X^{p-1}}$$

(where $X^{-1} := \{0\}$). The sequence (2.1) is called exact if $H^p(X) = \{0\}$ for all $p \geq 0$.

We always assume that the spaces X^p are finitely dimensional. Applying the nullity-rank theorem to the mapping $d : X^p \rightarrow X^{p+1}$, we obtain the following identity:

Lemma 2.1 *We have for any $p \geq 0$*

$$\dim H^p(X) = \dim X^p - \dim dX^p - \dim dX^{p-1}. \tag{2.2}$$

It implies the following.

Lemma 2.2 *For a finite cochain complex*

$$0 \;\rightarrow\; X^0 \;\xrightarrow{d}\; X^1 \;\xrightarrow{d}\; \ldots \;\xrightarrow{d}\; X^{n-1} \;\xrightarrow{d}\; X^n \;\xrightarrow{d} 0\;, \tag{2.3}$$

the following identity is satisfied

$$\sum_{k=0}^{n} (-1)^k \dim H^k(X) = \sum_{k=0}^{n} (-1)^k \dim X^k. \tag{2.4}$$

In particular, if the sequence (2.3) *is exact, then*

$$\sum_{k=0}^{n} (-1)^k \dim X^k = 0. \tag{2.5}$$

For any finite cochain complex (2.3), define its Euler characteristic by

$$\chi(X) = \sum_{p=0}^{n} (-1)^p \dim X^p.$$

Then (2.4) implies

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \dim H^k(X).$$

2.2 Chain complexes

Given a cochain complex (2.1) with finite-dimensional spaces X^p, denote by X_p the dual space to X^p and by ∂ the dual operator to d. Then we obtain a chain complex

$$0 \leftarrow X_0 \overset{\partial}{\leftarrow} X_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} X_{p-1} \overset{\partial}{\leftarrow} X_p \overset{\partial}{\leftarrow} \dots \tag{2.6}$$

Denoting by $(\cdot,\cdot)$ the natural pairing of dual spaces, we obtain by definition

$$(d\omega, v) = (\omega, \partial v)$$

for all $\omega \in X^p$ and $v \in X_{p+1}$. Since $d^2 = 0$, it follows that also $\partial^2 = 0$. Hence, one can define the *homologies* of the chain complex (2.6) by

$$H_p(X) = \ker \partial|_{X_p} / \operatorname{Im} \partial|_{X_{p+1}} .$$

By duality we have

$$\ker \partial|_{X_p} = \left(\operatorname{Im} d|_{X^{p-1}}\right)^{\perp}, \qquad \ker d|_{X^p} = \left(\operatorname{Im} \partial|_{X_{p+1}}\right)^{\perp}, \tag{2.7}$$

where $\perp$ refers to the annihilator in the dual space, which implies the following.

Lemma 2.3 *The spaces $H^p(X)$ and $H_p(X)$ are dual. In particular,* $\dim H^p(X) = \dim H_p(X)$.

Lemma 2.4 *We have for any $p \geq 0$*

$$\dim H_p(X) = \dim X_p - \dim \partial X_p - \dim \partial X^{p+1}.$$

2.3 Sub-complexes and quotient complexes

Let X be a cochain complex as in (2.1), and assume that each X^p has a subspace J^p so that d is invariant on $\{J^p\}$, that is, $dJ^p \subset J^{p+1}$. Then we have a cochain *sub-complex* J as follows:

$$0 \rightarrow J^0 \overset{d}{\rightarrow} J^1 \overset{d}{\rightarrow} \ldots \overset{d}{\rightarrow} J^{p-1} \overset{d}{\rightarrow} J^p \overset{d}{\rightarrow} \ldots \tag{2.8}$$

Since the operator d is well defined also on the quotient spaces X^p/J^p, we obtain also a cochain *quotient complex* X/J:

$$0 \rightarrow X^0/J^0 \overset{d}{\rightarrow} X^1/J^1 \overset{d}{\rightarrow} \ldots \overset{d}{\rightarrow} X^{p-1}/J^{p-1} \overset{d}{\rightarrow} X^p/J^p \overset{d}{\rightarrow} \ldots \tag{2.9}$$

Consider the annihilator of J^p, that is the space

$$\left(J^p\right)^{\perp} = \{v \in X_p : v \perp J^p\}\,.$$

Lemma 2.5 *The dual operator ∂ of d is invariant on $\left\{\left(J^p\right)^{\perp}\right\}$, and the chain sub-complex*

$$0 \leftarrow \left(J^0\right)^{\perp} \overset{\partial}{\leftarrow} \left(J^1\right)^{\perp} \overset{\partial}{\leftarrow} \ldots \overset{\partial}{\leftarrow} \left(J^{p-1}\right)^{\perp} \overset{\partial}{\leftarrow} \left(J^p\right)^{\perp} \overset{\partial}{\leftarrow} \ldots \tag{2.10}$$

is dual to the cochain quotient complex (2.9)*.*

Proof. If $v \in (J^p)^\perp$ then, for any $\omega \in J^{p-1}$, we have $d\omega \in J^p$ and, hence,

$$(\omega, \partial v) = (d\omega, v) = 0,$$

which implies $\partial v \in (J^{p-1})^\perp$. Hence, ∂ maps $(J^p)^\perp$ to $(J^{p-1})^\perp$, so that the complex (2.10) is well-defined.

To prove the duality of (2.9) and (2.10), observe that $(J^p)^\perp$ is naturally isomorphic to the dual space $(X^p/J^p)'$. Indeed, each $v \in (J^p)^\perp$ defines a linear functional on X^p/J^p simply by $\omega \mapsto (\omega, v)$ where $\omega \in X^p$ is a representative of an element of X^p/J^p. If $\omega_1 = \omega_2 \mod J^p$ then $\omega_1 - \omega_2 \in J^p$ whence $(\omega_1 - \omega_2, v) = 0$ and $(\omega_1, v) = (\omega_2, v)$. Clearly, the mapping $(J^p)^\perp \to (X^p/J^p)'$ is injective and, hence, surjective because of the identity of the dimensions of the two spaces. Finally, the duality of the operators d and ∂ on the complexes (2.9) and (2.10) is a trivial consequence of their duality on the complexes $X^{\cdot}$ and X. ■

Let us describe a specific method of constructing of d-invariant subspaces.

Lemma 2.6 *Given any subspace S^p of X^p, set*

$$J^p = S^p + dS^{p-1}. \tag{2.11}$$

Then d is invariant on $\{J^p\}$. Besides, we have the following identity

$$(J^p)^\perp = \left\{ v \in (S^p)^\perp : \partial v \in \left(S^{p-1}\right)^\perp \right\}. \tag{2.12}$$

Proof. The first claim follows from $d^2 = 0$ since

$$dJ^p \subset dS^p + d^2 S^{p-1} = dS^p \subset J^{p+1}.$$

The condition $v \in (J^p)^\perp$ means that

$$v \perp S^p \quad \text{and} \quad v \perp dS^{p-1}. \tag{2.13}$$

Clearly, the first condition here is equivalent to $v \in (S^p)^\perp$, while the second condition is equivalent to

$$(d\omega, v) = 0 \;\forall \omega \in S^{p-1} \Leftrightarrow (\omega, \partial v) = 0 \,\forall \omega \in S^{p-1} \Leftrightarrow \partial v \perp S^{p-1} \Leftrightarrow \partial v \in \left(S^{p-1}\right)^\perp,$$

which proves (2.12). ■

2.4 Zigzag Lemma

Consider now three cochain complexes X, Y, Z connected by vertical linear mappings as on the diagram:

$$
\begin{array}{ccccccccc}
0 & \to & 0 & \to & 0 & \to & 0 & \to & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Y^0 & \stackrel{d}{\to} & Y^1 & \stackrel{d}{\to} & Y^2 & \to & \dots \\
\downarrow & & \downarrow^{\alpha} & & \downarrow^{\alpha} & & \downarrow^{\alpha} & & \\
0 & \to & X^0 & \stackrel{d}{\to} & X^1 & \stackrel{d}{\to} & X^2 & \to & \dots \\
\downarrow & & \downarrow^{\alpha} & & \downarrow^{\alpha} & & \downarrow^{\alpha} & & \\
0 & \to & Z^0 & \stackrel{d}{\to} & Z^1 & \stackrel{d}{\to} & Z^2 & \to & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & \dots
\end{array}
\tag{2.14}
$$

Each horizontal mapping is denoted by d and each vertical mapping is denoted by α. We assume that the diagram is commutative. Let us also assume that each column in (2.14) is an exact cochain complex, that is, the mapping $\alpha : Y^p \to X^p$ is an injection, and $\alpha : X^p \to Z^p$ a surjection with the kernel X^p. In this case we can identify Y^p with a subspace of

X^p and Z^p with the quotient X^p/Y^p.

Proposition 2.7 (Zigzag Lemma) *Under the above conditions the sequence*

$$0 \to H^0(Y) \to H^0(X) \to H^0(Z) \to \cdots \to H^p(Y) \to H^p(X) \to H^p(Z) \to H^{p+1}(Y) \tag{2.15}$$

is exact (more precisely, the mappings in (2.15) can be defined so that the sequence is exact).

The sequence (2.15) is called a *long exact sequence in cohomology.* A similar result holds for homologies of chain complexes.

We will normally apply Proposition 2.7 in the following form: if X is a cochain complex (2.1) and J is its sub-complex (2.8), then the following long sequence is exact:

$$0 \to \cdots \to H^p(J) \to H^p(X) \to H^p(X/J) \to H^{p+1}(J) \to \dots \tag{2.16}$$

Similarly, if X is a chain complex (2.6) and J its sub-complex, then the following long sequence is exact:

$$0 \leftarrow \cdots \leftarrow H_p(X/J) \leftarrow H_p(X) \leftarrow H_p(J) \leftarrow H_{p+1}(X/J) \leftarrow \dots \tag{2.17}$$

2.5 Reduced cohomologies and homologies

In the cochain and chain complexes (2.1) and (2.6) one naturally defines the spaces X^{-1} and X_{-1} as $\{0\}$. In a number of situations there is a need in another choice of X^{-1} and X_{-1}.

Assume that there is a injection $\widetilde{d}:\mathbb{K}\rightarrow X^0$ that satisfies the relation $d\widetilde{d}=0$. Setting $\widetilde{X}^{-1}\equiv\mathbb{K}$, we obtain an extended cochain complex

$$0\ \rightarrow\ \widetilde{X}^{-1}\ \stackrel{\widetilde{d}}{\rightarrow}\ X^0\ \stackrel{d}{\rightarrow}\ X^1\ \stackrel{d}{\rightarrow}\ \dots\ \stackrel{d}{\rightarrow}\ X^{p-1}\ \stackrel{d}{\rightarrow}\ X^p\ \stackrel{d}{\rightarrow}\dots \tag{2.18}$$

The cohomologies of the complex (2.18) are denoted by $\widetilde{H}^p(X)$ and are called the *reduced cohomologies*. Obviously, we have

$$\widetilde{H}^p(X)=\begin{cases} H^p(X)\,, & p\geq 1,\\ H^0(X)\,/\,\mathrm{const}, & p=0.\end{cases} \tag{2.19}$$

The dual space $\widetilde{X}_{-1}$ is also $\mathbb{K}$, and the dual operator $\widetilde{\partial}:X_0\rightarrow\mathbb{K}$ of $\widetilde{d}$ is given by $\widetilde{\partial}v=(\widetilde{d}1,v)$ for any $v\in X_0$. Hence, we obtain an extended chain complex

$$0\ \leftarrow\ \widetilde{X}_{-1}\ \stackrel{\widetilde{\partial}}{\leftarrow}\ X_0\ \stackrel{\partial}{\leftarrow}\ X_1\ \stackrel{\partial}{\leftarrow}\ \dots\ \stackrel{\partial}{\leftarrow}\ X_{p-1}\ \stackrel{\partial}{\leftarrow}\ X_p\ \stackrel{\partial}{\leftarrow}\dots \tag{2.20}$$

and the reduced homologies $\widetilde{H}_p(X)$.

For example, let X^0 be a space of $\mathbb{K}$-valued functions over a finite set V and assume that $d\,\mathrm{const} = 0$. Define a mapping $\widetilde{d} : \mathbb{K} \to X^0$ as follows: for any $c \in \mathbb{K}$, $\widetilde{d}c$ is the constant function on V taking the value c. It follows that $d\widetilde{d} = 0$ so that the reduced cohomologies are well-defined. In this case $\widetilde{\partial} v = (1, v)$ where 1 is regarded as a constant function on V.

Brief summary

Given a finite set V, we define a p-form ω on V as $\mathbb{K}$-valued function on V^{p+1}. The set of all p-forms is a linear space over $\mathbb{K}$ that is denoted by Λ^p. It has a canonical basis $e^{i_0 \dots i_p}$. For any $\omega \in \Lambda^p$ we have

$$\omega = \sum_{i_0,\dots,i_p \in V} \omega_{i_0 \dots i_p} e^{i_0 \dots i_p}$$

where $\omega_{i_0 \dots i_p} = \omega\left(i_0, \dots, i_p\right)$. The exterior derivative $d : \Lambda^p \to \Lambda^{p+1}$ is defined by

$$(d\omega)_{i_0 \dots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \, \omega_{i_0 \dots \widehat{i_q} \dots i_{p+1}}$$

and satisfies $d^2 = 0$. The concatenation of forms $\varphi \in \Lambda^p$ and $\psi \in \Lambda^q$ is a form $\varphi\psi \in \Lambda^{p+q}$ defined by

$$(\varphi\psi)_{i_0 \dots i_{p+q}} = \varphi_{i_0 \dots i_p} \psi_{i_p i_{p+1} \dots i_{p+q}}.$$

Then $d\left(\varphi\psi\right) = \left(d\varphi\right)\psi + (-1)^p \varphi d\psi$.

We have defined a subspace $\mathcal{R}^p \subset \Lambda^p$ of regular forms that is spanned by $e^{i_0...i_p}$ with regular paths $i_0...i_p$ (when $i_k \neq i_{l+1}$), and observed that the spaces $\mathcal{R}^p$ are invariant for d and for concatenation.

A p-path on V is a formal linear combination of the elementary p-paths $e_{i_0...i_p} \equiv i_0...i_p$, and the linear space of all p-paths is denoted by Λ_p. For any $v \in \Lambda_p$ we have

$$v = \sum_{i_0,...,i_p \in V} v^{i_0...i_p} e_{i_0...i_p}$$

and a pairing with a p-path ω:

$$(\omega, v) = \sum_{i_0,...,i_p} \omega_{i_0...i_p} v^{i_0...i_p}.$$

The dual operator $\partial : \Lambda_{p+1} \to \Lambda_p$ is given by

$$\partial e_{i_0...i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \, e_{i_0...\widehat{i_q}...i_{p+1}}.$$

The product of two paths $u \in \Lambda_p$ and $v \in \Lambda_q$ is a paths $uv \in \Lambda^{p+q+1}$ defined by

$$(uv)^{i_0...i_p j_0...j_q} = u^{i_0...i_p} v^{j_0...j_q}.$$

It satisfies the product rule

$$\partial (uv) = (\partial u)v + (-1)^{p+1} u\partial v$$

where ∂v in the case $v \in \Lambda_0$ is a constant $\sum_i v_i$ (that is equivalent to $\partial e_i = 1$).

Let I_p be the subspace of Λ_p that is spanned by $e_{i_0 \dots i_p}$ with irregular paths $i_0 \dots i_p$. Then the spaces I_p are invariant for ∂ and for product, which allows to define ∂ and product on the quotient spaces $\mathcal{R}_p = \Lambda_p / I_p$. For simplicity of notation we identify the elements of $\mathcal{R}_p$ with their representatives that are regular p-paths. Then $e_{i_0 \dots i_p}$ with irregular $i_0 \dots i_p$ are treated as zeros.

3 Forms and paths on digraphs

A digraph is a pair (V, E) where V is an arbitrary set and E is a subset of $V \times V \setminus \operatorname{diag}$. The elements of V are called *vertices* and the elements of E are called (*directed*) *edges*. The set V will be always assumed non-empty and finite.

3.1 Allowed paths

Let $i_0...i_p$ be an elementary regular p-path on V. It is called *allowed* if $i_k i_{k+1} \in E$ for any $k = 0, ..., p-1$, and *non-allowed* otherwise. The set of all allowed elementary p-paths will be denoted by E_p, and non-allowed – by N_p. For example, $E_0 = V$ and $E_1 = E$.

Denote by $\mathcal{A}_p = \mathcal{A}_p(V, E)$ the subspace of $\mathcal{R}_p$ spanned by the allowed elementary p-paths, that is,

$$\mathcal{A}_p = \operatorname{span}\left\{e_{i_0...i_p} : i_0...i_p \in E_p\right\} = \left\{v \in \mathcal{R}_p : v^{i_0...i_p} = 0 \ \forall i_0...i_p \in N_p\right\}. \tag{3.1}$$

The elements of $\mathcal{A}_p$ are called allowed p-paths.

Similarly, denote by $\mathcal{N}^p$ the subspace of $\mathcal{R}^p$, spanned by the non-allowed elementary p-forms, that is,

$$\mathcal{N}^p = \operatorname{span}\left\{e^{i_0...i_p} : i_0...i_p \in N_p\right\} = \left\{\omega \in \mathcal{R}^p : \omega_{i_0...i_p} = 0 \ \forall i_0...i_p \in E_p\right\}.$$

Clearly, we have $\mathcal{A}_p = (\mathcal{N}^p)^\perp$ where $\perp$ refers to the annihilator subspace with respect to the couple $(\mathcal{R}^p, \mathcal{R}_p)$ of dual spaces.

3.2 The space p-forms on a digraph

We would like to reduce the space $\mathcal{R}^p$ of regular p-forms so that the non-allowed forms can be treated as zeros. Consider the following subspaces of spaces $\mathcal{R}^p$

$$\boxed{\mathcal{J}^p \equiv \mathcal{J}^p(V,E) := \mathcal{N}^p + d\mathcal{N}^{p-1},} \tag{3.2}$$

that are d-invariant by Lemma 2.6, and define the space Ω^p of p-forms on the digraph (V,E) by

$$\boxed{\Omega^p \equiv \Omega^p(V,E) := \mathcal{R}^p / \mathcal{J}^p}. \tag{3.3}$$

Then d is well-defined on Ω^p and we obtain a cochain complex

$$0 \longrightarrow \Omega^0 \xrightarrow{d} \dots \xrightarrow{d} \Omega^p \xrightarrow{d} \Omega^{p+1} \xrightarrow{d} \dots \tag{3.4}$$

Shortly we write $\Omega^{\cdot} = \mathcal{R}^{\cdot}/\mathcal{J}$ where $\Omega^{\cdot}$ is the complex (3.4) and $\mathcal{R}^{\cdot}$ and $\mathcal{J}$ refer to the corresponding cochain complexes.

If the digraph (V,E) is complete, that is, $E = V \times V \setminus \mathrm{diag}$ then the spaces $\mathcal{N}^p$ and $\mathcal{J}^p$ are trivial, and $\Omega^p = \mathcal{R}^p$.

Let us show that the concatenation is also well-defined on the spaces Ω^p.

Lemma 3.1 *Let $\varphi \in \mathcal{R}^p$ and $\psi \in \mathcal{R}^q$. If $\varphi \in \mathcal{J}^p$ or $\psi \in \mathcal{J}^q$ then $\varphi\psi \in \mathcal{J}^{p+q}$, that is, $\{\mathcal{J}^p\}$ is a graded ideal for the concatenation. Consequently, the concatenation of two forms is well-defined on the spaces $\mathcal{J}^p$ as well as on Ω^p, and it satisfies the product rule (1.5).*

Proof. Observe first that if $\varphi \in \mathcal{N}^p$ then $\varphi\psi \in \mathcal{N}^{p+q}$. Indeed, it suffices to prove this for elementary forms $\varphi = e^{i_0 \ldots i_p}$ and $\psi = e^{j_0 \ldots j_q}$ where the claim is obvious: if the p-path $i_0 \ldots i_p$ is non-allowed then so is the concatenated $(p+q)$-path $i_0 \ldots i_p j_1 \ldots j_q$.

If $\varphi \in \mathcal{J}^p$ then $\varphi = \varphi_0 + d\varphi_1$ where $\varphi_0 \in \mathcal{N}^p$ and $\varphi_1 \in \mathcal{N}^{p-1}$. Then we have

$$\begin{aligned} \varphi\psi &= \varphi_0\psi + (d\varphi_1)\,\psi \\ &= \varphi_0\psi + d\,(\varphi_1\psi) - (-1)^{p-1}\,\varphi_1 d\psi. \end{aligned}$$

By the above observation, all the forms $\varphi_0\psi$, $\varphi_1\psi$, $\varphi_1 d\psi$ are in $\mathcal{N}^{\cdot}$. It follows that $d\,(\varphi_1\psi) \in \mathcal{J}^{p+q}$ and, hence, $\varphi\psi \in \mathcal{J}^{p+q}$. In the same way one handles the case $\psi \in \mathcal{J}^q$.

To prove that concatenation is well defined on Ω^p, we need to verify that if $\varphi = \varphi' \operatorname{mod} \mathcal{J}^p$ and $\psi = \psi' \operatorname{mod} \mathcal{J}^q$ then $\varphi\psi = \varphi\psi \operatorname{mod} \mathcal{J}^{p+q}$.

Indeed, we have

$$\varphi\psi - \varphi'\psi' = \varphi\,(\psi - \psi') + (\varphi - \varphi')\,\psi',$$

and each of the terms in the right hand side belong to J^{p+q} by the first part. Finally, the Leibniz formula for equivalence classes follows from that for their representatives. ■

Frequently it will be convenient to use the following notation. For p-forms $\omega', \omega'' \in \mathcal{R}^p$ we write

$$\omega' \simeq \omega'' \ \text{ if } \ \omega' = \omega'' \bmod \mathcal{J}^p.$$

Then the equivalence classes of $\simeq$ are exactly the elements of Ω^p.

As it follows from Lemmas 2.6 and 3.1, $\omega \simeq 0$ implies $d\omega \simeq 0$, and if $\varphi \simeq 0$ or $\psi \simeq 0$ then $\varphi\psi \simeq 0$.

3.3 The space of ∂-invariant paths

Consider the following subspaces of $\mathcal{A}_p$

$$\boxed{\Omega_p \equiv \Omega_p (V, E) = \{v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1}\}} \tag{3.5}$$

that are ∂-invariant. Indeed, $v \in \Omega_p \Rightarrow \partial v \in \mathcal{A}_{p-1} \subset \Omega_{p-1}$. The elements of Ω_p are called *∂-invariant* p-paths.

We obtain a chain complex Ω.

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots$$

that, in fact, is dual to $\Omega^{\cdot}$. Indeed, by Lemma 2.5, the dual to the cochain complex $\Omega^{\cdot} = \mathcal{R}^{\cdot}/\mathcal{J}$ is

$$0 \leftarrow (\mathcal{J}^0)^{\perp} \overset{\partial}{\leftarrow} (\mathcal{J}^1)^{\perp} \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} (\mathcal{J}^{p-1})^{\perp} \overset{\partial}{\leftarrow} (\mathcal{J}^p)^{\perp} \overset{\partial}{\leftarrow} \dots$$

while by Lemma 2.6 we have

$$\begin{aligned} (\mathcal{J}^p)^{\perp} &= \left\{ v \in (\mathcal{N}^p)^{\perp} : \partial v \in \left(\mathcal{N}^{p-1}\right)^{\perp} \right\} \\ &= \{v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1}\} = \Omega_p. \end{aligned}$$

By construction we have $\Omega_0 = \mathcal{A}_0$ and $\Omega_1 = \mathcal{A}_1$ so that

$$\dim \Omega_0 = |V| \quad \text{and} \quad \dim \Omega_1 = |E|\,,$$

while in general $\Omega_p \subset \mathcal{A}_p$.

Note that, unlike the operation of concatenation of forms, the operation of product of paths is not invariant in spaces $\mathcal{A}_p$ or Ω_p.

Let us define the (co)homologies of the digraph (V, E) by

$$H^p\,(V, E) := H^p\,(\Omega) \;\text{ and }\; H_p\,(V, E) := H_p\,(\Omega)\,.$$

Recall that by Lemma 2.3 the spaces $H^p\,(V, E)$ and $H_p\,(V, E)$ are dual and, hence, their dimensions are the same. The values of $\dim H_p\,(V, E)$ can be regarded as invariants of the digraph (V, E).

By Lemma 2.4, we have for any $p \geq 0$

$$\dim H_p\,(\Omega) = \dim \Omega_p - \dim \partial \Omega_p - \dim \partial \Omega_{p+1}. \tag{3.6}$$

Let us define the Euler characteristic of the digraph (V, E) by

$$\chi\,(V, E) = \sum_{p=0}^{n} (-1)^p \dim H_p\,(\Omega) \tag{3.7}$$

provided n is so big that

$$\dim H_p(\Omega) = 0 \text{ for all } p > n. \tag{3.8}$$

We do not know if such an n exists for any finite digraph. Hence, $\chi(V, E)$ is defined only if the digraph satisfied (3.8).

If $\dim \Omega_p = 0$ for $p > n$, then by Lemma 2.2

$$\chi(V, E) = \sum_{p=0}^{n} (-1)^p \dim \Omega_p. \tag{3.9}$$

The definition (3.7) has an advantage that it may work even when all $\dim \Omega_p > 0$.

3.4 Computation of $\dim H^0$

Proposition 3.2 *We have*

$$\dim H^0(\Omega) = C, \tag{3.10}$$

where C is the number of (undirected) connected components of the digraph (V, E).

Proof. By definition,

$$H^0(\Omega) = \ker d|_{\Omega^0} = \left\{ f \in \Omega^0 : df \simeq 0 \right\}.$$

The condition $df \simeq 0$ means that $(df)_{ij} = 0$ for all $ij \in E$, that is, $f_i = f_j$ for all edges ij. The latter is equivalent to the fact that $f = \mathrm{const}$ on any connected component of (V, E), and the dimension of this space of functions is clearly C. ■

3.5 Some condition for $\dim \Omega^p = 0$

Proposition 3.3 *If $\dim \Omega^n \leq 1$ then $\dim \Omega^p = 0$ for all $p > n$.*

Proof. Assume $\dim \Omega^n = 0$. Any regular p-form $e^{i_0 \dots i_p}$ with $p > n$ is a concatenation of an n-form and a $(p-n)$-form:

$$e^{i_0 \dots i_p} = e^{i_0 \dots i_n} e^{i_n \dots i_p}.$$

Since $e^{i_0 \dots i_n} \simeq 0$ by hypothesis, it follows by Lemma 3.1 that also $e^{i_0 \dots i_p} \simeq 0$, whence $\dim \Omega^p = 0$.

Let now $\dim \Omega^n = 1$. We have

$$e^{i_0 \dots i_p} = e^{i_0 \dots i_n} e^{i_n \dots i_p} = e^{i_0 i_1} e^{i_1 \dots i_{n+1}} e^{i_{n+1} \dots i_p}$$

We claim that

$$\text{either} \quad e^{i_0 \dots i_n} \simeq 0 \quad \text{or} \quad e^{i_1 \dots i_{n+1}} \simeq 0, \tag{3.11}$$

which would imply that $e^{i_0 \dots i_p} \simeq 0$ and $\dim \Omega^p = 0$. Indeed, if (3.11) fails then both forms $e^{i_0 \dots i_n}$ and $e^{i_1 \dots i_{n+1}}$ belong to non-zero equivalence classes modulo $\mathcal{J}^n$. Since the latter has dimension 1, it follows that

$$e^{i_0 \dots i_n} = \text{const}\, e^{i_1 \dots i_{n+1}} \mod \mathcal{J}^n.$$

Clearly, this identity is only possible if $e^{i_0 \dots i_n} = e^{i_1 \dots i_{n+1}}$ whence $i_0 = i_1 \dots = i_{n+1}$, which contradicts the regularity. ∎

3.6 Poincaré lemma for star-like graphs

We say that a digraph (V, E) is *star-like* if there is a vertex $a \in V$ (called a star center) such that $ai \in E$ for all $i \neq a$. For example, here is a star-like digraph:

$$\begin{array}{ccc} a\bullet & \longrightarrow & \bullet \\ \downarrow & \searrow & \vdots \\ \bullet & \cdots & \bullet \end{array}$$

Clearly, a complete digraph is star-like.

Theorem 3.4 *If (V, E) is a star-like digraph, then $H_p(V, E) = \{0\}$ for any $p \geq 1$. Consequently, $\chi(V, E) = 1$.*

Proof. We prove that $H_p(V, E) = \{0\}$. For that we need to prove that if $v \in \Omega_p$ and $\partial v = 0$ then $v = \partial\omega$ for some $\omega \in \Omega_{p+1}$. Set $\omega = e_a v$. We claim that $\omega \in \mathcal{A}_{p+1}$. Since v is a linear combination of allowed paths $e_{i_0 \ldots i_p}$, it suffices to show that $e_{ai_0 \ldots i_p} \in \mathcal{A}_{p+1}$ for any allowed path $e_{i_0 \ldots i_p}$. Indeed, if $i_0 = a$ then $e_{ai_0 \ldots i_p} = 0 \in \mathcal{A}_{p+1}$. If $i_0 \neq a$ then $e_{ai_0 \ldots i_p}$ is allowed by the star condition. Hence, we have $\omega \in \mathcal{A}_{p+1}$.

By the product rule (1.14) we have

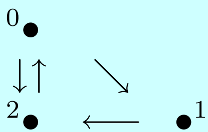
$$\partial\omega = \partial\left(e_a v\right) = v - e_a\partial v = v,$$

where we have used $\partial v = 0$. It follows that $\partial\omega \in \mathcal{A}_p$ and, hence, $\omega \in \Omega_{p+1}$, which finishes the proof of $H_p\left(V,E\right) = \{0\}$.

Since the graph (V,E) is connected, we have also $\dim H_0\left(V,E\right) = 1$ by Proposition 3.2. It follows that $\chi = 1$. ■

Remark. In a similar manner one can handle the *inverse* star-like graphs, that is, when the requirement $ai \in E$ in the definition of a start property is replaced by $ia \in E$. Using the right multiplication with e_a, one proves in the same way that the statement of Theorem 3.4 remains true for inverse star-like graph.

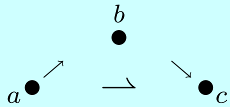
Example. The graph



is star like with the star center 0. Hence, $\chi = 1$.

3.7 Computation of $\dim \Omega_2$

Recall that $\dim \Omega_0 = \dim \mathcal{A}_0 = |V|$ and $\dim \Omega_1 = \dim \mathcal{A}_1 = |E|$. Here we compute $\dim \Omega_2$. We say that a pair $ac \in V \times V \setminus \mathrm{diag}$ is a *semi-edge* if ac is not an edge, but there is $b \in V$ such that both ab and bc are edges:



Denote by $\mathcal{S}$ the set of all semi-edges of a digraph (V, E).

Proposition 3.5 *We have*

$$\dim \Omega_2 = \dim \mathcal{A}_2 - |\mathcal{S}| = |E_2| - |S| . \tag{3.12}$$

Proof. Recall that

$$\mathcal{A}_2 = \mathrm{span}\{e_{abc} : abc \text{ is allowed}\}, \quad \dim \mathcal{A}_2 = |E_2|,$$

and

$$\Omega_2 = \{v \in \mathcal{A}_2 : \partial v \in \mathcal{A}_1\} = \{v \in \mathcal{A}_2 : \partial v = 0 \bmod \mathcal{A}_1\}.$$

If abc is allowed then ab and bc are edges, whence

$$\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} = -e_{ac} \operatorname{mod} \mathcal{A}_1.$$

If ac is an edge then $e_{ac} = 0 \operatorname{mod} \mathcal{A}_1$. If ac is not an edge then ac is a semi-edge, and in this case

$$\partial e_{abc} \neq 0 \mod \mathcal{A}_1.$$

For any $v \in \Omega_2$, we have

$$v = \sum_{\{abc \text{ is allowed}\}} v^{abc} e_{abc}$$

hence it follows that

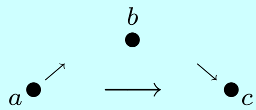
$$\partial v = - \sum_{\{abc\colon\ ac \text{ is semi-edge}\}} v^{abc} e_{ac} \mod \mathcal{A}_1.$$

The condition $\partial v = 0 \operatorname{mod} \mathcal{A}_1$ is equivalent to

$$\sum_{\{abc\colon\ ac \text{ is semi-edge}\}} v^{abc} e_{ac} = 0 \operatorname{mod} \mathcal{A}_1,$$

which is equivalent to $\sum_b v^{abc} = 0$ for all semi-edges ac. The number of these conditions is exactly $|\mathcal{S}|$, and they all are independent for different semi-edges, because a triple abc determines at most one semi-edge. Hence, Ω_2 is obtained from $\mathcal{A}_2$ by imposing $|\mathcal{S}|$ linearly independent conditions, which implies (3.12). ■

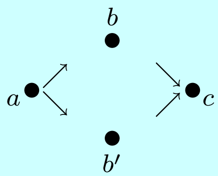
Let us call by a *triangle* a sequence of three distinct vertices $a, b, c \in V$ such that ab, bc, ac are edges:



Note that a triangle determines a 2-path $e_{abc} \in \Omega_2$ as $e_{abc} \in \mathcal{A}_2$ and

$$\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in \mathcal{A}_1.$$

Let us called by a *square* a sequence of four distinct vertices $a, b, b', c \in V$ such that $ab, bc, ab', b'c$ are edges:

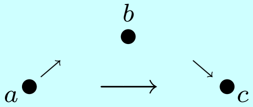


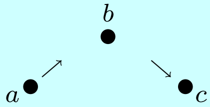
Note that a square determines a 2-path $v := e_{abc} - e_{ab'c} \in \Omega_2$ as $v \in \mathcal{A}_2$ and

$$\begin{aligned} \partial v &= (e_{bc} - e_{ac} + e_{ab}) - (e_{b'c} - e_{ac} + e_{ab'}) \\ &= e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in \mathcal{A}_1. \end{aligned}$$

Corollary 3.6 *If* (V, E) *contains no squares then* $\dim \Omega_2$ *is equal to the number of distinct triangles in* (V, E). *In particular, if* (V, E) *contains neither triangles nor squares then* $\dim \Omega_2 = 0$.

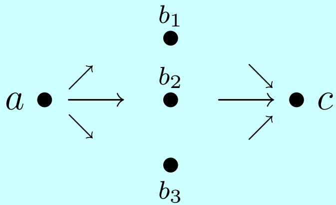
Proof. Let us split the family E_2 of allowed 2-paths into two subsets: an allowed path abc is of the first kind if ac is an edge and of the second kind otherwise:

1^{st} kind: 

, 2^{nd} kind: 

Clearly, the paths of the first kind are in one-to-one correspondence with triangles. Each path abc of the second kind determines a semi-edge ac. The mapping of $abc \mapsto ac$ from the paths of second kind to semi-edges is also one-to-one: if $abc \mapsto ac$ and $ab'c \mapsto ac$ then we obtain a square a, b, b', c which contradicts the hypotheses. Hence, the number of the path of the second kind is equal to $|\mathcal{S}|$, which implies that the number of the paths of the first kind is equal to $|E_2| - |\mathcal{S}|$, and so is the number of triangles. Comparing with (3.12) we finish the proof. ■

In the presence of squares one cannot relate directly $\dim \Omega_2$ to the number of squares and triangles as there may be a linear dependence between them. Indeed, in the following digraph



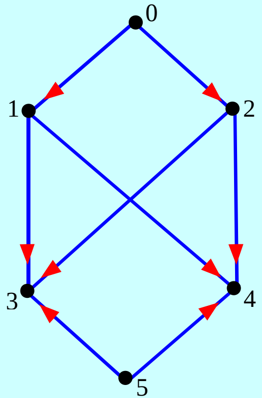
there are three squares a, b_i, b_j, c, which determine three paths

$$ab_1c - ab_2c, \quad ab_2c - ab_3c, \quad ab_3c - ab_1c$$

that are linearly dependent (the sum is equal to 0). In fact, $\dim \Omega_2 = 2$ as $|E_2| = 3$ and there is only one semi-edge ac.

3.8 An example of direct computation of $\dim H_p$

Consider the graph of 6 vertices $V = \{0, 1, 2, 3, 5\}$ with 8 edges $E = \{01, 02, 13, 14, 23, 24, 53, 54\}$.



Let us compute the spaces Ω_p and the homologies $H_p(\Omega)$. We have

$$\begin{aligned}
\Omega_0 &= \mathcal{A}_0 = \operatorname{span}\{e_0, e_1, e_2, e_3, e_4, e_5\}, \quad \dim \Omega_0 = 6 \\
\Omega_1 &= \mathcal{A}_1 = \operatorname{span}\{e_{01}, e_{02}, e_{13}, e_{14}, e_{23}, e_{24}, e_{53}, e_{54}\}, \quad \dim \Omega_1 = 8 \\
\mathcal{A}_2 &= \operatorname{span}\{e_{013}, e_{014}, e_{023}, e_{024}\}, \quad \dim \mathcal{A}_2 = 4.
\end{aligned}$$

The set of semi-edges is $\mathcal{S}=\{e_{03},e_{04}\}$ so that $\dim\Omega_2=\dim\mathcal{A}_2-|S|=2.$ The basis in Ω_2 can be easily spotted as each of two squares $0,1,2,3$ and $0,1,2,4$ determine a ∂-invariant 2-paths, whence

$$\Omega_2=\operatorname{span}\{e_{013}-e_{023},e_{014}-e_{024}\}.$$

Since there are no allowed 3-paths, we see that $\mathcal{A}_3=\Omega_3=\{0\}$. It follows that

$$\chi=\dim\Omega_0-\dim\Omega_1+\dim\Omega_2=6-8+2=0.$$

Let us compute $\dim H_2$ by (3.6):

$$\dim H_2=\dim\Omega_2-\dim\partial\Omega_2-\dim\partial\Omega_3=2-\dim\partial\Omega_2.$$

The image $\partial\Omega_2$ is spanned by two 1-paths

$$\begin{aligned}\partial\left(e_{013}-e_{023}\right) &= e_{13}-e_{03}+e_{01}-\left(e_{23}-e_{03}+e_{02}\right)=e_{13}+e_{01}-e_{23}-e_{02}\\ \partial\left(e_{014}-e_{024}\right) &= e_{14}-e_{04}+e_{01}-\left(e_{24}-e_{04}+e_{02}\right)=e_{14}+e_{01}-e_{24}-e_{02}\end{aligned}$$

that are clearly linearly independent. Hence, $\dim\partial\Omega_2=2$ whence $\dim H_2=0$. The dimension of H_1 can be computed similarly, but we can do easier using the Euler characteristic:

$$\dim H_0-\dim H_1+\dim H_2=\chi=0$$

whence $\dim H_1 = 1$.

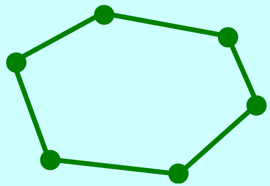
In fact, a non-trivial element of H_1 is determined by 1-path

$$v = e_{13} - e_{14} - e_{53} + e_{54}.$$

Indeed, by a direct computation $\partial v = 0$, so that $v \in \ker \partial|_{\Omega_1}$ while for v to be in $\operatorname{Im} \partial|_{\Omega_2}$ it should be a linear combination of $\partial (e_{013} - e_{023})$ and $\partial (e_{014} - e_{024})$, which is not possible since they do not have the term e_{54}.

3.9 Cycle graphs

We say that a digraph (V, E) is a (undirected) *cycle* it is connected and every vertex had the degree 2.



For a cycle graph we have $\dim H_0 = 1$ and

$$\dim \Omega_0 = |V| = |E| = \dim \Omega_1. \tag{3.13}$$

Proposition 3.7 *Let (V, E) be a cycle graph. Then*

$$\begin{aligned} \dim \Omega_p &= 0 \text{ for all } p \geq 3 \\ \dim H_p(\Omega) &= 0 \text{ for all } p \geq 2. \end{aligned}$$

If (V,E) *is a triangle or a square then*

$$\dim\Omega_2 = 1,\ \dim H_1(\Omega) = 0, \quad \chi = 1$$

whereas otherwise

$$\dim\Omega_2 = 0, \quad \dim H_1(\Omega) = 1, \quad \chi = 0.$$

Proof. Observe first that $\dim\Omega_2 \le 1$ will imply $\dim\Omega_p = 0$ for all $p \ge 3$ by Proposition 3.3, whence $\dim H_p = 0$ for $p \ge 3$. Hence, we need only to handle the cases $p = 1, 2$.

Using two equivalent definition of the Euler characteristic, we have

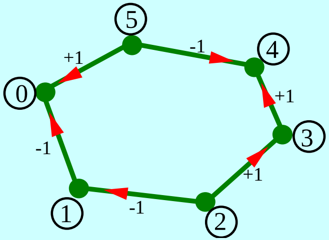
$$\begin{aligned}\chi &= \dim H_0 - \dim H_1 + \dim H_2 \\ &= \dim\Omega_0 - \dim\Omega_1 + \dim\Omega_2\end{aligned}$$

whence

$$\chi = \dim\Omega_2 = 1 - \dim H_1 + \dim H_2. \tag{3.14}$$

Assume first that (V,E) is neither a triangle nor a square. Then (V,E) contains neither a triangle nor a square. By Corollary 3.6 $\dim\Omega_2 = 0$ whence $\dim H_2 = 0$ and by (3.14) $\chi = 0$ and $\dim H_1 = 1$.

Let us construct an 1-path spanning H_1. For that let us identify V with $\mathbb{Z}_N$ where $N = |V|$ so that in the unoriented graph based on (V, E), the edges are $i\,(i+1)$. Hence, in the digraph (V, E) either $i\,(i+1)$ or $(i+1)\,i$ is an edge. Consider an allowed 1-path v with components $v^{i(i+1)} = 1$ if $i\,(i+1)$ is an edge, and $v^{(i+1)i} = -1$ if $(i+1)\,i$ is an edge (and all other components of v vanish):

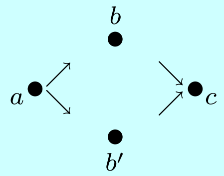


Since $v \neq 0$, v is not in $\operatorname{Im} \partial|_{\Omega_2}$. However, $v \in \ker \partial_{\Omega_1}$ because by construction $v^{i(i+1)} - v^{(i+1)i} \equiv 1$ whence for any i

$$(\partial v)^i = \sum_{j \in V} \left(v^{ji} - v^{ij}\right) = v^{(i-1)i} + v^{(i+1)i} - v^{i(i-1)} - v^{i(i+1)} = 1 - 1 = 0.$$

Let (V,E) be a triangle, say, with vertices a,b,c then $\dim \mathcal{A}_2 = 1$, $\mathcal{S} = \emptyset$ whence $\dim \Omega_2 = 1$ and $\chi = 1$. Note that in this case $\Omega_2 = \operatorname{span}\{e_{abc}\}$. Since a triangle is star-like, we have by Theorem 3.4 $\dim H_p = 0$ for all $p \geq 1$.

Let (V,E) be a square, say a,b,b',c:



Then

$$\mathcal{A}_2 = \operatorname{span}\{e_{abc}, e_{ab'c}\}, \quad \mathcal{S} = \{ac\}$$

whence $\dim \Omega_2 = 2 - 1 = 1$ and $\chi = 1$. Note that in this case

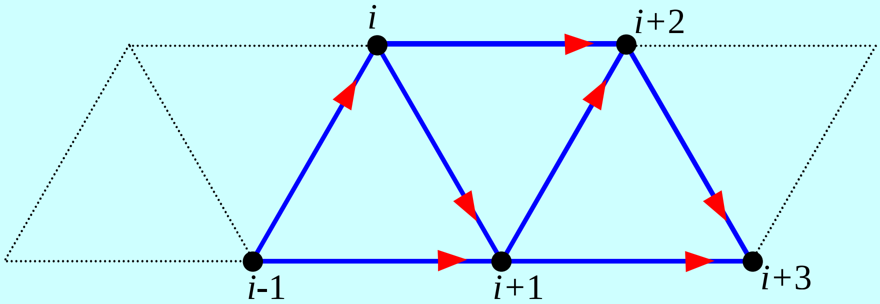
$$\Omega_2 = \operatorname{span}\{e_{abc} - e_{ab'c}\}.$$

For $v = e_{abc} - e_{ab'c}$ we have $\partial v = e_{bc} - e_{b'c} + e_{ab} - e_{ab'} \neq 0$ so that $\ker \partial|_{\Omega_2} = 0$. It follows that $\dim H_2 = 0$. Then by (3.14) $\dim H_1 = 0$. ∎

3.10 Examples of ∂-invariant paths

3.10.1 Snake and simplex

A *snake* of length p is a subgraph of $p+1$ vertices, say, $0, 1, ..., p$ such that $i\,(i+1)$ and $i\,(i+2)$ are edges, which is equivalent to say that any triple $i\,(i+1)\,(i+2)$ is a triangle.

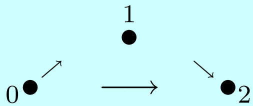


Any snake gives rise to a ∂-invariant p-path $v = e_{01...p}$. This path is obviously allowed, its boundary

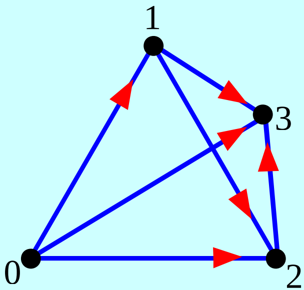
$$\partial v = e_{1...p} - e_{02...p} + e_{013...p} - ... + (-1)^p\, e_{01....p-1}$$

is also allowed, so that indeed $v \in \Omega_p$.

A *simplex* of dimension p is a subgraph of $p+1$ vertices, say $0, 1, ..., p$ so that any pair ij with $i < j$ is an edge. For example, a simplex of dimension 2 is a triangle



a simplex of dimension 3 is shown here:



Since the simplex contains a snake as a subgraph, the p-path $v = e_{01...p}$ is ∂-invariant also on a simplex.

3.10.2 Cylinder and hypercube

For any graph (V, E) consider its product with graph ${}^0\bullet \to \bullet^1$ that will be denoted by $(\widehat{V}, \widehat{E})$ where $\widehat{V} = V \times \{0, 1\}$ and the set of edges $\widehat{E}$ is defined by $(x, a) \rightsquigarrow (y, b)$ if and only if either $x \rightsquigarrow y$ in (V, E) and $a = b$ or $x = y$ and $a \rightsquigarrow b$:

$$\begin{array}{cccccc} 1\bullet & \dots & \overset{(x,1)}{\bullet} & \longrightarrow & \overset{(y,1)}{\bullet} & \dots \\ \uparrow & & \uparrow & & \uparrow & \\ 0\bullet & \dots & \overset{(x,0)}{\bullet} & \longrightarrow & \overset{(y,0)}{\bullet} & \dots \\ & & & & & \\ V & \dots & \underset{x}{\bullet} & \longrightarrow & \underset{y}{\bullet} & \dots \end{array}$$

The graph $(\widehat{V}, \widehat{E})$ is a *cylinder* over (V, E). We mark by the hat $\widehat{}$ all the notions related to the graph $(\widehat{V}, \widehat{E})$.

It will be convenient to identify $V \times \{0, 1\}$ with $V \sqcup V'$ where V' is a copy of V, and set the notation $(x, 0) \equiv x$ and $(x, 1) \equiv x'$. Define the operation of lifting paths from V to $\widehat{V}$ as follows. If $v = e_{i_0 \dots i_p}$ then $\widehat{v}$ is a $(p+1)$-path in $(\widehat{V},\widehat{E})$ defined by

$$\widehat{v}=\sum_{k=0}^{p}(-1)^k\, e_{i_0\dots i_k i'_k \dots i'_p}.$$

Clearly, if $i_0\dots i_p$ is allowed in (V,E) then $i_0\dots i_k i'_k\dots i'_p$ is allowed in $(\widehat{V},\widehat{E})$:

$$\begin{array}{ccccccccccc}
 & & & & \cdots & \overset{i'_k}{\bullet} & \longrightarrow & \overset{i'_{k+1}}{\bullet} & \longrightarrow & \cdots & \longrightarrow \overset{i'_p}{\bullet} \\
 & & & & & \uparrow & & \uparrow & & & \\
\overset{i_0}{\bullet} & \longrightarrow & \cdots & \longrightarrow & & \overset{i_k}{\bullet} & \longrightarrow & \overset{i_{k+1}}{\bullet} & & \cdots &
\end{array}$$

Extending by linearity this definition for a general p-path v on (V,E), we obtain that if $v\in\mathcal{A}_p$ then $\widehat{v}\in\widehat{\mathcal{A}}_{p+1}$.

Proposition 3.8 *If $v\in\Omega_p$ then $\widehat{v}\in\widehat{\Omega}_{p+1}$.*

Proof. We need to prove that if $v\in\mathcal{A}_p$ and $\partial v\in\mathcal{A}_{p-1}$ then $\partial\widehat{v}\in\widehat{\mathcal{A}}_p$. Let us prove first some properties of the lifting. For any path v in (V,E) define its image v' in (V',E') by

$$\left(e_{i_0\dots i_p}\right)'=e_{i'_0\dots i'_p}.$$

Let us show first that for any p-path u and q-path v on (V,E), the following identity holds:

$$\widehat{uv} = \widehat{u}v' + (-1)^{p+1} u\widehat{v} \tag{3.15}$$

It suffices to prove it for $u = e_{i_0\ldots i_p}$ and $v = e_{j_0\ldots j_q}$. Then $uv = e_{i_0\ldots i_p j_0\ldots j_q}$ and

$$\begin{aligned}\widehat{uv} &= \sum_{k=0}^{p} (-1)^k e_{i_0\ldots i_k i'_k\ldots i'_p j'_0\ldots j'_q} + \sum_{k=0}^{q} (-1)^{k+p+1} e_{i_0\ldots i_p j_0\ldots j_k j'_k\ldots j'_q} \\ &= \widehat{u}v' + (-1)^{p+1} u\widehat{v}.\end{aligned}$$

Now let us show that, for any p-path v with $p \geq 1$,

$$\partial\widehat{v} = -\widehat{\partial v} + v' - v. \tag{3.16}$$

It suffices to prove it for $v = e_{i_0\ldots i_p}$, which will be done by induction in p. For $p = 1$ write $v = e_{ab}$ so that $\widehat{v} = e_{aa'b'} - e_{abb'}$ and

$$\begin{aligned}\partial\widehat{v} &= (e_{a'b'} - e_{ab'} + e_{aa'}) - (e_{bb'} - e_{ab'} + e_{ab}) \\ &= e_{aa'} - e_{bb'} + e_{a'b'} - e_{ab} \\ &= -(e_b - e_a)^{\widehat{}} + v' - v \\ &= -\widehat{\partial v} + v' - v.\end{aligned}$$

For $p>1$ write $v=ue_{i_p}$ where $u=e_{i_0\dots i_{p-1}}$. Using (3.15) and the inductive hypothesis with the $(p-1)$-path u we obtain

$$\begin{aligned}
\partial\widehat{v} &= \partial\left(\widehat{u}e_{i_p'}+(-1)^p\,ue_{i_pi_p'}\right)\\
&= (\partial\widehat{u})\,e_{i_p'}+(-1)^{p+1}\,\widehat{u}+(-1)^p\,(\partial u)\,e_{i_pi_p'}+u\left(e_{i_p'}-e_{i_p}\right)\\
&= \left[-\widehat{\partial u}+u'-u\right]e_{i_p'}+(-1)^{p+1}\,\widehat{u}+(-1)^p\,(\partial u)\,e_{i_pi_p'}+ue_{i_p'}-v\\
&= -(\widehat{\partial u})e_{i_p'}+v'+(-1)^{p+1}\,\widehat{u}+(-1)^p\,(\partial u)\,e_{i_pi_p'}-v
\end{aligned}$$

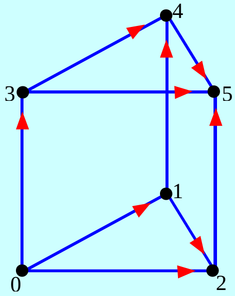
On the other hand,

$$\begin{aligned}
\widehat{\partial v} &= \left((\partial u)\,e_{i_p}+(-1)^p\,u\right)^{\wedge}\\
&= (\widehat{\partial u})e_{i_p'}+(-1)^{p-1}\,(\partial u)\,e_{i_pi_p'}+(-1)^p\,\widehat{u},
\end{aligned}$$

whence it follows that $\partial\widehat{v}+\widehat{\partial v}=v'-v$, which finishes the proof of (3.16).

Finally, if $v\in\mathcal{A}_p$ and $\partial v\in\mathcal{A}_{p-1}$ then v' and $\widehat{\partial v}$ belong to $\widehat{\mathcal{A}}_p$ whence it follows from (3.16) also $\partial\widehat{v}\in\widehat{\mathcal{A}}_p$. This proves that $\widehat{v}\in\widehat{\mathcal{A}}_{p+1}$. ■

Example. The cylinder over a triangle 012 is the following graph:



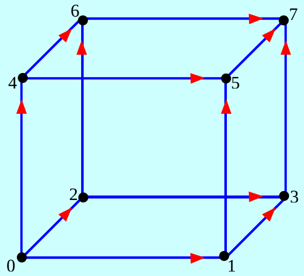
Since 2-path e_{012} is ∂-invariant on the triangle, lifting it to the cylinder, we obtain a ∂-invariant 3-path $e_{00'1'2'} - e_{011'2'} + e_{0122'}$, that can be written in the form $e_{0345} - e_{0145} + e_{0125}$.

Example. The cylinder over the graph ${}^0\bullet \rightarrow \bullet^1$ is a square

$$\begin{array}{ccc} {}^2\bullet & \longrightarrow & \bullet^3 \\ \uparrow & & \uparrow \\ {}^0\bullet & \longrightarrow & \bullet^1 \end{array}$$

Lifting a ∂-invariant 1-path $e_{01} \in \Omega_1$ we obtain a ∂-invariant 2-path on the square $e_{00'1'} - e_{011'}$ that we rewrite in the form $e_{023} - e_{013}$.

The cylinder over a square is a 3-cube:



Lifting the 2-path $e_{023} - e_{013}$ we obtain a ∂-invariant 3-path

$$e_{00'2'3'} - e_{022'3'} + e_{0233'} - e_{00'1'3'} + e_{011'3'} - e_{0133'}$$

that we can rewrite in the form

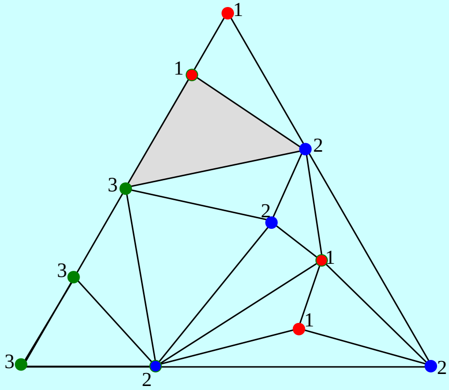
$$e_{0467} - e_{0267} + e_{0237} - e_{0457} + e_{0157} - e_{0137}.$$

Similarly, any binary hypercube of dimension p determines a ∂-invariant p-path that is an alternating sum of $p!$ terms.

3.11 Lemma of Sperner revisited

Consider a triangle ABC on the place, and its triangulation T. The set S of vertices of T is colored with three colors $1, 2, 3$ in such a way that

- the vertices A, B, C are colored with $1, 2, 3$ respectively;
- each vertex on an edge of ABC is colored only with one of the two colors of the ends of its edge.

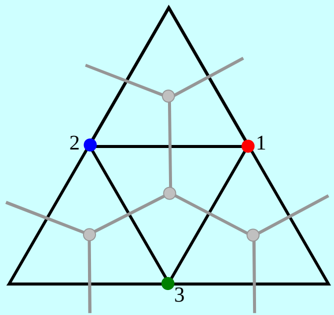


The classical lemma of Sperner says that then there exists a 3-color triangle from T, that is, a triangle, whose vertices are colored with the three different colors. Moreover, the number of such triangles is odd.

We give here a new proof using the boundary operator ∂ for 1-paths. Let us first do some reduction. Firstly, let us change the triangulation T so that there are no vertices on the edges AB, AC, BC except for A, B, C. Indeed, if X is a vertex on AB then move X a bit inside the triangle ABC. This gives rise to a new triangle in the triangulation T that is formed by X and its former neighbors, say Y and Z, on the edge AB. However, since all X, Y, Z are colored with two colors, no 3-color triangle appears after this move. By induction, we remove all vertices from the edges of ABC.

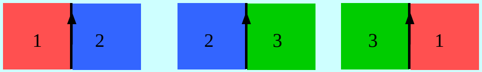
Secondly, we project the triangle ABC and the triangulation T onto the sphere $\mathbb{S}^2$ and add to the set T the triangle ABC itself from the other side of the sphere. Then we obtain a triangulation of $\mathbb{S}^2$, denote it again by T, and we need to prove that the number of 3-color triangles is *even*. Indeed, since we know that one of the triangles, namely, ABC is 3-color, this would imply that the number of 3-color triangles in the original triangulation was odd.

Let us regard T as a graph on $\mathbb{S}^2$ and construct a dual graph V. Chose at each face of T a point and regard them as vertices of the dual graph V. The vertices in V are connected if the corresponding triangles in T have a common edge. Then the faces of V are in one-to-one correspondence to the vertices of T.

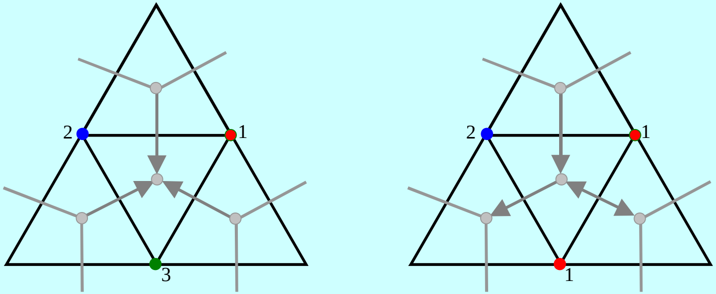


Hence, given a graph V on $\mathbb{S}^2$ such that each vertex has degree 3 and each face is colored with one of the colors $1, 2, 3$, prove that the number of 3-color vertices (that is, the vertices, whose adjacent faces have all three colors) is even.

Let us make V into a digraph as follows. Each edge ξ in V has two adjacent faces. Choose the orientation on ξ so that the color from the left hand side and the color from the right hand side of ξ form one of the following pairs: $(1,2)\,,(2,3)\,,(3,1)$ (if the colors are the same then allow both orientations of ξ).



For example:



Denote by E the set of the oriented edges and set $v = \sum_{\{ab \in E\}} e_{ab}$. We have for any $a \in V$

$$(\partial v)_a = \sum_b v^{ba} - \sum_c v^{ac} = \#\{\text{incoming edges}\} - \#\{\text{outcoming edges}\}.$$

If a is 3-color, then either all three edges at a are outcoming or all are incoming whence

$$(\partial v)_a = \pm 3 = 1 \bmod 2.$$

Otherwise $(\partial v)_a = 0$ (see the above pictures). Denoting by n the number of 3-color vertices, we obtain

$$(\partial v, 1) = \sum_{a \in V} (\partial v)_a = n \bmod 2.$$

On the other hand, $(\partial v, 1) = (v, d1) = 0$ whence we conclude that

$$n = 0 \bmod 2.$$

Brief summary

A p-path on finite set V is a formal linear combination of the elementary p-paths $e_{i_0...i_p} \equiv i_0...i_p$, where $i_k \in V$, and the linear space of all p-paths is denoted by Λ_p. For any $v \in \Lambda_p$ we write

$$v = \sum_{i_0,\dots,i_p \in V} v^{i_0\dots i_p} e_{i_0\dots i_p}.$$

The boundary operator $\partial : \Lambda_{p+1} \to \Lambda_p$ is defined by

$$\partial e_{i_0\dots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \, e_{i_0\dots\widehat{i_q}\dots i_{p+1}}.$$

It satisfies $\partial^2 = 0$.

The product of two paths $u \in \Lambda_p$ and $v \in \Lambda_q$ is a paths $uv \in \Lambda^{p+q+1}$ defined by

$$(uv)^{i_0\dots i_p j_0\dots j_q} = u^{i_0\dots i_p} v^{j_0\dots j_q}.$$

It satisfies the product rule

$$\partial\,(uv) = (\partial u)v + (-1)^{p+1}\, u\partial v.$$

Let I_p be the subspace of Λ_p that is spanned by irregular $e_{i_0...i_p}$ (a path $i_0...i_p$ is irregular if $i_k = i_{k+1}$ for some k). Then the spaces I_p are invariant for ∂ and for product, which allows to define ∂ and product on the quotient spaces $\mathcal{R}_p = \Lambda_p/I_p$. We identify the elements of $\mathcal{R}_p$ with their representatives that are regular p-paths. Then $e_{i_0...i_p}$ with irregular $i_0...i_p$ are treated as zeros.

Let (V, E) be a digraph, that is, $E \subset V \times V \setminus \text{diag}$ is a set of directed edges. An elementary regular path $e_{i_0...i_p}$ is called allowed if $i_k i_{k+1} \in E$ for all k, and non-allowed otherwise.

Let $\mathcal{A}_p$ be a subspace of $\mathcal{R}_p$ that is spanned by all allowed $e_{i_0...i_p}$. The elements of $\mathcal{A}_p$ are called allowed p-paths. For example, $\mathcal{A}_0$ consists of linear combinations of all vertices, and $\mathcal{A}_1$ consists of linear combinations of all edges.

In general, the spaces $\mathcal{A}_p$ is not ∂-invariant, so we introduce smaller spaces

$$\boxed{\Omega_p = \{v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1}\}},$$

that are ∂-invariant, that is, $\partial\Omega_p \subset \Omega_{p-1}$. The elements of Ω_p are called ∂-invariant paths.

Note that $\Omega_0 = \mathcal{A}_0$ and $\Omega_1 = \mathcal{A}_1$, but for $p \geq 2$, Ω_p can actually be smaller than $\mathcal{A}_p$. We obtain a chain complex

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots$$

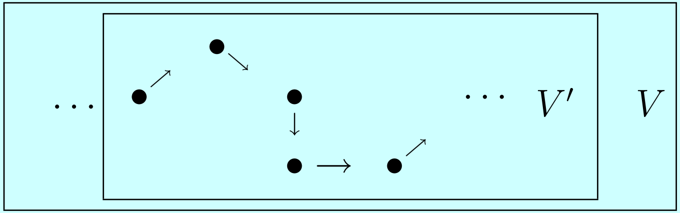
whose homologies $H_p(\Omega) = H_p(V,E)$ are the subject for our study. So far we know that $\dim H_0(V,E)$ is equal to the number of connected components of the graph.

4 Surgery of digraphs

4.1 Homologies of subgraphs

Let (V',E') be a subgraph of (V,E) in the sense that $V' \subset V$ and $E' \subset E$. Let us mark by the dash "′" all the notation related to the graph (V',E') rather than to (V,E).

As it was already observed, $\mathcal{R}'_p \subset \mathcal{R}_p$ and ∂ commutes with this inclusion. It is also obvious that if $e_{i_0 \dots i_p}$ is an allowed path in (V',E') then it is also allowed in (V,E), whence $\mathcal{A}'_p \subset \mathcal{A}_p$.



By the definition (3.5) of Ω_p, we obtain that $\Omega'_p \subset \Omega_p$ and ∂ commutes with this inclusion. Consequently, the chain complex

$$0 \leftarrow \Omega'_0 \overset{\partial}{\leftarrow} \Omega'_1 \overset{\partial}{\leftarrow} \Omega'_2 \overset{\partial}{\leftarrow} \Omega'_3 \overset{\partial}{\leftarrow} \dots$$

is a sub-complex of

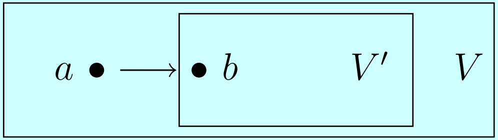
$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \Omega_2 \overset{\partial}{\leftarrow} \Omega_3 \overset{\partial}{\leftarrow} \dots$$

By Proposition 2.7 (cf. (2.17)) we obtain that the following long sequence is exact:

$$0 \leftarrow H_0(\Omega/\Omega') \leftarrow H_0(\Omega) \leftarrow H_0(\Omega') \leftarrow \cdots \leftarrow H_p(\Omega/\Omega') \leftarrow H_p(\Omega) \leftarrow H_p(\Omega') \leftarrow H_{p+1}(\Omega/\Omega') \leftarrow \dots \tag{4.1}$$

4.2 Removing a vertex of degree 1

Theorem 4.1 *Suppose that a graph* (V, E) *has a vertex* a *such that there is only one outcoming edge* ab *from* a *and no incoming edges to* a*. Let* $V' = V \setminus \{a\}$ *and* $E' = E \setminus \{ab\}$*.*



Then $H_p(V, E) \cong H_p(V', E')$ *for all* $p \geq 0$*.*

Remark. The same is true if the edge ab in the statement is replaced by ba.

Proof. Let us first prove that $\Omega'_p = \Omega_p$ for $p \geq 2$. Since always $\Omega'_p \subset \Omega_p$, it suffices to prove the opposite inclusion $\Omega_p \subset \Omega'_p$. Let us first show that, for all $p \geq 2$,

$$\Omega_p \subset \mathcal{A}'_p, \tag{4.2}$$

that is

$$v \in \mathcal{A}_p \text{ and } \partial v \in \mathcal{A}_{p-1} \Rightarrow v \in \mathcal{A}'_p.$$

Every elementary allowed p-path on (V, E) either is allowed on (V', E') or starts with ab, which implies that v can be represented in the form

$$v = e_{ab}u + v',$$

where $v' \in \mathcal{A}'_p$, while $u \in \mathcal{A}'_{p-2}$ is a linear combination of the paths $e_{i_0 \ldots i_{p-2}} \in \mathcal{A}'_{p-2}$ with $i_0 \neq b$. It follows that

$$\partial v = (e_b - e_a)\, u + e_{ab}\partial u + \partial v'. \tag{4.3}$$

Note that $e_a u$ is a linear combination of the elementary paths $e_{ai_0 \ldots i_{p-2}}$ where $i_0, ..., i_{p-2} \in V'$ and $i_0 \neq b$. Since ai_0 is not an edge, those elementary paths are not allowed in (V, E). No other terms in the right hand side of (4.3) has $e_{ai_0 \ldots i_{p-2}}$-component. Since ∂v is allows in (V, E), its $e_{ai_0 \ldots i_{p-2}}$-component is 0, which is only possible if $e_a u = 0$, that is, $u = 0$. It follows that $v = v' \in \mathcal{A}'_p$, which finishes the proof of (4.2).

Let us now show that $\Omega_p \subset \Omega'_p$ for all $p \geq 2$. Indeed, if $v \in \Omega_p$ then by definition $\partial v \in \mathcal{A}_{p-1}$ and by (4.2) $v \in \mathcal{A}'_p$, which together imply $\partial v \in \mathcal{A}'_{p-1}$. It follows that $v \in \Omega'_p$. Consequently, we have proved that

$$\Omega_p = \Omega'_p \text{ for all } \; p \geq 2. \tag{4.4}$$

It follows that, for all $p \geq 2$,

$$\dim H_p(\Omega') = \dim H_p(\Omega). \tag{4.5}$$

For $p = 0$ this identity also true as the number of connected components of (V, E) and (V', E') is the same.

We are left to treat the case $p = 1$. Observe that

$$\Omega_0 = \Omega_0' + \mathrm{span}\{e_a\} \quad \text{and} \quad \Omega_1 = \Omega_1' + \mathrm{span}\{e_{ab}\}. \tag{4.6}$$

By (4.4) and (4.6) the cochain complex Ω/Ω' has the form

$$0 \longleftarrow \mathrm{span}\{e_a\} \overset{\partial}{\longleftarrow} \mathrm{span}\{e_{ab}\} \longleftarrow 0 = \Omega_2/\Omega_2'.$$

Since

$$\partial e_{ab} = e_b - a_a = -e_a \mod \Omega_0',$$

it follows that $\mathrm{Im}\,\partial|_{\Omega_1/\Omega_1'} = \mathrm{span}\{e_a\}$, while $\ker \partial|_{\Omega_1/\Omega_1'} = 0$, whence

$$\dim H_0(\Omega/\Omega') = \dim H_1(\Omega/\Omega') = 0.$$

By (4.1) we have a long exact sequence

$$H_0(\Omega/\Omega') = 0 \longleftarrow H_1(\Omega) \longleftarrow H_1(\Omega') \longleftarrow 0 = H_1(\Omega/\Omega')$$

which implies that

$$\dim H_1(\Omega) = \dim H_1(\Omega'),$$

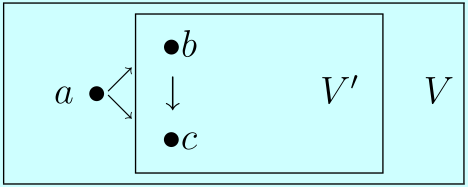
thus finishing the proof. ■

Corollary 4.2 *Let a digraph* (V, E) *be a tree (that is, the underlying undirected graph is a tree). Then* $H_p(V, E) = 0$ *for all* $p \geq 1$.

Proof. Induction in the number of edges $|E|$. If $|E| = 0$ then the claim is obvious. If $|E| > 0$ then there is a vertex $a \in V$ of degree 1 (indeed, if this is not the case then moving along undirected edges allows to produce a cycle). Removing this vertex and the adjacent edge, we obtain a tree (V', E') with $|E'| < |E|$. By the inductive hypothesis $H_p(V', E') = 0$ for $p \geq 1$, whence by Theorem 4.1 also $H_p(V, E) = 0$. ■

4.3 Removing of a vertex of degree 2

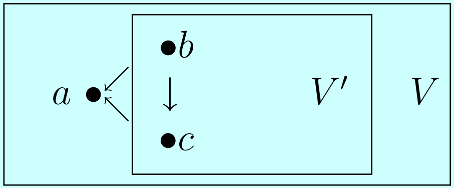
Theorem 4.3 *Suppose that a graph (V, E) has a vertex a with two outcoming edges ab and ac and no incoming edges. Assume also that either bc or cb (or both) is an edge:*



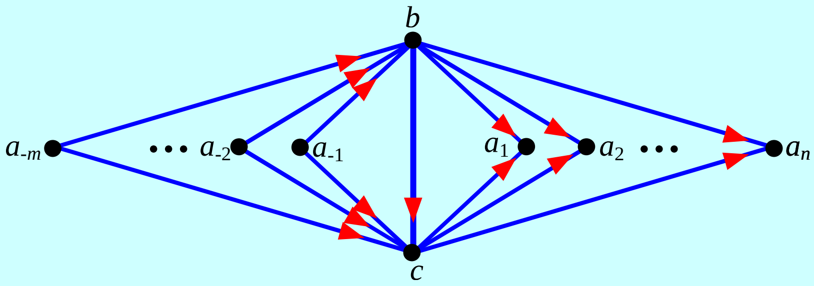
Let $V' = V \setminus \{a\}$ and $E' = E \setminus \{ab, ac\}$. Then, for any $p \geq 0$,

$$\dim H_p(V, E) = \dim H_p(V', E'). \tag{4.7}$$

The same is true if the vertex a has two incoming edges ba and ca and no outcoming edges, while either bc or cb is an edge:



Example. Consider a graph (V, E) as on the picture:



Each of the vertices a_i satisfies the hypotheses of Theorem 4.3 (either with incoming or outcoming edges). Removing these vertices successively, we see that all the homologies of (V, E) are the same as those of the remaining graph ${}^b\bullet \to \bullet^c$. Since it is a star-like graph, we obtain $\dim H_0 = 1$ and $\dim H_p = 0$ for all $p \geq 1$. In particular, $\chi = 1$.

Proof of Theorem 4.3. Without loss of generality assume that bc is an edge. Since the number of connected components of the graphs (V, E) and (V', E') is obviously the same, the identity (4.7) for $p = 0$ follows from Proposition 3.2.

For $p \geq 1$ consider the long exact sequence (4.1), that is,

$$\ldots \leftarrow H_p\left(\Omega/\Omega'\right) \leftarrow H_p\left(\Omega\right) \leftarrow H_p\left(\Omega'\right) \leftarrow H_{p+1}\left(\Omega/\Omega'\right) \leftarrow \ldots,$$

which implies the identity

$$\dim H_p\left(\Omega\right) = \dim H_p\left(\Omega'\right) \quad \text{for } p \geq 1,$$

if we prove that

$$\dim H_p\left(\Omega/\Omega'\right) = 0 \quad \text{for } p \geq 1. \tag{4.8}$$

The condition (4.8) means that

$$\ker \partial|_{\Omega_p/\Omega'_p} \subset \operatorname{Im} \partial|_{\Omega_{p+1}/\Omega'_{p+1}}$$

that is, if

$$v \in \Omega_p \quad \text{and} \quad \partial v = 0 \bmod \Omega'_{p-1} \tag{4.9}$$

then there exists $\omega \in \Omega_{p+1}$ such that

$$\partial \omega = v \bmod \Omega'_p. \tag{4.10}$$

In fact, it suffices to have $\omega \in \mathcal{A}_{p+1}$ because then the identity (4.10) implies $\partial \omega \in \Omega_p$ and, hence, $\omega \in \Omega_{p+1}$.

Consider first the case $p = 1$. Every 1-path $v \in \Omega_1$ has the form

$$v = v^{ab} e_{ab} + v^{ac} e_{ac} + v'$$

where $v' \in \mathcal{A}_1' = \Omega_1'$. Since $(\partial v)^a = 0$ and

$$(\partial v)^a = \sum_k \left(v^{ka} - v^{ak}\right) = -\left(v^{ab} + v^{ac}\right),$$

it follows that

$$v^{ab} + v^{ac} = 0,$$

whence

$$v = v^{ab} \left(e_{ab} - e_{ac}\right) \operatorname{mod} \Omega_1'.$$

For 2-form $\omega = v^{ab} e_{abc}$ we have

$$\partial \omega = v^{ab} \left(e_{bc} - e_{ac} + e_{ab}\right) = v \operatorname{mod} \Omega_1',$$

which finishes the proof of (4.8) in the case $p = 1$.

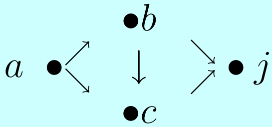
Consider now the case $p = 2$. For any $v \in \Omega_2$ and any vertex $j \neq a, b, c$, we have

$$(\partial v)^{aj} = \sum_{k \in V} \left(v^{kaj} - v^{akj} + v^{ajk}\right) = -\left(v^{abj} + v^{acj}\right),$$

because there are no incoming edges at a and only two outcoming edges ab and ac. By (4.9) we have $(\partial v)^{aj} = 0$ whence

$$v^{abj} + v^{acj} = 0. \tag{4.11}$$

Denote by J the set of vertices j such that either $j = c$ or both bj and cj are edges:



We claim that

$$j \in V \setminus J \Rightarrow v^{abj} = v^{acj} = 0.$$

If $j = a$ or b then this is trivial. Otherwise, $j \neq a, b, c$ and either bj or cj is not an edge. If bj is not an edge then $v^{abj} = 0$ whence by (4.11) also $v^{acj} = 0$, and the same is valid if cj is not an edge.

It follows that v can be represented in the form

$$\begin{aligned} v &= \sum_{j \in J} v^{abj} e_{abj} + \sum_{j \in J} v^{acj} e_{acj} + v' \\ &= \sum_{j \in J} v^{abj} \left(e_{abj} - e_{acj}\right) + v', \end{aligned} \tag{4.12}$$

where $v' \in \mathcal{A}_2'$. In the last line we have used (4.11) for $j \neq c$ and $e_{acj} = 0$ for $j = c$.

For any $j \in J$, we have

$$\begin{aligned} \partial \left(e_{abj} - e_{acj} \right) &= \left(e_{bj} - e_{aj} + e_{ab} \right) - \left(e_{cj} - e_{aj} + e_{ac} \right) \\ &= e_{bj} - e_{cj} + e_{ab} - e_{ac} \in \mathcal{A}_1. \end{aligned}$$

Since $\partial v \in \mathcal{A}_1'$, it follows from (4.12) that $\partial v' \in \mathcal{A}_1$. Since $v' \in \mathcal{A}_2'$, we conclude that $\partial v' \in \mathcal{A}_1'$ whence $v' \in \Omega_2'$. Therefore,

$$v = \sum_{j \in J} v^{abj} \left(e_{abj} - e_{acj} \right) \mod \Omega_2'. \tag{4.13}$$

Since $\left(\partial v \right)^{ab} = 0$ and

$$\left(\partial v \right)^{ab} = \sum_{j \in V} \left(v^{jab} - v^{ajb} + v^{abj} \right) = \sum_{j \in J} v^{abj},$$

it follows that

$$\sum_{j \in J} v^{abj} = 0. \tag{4.14}$$

Consider the 3-path

$$\omega = \sum_{j\in J} v^{abj} e_{abcj}.$$

For any $j \in J\setminus\{c\}$ we have $e_{abcj} \in E_3$ whereas for $j = c$ we have $e_{abcj} = 0$. Hence, $\omega \in \mathcal{A}_3$. Since

$$\partial e_{abcj} = e_{bcj} - e_{acj} + e_{abj} - e_{abc}$$

and $e_{bcj} \in \Omega_2'$, it follows from (4.13) and (4.14) that

$$\partial\omega = \sum_{j\in J} v^{abj}\left(e_{abj} - e_{acj} - e_{abc}\right) = v \mod \Omega_2',$$

which finishes the proof in the case $p = 2$.

Consider the case $p \geq 3$. Any p-path $v \in \Omega_p$ has the form

$$v = \sum_{\gamma\in E_{p-2}'} v^{ab\gamma} e_{ab\gamma} + \sum_{\gamma\in E_{p-2}'} v^{ac\gamma} e_{ac\gamma} + v', \tag{4.15}$$

where $v' \in \mathcal{A}'_p$. Using product of paths and (1.14), we obtain

$$\begin{aligned} \partial e_{ab\gamma} &= \partial\left(e_{ab}e_\gamma\right) = \left(\partial e_{ab}\right)e_\gamma + e_{ab}\partial e_\gamma \\ &= \left(e_b - e_a\right)e_\gamma + e_{ab}\partial e_\gamma \\ &= e_{b\gamma} - e_{a\gamma} + e_{ab}\partial e_\gamma \end{aligned}$$

and a similar formula for $\partial e_{ac\gamma}$, whence it follows that

$$\partial v = \sum_{\gamma \in E'_{p-2}} \left(v^{ab\gamma}e_{b\gamma} + v^{ac\gamma}e_{c\gamma} - \left(v^{ab\gamma} + v^{ac\gamma}\right)e_{a\gamma}\right) \tag{4.16}$$

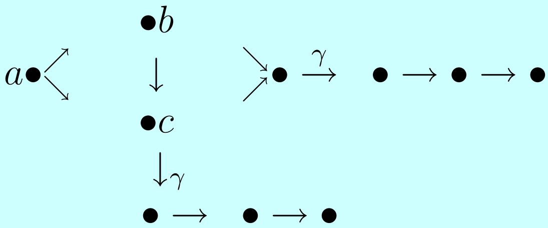
$$+e_{ab}\sum_{\gamma \in E'_{p-2}} v^{ab\gamma}\partial e_\gamma + e_{ac}\sum_{\gamma \in E'_{p-2}} v^{ac\gamma}\partial e_\gamma + \partial v'. \tag{4.17}$$

Let $\gamma = \gamma_0 ... \gamma_{p-2}$ where $\gamma_i \in V'$. We claim that if $\gamma_0 \neq c$ then

$$v^{ab\gamma} + v^{ac\gamma} = 0. \tag{4.18}$$

If $\gamma_0 = a$ or b then we have trivially $v^{ab\gamma} = v^{ac\gamma} = 0$. Otherwise let us look at the component $e_{a\gamma}$ in (4.16)-(4.17). Since it occurs only once, namely in the last term of (4.16), while $(\partial v)^{a\gamma} = 0$, we obtain (4.18). Note that if $\gamma_0 = c$ then $v^{ac\gamma} = 0$ but $v^{ab\gamma}$ may be non-zero, so that (4.18) may not be valid.

Denote by Γ the set of paths $\gamma \in E'_{p-2}$ such that either $\gamma_0 = c$ or both $b\gamma$ and $c\gamma$ are in E'_{p-1}:



It follows from (4.18) that if $\gamma \in E'_{p-2} \setminus \Gamma$ then both $v^{ab\gamma}$ and $v^{ac\gamma}$ vanish. Indeed, since $\gamma_0 \neq c$, we have (4.18). Since $b\gamma$ or $c\gamma$ is not in E'_{p-1}, one of the terms $v^{ab\gamma}, v^{ac\gamma}$ vanish, whence the second term also vanish by (4.18). Hence, the summation in (4.15) can be restricted to $\gamma \in \Gamma$:

$$\begin{aligned} v &= \sum_{\gamma \in \Gamma} v^{ab\gamma} e_{ab\gamma} + \sum_{\gamma \in \Gamma} v^{ac\gamma} e_{ac\gamma} + v' \\ &= \sum_{\gamma \in \Gamma} v^{ab\gamma} \left(e_{ab\gamma} - e_{ac\gamma} \right) + v', \end{aligned} \tag{4.19}$$

where in the second line we have used (4.18) for $\gamma_0 \neq c$ and $e_{ac\gamma} = 0$ for $\gamma_0 = c$. Set

$$u = \sum_{\gamma \in \Gamma} v^{ab\gamma} e_\gamma, \tag{4.20}$$

so that we can rewrite (4.19) in the form

$$v = (e_{ab} - e_{ac})\, u + v' \tag{4.21}$$

whence

$$\partial v = (e_b - e_c)\, u + (e_{ab} - e_{ac})\, \partial u + \partial v'. \tag{4.22}$$

Let us show that $\partial u = 0$. Indeed, since the $(p-1)$-paths ∂v, $e_b u$, $e_c u$, and $\partial v'$ are in $\mathcal{R}'_{p-1}$, it follows from (4.22) that also

$$(e_{ab} - e_{ac})\, \partial u \in \mathcal{R}'_{p-1}.$$

We have the identity

$$(e_{ab} - e_{ac})\, \partial u = \sum_{i_0, \ldots, i_{p-3} \in V'} (\partial u)^{i_0 \ldots i_{p-3}} \left(e_{abi_0 \ldots i_{p-3}} - e_{aci_0 \ldots i_{p-3}} \right).$$

If $i_0 \neq b$ then $e_{abi_0 \ldots i_{p-3}} \notin \mathcal{R}'_{p-1}$ so that the coefficient $(\partial u)^{i_0 \ldots i_{p-3}}$ must vanish. If $i_0 = b$ then $i_0 \neq c$, $e_{aci_0 \ldots i_{p-3}} \notin \mathcal{R}'_{p-1}$ and again $(\partial u)^{i_0 \ldots i_{p-3}} = 0$. Hence, we conclude that $\partial u = 0$, which was claimed.

It follows that

$$\partial v = (e_b - e_c)\, u + \partial v'.$$

Since $\partial v, e_b u, e_c u \in \mathcal{A}'_{p-1}$, it follows that $\partial v' \in \mathcal{A}'_{p-1}$ whence $v' \in \Omega'_{p-1}$. Substituting this into (4.21) , we obtain

$$v = (e_{ab} - e_{ac})\, u \mod \Omega'_p. \tag{4.23}$$

Consider a $(p+1)$-path $\omega = e_{abc} u$. Since

$$\partial (e_{abc} u) = (\partial e_{abc})\, u - e_{abc} \partial u = (e_{bc} - e_{ac} + e_{ab})\, u,$$

we have

$$\partial (e_{abc} u) = e_{bc} u + v \mod \Omega'_p. \tag{4.24}$$

We are left to show that $e_{bc} u \in \Omega'_p$. That $e_{bc} u \in \mathcal{A}'_p$ follows from the definition of Γ and (4.20). Next we have

$$\partial (e_{bc} u) = (e_b - e_c)\, u + e_{bc} \partial u = e_b u - e_c u \in \mathcal{A}'_{p-1},$$

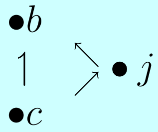
which implies that $e_{bc}u \in \Omega'_p$. From (4.24) we conclude that

$$\partial\left(e_{abc}u\right) = v \bmod \Omega'_p,$$

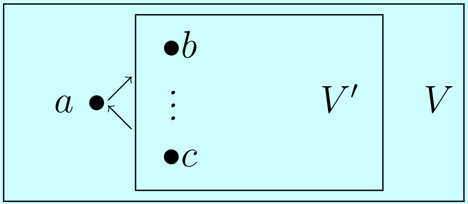
which finishes the proof. ■

4.4 Removing a vertex of degree $1+1$

Recall that a pair cb of distinct vertices on a graph is a *semi-edge* if cb is not an edge but there is a vertex j such that cjb is an edge:



Theorem 4.4 *Suppose that a graph (V, E) has a vertex a such that there is only one outcoming edge ab from a and only one incoming edge ca, where $b \neq c$. Let $V' = V \setminus \{a\}$ and $E' = E \setminus \{ab, ca\}$.*



Then the following is true.

(a) *For any $p \geq 2$,*

$$\dim H_p(V, E) = \dim H_p(V', E'). \tag{4.25}$$

(b) *If cb is an edge or a semi-edge in* (V', E') *then* (4.25) *is satisfied also for* $p = 0, 1$, *that is, for all* $p \geq 0$.

(c) *If cb is neither edge nor semi-edge in* (V', E'), *but* b, c *belong to the same connected component of* (V', E') *then*

$$\dim H_1(V, E) = \dim H_1(V', E') + 1$$

and $\dim H_0(V, E) = \dim H_0(V', E')$.

(d) *If* b, c *belong to different connected components of* (V', E') *then*

$$\dim H_1(V, E) = \dim H_1(V', E')$$

and $\dim H_0(V, E) = \dim H_0(V', E') - 1$.

Consequently, in the case (b), $\chi(V, E) = \chi(V', E')$, *whereas in the cases* (c) *and* (d), $\chi(V, E) = \chi(V', E') - 1$.

Example. Consider the graphs

$$(V,E)=\begin{array}{ccccc} & & b\bullet & & \\ & \nearrow & & \nwarrow & \\ a\bullet & & \downarrow & & \bullet d \\ & \nwarrow & & \nearrow & \\ & & c\bullet & & \end{array} \quad \text{and} \quad (V',E')=\begin{array}{ccc} b\bullet & & \\ & \nwarrow & \\ \downarrow & & \bullet d \\ & \nearrow & \\ c\bullet & & \end{array}$$

Since cb is semi-edge in (V',E') we have case (b) so that all homologies of (V,E) and (V',E') are the same. Removing further vertex d we obtain a digraph ${}^{b}\bullet \longrightarrow \bullet^{c}$ that will be denoted by (V'',E''). It is a star-like graph with all $\dim H_p(V'',E'')=0$ for $p\geq 1$. Since cb is neither edge nor semi-edge in (V'',E''), but the graph is connected, we conclude by case (c) that

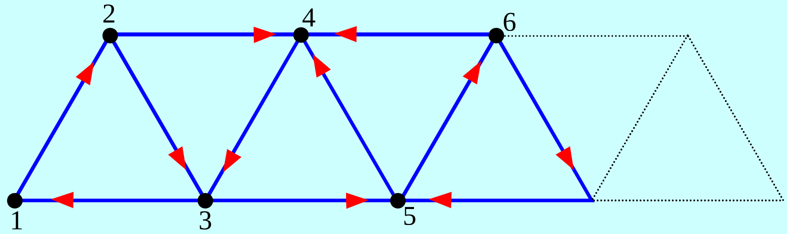
$$H_p(V',E')=H_p(V'',E'') \text{ for } p\geq 2,$$

and

$$\dim H_1(V',E')=\dim H_1(V'',E'')+1=1.$$

It follows that $\dim H_p(V,E)=0$ for $p\geq 2$ and $\dim H_1(V,E)=1$.

Example. Consider a digraph (a kind of anti-snake):



We start building this graph with $1 \to 2$. Since 21 is neither edge nor semi-edge, adding a path $2 \to 3 \to 1$ increases $\dim H_1$ by 1 and preserves other homologies. Since 23 is an edge, adding a path $2 \to 4 \to 3$ preserves all homologies. Since 34 is neither edge nor semi-edge, adding a path $3 \to 5 \to 4$ increases $\dim H_1$ by 1 and preserves other homologies. Similarly, adding a path $5 \to 6 \to 4$ preserves all homologies.

One can repeat this pattern arbitrarily many times. By doing so we construct a digraph with a prescribed value of $\dim H_1$ while keeping $\dim H_p = 0$ for all $p \geq 2$. Consequently, the Euler characteristic χ can take arbitrary negative values.

Proof of Theorem 4.4. *Proof of* (a). The identity (4.25) for $p \geq 2$ will follow if if prove that

$$\dim H_p\left(\Omega/\Omega'\right) = 0 \text{ for } p \geq 2. \tag{4.26}$$

In order to prove (4.26) it suffices to show that

$$\ker \partial|_{\Omega_p/\Omega'_p} = 0,$$

which is equivalent to

$$v \in \Omega_p, \quad \partial v = 0 \bmod \Omega'_{p-1} \Rightarrow v = 0 \bmod \Omega'_p. \tag{4.27}$$

By the definition (3.5) of Ω_p, (4.27) is equivalent to

$$v \in \mathcal{A}_p \text{ and } \partial v \in \mathcal{A}'_{p-1} \Rightarrow v \in \mathcal{A}'_p. \tag{4.28}$$

Hence, let us prove (4.28) for all $p \geq 2$.

Every elementary allowed p-path on (V, E) either contains one of the edges ab, ca or is allowed in (V', E'). Let us show that, for any v as in (4.28), its components $v^{\dots ab \dots}$ and $v^{\dots ca \dots}$ vanish, which will imply that

$v \in \mathcal{A}'_p$. Any such component can be written in the form $v^{\alpha ab\beta}$ or $v^{\gamma ca\beta}$ where α, β, γ are some paths. Consider the following cases. For further applications, in the Cases 1,2 we assume only that $v \in \Omega_p$ (whereas in the Case 3 v is as in (4.28)).

Case 1. Let us consider first the component $v^{\alpha ab\beta}$ where β is non-empty. If $\alpha ab\beta$ is not allowed in (V, E) then $v^{\alpha ab\beta} = 0$ by definition. Let $\alpha ab\beta$ be allowed in (V, E). The path $\alpha a\beta$ is not allowed because the only outcoming edge from a is ab. Since $\partial v \in \mathcal{A}_{p-1}$, we have

$$(\partial v)^{\alpha a\beta} = 0.$$

Let us show that

$$(\partial v)^{\alpha a\beta} = \pm v^{\alpha ab\beta}, \tag{4.29}$$

which will imply $v^{\alpha ab\beta} = 0$. Indeed, by (1.6) $(\partial v)^{\alpha a\beta}$ is the sum of the terms $\pm v^{\omega}$ where ω is a p-path that is obtained from $\alpha a\beta$ by inserting one vertex. Since there is no edge from a to β, the only way ω can be allowed is when $\omega = \alpha ab\beta$. Since for any other ω we have $v^{\omega} = 0$, we obtain (4.29), which implies that $v^{\alpha ab\beta} = 0$.

Case 2. In the same way one proves that $v^{\gamma ca\beta} = 0$ provided γ is non-empty, using the fact that the only incoming edge in a is ca.

Case 3. Consider now an arbitrary component $v^{\alpha ab\beta}$. If β is non-empty then $v^{\alpha ab\beta} = 0$ by Case 1. Let β be empty. Then α must have the form $\alpha = \gamma c$ so that $v^{\alpha ab\beta} = v^{\gamma cab}$. If γ is non-empty then $v^{\gamma cab} = 0$ by Case 2. Finally, let γ be also empty so that $v^{\alpha ab\beta} = v^{cab}$ (which is only possible if $p = 2$). Since $\partial v \in \mathcal{A}_1'$, we have

$$(\partial v)^{ab} = 0.$$

On the other hand,

$$(\partial v)^{ab} = \sum_{i \in V} v^{iab} - v^{aib} + v^{abi}.$$

Here all the terms of the form v^{iab} vanish, except possibly for v^{cab}, because ia is not an edge unless $i = c$. All the terms v^{aib} vanish because ai is not an edge. All the terms v^{abi} vanish by Case 1. Hence, we obtain

$$(\partial v)^{ab} = v^{cab}$$

whence $v^{cab} = 0$ follows, thus finishing the proof of the part (a).

Proof of $(b), (c), (d)$. If b, c belong to the same connected component of (V', E') then the number of connected components of (V, E) and that of (V', E') are the same, so that

$$\dim H_0(\Omega) = \dim H_0(\Omega'), \tag{4.30}$$

whereas if b, c belong to different connected components of (V', E') then after joining them by a the number of connected components reduces by 1, so that

$$\dim H_0(\Omega) = \dim H_0(\Omega') - 1. \tag{4.31}$$

To handle H_1 we use the long exact sequence (4.1) that by (4.26) has the form

$$0 \leftarrow H_0(\Omega/\Omega') \leftarrow H_0(\Omega) \leftarrow H_0(\Omega') \leftarrow H_1(\Omega/\Omega') \leftarrow H_1(\Omega) \leftarrow H_1(\Omega') \leftarrow 0. \tag{4.32}$$

Since we know already the relation between $H_0(\Omega')$ and $H_0(\Omega)$, to obtain the relation between $H_1(\Omega')$ and $H_1(\Omega)$ we need to compute $\dim H_0(\Omega/\Omega')$ and $\dim H_1(\Omega/\Omega')$ from the quotient complex Ω/Ω'. Observe that

$$\Omega_0 = \Omega_0' + \operatorname{span}\{e_a\}, \quad \Omega_1 = \Omega_1' + \operatorname{span}\{e_{ab}, e_{ca}\} \tag{4.33}$$

so that the quotient complex Ω/Ω' has the form

$$0 \longleftarrow \operatorname{span}\{e_a\} \xleftarrow{\partial} \operatorname{span}\{e_{ab}, e_{ca}\} \xleftarrow{\partial} \Omega_2/\Omega_2' \xleftarrow{\partial} \dots$$

We need to determine $\operatorname{Im}\partial|_{\Omega_1/\Omega_1'}$, $\ker\partial|_{\Omega_1/\Omega_1'}$, $\operatorname{Im}\partial|_{\Omega_2/\Omega_2'}$. Since

$$\partial e_{ab} = e_b - e_a = -e_a \mod \Omega_0',$$

it follows that

$$\operatorname{Im}\partial|_{\Omega_1/\Omega_1'} = \Omega_0/\Omega_0',$$

whence

$$\dim H_0(\Omega/\Omega') = 0. \tag{4.34}$$

For any scalars $k, l \in \mathbb{K}$, we have

$$\partial(ke_{ab} + le_{ca}) = (l-k)e_a \mod \Omega_0',$$

so that $\partial(ke_{ab} + le_{ca}) = 0$ if and only if $k = l$, that is

$$\ker\partial|_{\Omega_1/\Omega_1'} = \operatorname{span}(e_{ab} + e_{ca}) \mod \Omega_1'. \tag{4.35}$$

Let us now compute $\operatorname{Im}\partial|_{\Omega_2/\Omega_2'}$. For any $v\in\Omega_2$ we have by the above Cases 1,2 that $v^{abi}=v^{jca}=0$, which implies that v has the form

$$v=v'+v^{cab}e_{cab}, \tag{4.36}$$

where $v'\in\mathcal{A}_2'$. It follows that

$$\partial v=\partial v'+v^{cab}\left(e_{ab}-e_{cb}+e_{ca}\right). \tag{4.37}$$

Since all 1-paths ∂v, e_{ab} and e_{ca} belong to $\mathcal{A}_1$, it follows that $\partial v'-v^{cab}e_{cb}\in\mathcal{A}_1$ whence also $\partial v'-v^{cab}e_{cb}\in\mathcal{A}_1'$. Therefore,

$$\partial v=v^{cab}\left(e_{ab}+e_{ca}\right)\operatorname{mod}\Omega_1'. \tag{4.38}$$

Next consider two cases.

(i) Let Ω_2 contain an element v with $v^{cab}\neq 0$. Then by (4.38)

$$\operatorname{Im}\partial|_{\Omega_2/\Omega_2'}=\operatorname{span}\left(e_{ab}+e_{ca}\right)\operatorname{mod}\Omega_1', \tag{4.39}$$

which together with (4.35) implies

$$\dim H_1\left(\Omega/\Omega'\right)=0. \tag{4.40}$$

Substituting (4.34) and (4.40) into the exact sequence (4.32), we obtain that the identity

$$\dim H_p(\Omega') = \dim H_p(\Omega)$$

holds for all $p \geq 0$.

(ii) Assume that $v^{cab} = 0$ for all $v \in \Omega_2$. Then by (4.38)

$$\operatorname{Im} \partial|_{\Omega_2/\Omega'_2} = 0,$$

which together with (4.35) implies

$$\dim H_1(\Omega/\Omega') = 1. \tag{4.41}$$

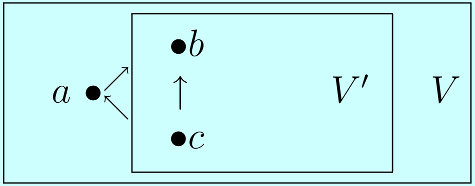
Using again the exact sequence (4.32), that is,

$$0 \leftarrow H_0(\Omega) \leftarrow H_0(\Omega') \leftarrow H_1(\Omega/\Omega') \leftarrow H_1(\Omega) \leftarrow H_1(\Omega') \leftarrow 0,$$

we obtain by (2.5) and (4.41)

$$\dim H_1(\Omega') - \dim H_1(\Omega) + 1 - \dim H^0(\Omega') + \dim H^0(\Omega) = 0 \tag{4.42}$$

Let us now specify when (i) or (ii) occur. Assume first that cb is an edge:

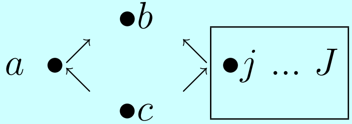


Then

$$\partial e_{cab} = e_{ab} - e_{cb} + e_{ca} \in \mathcal{A}_1,$$

whence it follows that $e_{cab} \in \Omega_2$. Hence, we have the case (i) with $v = e_{cab}$.

Assume now that cb is not an edge. Denote by J the set of vertices $j \in V'$ such that the 2-path cjb is allowed in (V', E'):



Assume first that J is non-empty, that is, cb is a semi-edge, and set

$$v = e_{cab} - \frac{1}{|J|}\sum_{j\in J} e_{cjb},$$

where $|J|$ is the number of elements in J. It is clear that $v \in \mathcal{A}_2$. We have

$$\begin{aligned}
\partial v &= (e_{ab} - e_{cb} + e_{ca}) - \frac{1}{|J|} \sum_{j \in J} (e_{jb} - e_{cb} + e_{cj}) \\
&= (e_{ab} + e_{ca}) - \frac{1}{|J|} \sum_{j \in J} (e_{jb} + e_{cj}), \qquad (4.43)
\end{aligned}$$

where the term e_{cb} has cancelled out. It follows from (4.43) that $\partial v \in \mathcal{A}_1$ whence $v \in \Omega_2$, and we obtain again the case (i). This finishes the proof of (b).

Let us show that if $J = \emptyset$ (that is, when cb is neither edge nor semi-edge) then we have the case (ii). Any 2-path $v \in \Omega_2$ has the form (4.36) and ∂v is given by (4.37). It follows that

$$(\partial v)^{cb} = (\partial v')^{cb} - v^{cab}.$$

Since $\partial v \in \mathcal{A}_1$ and cb is not an edge, we have $(\partial v)^{cb} = 0$. We have by (1.6)

$$(\partial v')^{cb} = \sum_{j \in V'} (v')^{jcb} - (v')^{cjb} + (v')^{cbj},$$

which implies that $(\partial v')^{cb} = 0$ as no elementary 2-path of the form jcb, cjb, cbj is allowed in (V', E'), whereas $v' \in \mathcal{A}_2'$. It follows that $v^{cab} = 0$ so that we have the case (ii).

If in addition b, c belong to the same connected component of (V', E') then we have (4.30), that is,

$$\dim H^0(\Omega) = \dim H^0(\Omega').$$

Substituting into (4.42), we obtain

$$\dim H_1(\Omega) = \dim H_1(\Omega') + 1.$$

which proves part (c).

If b, c belong to different components of (V', E') then we have by (4.31)

$$\dim H^0(\Omega) = \dim H^0(\Omega') - 1,$$

whence by (4.42)

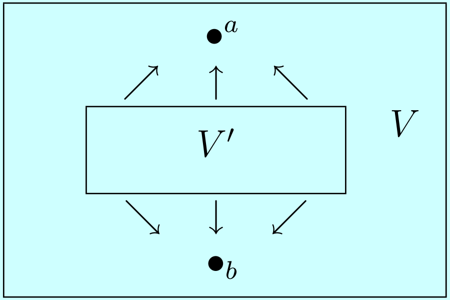
$$\dim H_1(\Omega) = \dim H_1(\Omega'),$$

which finishes the proof of part (d).

Finally, the identities for the Euler characteristic follows easily from the relations between $\dim H_p(\Omega)$ and $\dim H_p(\Omega')$. ■

4.5 Suspension

Let a digraph (V, E) have a subgraph (V', E') such that $V \setminus V' = \{a, b\}$ and $E \setminus E' = \{ia, ib, i \in V'\}$:



The digraph (V, E) is called a *suspension* of (V', E') and is denoted by $\operatorname{Sus}(V', E')$. Similarly, if a and b have outcoming edges then (V, E) is an inverse suspension of (V', E') .

The next theorem determines the homologies of a suspension.

Theorem 4.5 *If (V, E) is a suspension (or inverse suspension) of (V', E') then, for any $p \geq 1$,*

$$H_p(V, E) \cong \widetilde{H}_{p-1}(V', E') . \tag{4.44}$$

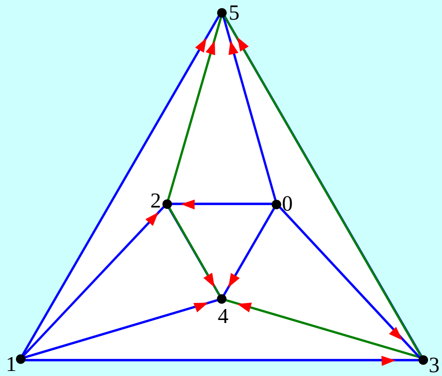
Here $\widetilde{H}_p$ is a reduced homology: $\widetilde{H}_p = H_p$ for $p \geq 1$ and $\widetilde{H}_0 \cong H_0 / \operatorname{const}$.

Denoting the digraph (V', E') by G, we can write the identity (4.44) in the functorial form as follows:

$$H_p(\operatorname{Sus} G) = \widetilde{H}_{p-1}(G).$$

It follows that $\chi(\operatorname{Sus} G) = 2 - \chi(G)$.

Example. Consider the digraph $G = (V, E)$ as follows:



Clearly, $G = \operatorname{Sus} G'$ where G' is the subgraph with vertices $\{2, 3, 4, 5\}$. Also, $G' = \operatorname{Sus} G''$ where G'' is a subgraph with vertices $\{4, 5\}$. Since $\dim H_0(G'') = 2$ and $\dim H_p(G'') = 0$ for $p \geq 1$, we obtain by (4.44)

$$\begin{aligned} \dim H_0(G') &= 1, \ \ \dim H_1(G') = 1, \ \dim H_p(G') = 0 \text{ for } p \geq 2, \\ \dim H_0(G) &= 1, \ \dim H_1(G) = 0, \ \ \dim H_2(G) = 1, \ \dim H_p(G) = 0 \ \text{ for } p \geq 3. \end{aligned}$$

Consequently, $\chi(G) = 2$.

In the digraph G we have

$$\dim \Omega_0 = |V| = 6 \quad \text{and} \quad \dim \Omega_1 = |E| = 12$$

and

$$\mathcal{A}_2 = \operatorname{span}\{e_{024}, e_{025}, e_{034}, e_{035}, e_{124}, e_{125}, e_{134}, e_{135}\}.$$

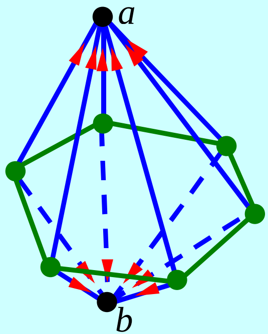
The set of semi-edges is empty, whence $\dim \Omega_2 = \dim \mathcal{A}_2 = 8$ and, hence, $\Omega_2 = \mathcal{A}_2$. Alternatively, one can see that because all the 2-paths spanning $\mathcal{A}_2$ are triangles so that all they are ∂-invariant. Also, there are no allowed 3-paths, so that $\mathcal{A}_3 = \{0\}$ whence $\dim \Omega_p = 0$ for all $p \geq 3$.

A spanning element in $H^2(G)$ is

$$v = e_{024} - e_{025} - e_{034} + e_{035} - e_{124} + e_{125} + e_{134} - e_{135}$$

as $v \neq 0$ and $\partial v = 0$.

Example. Let S be any cycle graph that is neither triangle nor square. We regards S as a circle. Define S_n inductively by $S_1 = S$ and $S_{n+1} = \operatorname{Sus} S_n$. Then S_n can be regarded as n-dimensional sphere. Here is an example of S_2:



Since $\chi(S) = 0$ by Proposition 3.7, it follows that $\chi(S_n) = 0$ if n is odd and $\chi(S_n) = 2$ if n is even. Theorem 4.5 also implies that $\dim H_n(S_n) = \dim H_1(S) = 1$, which gives an example of a non-trivial H_n with an arbitrary n.

Proof of Theorem 4.5. For any $p \geq 0$ consider a linear mapping

$$\tau : \mathcal{A}'_p \to \mathcal{A}_{p+1},$$

defined by

$$\tau v = v\left(e_a - e_b\right). \tag{4.45}$$

Since every vertex from V' is connected to a and b, the path τv is indeed allowed. By the product rule (1.14) we have

$$\partial\left(\tau v\right) = \left(\partial v\right)\left(e_b - e_a\right) + \left(-1\right)^{p+1} v\partial\left(e_a - e_b\right) = \tau\partial v$$

so that the operators ∂ and τ commute. It follows that

$$\tau\left(\Omega'_p\right) \subset \Omega_{p+1}.$$

Indeed, if $v \in \Omega'_p$ then

$$v \in \mathcal{A}'_p \quad \text{and} \quad \partial v \in \mathcal{A}'_{p-1}$$

whence

$$\tau v \in \mathcal{A}_{p+1} \quad \text{and} \quad \partial\left(\tau v\right) = \tau\left(\partial v\right) \in \mathcal{A}_p,$$

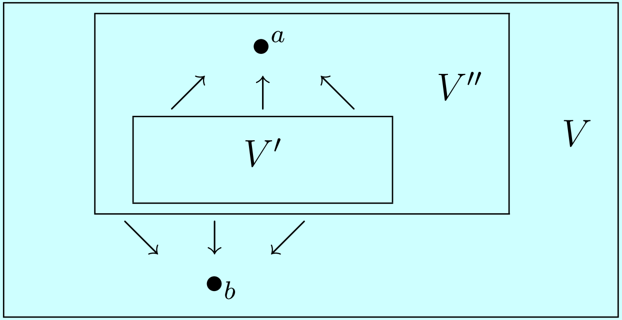
whence $\tau v \in \Omega_{p+1}$. Hence, we have the commutative following diagram for all $p \geq 1$:

$$\begin{array}{ccc} \Omega'_{p-1} & \overset{\partial}{\longleftarrow} & \Omega'_{p} \\ \downarrow^{\tau} & & \downarrow^{\tau} \\ \Omega_{p} & \overset{\partial}{\longleftarrow} & \Omega_{p+1} \end{array} \tag{4.46}$$

Let us extend it to the case $p = 0$. Set $\Omega'_{-1} = \mathbb{K}$ as in the case of reduced homology. The operator $\tau : \mathbb{K} \to \Omega_0$ is also defined by (4.45), which now amounts to $\tau 1 = e_a - e_b$. The operator ∂ should be replaced by $\widetilde{\partial} : \Omega'_0 \to \mathbb{K}$ where $\widetilde{\partial} e_i = 1$ (this is the same operator $\widetilde{\partial}$ that is used in the reduced homologies and in the product rule). The above argument, based on the product rule, remains valid. Hence, the diagram (4.46) remains commutative also for $p = 0$, where it takes the form

$$\begin{array}{ccc} \mathbb{K} & \overset{\widetilde{\partial}}{\longleftarrow} & \Omega'_{0} \\ \downarrow^{\tau} & & \downarrow^{\tau} \\ \Omega_{0} & \overset{\partial}{\longleftarrow} & \Omega_{1} \end{array}$$

Consider the digraph (V'', E'') that is obtained by adding to (V', E') the vertex a and all the edges ia with $i \in V'$, that is, (V'', E'') is a *cone* over (V', E'):



Let us mark by a double dash $''$ all the notation related to this digraph. For any $p \geq 0$, define a linear mapping $\rho : \mathcal{A}_p \to \mathcal{A}''_p$ by

$$\rho e_{i_0 \ldots i_p} = \begin{cases} e_{i_0 \ldots i_p}, & \text{if } i_p \neq b \\ e_{i_0 \ldots i_{p-1} a}, & \text{if } i_p = b. \end{cases} \tag{4.47}$$

Clearly, ρ is surjective. Let us show that ρ commutes with ∂. If $v = e_{i_0 \ldots i_p}$ with $i_p \neq b$ then $\rho v = v$ and $\rho(\partial v) = \partial v$ so that $\rho(\partial v) = \partial(\rho v)$. If $i_p = b$ then, setting $u = e_{i_0 \ldots i_{p-1}}$, we obtain $\rho v = u e_a$ and

$$\partial(\rho v) = (\partial u) e_a + (-1)^p u.$$

On the other hand, we have

$$\partial v = (\partial u)\, e_a + (-1)^p\, u$$

whence it follows that

$$\rho\,(\partial v) = (\partial u)\, e_b + (-1)^p\, u,$$

which proves that $\rho\,(\partial v) = \partial\,(\rho v)$.

It follows that ρ maps Ω_p to Ω''_p and the following diagram is commutative for any $p \geq 0$:

$$\begin{array}{ccc} \Omega_p & \overset{\partial}{\longleftarrow} & \Omega_{p+1} \\ \downarrow^{\rho} & & \downarrow^{\rho} \\ \Omega''_p & \overset{\partial}{\longleftarrow} & \Omega''_{p+1} \end{array} \tag{4.48}$$

We will merge the diagrams (4.48) and (4.46), and for that we need to verify that the following sequence is exact for all $p \geq -1$:

$$0 \longrightarrow \Omega'_p \overset{\tau}{\longrightarrow} \Omega_{p+1} \overset{\rho}{\longrightarrow} \Omega''_{p+1} \longrightarrow 0. \tag{4.49}$$

Since τ is injective and ρ is surjective, it suffices to show that $\operatorname{Im}\tau = \ker\rho$. We have

$$\tau e_{i_0 \ldots i_p} = e_{i_0 \ldots i_p a} - e_{i_0 \ldots i_p b}$$

so that $\operatorname{Im}\tau$ consists of all p-paths of the form

$$\sum_{i_0,\dots,i_p\in V'} c^{i_0\dots i_p}\left(e_{i_0\dots i_p a}-e_{i_0\dots i_p b}\right) \tag{4.50}$$

with arbitrary coefficients $c^{i_0\dots i_p}$. Observe that, for any $u\in\Omega_{p+1}$,

$$\rho u=\sum_{i_0,\dots,i_{p+1}\in V'} u^{i_0\dots i_{p+1}}e_{i_0\dots i_p i_{p+1}}+\sum_{i_0,\dots,i_p\in V'}\left(u^{i_0\dots i_p a}+u^{i_0\dots i_p b}\right)e^{i_0\dots i_p a}. \tag{4.51}$$

Then the equation $\rho u=0$ that amounts to the system

$$\begin{cases} u^{i_0\dots i_{p+1}}=0, & \text{for all } i_0\dots i_{p+1}\in V', \\ u^{i_0\dots i_p a}+u^{i_0\dots i_p b}=0, & \text{for all } i_0\dots i_p\in V', \end{cases} \tag{4.52}$$

that is, to the identity

$$u=\sum_{i_0,\dots,i_p\in V'} u^{i_0\dots i_p a}\left(e_{i_0\dots i_p a}-e_{i_0\dots i_p b}\right). \tag{4.53}$$

Comparing with (4.50) we see that $\operatorname{Im}\tau=\ker\rho$.

Hence, we have constructed the following commutative diagram where the rows are chain complexes and the columns are exact:

$$
\begin{array}{ccccccccccccc}
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow \dots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & & \downarrow & \\
0 & \longleftarrow & \Omega'_{-1} & \stackrel{\widetilde{\partial}}{\longleftarrow} & \Omega'_0 & \stackrel{\partial}{\longleftarrow} & \Omega'_1 & \stackrel{\partial}{\longleftarrow} \dots & \stackrel{\partial}{\longleftarrow} & \Omega'_{p-1} & \stackrel{\partial}{\longleftarrow} & \Omega'_p & \stackrel{\partial}{\longleftarrow} \dots \\
\downarrow & & \downarrow^{\tau} & & \downarrow^{\tau} & & \downarrow^{\tau} & & & \downarrow^{\tau} & & \downarrow^{\tau} & \\
0 & \longleftarrow & \Omega_0 & \stackrel{\partial}{\longleftarrow} & \Omega_1 & \stackrel{\partial}{\longleftarrow} & \Omega_2 & \stackrel{\partial}{\longleftarrow} \dots & \stackrel{\partial}{\longleftarrow} & \Omega_p & \stackrel{\partial}{\longleftarrow} & \Omega_{p+1} & \stackrel{\partial}{\longleftarrow} \dots \\
\downarrow & & \downarrow^{\rho} & & \downarrow^{\rho} & & \downarrow^{\rho} & & & \downarrow^{\rho} & & \downarrow^{\rho} & \\
0 & \longleftarrow & \Omega''_0 & \stackrel{\partial}{\longleftarrow} & \Omega''_1 & \stackrel{\partial}{\longleftarrow} & \Omega''_2 & \stackrel{\partial}{\longleftarrow} \dots & \stackrel{\partial}{\longleftarrow} & \Omega''_p & \stackrel{\partial}{\longleftarrow} & \Omega''_{p+1} & \stackrel{\partial}{\longleftarrow} \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & & \downarrow & \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow \dots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow \dots
\end{array}
\tag{4.54}
$$

The homologies of the first chain complex in (4.54) are the reduced homologies $\widetilde{H}_{\cdot}(\Omega')$, while the second and the third complexes yield the homologies $H_{\cdot}(\Omega)$ and $H_{\cdot}(\Omega'')$ respectively. By (2.17) we obtain a long exact sequence

$$
0 \leftarrow \dots \leftarrow H_p(\Omega'') \leftarrow H_p(\Omega) \leftarrow \widetilde{H}_{p-1}(\Omega') \leftarrow H_{p+1}(\Omega'') \leftarrow \dots
$$

Since (V'', E'') is a star-like, we have by Theorem 3.4 $H_p(\Omega'') = \{0\}$ for any $p \geq 1$, whence it follows that

$$\dim H_p(\Omega) = \dim \widetilde{H}_{p-1}(\Omega'),$$

which was to be proved. ■