Analysis and Geometry on Graphs
Part 2. Differential forms on digraphs

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1 Differential forms on a finite set

Let $V$ be a non-empty finite set. Denote by $\Lambda^0 = \Lambda^0(V)$ the linear space of all $\mathbb{K}$-valued functions on $V$, where $\mathbb{K}$ is a fixed scalar field, say $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. More generally, for any integer $p \geq 0$, denote by $\Lambda^p = \Lambda^p(V)$ the linear space of all $\mathbb{K}$-valued functions on $V^{p+1} = \underbrace{V \times \ldots \times V}_{p+1}$. Clearly, $\dim \Lambda^p = |V|^{p+1}$.

**Definition.** Elements of $\Lambda^p$ are referred to as $p$-forms on $V$.

The value of a $p$-form $\omega$ at a point $(i_0, i_1, \ldots, i_p) \in V^{p+1}$ will be denoted by $\omega_{i_0i_1\ldots i_p}$. In particular, the value of a function $f \in \Lambda^0(V)$ at $i \in V$ will be denoted by $f_i$.

Denote by $e^{j_0\ldots j_p}$ a $p$-form that takes value 1 $\in \mathbb{K}$ at the point $(j_0, j_1, \ldots, j_p)$ and 0 at all other points. For example, $e^j$ is a function on $V$ that is equal to 1 at $j$ and 0 away from $j$. Let us refer to $e^{j_0\ldots j_p}$ as an elementary $p$-form. Clearly, the family $\{e^{j_0\ldots j_p}\}$ of all elementary $p$-forms is a basis in the linear space $\Lambda^p$ and, for any $\omega \in \Lambda^p$,

$$\omega = \sum_{j_0, \ldots, j_p \in V} \omega_{j_0\ldots j_p} e^{j_0\ldots j_p}.$$
1.1 Exterior derivative

Definition. Define the exterior derivative \(d : \Lambda^p \to \Lambda^{p+1}\) by

\[
(d\omega)_{i_0\ldots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \omega_{i_0\ldots \hat{i}_q\ldots i_{p+1}},
\]

for any \(\omega \in \Lambda^p\), where the hat \(\hat{i}_q\) means omission of the index \(i_q\).

For example, for a function \(f \in \Lambda^0\) we have

\[
(df)_{ij} = f_j - f_i,
\]

for 1-form \(\omega\)

\[
(d\omega)_{ijk} = \omega_{jk} - \omega_{ik} + \omega_{ij},
\]

for a 2-form \(\omega\)

\[
(d\omega)_{ijkl} = \omega_{jkl} - \omega_{ikl} + \omega_{ijl} - \omega_{ijk}.
\]
It follows from (1.1) that

\[
(de^{j_0...j_p})_{i_0...i_p+1} = \sum_{q=0}^{p+1} (-1)^q e^{j_0...j_p}_{i_0...i_q...i_{p+1}}
\]

\[
= \sum_{q=0}^{p+1} (-1)^q e^{j_0...j_{q-1}i_qj_q...j_p}_{i_0...i_q...i_{p+1}}
\]

\[
= \sum_{q=0}^{p+1} (-1)^q \sum_{i} e^{j_0...j_{q-1}i_q...j_p}_{i_0...i_{p+1}}
\]

whence

\[
de^{j_0...j_p} = \sum_{i} \sum_{q=0}^{p+1} (-1)^q e^{j_0j_1...j_{q-1}j_q...j_p}_{i_0...i_{p+1}}.
\]  \hspace{1cm} (1.2)

For example,

\[
de^j = \sum_{i} (e^{ij} - e^{ji}),
\]

\[
de^{jk} = \sum_{i} (e^{ijk} - e^{jik} + e^{jki}).
\]
Lemma 1.1  For any $p \geq 0$ and all $\omega \in \Lambda^p$,

$$d^2 \omega = 0. \quad (1.3)$$

Proof. We have

$$(d^2 \omega)_{i_0...i_{p+2}} = \sum_{q=0}^{p+2} (-1)^q (d\omega)_{i_0...\hat{i}_q...i_{p+2}}$$

$$= \sum_{q=0}^{p+2} (-1)^q \left( \sum_{r=0}^{q-1} (-1)^r \omega_{i_0...\hat{i}_r...\hat{i}_q...i_{p+2}} + \sum_{r=q+1}^{p+2} (-1)^{r-1} \omega_{i_0...\hat{i}_q...\hat{i}_{r}...i_{p+2}} \right)$$

$$= \sum_{0 \leq r < q \leq p+2} (-1)^{q+r} \omega_{i_0...\hat{i}_r...\hat{i}_q...i_{p+2}}$$

$$- \sum_{0 \leq q < r \leq p+2} (-1)^{q+r} \omega_{i_0...\hat{i}_q...\hat{i}_{r}...i_{p+2}}.$$ 

After switching $q$ and $r$ in the second sum we see that it is equal to the first one, whence $d^2 \omega = 0$ follows. □
1.2 Concatenation and product rule

Definition. For forms $\varphi \in \Lambda^p$ and $\psi \in \Lambda^q$ denote by $\varphi \psi$ a form from $\Lambda^{p+q}$ that is defined by

\[
(\varphi \psi)_{i_0...i_p} = \varphi_{i_0...i_p} \psi_{i_p} \psi_{i_p+1...i_p+q}. \tag{1.4}
\]

The form $\varphi \psi$ is called the concatenation of $\varphi$ and $\psi$.

Clearly, $\varphi \psi$ is a bilinear operation with respect to $\varphi$, $\psi$. For example, if $\varphi$ is a function, that is, $p = 0$, then $\varphi \psi \in \Lambda^q$ and

\[
(\varphi \psi)_{i_0...i_q} = \varphi_{i_0} \psi_{i_0...i_q}.
\]

Also $\psi \varphi \in \Lambda^q$ and

\[
(\psi \varphi)_{i_0...i_q} = \psi_{i_0...i_q} \varphi_{i_q}.
\]

For the elementary forms $e_{i_0...i_p}$ and $e_{j_0...j_q}$ we have

\[
e_{i_0...i_p} e_{j_0...j_q} = \begin{cases} 
0, & i_p \neq j_0, \\
\epsilon_{i_0...i_p j_1...i_q}, & i_p = j_0.
\end{cases}
\]
Lemma 1.2 For all $\varphi \in \Lambda^p$ and $\psi \in \Lambda^q$, we have
\[ d(\varphi \psi) = (d\varphi) \psi + (-1)^p \varphi d\psi. \quad (1.5) \]

Proof. Denoting $\omega = \varphi \psi$, we have
\[ (d\omega)_{i_0...i_{p+q+1}} = \sum_{r=0}^{p+q+1} (-1)^r \omega_{i_0...\hat{i}_r...i_{p+q+1}} \]
\[ = \sum_{r=0}^{p} (-1)^r \omega_{i_0...\hat{i}_r...i_{p+1}...i_{p+q+1}} + \sum_{r=p+1}^{p+q+1} (-1)^r \omega_{i_0...i_{p+1}...\hat{i}_r...i_{p+q+1}} \]
\[ = \sum_{r=0}^{p} (-1)^r \varphi_{i_0...\hat{i}_r...i_{p+1}} \psi_{i_{p+1}...i_{p+q+1}} + \sum_{r=p+1}^{p+q+1} (-1)^r \varphi_{i_0...i_{p+1}} \psi_{i_{p+1}...\hat{i}_r...i_{p+q+1}}. \]

Noticing that
\[ (d\varphi)_{i_0...i_{p+1}} = \sum_{r=0}^{p+1} (-1)^r \varphi_{i_0...\hat{i}_r...i_{p+1}} \]
and
\[ (d\psi)_{i_{p+1}...i_{p+q+1}} = \sum_{r=p}^{p+q+1} (-1)^{r-p} \psi_{i_{p+1}...\hat{i}_r...i_{p+q+1}}, \]
we obtain
\[
(d\omega)_{i_0...i_p+q+1} = \left[(d\varphi)_{i_0...i_p+1} - (-1)^{p+1} \varphi_{i_0...i_p}\right] \psi_{i_p+1...i_p+q+1} \\
+ (-1)^{p} \varphi_{i_0...i_p} \left[(d\psi)_{i_p...i_p+q+1} - \psi_{i_p+1...i_p+q+1}\right] \\
= ((d\varphi \psi)_{i_0...i_p+q+1} + (-1)^{p} (\varphi d\psi)_{i_0...i_p+q+1}
\]
which was to be proved. ■

1.3 Spaces of paths and Stokes’s formula

An elementary \textit{p-path} is any (ordered) sequence \(i_0, ..., i_p\) of \(p+1\) vertices of \(V\) that will be denoted simply by \(i_0...i_p\) or by \(e_{i_0...i_p}\). We use the notation \(e_{i_0...i_p}\) when we consider the elementary path as an element of a linear space \(\Lambda_p = \Lambda_p(V)\) that consists of all formal linear combination of all elementary \(p\)-paths. The elements of \(\Lambda_p\) are called \textit{p-paths}. Each \(p\)-path has a form
\[
v = \sum_{i_0i_1...i_p} v^{i_0i_1...i_p} e_{i_0i_1...i_p},
\]
with arbitrary scalars \(v^{i_0i_1...i_p}\), that are called the coefficients of \(v\).
We have a natural pairing of $p$-forms and $p$-paths as follows:

$$(\omega, v) := \sum_{i_0...i_p} \omega_{i_0...i_p} v^{i_0...i_p}$$

for all $\omega \in \Lambda^p$ and $v \in \Lambda_p$. It follows that the spaces $\Lambda^p$ and $\Lambda_p$ are dual.

**Definition.** Define the boundary operator $\partial : \Lambda_{p+1} \to \Lambda_p$ by

$$(\partial v)^{i_0...i_p} = \sum_{k, q=0}^{p+1} (-1)^q v^{i_0...i_q-1ki_q...i_p}$$

(1.6)

where the index $k$ is inserted so that it is preceded by $q$ indices.

This definition is valid for $p \geq 0$. Sometimes we need also the operator $\partial : \Lambda_0 \to \Lambda_{-1}$ where we set $\Lambda_{-1} = \{\emptyset\}$, so that $\Lambda_{-1}$ can be understood as a 0-dimensional linear space. Then by definition $\partial v = \emptyset$ for all $v \in \Lambda_0$.

If $v$ is an 1-path, then $\partial v$ is given by

$$(\partial v)^i = \sum_k \left( v^{ki} - v^{ik} \right).$$
If $v$ is a 2-path then

$$(\partial v)^{ij} = \sum_{k} (v^{kj} - v^{ji} + v^{ik}).$$

It follows from (1.6) that

$$(\partial e_{j_0...j_{p+1}})^{i_0...i_p} = \sum_{q=0}^{p+1} (-1)^q e_{j_0...j_q...i_0...i_{q-1}i_q...i_p}$$

$$= \sum_{q=0}^{p+1} \sum_{k} (-1)^q e_{j_0...j_q...i_q...i_p}$$

$$= \sum_{q=0}^{p+1} (-1)^q e_{j_0...j_q-1j_q+1...j_{p+1}}$$

whence

$$\partial e_{j_0...j_{p+1}} = \sum_{q=0}^{p+1} (-1)^q e_{j_0...\hat{j}_q...j_{p+1}}. \tag{1.7}$$

For example, $\partial e_{ij} = e_j - e_i$ and $\partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}$. 
Lemma 1.3  For any p-form $\omega$ and any $(p+1)$-path $v$ the following identity holds

$$(d\omega, v) = (\omega, \partial v).$$

Hence, the operators $d : \Lambda^p \to \Lambda^{p+1}$ and $\partial : \Lambda_{p+1} \to \Lambda_p$ are dual. 

Proof. It suffices to prove this for $v = e_{i_0 \ldots i_{p+1}}$. We have

$$(d\omega, v) = (d\omega)_{i_0 \ldots i_{p+1}} = (d\omega)_{i_0 \ldots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \omega_{i_0 \hat{i}_q \ldots i_{p+1}}$$

while

$$(\omega, \partial v) = \left(\omega, \sum_{q=0}^{p+1} (-1)^q e_{i_0 \hat{i}_q \ldots i_{p+1}}\right) = \sum_{q=0}^{p+1} (-1)^q \omega_{i_0 \hat{i}_q \ldots i_{p+1}},$$

whence the identity of the two expressions follows. $\blacksquare$

Corollary 1.4  For any $v \in \Lambda_p$, we have $\partial^2 v = 0$. 

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1.4 Product of paths

For any two paths $u \in \Lambda_p$ and $v \in \Lambda_q$ define the product $uv \in \Lambda_{p+q+1}$ as follows:

\[
(uv)^{i_0 \ldots i_p j_0 \ldots j_q} = u^{i_0 \ldots i_p} v^{j_0 \ldots j_q}.
\] (1.8)

For example, if $u = e_{i_0 \ldots i_p}$ and $v = e_{j_0 \ldots j_q}$, then

\[
e_{i_0 \ldots i_p} e_{j_0 \ldots j_q} = e_{i_0 \ldots i_p j_0 \ldots j_q}.
\] (1.9)

This definition is valid for all $p, q \geq 0$.

To state a product rule for $\partial (uv)$ we need also the notion of a product also for $p = -1$ or $q = -1$. For that consider instead of $\Lambda_{-1} = \emptyset$ a modified space $\tilde{\Lambda}_{-1} \equiv \mathbb{K}$ so that any $u \in \tilde{\Lambda}_{-1}$ is just a scalar. Then (1.8) can be used again to define the product $uv$ for $u \in \tilde{\Lambda}_{-1}$ (or $v \in \tilde{\Lambda}_{-1}$) because the right hand side of (1.8) amounts to multiplying by the scalar $u$ (resp. $v$). That is, if $p = -1$ then $uv$ is just the multiple of $v$ with the coefficient $u$.

We need then a modified version of $\partial$ when acting from $\Lambda_0$ to $\tilde{\Lambda}_{-1}$. Define the operator $\tilde{\partial} : \Lambda_p \rightarrow \Lambda_{p-1}$ as follows. If $p > 0$ then $\tilde{\partial} \equiv \partial$, and for $p = 0$ define $\tilde{\partial} : \Lambda_0 \rightarrow \mathbb{K}$ by setting $\tilde{\partial} e_i = 1$ and extending to all $v \in \Lambda_0$ by
linearity. In other word, for \( v \in \Lambda_0 \) we have \( \tilde{\partial}v = (1, v) \). This definition of \( \tilde{\partial} \) is the same as the one used in the extended chain complex (2.20). It is easy to see that \( \tilde{\partial}^2 = 0 \).

**Lemma 1.5** For any paths \( u \in \Lambda_p \) and \( v \in \Lambda_q \) with \( p, q \geq 0 \), we have

\[
\partial (uv) = (\tilde{\partial}u)v + (-1)^{p+1} u\tilde{\partial}v. \tag{1.10}
\]

**Proof.** By bilinearity it suffices to prove (1.10) for \( u = e_{i_0...i_p} \) and \( v = e_{j_0...j_q} \). Consider first the case \( p = q = 0 \). Then \( u = e_i, v = e_j \) and \( \tilde{\partial}u = \tilde{\partial}v = 1 \) and

\[
\partial (uv) = \partial (e_{ij}) = e_j - e_i = (\tilde{\partial}u)v - u(\tilde{\partial}v),
\]

which proves (1.10) in this case.

If \( p = 0 \) and \( q \geq 1 \) then \( u = e_i \) and \( v = e_{j_0...j_q} \), whence

\[
\partial (uv) = \partial e_{i_0...j_q} = e_{j_0...j_q} - e_{i_1j_2...j_q} + e_{i_0j_2...j_q} - ... = v - e_i(\partial v) = (\tilde{\partial}u)v - u(\tilde{\partial}v)
\]

which proves (1.10) in this case.
If $p \geq 1$ and $q = 0$ then $u = u_{i_0...i_p}$, $v = e_j$, whence
\[
\partial (uv) = \partial e_{i_0...i_qj} = e_{i_1...i_qj} - e_{i_0i_2...i_qj} - \ldots + (-1)^{p+1} e_{i_0...i_p}
\]
\[
= (\partial u) e_j + (-1)^{p+1} u = (\tilde{\partial} u) v + (-1)^{p+1} u \tilde{\partial} v.
\]

Finally, if $p \geq 1$ and $q \geq 1$ then
\[
\partial (uv) = \partial e_{i_0...i_p j_0...j_q} = e_{i_1...i_p j_0...j_q} - e_{i_0i_2...i_p j_0...j_q} + \ldots + (-1)^{p+1} (e_{i_0...i_p j_1...j_q} - e_{i_0...i_p j_0j_2...j_q} + \ldots)
\]
\[
= (\partial u) v + (-1)^{p+1} u (\partial v),
\]

which finishes the proof. ■

1.5 Regular forms

We say that a path $i_0...i_p$ is regular if $i_k \neq i_{k+1}$ for all $k = 0, \ldots, p - 1$, and irregular otherwise. Consider the following subspace of $\Lambda^p$:
\[
\mathcal{R}^p = \text{span} \{ e^{i_0...i_p} : i_0...i_p \text{ is regular} \}
\]
\[
= \{ \omega \in \Lambda^p : \omega_{i_0...i_p} = 0 \text{ if } i_0...i_p \text{ is irregular} \}.
\]
The elements of $\mathcal{R}^p$ are called regular $p$-forms. For example, $\omega \in \mathcal{R}^1$ if $\omega_{ii} \equiv 0$ and $\omega \in \mathcal{R}^2$ if $\omega_{iij} \equiv \omega_{jii} \equiv 0$. The condition $f \in \mathcal{R}^0$ has no additional restriction so that $\mathcal{R}^0 = \Lambda^0$.

The operations of exterior derivative and concatenation can be restricted to regular forms.

**Lemma 1.6** If $\omega \in \mathcal{R}^p$ then $d\omega \in \mathcal{R}^{p+1}$. If $\varphi \in \mathcal{R}^p$ and $\psi \in \mathcal{R}^q$ then $\varphi \psi \in \mathcal{R}^{p+q}$.

**Proof.** Let $\omega \in \mathcal{R}^p$. To prove that $d\omega \in \mathcal{R}^{p+1}$, we must show that

$$
(d\omega)_{i_0...i_{p+1}} = 0 \quad (1.11)
$$

whenever $i_0...i_{p+1}$ is irregular, say $i_k = i_{k+1}$. We have by (1.1)

$$
(d\omega)_{i_0...i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \omega_{i_0...\hat{i}_q...i_{p+1}}
$$

If $q \neq k, k + 1$ then both $i_k, i_{k+1}$ are present in $\omega_{i_0...\hat{i}_q...i_{p+1}}$ which makes this term equal to 0 since $\omega$ is regular. In the remaining two cases $q = k$...
and $q = k + 1$ the term $\omega_{i_0\ldots\hat{i}_q\ldots i_{p+1}}$ has the same values (because the sequences $i_0\ldots\hat{i}_q\ldots i_{p+1}$ are the same) but the signs $(-1)^q$ are opposite. Hence, they cancel out, which proves (1.11).

Let us prove that $\phi\psi$ is regular provided so are $\phi$ and $\psi$. By (1.4), we have

$$(\phi\psi)_{i_0\ldots i_{p+q}} = \phi_{i_0\ldots i_p} \psi_{i_p\ldots i_{p+q}}.$$

If the sequence $i_0\ldots i_{p+q}$ is irregular, say $i_k = i_{k+1}$ then the both indices $i_k, i_{k+1}$ are present either in the sequence $i_0\ldots i_p$ or in $i_p\ldots i_{p+q}$, which implies that one of the terms $\phi_{i_0\ldots i_p}, \psi_{i_p\ldots i_{p+q}}$ vanishes and, hence,

$$(\phi\psi)_{i_0\ldots i_{p+q}} = 0.$$ 

1.6 Regular paths

We say that an elementary $p$-path $e_{i_0\ldots i_p}$ is regular (or irregular) if the path $i_0\ldots i_p$ is regular (resp. irregular). We would like to define the boundary operator $\partial$ on the subspace of $\Lambda_p$ spanned by regular elementary paths. Just restriction of $\partial$ to the subspace does not work as $\partial$ is not invariant on this subspace, so that we have to consider a quotient space instead.
Let $I_p$ be the subspace of $\Lambda_p$ that is spanned by irregular $e_{i_0...i_p}$. Consider the quotient spaces

$$\mathcal{R}_p := \Lambda_p/I_p.$$ 

The elements of $\mathcal{R}_p$ are the equivalence classes $v \mod I_p$ where $v \in \Lambda_p$, and they are called *regularized* $p$-paths. The next lemma shows that the boundary operator $\partial$, the product and the pairing are well-defined for regularized paths.

**Lemma 1.7**  

(a) If $v_1, v_2 \in \Lambda_p$ and $v_1 = v_2 \mod I_p$ then $\partial v_1 = \partial v_2 \mod I_{p-1}$.

(b) If $\omega \in \mathcal{R}^p$, $v_1, v_2 \in \Lambda_p$ and $v_1 = v_2 \mod I_p$ then $(\omega, v_1) = (\omega, v_2)$.

(b) Let $u_1, u_2 \in \Lambda_p$ and $v_1, v_2 \in \Lambda_q$. If $u_1 = u_2 \mod I_p$ and $v_1 = v_2 \mod I_q$ then $u_1 v_1 = u_2 v_2 \mod I_{p+q+1}$.

**Proof.**  

(a) It suffices to prove that if $v = 0 \mod I_p$ then $\partial v = 0 \mod I_{p-1}$. Since $v$ is a linear combination of irregular paths, it suffices to prove that $\partial e_{i_0...i_p}$ is irregular provided $e_{i_0...i_p}$ is irregular. If $e_{i_0...i_p}$ is irregular then there exists an index $k$ such that $i_k = i_{k+1}$. Then we
have

\[ \partial e_{i_0 \ldots i_p} = e_{i_1 \ldots i_p} - e_{i_0 i_2 \ldots i_p} + \ldots + (-1)^k e_{i_0 \ldots i_{k-1} i_{k+1} i_{k+2} \ldots i_p} + (-1)^{k+1} e_{i_0 \ldots i_{k-1} i_k i_{k+2} \ldots i_p} \] \hspace{1cm} (1.12)

+ \ldots + (-1)^p e_{i_0 \ldots i_{p-1}}.

By \( i_k = i_{k+1} \) the two terms in the middle line of (1.12) cancel out, whereas all other terms are irregular, whence, \( \partial e_{i_0 \ldots i_p} \in I_{p-1} \).

(b) Indeed, \( v_1 - v_2 \in I_p \) is a linear combination of irregular paths \( e_{i_0 \ldots i_p} \). Since \( (\omega, e_{i_0 \ldots i_p}) = 0 \) for irregular paths, it follows that \( (\omega, v_1 - v_2) = 0 \) and \( (\omega, v_1) = (\omega, v_2) \).

(c) Observe first that if \( u \in \Lambda_p, v \in \Lambda_q \) then \( uv = 0 \mod I_{p+q+1} \) provided \( u = 0 \mod I_p \) or \( v = 0 \mod I_q \). Indeed, if for example \( u = 0 \mod I_p \) then \( u \) is a linear combination of irregular paths \( e_{i_0 \ldots i_p} \), and the product of an irregular path with any path is irregular. Since

\[ u_1 v_1 - u_2 v_2 = (u_1 - u_2) v_1 + u_2 (v_1 - v_2) \]

and \( u_1 - u_2 = 0 \mod I_p, v_1 - v_2 = 0 \mod I_q \), we conclude that

\[ u_1 v_1 = u_2 v_2 \mod I_{p+q+1}. \]
It follows from Lemma 1.3 that, for all \( \omega \in \mathcal{R}^p \) and \( v \in \mathcal{R}_{p+1} \),

\[
(d\omega, v) = (\omega, \partial v).
\] (1.13)

By Lemma 1.5, we obtain that, for all \( u \in \mathcal{R}_p \) and \( v \in \mathcal{R}_q \),

\[
\partial (uv) = (\partial u)v + (-1)^{p+1}u\partial v.
\] (1.14)

Clearly, \( \mathcal{R}_p \) is linearly isomorphic to the space of regular paths:

\[
\text{span} \{ e_{i_0...i_p} : i_0...i_p \text{ is regular} \}.
\]

For simplicity of notation, we will identify \( \mathcal{R}_p \) with this space, by setting all irregular \( p \)-paths to be equal to 0. Hence, when applying the formulas for \( \partial \) and for the product in the spaces \( \mathcal{R}_p \), one should make the following adjustments: all elementary irregular paths \( e_{i_0...i_p} \) are equal to zero, and the components \( v^{i_0...i_p} \) for irregular paths \( i_0...i_p \) vanish by definition. In particular, the formula (1.6) for the component \( (\partial v)^{i_0...i_p} \) is valid only for regular \( i_0...i_p \), whereas the formula (1.7) for \( \partial e_{j_0...j_{p+1}} \) remains valid for all \( j_0...j_{p+1} \).
Let $V'$ be a subset of $V$. Clearly, every elementary regular $p$-path $e_{i_0...i_p}$ on $V'$ is also a regular $p$-path on $V$, so that we have a natural inclusion

$$\mathcal{R}_p (V') \subset \mathcal{R}_p (V).$$

(1.15)

By (1.7), $\partial e_{i_0...i_p}$ has the same expression in the both spaces $\mathcal{R}_p (V')$, $\mathcal{R}_p (V)$ so that $\partial$ commutes with the inclusion (1.15).

Note for comparison that for $p$-forms the inclusion $\mathcal{R}^p (V') \subset \mathcal{R}^p (V)$ is also valid, but the operator $d$ does not commute with it. For example, in the formula

$$de^i = \sum_j (e^{ji} - e^{ij})$$

the summation index $j$ on the right hand side runs over all vertices, and the result depends on the set of vertices.
2 Elements of homological algebra

2.1 Cochain complexes

A cochain complex $X$ is a sequence

$$0 \to X^0 \xrightarrow{d} X^1 \xrightarrow{d} \cdots \xrightarrow{d} X^{p-1} \xrightarrow{d} X^p \xrightarrow{d} \cdots$$  \tag{2.1}

of vector spaces $\{X^p\}_{p=0}^{\infty}$ over a field $\mathbb{K}$ and linear mappings $d : X^p \to X^{p+1}$ with the property that $d^2 = 0$ at each level. The latter means that $\text{Im} \ d|_{X^{p-1}} \subset \ker d|_{X^p}$ that allows to define the de Rham cohomologies of the complex $X$ by

$$H^p (X) = \ker d|_{X^p} / \text{Im} \ d|_{X^{p-1}}$$

(where $X^{-1} := \{0\}$). The sequence (2.1) is called exact if $H^p (X) = \{0\}$ for all $p \geq 0$.

We always assume that the spaces $X^p$ are finitely dimensional. Applying the nullity-rank theorem to the mapping $d : X^p \to X^{p+1}$, we obtain the following identity:
Lemma 2.1 We have for any $p \geq 0$

$$\dim H^p (X) = \dim X^p - \dim dX^p - \dim dX^{p-1}. \quad (2.2)$$

It implies the following.

Lemma 2.2 For a finite cochain complex

$$0 \rightarrow X^0 \xrightarrow{d} X^1 \xrightarrow{d} \ldots \xrightarrow{d} X^{n-1} \xrightarrow{d} X^n \xrightarrow{d} 0, \quad (2.3)$$

the following identity is satisfied

$$\sum_{k=0}^{n} (-1)^k \dim H^k (X) = \sum_{k=0}^{n} (-1)^k \dim X^k. \quad (2.4)$$

In particular, if the sequence (2.3) is exact, then

$$\sum_{k=0}^{n} (-1)^k \dim X^k = 0. \quad (2.5)$$
For any finite cochain complex (2.3), define its Euler characteristic by

\[ \chi(X) = \sum_{p=0}^{n} (-1)^p \dim X^p. \]

Then (2.4) implies

\[ \chi(X) = \sum_{k=0}^{n} (-1)^k \dim H^k(X). \]

2.2 Chain complexes

Given a cochain complex (2.1) with finite-dimensional spaces \( X^p \), denote by \( X_p \) the dual space to \( X^p \) and by \( \partial \) the dual operator to \( d \). Then we obtain a chain complex

\[ \begin{array}{ccccccccc}
0 & \leftarrow & X_0 & \leftarrow & X_1 & \leftarrow & \ldots & \leftarrow & X_{p-1} & \leftarrow & X_p & \leftarrow & \ldots \\
\partial & & \partial & & \partial & & \ldots & & \partial & & \partial & & \ldots
\end{array} \] \quad (2.6)

Denoting by \( (\cdot, \cdot) \) the natural pairing of dual spaces, we obtain by definition

\[ (d\omega, v) = (\omega, \partial v) \]
for all $\omega \in X^p$ and $v \in X_{p+1}$. Since $d^2 = 0$, it follows that also $\partial^2 = 0$. Hence, one can define the homologies of the chain complex (2.6) by

$$H_p (X) = \ker \partial|_{X^p} / \text{Im} \partial|_{X^{p+1}}.$$  

By duality we have

$$\ker \partial|_{X^p} = (\text{Im} d|_{X^{p-1}})^\perp, \quad \ker d|_{X^p} = (\text{Im} \partial|_{X^{p+1}})^\perp,$$

where $\perp$ refers to the annihilator in the dual space, which implies the following.

**Lemma 2.3** The spaces $H^p (X)$ and $H_p (X)$ are dual. In particular, $\dim H^p (X) = \dim H_p (X)$.

**Lemma 2.4** We have for any $p \geq 0$

$$\dim H_p (X) = \dim X_p - \dim \partial X_p - \dim \partial X^{p+1}.$$
2.3 Sub-complexes and quotient complexes

Let $X$ be a cochain complex as in (2.1), and assume that each $X^p$ has a subspace $J^p$ so that $d$ is invariant on $\{J^p\}$, that is, $dJ^p \subset J^{p+1}$. Then we have a cochain sub-complex $J$ as follows:

$$
0 \rightarrow J^0 \xrightarrow{d} J^1 \xrightarrow{d} \ldots \xrightarrow{d} J^{p-1} \xrightarrow{d} J^p \xrightarrow{d} \ldots \quad (2.8)
$$

Since the operator $d$ is well defined also on the quotient spaces $X^p/J^p$, we obtain also a cochain quotient complex $X/J$:

$$
0 \rightarrow X^0/J^0 \xrightarrow{d} X^1/J^1 \xrightarrow{d} \ldots \xrightarrow{d} X^{p-1}/J^{p-1} \xrightarrow{d} X^p/J^p \xrightarrow{d} \ldots \quad (2.9)
$$

Consider the annihilator of $J^p$, that is the space

$$(J^p)^\perp = \{v \in X_p : v \perp J^p\}.$$ 

**Lemma 2.5** The dual operator $\partial$ of $d$ is invariant on $\{(J^p)^\perp\}$, and the chain sub-complex

$$
0 \leftarrow (J^0)^\perp \xleftarrow{\partial} (J^1)^\perp \xleftarrow{\partial} \ldots \xleftarrow{\partial} (J^{p-1})^\perp \xleftarrow{\partial} (J^p)^\perp \xleftarrow{\partial} \ldots \quad (2.10)
$$
is dual to the cochain quotient complex (2.9).

**Proof.** If \( v \in (J^p)^\perp \) then, for any \( \omega \in J^{p-1} \), we have \( d\omega \in J^p \) and, hence,

\[
(\omega, \partial v) = (d\omega, v) = 0,
\]
which implies \( \partial v \in (J^{p-1})^\perp \). Hence, \( \partial \) maps \( (J^p)^\perp \) to \( (J^{p-1})^\perp \), so that the complex (2.10) is well-defined.

To prove the duality of (2.9) and (2.10), observe that \( (J^p)^\perp \) is naturally isomorphic to the dual space \( (X^p/J^p)' \). Indeed, each \( v \in (J^p)^\perp \) defines a linear functional on \( X^p/J^p \) simply by \( \omega \mapsto (\omega, v) \) where \( \omega \in X^p \) is a representative of an element of \( X^p/J^p \). If \( \omega_1 = \omega_2 \mod J^p \) then \( \omega_1 - \omega_2 \in J^p \) whence \( (\omega_1 - \omega_2, v) = 0 \) and \( (\omega_1, v) = (\omega_2, v) \). Clearly, the mapping \( (J^p)^\perp \to (X^p/J^p)' \) is injective and, hence, surjective because of the identity of the dimensions of the two spaces. Finally, the duality of the operators \( d \) and \( \partial \) on the complexes (2.9) and (2.10) is a trivial consequence of their duality on the complexes \( X^\bullet \) and \( X \).

Let us describe a specific method of constructing of \( d \)-invariant subspaces.

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Lemma 2.6  Given any subspace $S^p$ of $X^p$, set

$$J^p = S^p + dS^{p-1}. \quad (2.11)$$

Then $d$ is invariant on $\{J^p\}$. Besides, we have the following identity

$$(J^p)^\perp = \left\{ v \in (S^p)^\perp : \partial v \in (S^{p-1})^\perp \right\}. \quad (2.12)$$

Proof. The first claim follows from $d^2 = 0$ since

$$dJ^p \subset dS^p + d^2S^{p-1} = dS^p \subset J^{p+1}.$$ 

The condition $v \in (J^p)^\perp$ means that

$$v \perp S^p \quad \text{and} \quad v \perp dS^{p-1}. \quad (2.13)$$

Clearly, the first condition here is equivalent to $v \in (S^p)^\perp$, while the second condition is equivalent to

$$(d\omega, v) = 0 \ \forall \omega \in S^{p-1} \Leftrightarrow (\omega, \partial v) = 0 \ \forall \omega \in S^{p-1} \Leftrightarrow \partial v \perp S^{p-1} \Leftrightarrow \partial v \in (S^{p-1})^\perp,$$

which proves (2.12). $\blacksquare$
2.4 Zigzag Lemma

Consider now three cochain complexes $X, Y, Z$ connected by vertical linear mappings as on the diagram:

$$
\begin{array}{c}
0 \\ \downarrow \\ 0 \\
d \to Y^0 \to Y^1 \to Y^2 \to \ldots \\
\downarrow \alpha \downarrow \alpha \downarrow \alpha \\
0 \\
d \to X^0 \to X^1 \to X^2 \to \ldots \\
\downarrow \alpha \downarrow \alpha \downarrow \alpha \\
0 \\
d \to Z^0 \to Z^1 \to Z^2 \to \ldots \\
\downarrow \downarrow \downarrow \\
0 \\
\end{array}
$$

Each horizontal mapping is denoted by $d$ and each vertical mapping is denoted by $\alpha$. We assume that the diagram is commutative. Let us also assume that each column in (2.14) is an exact cochain complex, that is, the mapping $\alpha : Y^p \to X^p$ is an injection, and $\alpha : X^p \to Z^p$ a surjection with the kernel $X^p$. In this case we can identify $Y^p$ with a subspace of...
$X^p$ and $Z^p$ with the quotient $X^p/Y^p$.

**Proposition 2.7** (Zigzag Lemma) *Under the above conditions the sequence*

$$0 \to H^0(Y) \to H^0(X) \to H^0(Z) \to \cdots \to H^p(Y) \to H^p(X) \to H^p(Z) \to H^{p+1}(Y) \quad (2.15)$$

*is exact* (more precisely, the mappings in (2.15) can be defined so that the sequence is exact).

The sequence (2.15) is called a *long exact sequence in cohomology*. A similar result holds for homologies of chain complexes.

We will normally apply Proposition 2.7 in the following form: if $X$ is a cochain complex (2.1) and $J$ is its sub-complex (2.8), then the following long sequence is exact:

$$0 \to \cdots \to H^p(J) \to H^p(X) \to H^p(X/J) \to H^{p+1}(J) \to \cdots \quad (2.16)$$

Similarly, if $X$ is a chain complex (2.6) and $J$ its sub-complex, then the following long sequence is exact:

$$0 \leftarrow \cdots \leftarrow H_p(X/J) \leftarrow H_p(X) \leftarrow H_p(J) \leftarrow H_{p+1}(X/J) \leftarrow \cdots \quad (2.17)$$
2.5 Reduced cohomologies and homologies

In the cochain and chain complexes (2.1) and (2.6) one naturally defines the spaces $X^{-1}$ and $X_{-1}$ as $\{0\}$. In a number of situations there is a need in another choice of $X^{-1}$ and $X_{-1}$.

Assume that there is an injection $\tilde{d} : \mathbb{K} \rightarrow X^0$ that satisfies the relation $\tilde{d} \tilde{d} = 0$. Setting $\tilde{X}^{-1} \equiv \mathbb{K}$, we obtain an extended cochain complex

$$
0 \rightarrow \tilde{X}^{-1} \xrightarrow{\tilde{d}} X^0 \xrightarrow{d} X^1 \xrightarrow{d} \ldots \xrightarrow{d} X^{p-1} \xrightarrow{d} X^p \xrightarrow{d} \ldots
$$

The cohomologies of the complex (2.18) are denoted by $\tilde{H}^p(X)$ and are called the reduced cohomologies. Obviously, we have

$$
\tilde{H}^p(X) = \begin{cases} 
H^p(X), & p \geq 1, \\
H^0(X) / \text{const}, & p = 0.
\end{cases}
$$

The dual space $\tilde{X}_{-1}$ is also $\mathbb{K}$, and the dual operator $\tilde{\partial} : X_0 \rightarrow \mathbb{K}$ of $\tilde{d}$ is given by $\tilde{\partial}v = (\tilde{d}1, v)$ for any $v \in X_0$. Hence, we obtain an extended chain complex

$$
0 \leftarrow \tilde{X}_{-1} \xleftarrow{\tilde{\partial}} X_0 \xleftarrow{\partial} X_1 \xleftarrow{\partial} \ldots \xleftarrow{\partial} X_{p-1} \xleftarrow{\partial} X_p \xleftarrow{\partial} \ldots
$$

(2.20)
and the reduced homologies $\tilde{H}_p(X)$.

For example, let $X^0$ be a space of $\mathbb{K}$-valued functions over a finite set $V$ and assume that $d \text{ const} = 0$. Define a mapping $\tilde{d} : \mathbb{K} \to X^0$ as follows: for any $c \in \mathbb{K}$, $\tilde{d}c$ is the constant function on $V$ taking the value $c$. It follows that $d\tilde{d} = 0$ so that the reduced cohomologies are well-defined. In this case $\tilde{\partial}v = (1, v)$ where $1$ is regarded as a constant function on $V$. 
Given a finite set $V$, we define a $p$-form $\omega$ on $V$ as $\mathbb{K}$-valued function on $V^{p+1}$. The set of all $p$-forms is a linear space over $\mathbb{K}$ that is denoted by $\Lambda^p$. It has a canonical basis $e^{i_0...i_p}$. For any $\omega \in \Lambda^p$ we have

$$\omega = \sum_{i_0,...,i_p \in V} \omega_{i_0...i_p} e^{i_0...i_p}$$

where $\omega_{i_0...i_p} = \omega(i_0,...,i_p)$. The exterior derivative $d : \Lambda^p \to \Lambda^{p+1}$ is defined by

$$(d\omega)_{i_0...i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \omega_{i_0...\hat{i}_q...i_{p+1}}$$

and satisfies $d^2 = 0$. The concatenation of forms $\varphi \in \Lambda^p$ and $\psi \in \Lambda^q$ is a form $\varphi\psi \in \Lambda^{p+q}$ defined by

$$(\varphi\psi)_{i_0...i_{p+q}} = \varphi_{i_0...i_p} \psi_{i_p i_{p+1}...i_{p+q}}.$$ 

Then $d(\varphi\psi) = (d\varphi)\psi + (-1)^p \varphi d\psi$. 

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We have defined a subspace $\mathcal{R}^p \subset \Lambda^p$ of regular forms that is spanned by $e^{i_0...i_p}$ with regular paths $i_0...i_p$ (when $i_k \neq i_{k+1}$), and observed that the spaces $\mathcal{R}^p$ are invariant for $d$ and for concatenation.

A $p$-path on $V$ is a formal linear combination of the elementary $p$-paths $e_{i_0...i_p} \equiv i_0...i_p$, and the linear space of all $p$-paths is denoted by $\Lambda_p$. For any $v \in \Lambda_p$ we have

$$v = \sum_{i_0,...,i_p \in V} v^{i_0...i_p} e_{i_0...i_p}$$

and a pairing with a $p$-path $\omega$:

$$(\omega, v) = \sum_{i_0,...,i_p} \omega_{i_0...i_p} v^{i_0...i_p}.$$ 

The dual operator $\partial : \Lambda_{p+1} \rightarrow \Lambda_p$ is given by

$$\partial e_{i_0...i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q e_{i_0...\hat{i}_q...i_{p+1}}.$$ 

The product of two paths $u \in \Lambda_p$ and $v \in \Lambda_q$ is a paths $uv \in \Lambda^{p+q+1}$ defined by

$$(uv)^{i_0...i_{p}j_0...j_q} = u^{i_0...i_p} v^{j_0...j_q}.$$
It satisfies the product rule

$$\partial (uv) = (\partial u)v + (-1)^{p+1} u\partial v$$

where $\partial v$ in the case $v \in \Lambda_0$ is a constant $\sum_i v_i$ (that is equivalent to $\partial e_i = 1$).

Let $I_p$ be the subspace of $\Lambda_p$ that is spanned by $e_{i_0...i_p}$ with irregular paths $i_0...i_p$. Then the spaces $I_p$ are invariant for $\partial$ and for product, which allows to define $\partial$ and product on the quotient spaces $\mathcal{R}_p = \Lambda_p/I_p$. For simplicity of notation we identify the elements of $\mathcal{R}_p$ with their representatives that are regular $p$-paths. Then $e_{i_0...i_p}$ with irregular $i_0...i_p$ are treated as zeros.

**3 Forms and paths on digraphs**

A digraph is a pair $(V, E)$ where $V$ is an arbitrary set and $E$ is a subset of $V \times V \setminus \text{diag}$. The elements of $V$ are called vertices and the elements of $E$ are called (directed) edges. The set $V$ will be always assumed non-empty and finite.
3.1 Allowed paths

Let $i_0 \ldots i_p$ be an elementary regular $p$-path on $V$. It is called allowed if $i_k i_{k+1} \in E$ for any $k = 0, \ldots, p-1$, and non-allowed otherwise. The set of all allowed elementary $p$-paths will be denoted by $E_p$, and non-allowed – by $N_p$. For example, $E_0 = V$ and $E_1 = E$.

Denote by $\mathcal{A}_p = \mathcal{A}_p (V, E)$ the subspace of $\mathcal{R}_p$ spanned by the allowed elementary $p$-paths, that is,

$$\mathcal{A}_p = \text{span} \left\{ e_{i_0 \ldots i_p} : i_0 \ldots i_p \in E_p \right\} = \left\{ v \in \mathcal{R}_p : v^{i_0 \ldots i_p} = 0 \ \forall i_0 \ldots i_p \in N_p \right\}.$$ \hspace{1cm} (3.1)

The elements of $\mathcal{A}_p$ are called allowed $p$-paths.

Similarly, denote by $\mathcal{N}^p$ the subspace of $\mathcal{R}^p$, spanned by the non-allowed elementary $p$-forms, that is,

$$\mathcal{N}^p = \text{span} \left\{ e^{i_0 \ldots i_p} : i_0 \ldots i_p \in N_p \right\} = \left\{ \omega \in \mathcal{R}^p : \omega_{i_0 \ldots i_p} = 0 \ \forall i_0 \ldots i_p \in E_p \right\}.$$ 

Clearly, we have $\mathcal{A}_p = (\mathcal{N}^p)^\perp$ where $\perp$ refers to the annihilator subspace with respect to the couple $(\mathcal{R}^p, \mathcal{R}_p)$ of dual spaces.
3.2 The space $p$-forms on a digraph

We would like to reduce the space $\mathcal{R}^p$ of regular $p$-forms so that the non-allowed forms can be treated as zeros. Consider the following subspaces of spaces $\mathcal{R}^p$

$$\mathcal{J}^p \equiv \mathcal{J}^p (V, E) := \mathcal{N}^p + d\mathcal{N}^{p-1}, \quad (3.2)$$

that are $d$-invariant by Lemma 2.6, and define the space $\Omega^p$ of $p$-forms on the digraph $(V, E)$ by

$$\Omega^p \equiv \Omega^p (V, E) := \mathcal{R}^p / \mathcal{J}^p. \quad (3.3)$$

Then $d$ is well-defined on $\Omega^p$ and we obtain a cochain complex

$$0 \rightarrow \Omega^0 \overset{d}{\rightarrow} \ldots \overset{d}{\rightarrow} \Omega^p \overset{d}{\rightarrow} \Omega^{p+1} \overset{d}{\rightarrow} \ldots \quad (3.4)$$

Shortly we write $\Omega^\cdot = \mathcal{R}^\cdot / \mathcal{J}$ where $\Omega^\cdot$ is the complex (3.4) and $\mathcal{R}^\cdot$ and $\mathcal{J}$ refer to the corresponding cochain complexes.

If the digraph $(V, E)$ is complete, that is, $E = V \times V \setminus \text{diag}$ then the spaces $\mathcal{N}^p$ and $\mathcal{J}^p$ are trivial, and $\Omega^p = \mathcal{R}^p$.

Let us show that the concatenation is also well-defined on the spaces $\Omega^p$. 

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Lemma 3.1 Let $\varphi \in R^p$ and $\psi \in R^q$. If $\varphi \in J^p$ or $\psi \in J^q$ then $\varphi \psi \in J^{p+q}$, that is, $\{J^p\}$ is a graded ideal for the concatenation. Consequently, the concatenation of two forms is well-defined on the spaces $J^p$ as well as on $\Omega^p$, and it satisfies the product rule (1.5).

Proof. Observe first that if $\varphi \in N^p$ then $\varphi \psi \in N^{p+q}$. Indeed, it suffices to prove this for elementary forms $\varphi = e^{i_0\cdots i_p}$ and $\psi = e^{j_0\cdots j_q}$ where the claim is obvious: if the $p$-path $i_0\ldots i_p$ is non-allowed then so is the concatenated $(p + q)$-path $i_0\ldots i_p j_1\ldots j_q$.

If $\varphi \in J^p$ then $\varphi = \varphi_0 + d\varphi_1$ where $\varphi_0 \in N^p$ and $\varphi_1 \in N^{p-1}$. Then we have

$$\varphi \psi = \varphi_0 \psi + (d\varphi_1) \psi$$
$$\phantom{\varphi \psi} = \varphi_0 \psi + d(\varphi_1 \psi) - (-1)^{p-1} \varphi_1 d\psi.$$

By the above observation, all the forms $\varphi_0 \psi$, $\varphi_1 \psi$, $\varphi_1 d\psi$ are in $N$. It follows that $d(\varphi_1 \psi) \in J^{p+q}$ and, hence, $\varphi \psi \in J^{p+q}$. In the same way one handles the case $\psi \in J^q$.

To prove that concatenation is well defined on $\Omega^p$, we need to verify that if $\varphi = \varphi' \mod J^p$ and $\psi = \psi' \mod J^q$ then $\varphi \psi = \varphi \psi' \mod J^{p+q}$. 

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Indeed, we have

\[
\varphi \psi - \varphi' \psi' = \varphi (\psi - \psi') + (\varphi - \varphi') \psi',
\]

and each of the terms in the right hand side belong to \( J^{p+q} \) by the first part. Finally, the Leibniz formula for equivalence classes follows from that for their representatives. ■

Frequently it will be convenient to use the following notation. For \( p \)-forms \( \omega', \omega'' \in \mathcal{R}^p \) we write

\[
\omega' \simeq \omega'' \quad \text{if} \quad \omega' = \omega'' \mod J^p.
\]

Then the equivalence classes of \( \simeq \) are exactly the elements of \( \Omega^p \).

As it follows from Lemmas 2.6 and 3.1, \( \omega \simeq 0 \) implies \( d\omega \simeq 0 \), and if \( \varphi \simeq 0 \) or \( \psi \simeq 0 \) then \( \varphi \psi \simeq 0 \).
3.3 The space of $\partial$-invariant paths

Consider the following subspaces of $A_p$

$$\Omega_p \equiv \Omega_p (V, E) = \{ v \in A_p : \partial v \in A_{p-1} \} \quad (3.5)$$

that are $\partial$-invariant. Indeed, $v \in \Omega_p \Rightarrow \partial v \in A_{p-1} \subset \Omega_{p-1}$. The elements of $\Omega_p$ are called $\partial$-invariant $p$-paths.

We obtain a chain complex $\Omega$.

$$0 \leftarrow \Omega_0 \leftarrow \Omega_1 \leftarrow \ldots \leftarrow \Omega_{p-1} \leftarrow \Omega_p \leftarrow \ldots$$

that, in fact, is dual to $\Omega^\ast$. Indeed, by Lemma 2.5, the dual to the cochain complex $\Omega^\ast = R^\ast / \mathcal{J}$ is

$$0 \leftarrow (\mathcal{J}^0)^\perp \leftarrow (\mathcal{J}^1)^\perp \leftarrow \ldots \leftarrow (\mathcal{J}^{p-1})^\perp \leftarrow (\mathcal{J}^p)^\perp \leftarrow \ldots$$

while by Lemma 2.6 we have

$$(\mathcal{J}^p)^\perp = \left\{ v \in (\mathcal{N}^p)^\perp : \partial v \in (\mathcal{N}^{p-1})^\perp \right\}$$

$$= \left\{ v \in A_p : \partial v \in A_{p-1} \right\} = \Omega_p.$$
By construction we have $\Omega_0 = A_0$ and $\Omega_1 = A_1$ so that

$$\dim \Omega_0 = |V| \quad \text{and} \quad \dim \Omega_1 = |E|,$$

while in general $\Omega_p \subset A_p$.

Note that, unlike the operation of concatenation of forms, the operation of product of paths is not invariant in spaces $A_p$ or $\Omega_p$.

Let us define the (co)homologies of the digraph $(V, E)$ by

$$H^p (V, E) := H^p (\Omega) \quad \text{and} \quad H_p (V, E) := H_p (\Omega).$$

Recall that by Lemma 2.3 the spaces $H^p (V, E)$ and $H_p (V, E)$ are dual and, hence, their dimensions are the same. The values of $\dim H_p (V, E)$ can be regarded as invariants of the digraph $(V, E)$.

By Lemma 2.4, we have for any $p \geq 0$

$$\dim H_p (\Omega) = \dim \Omega_p - \dim \partial \Omega_p - \dim \partial \Omega_{p+1}.$$

Let us define the Euler characteristic of the digraph $(V, E)$ by

$$\chi (V, E) = \sum_{p=0}^{n} (-1)^p \dim H_p (\Omega)$$
provided \( n \) is so big that

\[
\dim H_p(\Omega) = 0 \text{ for all } p > n. \tag{3.8}
\]

We do not know if such an \( n \) exists for any finite digraph. Hence, \( \chi(V, E) \) is defined only if the digraph satisfied (3.8).

If \( \dim \Omega_p = 0 \) for \( p > n \), then by Lemma 2.2

\[
\chi(V, E) = \sum_{p=0}^{n} (-1)^p \dim \Omega_p. \tag{3.9}
\]

The definition (3.7) has an advantage that it may work even when all \( \dim \Omega_p > 0 \).
3.4 Computation of $\dim H^0$

Proposition 3.2 We have

$$\dim H^0(\Omega) = C,$$

(3.10)

where $C$ is the number of (undirected) connected components of the digraph $(V, E)$.

Proof. By definition,

$$H^0(\Omega) = \ker d|_{\Omega^0} = \{f \in \Omega^0 : df \simeq 0\} .$$

The condition $df \simeq 0$ means that $(df)_{ij} = 0$ for all $ij \in E$, that is, $f_i = f_j$ for all edges $ij$. The latter is equivalent to the fact that $f = \text{const}$ on any connected component of $(V, E)$, and the dimension of this space of functions is clearly $C$. ■

3.5 Some condition for $\dim \Omega^p = 0$

Proposition 3.3 If $\dim \Omega^n \leq 1$ then $\dim \Omega^p = 0$ for all $p > n$. 
**Proof.** Assume \( \dim \Omega^n = 0 \). Any regular \( p \)-form \( e^{i_0 \ldots i_p} \) with \( p > n \) is a concatenation of an \( n \)-form and a \((p-n)\)-form:

\[
e^{i_0 \ldots i_p} = e^{i_0 \ldots i_n} e^{i_n \ldots i_p}.
\]

Since \( e^{i_0 \ldots i_n} \simeq 0 \) by hypothesis, it follows by Lemma 3.1 that also \( e^{i_0 \ldots i_p} \simeq 0 \), whence \( \dim \Omega^p = 0 \).

Let now \( \dim \Omega^n = 1 \). We have

\[
e^{i_0 \ldots i_p} = e^{i_0 \ldots i_n} e^{i_n \ldots i_p} = e^{i_0 i_1} e^{i_1 \ldots i_{n+1}} e^{i_{n+1} \ldots i_p}.
\]

We claim that

\[
\text{either } e^{i_0 \ldots i_n} \simeq 0 \text{ or } e^{i_1 \ldots i_{n+1}} \simeq 0, \tag{3.11}
\]

which would imply that \( e^{i_0 \ldots i_p} \simeq 0 \) and \( \dim \Omega^p = 0 \). Indeed, if \( (3.11) \) fails then both forms \( e^{i_0 \ldots i_n} \) and \( e^{i_1 \ldots i_{n+1}} \) belong to non-zero equivalence classes modulo \( \mathcal{J}^n \). Since the latter has dimension 1, it follows that

\[
e^{i_0 \ldots i_n} = \text{const} \ e^{i_1 \ldots i_{n+1}} \mod \mathcal{J}^n.
\]

Clearly, this identity is only possible if \( e^{i_0 \ldots i_n} = e^{i_1 \ldots i_{n+1}} \) whence \( i_0 = i_1 \ldots = i_{n+1} \), which contradicts the regularity. \( \blacksquare \)
3.6 Poincaré lemma for star-like graphs

We say that a digraph \((V, E)\) is star-like if there is a vertex \(a \in V\) (called a star center) such that \(ai \in E\) for all \(i \neq a\). For example, here is a star-like digraph:

\[
\begin{array}{c}
a \bullet & \longrightarrow & \bullet \\
\downarrow & \searrow & \vdots \\
\bullet & \cdots & \bullet \\
\end{array}
\]

Clearly, a complete digraph is star-like.

**Theorem 3.4** If \((V, E)\) is a star-like digraph, then \(H_p (V, E) = \{0\}\) for any \(p \geq 1\). Consequently, \(\chi (V, E) = 1\).

**Proof.** We prove that \(H_p (V, E) = \{0\}\). For that we need to prove that if \(v \in \Omega_p\) and \(\partial v = 0\) then \(v = \partial \omega\) for some \(\omega \in \Omega_{p+1}\). Set \(\omega = e_a v\). We claim that \(\omega \in A_{p+1}\). Since \(v\) is a linear combination of allowed paths \(e_{i_0...i_p}\), it suffices to show that \(e_{ai_0...i_p} \in A_{p+1}\) for any allowed path \(e_{i_0...i_p}\). Indeed, if \(i_0 = a\) then \(e_{ai_0...i_p} = 0 \in A_{p+1}\). If \(i_0 \neq a\) then \(e_{ai_0...i_p}\) is allowed by the star condition. Hence, we have \(\omega \in A_{p+1}\).
By the product rule (1.14) we have

$$\partial \omega = \partial (e_a v) = v - e_a \partial v = v,$$

where we have used $\partial v = 0$. It follows that $\partial \omega \in \mathcal{A}_p$ and, hence, $\omega \in \Omega_{p+1}$, which finishes the proof of $H_p(V, E) = \{0\}$.

Since the graph $(V, E)$ is connected, we have also $\dim H_0(V, E) = 1$ by Proposition 3.2. It follows that $\chi = 1$. 

**Remark.** In a similar manner one can handle the *inverse* star-like graphs, that is, when the requirement $a_i \in E$ in the definition of a start property is replaced by $i a \in E$. Using the right multiplication with $e_a$, one proves in the same way that the statement of Theorem 3.4 remains true for inverse star-like graph.

**Example.** The graph

```
  0
   ↓↑ ↘
    2 ←− 1
```

is star like with the star center 0. Hence, $\chi = 1$. 

44
3.7 Computation of \( \dim \Omega_2 \)

Recall that \( \dim \Omega_0 = \dim \mathcal{A}_0 = |V| \) and \( \dim \Omega_1 = \dim \mathcal{A}_1 = |E| \). Here we compute \( \dim \Omega_2 \). We say that a pair \( ac \in V \times V \setminus \text{diag} \) is a semi-edge if \( ac \) is not an edge, but there is \( b \in V \) such that both \( ab \) and \( bc \) are edges:

\[
\begin{array}{c}
\quad b \\
\quad \downarrow \\
\quad \rightarrow \\
\quad \leftarrow \\
\quad \quad a \quad \rightarrow \quad \quad c
\end{array}
\]

Denote by \( \mathcal{S} \) the set of all semi-edges of a digraph \((V, E)\).

**Proposition 3.5** We have

\[
\dim \Omega_2 = \dim \mathcal{A}_2 - |\mathcal{S}| = |E_2| - |\mathcal{S}|.
\]  

(3.12)

**Proof.** Recall that

\[
\mathcal{A}_2 = \text{span} \{ e_{abc} : abc \text{ is allowed} \}, \quad \dim \mathcal{A}_2 = |E_2|,
\]

and

\[
\Omega_2 = \{ v \in \mathcal{A}_2 : \partial v \in \mathcal{A}_1 \} = \{ v \in \mathcal{A}_2 : \partial v = 0 \mod \mathcal{A}_1 \}.
\]
If $abc$ is allowed then $ab$ and $bc$ are edges, whence
\[ \partial e_{abc} = e_{bc} - e_{ac} + e_{ab} = -e_{ac} \mod A_1. \]

If $ac$ is an edge then $e_{ac} = 0 \mod A_1$. If $ac$ is not an edge then $ac$ is a semi-edge, and in this case
\[ \partial e_{abc} \neq 0 \mod A_1. \]

For any $v \in \Omega_2$, we have
\[ v = \sum_{\{abc \text{ is allowed}\}} v^{abc} e_{abc}, \]

hence it follows that
\[ \partial v = -\sum_{\{abc: \text{ac is semi-edge}\}} v^{abc} e_{ac} \mod A_1. \]

The condition $\partial v = 0 \mod A_1$ is equivalent to
\[ \sum_{\{abc: \text{ac is semi-edge}\}} v^{abc} e_{ac} = 0 \mod A_1, \]
which is equivalent to $\sum_b v^{abc} = 0$ for all semi-edges $ac$. The number of these conditions is exactly $|S|$, and they all are independent for different semi-edges, because a triple $abc$ determines at most one semi-edge. Hence, $\Omega_2$ is obtained from $A_2$ by imposing $|S|$ linearly independent conditions, which implies (3.12).

Let us call by a triangle a sequence of three distinct vertices $a, b, c \in V$ such that $ab, bc, ac$ are edges:

Note that a triangle determines a 2-path $e_{abc} \in \Omega_2$ as $e_{abc} \in A_2$ and

$$\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in A_1.$$

Let us called by a square a sequence of four distinct vertices $a, b, b', c \in V$ such that $ab, bc, ab', b'c$ are edges:
Note that a square determines a 2-path $v := e_{abc} - e_{ab'c} \in \Omega_2$ as $v \in A_2$ and

$$
\partial v = (e_{bc} - e_{ac} + e_{ab}) - (e_{b'c} - e_{ac} + e_{ab'}) = e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in A_1.
$$

**Corollary 3.6** If $(V, E)$ contains no squares then $\dim \Omega_2$ is equal to the number of distinct triangles in $(V, E)$. In particular, if $(V, E)$ contains neither triangles nor squares then $\dim \Omega_2 = 0$.

**Proof.** Let us split the family $E_2$ of allowed 2-paths into two subsets: an allowed path $abc$ is of the first kind if $ac$ is an edge and of the second kind otherwise:

1st kind: $\bullet_{\searrow} \quad \bullet_{\swarrow}$, 2nd kind: $\bullet_{\nearrow} \quad \bullet_{\swarrow}$

Clearly, the paths of the first kind are in one-to-one correspondence with triangles. Each path $abc$ of the second kind determines a semi-edge $ac$. The mapping of $abc \mapsto ac$ from the paths of second kind to semi-edges
is also one-to-one: if $abc \mapsto ac$ and $ab'c \mapsto ac$ then we obtain a square $a, b, b', c$ which contradicts the hypotheses. Hence, the number of the path of the second kind is equal to $|S|$, which implies that the number of the paths of the first kind is equal to $|E_2| - |S|$, and so is the number of triangles. Comparing with (3.12) we finish the proof. \[\blacksquare\]

In the presence of squares one cannot relate directly $\dim \Omega_2$ to the number of squares and triangles as there may be a linear dependence between them. Indeed, in the following digraph

there are three squares $a, b_i, b_j, c$, which determine three paths

$$ab_1c - ab_2c, \quad ab_2c - ab_3c, \quad ab_3c - ab_1c$$

that are linearly dependent (the sum is equal to 0). In fact, $\dim \Omega_2 = 2$ as $|E_2| = 3$ and there is only one semi-edge $ac$.\[49\]
3.8 An example of direct computation of $\dim H_p$

Consider the graph of 6 vertices $V = \{0, 1, 2, 3, 5\}$ with 8 edges $E = \{01, 02, 13, 14, 23, 24, 53, 54\}$.

Let us compute the spaces $\Omega_p$ and the homologies $H_p(\Omega)$. We have

$\Omega_0 = A_0 = \text{span} \{e_0, e_1, e_2, e_3, e_4, e_5\}, \quad \dim \Omega_0 = 6$

$\Omega_1 = A_1 = \text{span} \{e_{01}, e_{02}, e_{13}, e_{14}, e_{23}, e_{24}, e_{53}, e_{54}\}, \quad \dim \Omega_1 = 8$

$A_2 = \text{span} \{e_{013}, e_{014}, e_{023}, e_{024}\}, \quad \dim A_2 = 4.$
The set of semi-edges is $S = \{e_{03}, e_{04}\}$ so that $\dim \Omega_2 = \dim \mathcal{A}_2 - |S| = 2$. The basis in $\Omega_2$ can be easily spotted as each of two squares 0, 1, 2, 3 and 0, 1, 2, 4 determine a $\partial$-invariant 2-paths, whence

$$
\Omega_2 = \text{span} \{e_{013} - e_{023}, e_{014} - e_{024}\}.
$$

Since there are no allowed 3-paths, we see that $\mathcal{A}_3 = \Omega_3 = \{0\}$. It follows that

$$
\chi = \dim \Omega_0 - \dim \Omega_1 + \dim \Omega_2 = 6 - 8 + 2 = 0.
$$

Let us compute $\dim H_2$ by (3.6):

$$
\dim H_2 = \dim \Omega_2 - \dim \partial \Omega_2 - \dim \partial \Omega_3 = 2 - \dim \partial \Omega_2.
$$

The image $\partial \Omega_2$ is spanned by two 1-paths

$$
\partial (e_{013} - e_{023}) = e_{13} - e_{03} + e_{01} - (e_{23} - e_{03} + e_{02}) = e_{13} + e_{01} - e_{23} - e_{02}
$$

$$
\partial (e_{014} - e_{024}) = e_{14} - e_{04} + e_{01} - (e_{24} - e_{04} + e_{02}) = e_{14} + e_{01} - e_{24} - e_{02}
$$

that are clearly linearly independent. Hence, $\dim \partial \Omega_2 = 2$ whence $\dim H_2 = 0$. The dimension of $H_1$ can be computed similarly, but we can do easier using the Euler characteristic:

$$
\dim H_0 - \dim H_1 + \dim H_2 = \chi = 0
$$
whence $\dim H_1 = 1$.

In fact, a non-trivial element of $H_1$ is determined by 1-path

$$v = e_{13} - e_{14} - e_{53} + e_{54}.$$ 

Indeed, by a direct computation $\partial v = 0$, so that $v \in \ker \partial|_{\Omega_1}$ while for $v$ to be in $\text{Im} \partial|_{\Omega_2}$ it should be a linear combination of $\partial (e_{013} - e_{023})$ and $\partial (e_{014} - e_{024})$, which is not possible since they do not have the term $e_{54}$. 
3.9 Cycle graphs

We say that a digraph \((V, E)\) is a (undirected) cycle if it is connected and every vertex had the degree 2.

For a cycle graph we have \(\dim H_0 = 1\) and

\[
\dim \Omega_0 = |V| = |E| = \dim \Omega_1. \tag{3.13}
\]

**Proposition 3.7** Let \((V, E)\) be a cycle graph. Then

\[
\dim \Omega_p = 0 \text{ for all } p \geq 3
\]

\[
\dim H_p(\Omega) = 0 \text{ for all } p \geq 2.
\]
If \((V, E)\) is a triangle or a square then
\[
\dim \Omega_2 = 1, \quad \dim H_1 (\Omega) = 0, \quad \chi = 1
\]
whereas otherwise
\[
\dim \Omega_2 = 0, \quad \dim H_1 (\Omega) = 1, \quad \chi = 0.
\]

**Proof.** Observe first that \(\dim \Omega_2 \leq 1\) will imply \(\dim \Omega_p = 0\) for all \(p \geq 3\) by Proposition 3.3, whence \(\dim H_p = 0\) for \(p \geq 3\). Hence, we need only to handle the cases \(p = 1, 2\).

Using two equivalent definition of the Euler characteristic, we have
\[
\chi = \dim H_0 - \dim H_1 + \dim H_2
\]
\[
= \dim \Omega_0 - \dim \Omega_1 + \dim \Omega_2
\]
whence
\[
\chi = \dim \Omega_2 = 1 - \dim H_1 + \dim H_2. \tag{3.14}
\]

Assume first that \((V, E)\) is neither a triangle nor a square. Then \((V, E)\) contains neither a triangle nor a square. By Corollary 3.6 \(\dim \Omega_2 = 0\) whence \(\dim H_2 = 0\) and by (3.14) \(\chi = 0\) and \(\dim H_1 = 1\).
Let us construct an 1-path spanning $H_1$. For that let us identify $V$ with $\mathbb{Z}_N$ where $N = |V|$ so that in the unoriented graph based on $(V, E)$, the edges are $i \ (i + 1)$. Hence, in the digraph $(V, E)$ either $i \ (i + 1)$ or $(i + 1) \ i$ is an edge. Consider an allowed 1-path $v$ with components $v^{i(i+1)} = 1$ if $i \ (i + 1)$ is an edge, and $v^{(i+1)i} = -1$ if $(i + 1) \ i$ is an edge (and all other components of $v$ vanish):

Since $v \neq 0$, $v$ is not in $\text{Im} \ \partial|_{\Omega_2}$. However, $v \in \ker \partial_{\Omega_1}$ because by construction $v^{i(i+1)} - v^{(i+1)i} \equiv 1$ whence for any $i$

$$ (\partial v)^i = \sum_{j \in V} (v^{ji} - v^{ij}) = v^{(i-1)i} + v^{(i+1)i} - v^{i(i-1)} - v^{i(i+1)} = 1 - 1 = 0. $$
Let \((V, E)\) be a triangle, say, with vertices \(a, b, c\) then \(\dim A_2 = 1, \mathcal{S} = \emptyset\) whence \(\dim \Omega_2 = 1\) and \(\chi = 1\). Note that in this case \(\Omega_2 = \text{span} \{e_{abc}\}\).

Since a triangle is star-like, we have by Theorem 3.4 \(\dim H_p = 0\) for all \(p \geq 1\).

Let \((V, E)\) be a square, say \(a, b, b', c\):

\[
\begin{array}{cccc}
  & & b & \\
  & \bullet & & \\
a & \bullet & \nearrow & c \\
& & b' \\
\end{array}
\]

Then
\[
A_2 = \text{span} \{e_{abc}, e_{ab'c}\}, \quad \mathcal{S} = \{ac\}
\]
whence \(\dim \Omega_2 = 2 - 1 = 1\) and \(\chi = 1\). Note that in this case
\[
\Omega_2 = \text{span} \{e_{abc} - e_{ab'c}\}.
\]

For \(v = e_{abc} - e_{ab'c}\) we have \(\partial v = e_{bc} - e_{b'c} + e_{ab} - e_{ab'} \neq 0\) so that \(\ker \partial|_{\Omega_2} = 0\). It follows that \(\dim H_2 = 0\). Then by (3.14) \(\dim H_1 = 0\). ■
3.10 Examples of $\partial$-invariant paths

3.10.1 Snake and simplex

A *snake* of length $p$ is a subgraph of $p + 1$ vertices, say, $0, 1, \ldots, p$ such that $i(i + 1)$ and $i(i + 2)$ are edges, which is equivalent to say that any triple $i(i + 1)(i + 2)$ is a triangle.

![Diagram of a snake](image)

Any snake gives rise to a $\partial$-invariant $p$-path $v = e_{01\ldots p}$. This path is obviously allowed, its boundary

$$\partial v = e_{1\ldots p} - e_{02\ldots p} + e_{013\ldots p} - \ldots + (-1)^p e_{01\ldots p-1}$$
is also allowed, so that indeed \( v \in \Omega_p \).

A *simplex* of dimension \( p \) is a subgraph of \( p + 1 \) vertices, say \( 0, 1, \ldots, p \) so that any pair \( ij \) with \( i < j \) is an edge. For example, a simplex of dimension 2 is a triangle

\[
\begin{array}{c}
\bullet \rightarrow \bullet \\
0 & \rightarrow & 1 & \rightarrow & 2
\end{array}
\]

a simplex of dimension 3 is shown here:

\[
\begin{array}{c}
\bullet \rightarrow \bullet \\
0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 3
\end{array}
\]

Since the simplex contains a snake as a subgraph, the \( p \)-path \( v = e_{01\ldots p} \) is \( \partial \)-invariant also on a simplex.
3.10.2 Cylinder and hypercube

For any graph \((V, E)\) consider its product with graph \(0 \rightarrow 1\) that will be denoted by \((\hat{V}, \hat{E})\) where \(\hat{V} = V \times \{0, 1\}\) and the set of edges \(\hat{E}\) is defined by \((x, a) \rightsquigarrow (y, b)\) if and only if either \(x \rightsquigarrow y\) in \((V, E)\) and \(a = b\) or \(x = y\) and \(a \rightsquigarrow b\):

The graph \((\hat{V}, \hat{E})\) is a cylinder over \((V, E)\). We mark by the hat \(\hat{\cdot}\) all the notions related to the graph \((\hat{V}, \hat{E})\).

It will be convenient to identify \(V \times \{0, 1\}\) with \(V \sqcup V'\) where \(V'\) is a copy of \(V\), and set the notation \((x, 0) \equiv x\) and \((x, 1) \equiv x'\). Define the operation of lifting paths from \(V\) to \(\hat{V}\) as follows. If \(v = e_{i_0...i_p}\) then \(\hat{v}\) is
a \((p + 1)\)-path in \((\hat{V}, \hat{E})\) defined by

\[
\hat{v} = \sum_{k=0}^{p} (-1)^k e_{i_0...i_k i'_k...i'_p}.
\]

Clearly, if \(i_0...i_p\) is allowed in \((V, E)\) then \(i_0...i_k i'_k...i'_p\) is allowed in \((\hat{V}, \hat{E})\):

\[
\begin{array}{ccccccc}
\cdots & i'_k & \rightarrow & i'_{k+1} & \rightarrow & \cdots & i'_p \\
\uparrow & \uparrow & & & & \uparrow & \\
(\hat{v}) & \rightarrow & \cdots & \rightarrow & i_k & \rightarrow & \cdots
\end{array}
\]

Extending by linearity this definition for a general \(p\)-path \(v\) on \((V, E)\), we obtain that if \(v \in \mathcal{A}_p\) then \(\hat{v} \in \hat{\mathcal{A}}_{p+1}\).

**Proposition 3.8**  If \(v \in \Omega_p\) then \(\hat{v} \in \hat{\Omega}_{p+1}\).

**Proof.** We need to prove that if \(v \in \mathcal{A}_p\) and \(\partial v \in \mathcal{A}_{p-1}\) then \(\partial \hat{v} \in \hat{\mathcal{A}}_p\).

Let us prove first some properties of the lifting. For any path \(v\) in \((V, E)\) define its image \(v'\) in \((V', E')\) by

\[
(e_{i_0...i_p})' = e_{i'_0...i'_p}.
\]
Let us show first that for any $p$-path $u$ and $q$-path $v$ on $(V,E)$, the following identity holds:

$$\hat{uv} = \hat{uv}' + (-1)^{p+1} u\hat{v} \quad (3.15)$$

It suffices to prove it for $u = e_{i_0...i_p}$ and $v = e_{j_0...j_q}$. Then $uv = e_{i_0...i_pj_0...j_q}$ and

$$\hat{uv} = \sum_{k=0}^{p} (-1)^k e_{i_0...i_ki'_k...i'_pj_0...j'_q} + \sum_{k=0}^{q} (-1)^{k+p+1} e_{i_0...i_pj_0...j_kj'_k...j'_q}$$

$$= \hat{uv}' + (-1)^{p+1} u\hat{v}.$$ 

Now let us show that, for any $p$-path $v$ with $p \geq 1$,

$$\partial\hat{v} = -\hat{\partial v} + v' - v. \quad (3.16)$$

It suffices to prove it for $v = e_{i_0...i_p}$, which will be done by induction in $p$. For $p = 1$ write $v = e_{ab}$ so that $\hat{v} = e_{aa'b'} - e_{abb'}$ and

$$\partial\hat{v} = (e_{a'b'} - e_{ab'} + e_{aa'}) - (e_{bb'} - e_{ab'} + e_{ab})$$

$$= e_{aa'} - e_{bb'} + e_{a'b'} - e_{ab}$$

$$= - (e_{b} - e_{a}) + v' - v$$

$$= -\hat{\partial v} + v' - v.$$
For $p > 1$ write $v = u e_{i_p}$ where $u = e_{i_0 \ldots i_{p-1}}$. Using (3.15) and the inductive hypothesis with the $(p - 1)$-path $u$ we obtain

\[
\begin{align*}
\partial \hat{v} &= \partial \left( \hat{u} e_{i'_p} + (-1)^p u e_{i_p i'_p} \right) \\
&= (\partial \hat{u}) e_{i'_p} + (-1)^{p+1} \hat{u} + (-1)^p (\partial u) e_{i_p i'_p} + u (e_{i'_p} - e_{i_p}) \\
&= \left[ -\partial \hat{u} + u' - u \right] e_{i'_p} + (-1)^{p+1} \hat{u} + (-1)^p (\partial u) e_{i_p i'_p} + ue_{i'_p} - v \\
&= -(\partial \hat{u}) e_{i'_p} + v' + (-1)^{p+1} \hat{u} + (-1)^p (\partial u) e_{i_p i'_p} - v
\end{align*}
\]

On the other hand,

\[
\begin{align*}
\hat{\partial} v &= ( (\partial u) e_{i_p} + (-1)^p u ) \\
&= (\hat{\partial} u) e_{i'_p} + (-1)^{p-1} (\partial u) e_{i_p i'_p} + (-1)^p \hat{u},
\end{align*}
\]

whence it follows that $\partial \hat{v} + \hat{\partial} v = v' - v$, which finishes the proof of (3.16).

Finally, if $v \in \mathcal{A}_p$ and $\partial v \in \mathcal{A}_{p-1}$ then $v'$ and $\hat{\partial} v$ belong to $\hat{\mathcal{A}}_p$ whence it follows from (3.16) also $\partial \hat{v} \in \hat{\mathcal{A}}_p$. This proves that $\hat{v} \in \hat{\mathcal{A}}_{p+1}$. ■
Example. The cylinder over a triangle 012 is the following graph:

Since 2-path $e_{012}$ is $\partial$-invariant on the triangle, lifting it to the cylinder, we obtain a $\partial$-invariant 3-path $e_{00'1'2'} - e_{011'2'} + e_{0122'}$, that can be written in the form $e_{0345} - e_{0145} + e_{0125}$.

Example. The cylinder over the graph $0 \rightarrow 1$ is a square

2 \rightarrow 3

\uparrow \quad \uparrow

0 \rightarrow 1
Lifting a $\partial$-invariant 1-path $e_{01} \in \Omega_1$ we obtain a $\partial$-invariant 2-path on the square $e_{00'} - e_{011'}$ that we rewrite in the form $e_{023} - e_{013}$.

The cylinder over a square is a 3-cube:

Lifting the 2-path $e_{023} - e_{013}$ we obtain a $\partial$-invariant 3-path

$$e_{00'2'3'} - e_{022'3'} + e_{0233'} - e_{00'1'3'} + e_{011'3'} - e_{0133'}$$

that we can rewrite in the form

$$e_{0467} - e_{0267} + e_{0237} - e_{0457} + e_{0157} - e_{0137}.$$ 

Similarly, any binary hypercube of dimension $p$ determines a $\partial$-invariant $p$-path that is an alternating sum of $p!$ terms.
3.11 Lemma of Sperner revisited

Consider a triangle $ABC$ on the place, and its triangulation $T$. The set $S$ of vertices of $T$ is colored with three colors $1, 2, 3$ in such a way that

- the vertices $A, B, C$ are colored with $1, 2, 3$ respectively;
- each vertex on an edge of $ABC$ is colored only with one of the two colors of the ends of its edge.
The classical lemma of Sperner says that then there exists a 3-color triangle from $T$, that is, a triangle, whose vertices are colored with the three different colors. Moreover, the number of such triangles is odd.

We give here a new proof using the boundary operator $\partial$ for 1-paths. Let us first do some reduction. Firstly, let us change the triangulation $T$ so that there are no vertices on the edges $AB, AC, BC$ except for $A, B, C$. Indeed, if $X$ is a vertex on $AB$ then move $X$ a bit inside the triangle $ABC$. This gives rise to a new triangle in the triangulation $T$ that is formed by $X$ and its former neighbors, say $Y$ and $Z$, on the edge $AB$. However, since all $X, Y, Z$ are colored with two colors, no 3-color triangle appears after this move. By induction, we remove all vertices from the edges of $ABC$.

Secondly, we project the triangle $ABC$ and the triangulation $T$ onto the sphere $S^2$ and add to the set $T$ the triangle $ABC$ itself from the other side of the sphere. Then we obtain a triangulation of $S^2$, denote it again by $T$, and we need to prove that the number of 3-color triangles is even. Indeed, since we know that one of the triangles, namely, $ABC$ is 3-color, this would imply that the number of 3-color triangles in the original triangulation was odd.
Let us regard $T$ as a graph on $\mathbb{S}^2$ and construct a dual graph $V$. Chose at each face of $T$ a point and regard them as vertices of the dual graph $V$. The vertices in $V$ are connected if the corresponding triangles in $T$ have a common edge. Then the faces of $V$ are in one-to-one correspondence to the vertices of $T$.

Hence, given a graph $V$ on $\mathbb{S}^2$ such that each vertex has degree 3 and each face is colored with one of the colors 1, 2, 3, prove that the number of 3-color vertices (that is, the vertices, whose adjacent faces have all three colors) is even.
Let us make $V$ into a digraph as follows. Each edge $\xi$ in $V$ has two adjacent faces. Choose the orientation on $\xi$ so that the color from the left hand side and the color from the right hand side of $\xi$ form one of the following pairs: $(1, 2), (2, 3), (3, 1)$ (if the colors are the same then allow both orientations of $\xi$).

For example:
Denote by $E$ the set of the oriented edges and set $v = \sum_{ab \in E} e_{ab}$. We have for any $a \in V$

$$(\partial v)_a = \sum_b v^{ba} - \sum_c v^{ac} = \#\{\text{incoming edges}\} - \#\{\text{outcoming edges}\}. $$

If $a$ is 3-color, then either all three edges at $a$ are outcoming or all are incoming whence

$$(\partial v)_a = \pm 3 = 1 \mod 2. $$

Otherwise $(\partial v)_a = 0$ (see the above pictures). Denoting by $n$ the number of 3-color vertices, we obtain

$$(\partial v, 1) = \sum_{a \in V} (\partial v)_a = n \mod 2. $$

On the other hand, $(\partial v, 1) = (v, d1) = 0$ whence we conclude that

$$n = 0 \mod 2. $$
**Brief summary**

A $p$-path on finite set $V$ is a formal linear combination of the elementary $p$-paths $e_{i_0...i_p} \equiv i_0...i_p$, where $i_k \in V$, and the linear space of all $p$-paths is denoted by $\Lambda_p$. For any $v \in \Lambda_p$ we write

$$v = \sum_{i_0,...,i_p \in V} v^{i_0...i_p} e_{i_0...i_p}.$$  

The boundary operator $\partial : \Lambda_{p+1} \to \Lambda_p$ is defined by

$$\partial e_{i_0...i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q e_{i_0...\hat{i}_q...i_{p+1}}.$$  

It satisfies $\partial^2 = 0$.

The product of two paths $u \in \Lambda_p$ and $v \in \Lambda_q$ is a paths $uv \in \Lambda^{p+q+1}$ defined by

$$(uv)^{i_0...i_p j_0...j_q} = u^{i_0...i_p} v^{j_0...j_q}.$$  

It satisfies the product rule

$$\partial (uv) = (\partial u)v + (-1)^{p+1} u\partial v.$$
Let $I_p$ be the subspace of $\Lambda_p$ that is spanned by irregular $e_{i_0...i_p}$ (a path $i_0...i_p$ is irregular if $i_k = i_{k+1}$ for some $k$). Then the spaces $I_p$ are invariant for $\partial$ and for product, which allows to define $\partial$ and product on the quotient spaces $\mathcal{R}_p = \Lambda_p/I_p$. We identify the elements of $\mathcal{R}_p$ with their representatives that are regular $p$-paths. Then $e_{i_0...i_p}$ with irregular $i_0...i_p$ are treated as zeros.

Let $(V, E)$ be a digraph, that is, $E \subset V \times V \setminus \text{diag}$ is a set of directed edges. An elementary regular path $e_{i_0...i_p}$ is called allowed if $i_ki_{k+1} \in E$ for all $k$, and non-allowed otherwise.

Let $A_p$ be a subspace of $\mathcal{R}_p$ that is spanned by all allowed $e_{i_0...i_p}$. The elements of $A_p$ are called allowed $p$-paths. For example, $A_0$ consists of linear combinations of all vertices, and $A_1$ consists of linear combinations of all edges.

In general, the spaces $A_p$ is not $\partial$-invariant, so we introduce smaller spaces

$$\Omega_p = \{v \in A_p : \partial v \in A_{p-1}\},$$

that are $\partial$-invariant, that is, $\partial \Omega_p \subset \Omega_{p-1}$. The elements of $\Omega_p$ are called $\partial$-invariant paths.

Note that $\Omega_0 = A_0$ and $\Omega_1 = A_1$, but for $p \geq 2$, $\Omega_p$ can actually be
smaller than $A_p$. We obtain a chain complex

$$0 \leftarrow \Omega_0 \leftarrow \Omega_1 \leftarrow \ldots \leftarrow \Omega_{p-1} \leftarrow \Omega_p \leftarrow \ldots$$

whose homologies $H_p(\Omega) = H_p(V, E)$ are the subject for our study. So far we know that $\dim H_0(V, E)$ is equal to the number of connected components of the graph.

4 Surgery of digraphs

4.1 Homologies of subgraphs

Let $(V', E')$ be a subgraph of $(V, E)$ in the sense that $V' \subset V$ and $E' \subset E$. Let us mark by the dash "$'$" all the notation related to the graph $(V', E')$ rather than to $(V, E)$.

As it was already observed, $\mathcal{R}'_p \subset \mathcal{R}_p$ and $\partial$ commutes with this inclusion. It is also obvious that if $e_{i_0 \ldots i_p}$ is an allowed path in $(V', E')$ then it is also allowed in $(V, E)$, whence $A'_p \subset A_p$. 
By the definition (3.5) of $\Omega_p$, we obtain that $\Omega'_p \subset \Omega_p$ and $\partial$ commutes with this inclusion. Consequently, the chain complex

$$0 \leftarrow \Omega'_0 \overset{\partial}{\leftarrow} \Omega'_1 \overset{\partial}{\leftarrow} \Omega'_2 \overset{\partial}{\leftarrow} \Omega'_3 \overset{\partial}{\leftarrow} \ldots$$

is a sub-complex of

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \Omega_2 \overset{\partial}{\leftarrow} \Omega_3 \overset{\partial}{\leftarrow} \ldots$$

By Proposition 2.7 (cf. (2.17)) we obtain that the following long sequence is exact:

$$0 \leftarrow H_0(\Omega/\Omega') \leftarrow H_0(\Omega) \leftarrow H_0(\Omega') \leftarrow \cdots \leftarrow H_p(\Omega/\Omega') \leftarrow H_p(\Omega) \leftarrow H_p(\Omega') \leftarrow H_{p+1}(\Omega/\Omega') \leftarrow \cdots$$

(4.1)
4.2 Removing a vertex of degree 1

Theorem 4.1 Suppose that a graph $(V, E)$ has a vertex $a$ such that there is only one outcoming edge $ab$ from $a$ and no incoming edges to $a$. Let $V' = V \setminus \{a\}$ and $E' = E \setminus \{ab\}$.

![Diagram showing vertex $a$ with an outgoing edge to vertex $b$ and $V'$ and $V$ sets]

Then $H_p(V, E) \cong H_p(V', E')$ for all $p \geq 0$.

Remark. The same is true if the edge $ab$ in the statement is replaced by $ba$.

Proof. Let us first prove that $\Omega'_p = \Omega_p$ for $p \geq 2$. Since always $\Omega'_p \subset \Omega_p$, it suffices to prove the opposite inclusion $\Omega_p \subset \Omega'_p$. Let us first show that, for all $p \geq 2$,

$$\Omega_p \subset \mathcal{A}'_p,$$  \hspace{1cm} (4.2)

that is

$$v \in \mathcal{A}_p \text{ and } \partial v \in \mathcal{A}_{p-1} \Rightarrow v \in \mathcal{A}'_p.$$
Every elementary allowed $p$-path on $(V, E)$ either is allowed on $(V', E')$ or starts with $ab$, which implies that $v$ can be represented in the form

$$v = e_{ab}u + v',$$

where $v' \in A'_p$, while $u \in A'_{p-2}$ is a linear combination of the paths $e_{i_0\ldots i_{p-2}} \in A'_{p-2}$ with $i_0 \neq b$. It follows that

$$\partial v = (e_{b} - e_{a}) u + e_{ab} \partial u + \partial v'.$$  \hspace{1cm} (4.3)

Note that $e_{a}u$ is a linear combination of the elementary paths $e_{ai_0\ldots i_{p-2}}$ where $i_0, \ldots, i_{p-2} \in V'$ and $i_0 \neq b$. Since $ai_0$ is not an edge, those elementary paths are not allowed in $(V, E)$. No other terms in the right hand side of (4.3) has $e_{ai_0\ldots i_{p-2}}$-component. Since $\partial v$ is allows in $(V, E)$, its $e_{ai_0\ldots i_{p-2}}$-component is 0, which is only possible if $e_{a}u = 0$, that is, $u = 0$. It follows that $v = v' \in A'_p$, which finishes the proof of (4.2).

Let us now show that $\Omega_p \subset \Omega'_p$ for all $p \geq 2$. Indeed, if $v \in \Omega_p$ then by definition $\partial v \in A_{p-1}$ and by (4.2) $v \in A'_p$, which together imply $\partial v \in A'_{p-1}$. It follows that $v \in \Omega'_p$. Consequently, we have proved that

$$\Omega_p = \Omega'_p \text{ for all } p \geq 2.$$  \hspace{1cm} (4.4)
It follows that, for all \( p \geq 2 \),
\[
\dim H_p (\Omega') = \dim H_p (\Omega) .
\] (4.5)

For \( p = 0 \) this identity also true as the number of connected components of \((V, E)\) and \((V', E')\) is the same.

We are left to treat the case \( p = 1 \). Observe that
\[
\Omega_0 = \Omega'_0 + \text{span} \{e_a\} \quad \text{and} \quad \Omega_1 = \Omega'_1 + \text{span} \{e_{ab}\} .
\] (4.6)

By (4.4) and (4.6) the cochain complex \( \Omega/\Omega' \) has the form
\[
0 \leftarrow \text{span} \{e_a\} \leftarrow \partial \text{span} \{e_{ab}\} \leftarrow 0 = \Omega_2/\Omega'_2 .
\]

Since
\[
\partial e_{ab} = e_b - a_a = -e_a \text{ mod } \Omega'_0 ,
\]
it follows that \( \text{Im } \partial|_{\Omega_1/\Omega'_1} = \text{span} \{e_a\} \), while \( \ker \partial|_{\Omega_1/\Omega'_1} = 0 \), whence
\[
\dim H_0 (\Omega/\Omega') = \dim H_1 (\Omega/\Omega') = 0 .
\]

By (4.1) we have a long exact sequence
\[
H_0 (\Omega/\Omega') = 0 \leftarrow H_1 (\Omega) \leftarrow H_1 (\Omega') \leftarrow 0 = H_1 (\Omega/\Omega')
\]
which implies that

$$\dim H_1(\Omega) = \dim H_1(\Omega'),$$

thus finishing the proof. ■

**Corollary 4.2** Let a digraph $(V, E)$ be a tree (that is, the underlying undirected graph is a tree). Then $H_p(V, E) = 0$ for all $p \geq 1$.

**Proof.** Induction in the number of edges $|E|$. If $|E| = 0$ then the claim is obvious. If $|E| > 0$ then there is a vertex $a \in V$ of degree 1 (indeed, if this is not the case then moving along undirected edges allows to produce a cycle). Removing this vertex and the adjacent edge, we obtain a tree $(V', E')$ with $|E'| < |E|$. By the inductive hypothesis $H_p(V', E') = 0$ for $p \geq 1$, whence by Theorem 4.1 also $H_p(V, E) = 0$. ■
4.3 Removing of a vertex of degree 2

Theorem 4.3  Suppose that a graph \((V, E)\) has a vertex \(a\) with two out-coming edges \(ab\) and \(ac\) and no incoming edges. Assume also that either \(bc\) or \(cb\) (or both) is an edge:

Let \(V' = V \setminus \{a\}\) and \(E' = E \setminus \{ab, ac\}\). Then, for any \(p \geq 0\),

\[
\dim H_p (V, E) = \dim H_p (V', E').
\]  \(\text{(4.7)}\)

The same is true if the vertex \(a\) has two incoming edges \(ba\) and \(ca\) and no outcoming edges, while either \(bc\) or \(cb\) is an edge:
Example. Consider a graph \((V, E)\) as on the picture:

Each of the vertices \(a_i\) satisfies the hypotheses of Theorem 4.3 (either with incoming or outcoming edges). Removing these vertices successively, we see that all the homologies of \((V, E)\) are the same as those of the remaining graph \(b \rightarrow c\). Since it is a star-like graph, we obtain \(\dim H_0 = 1\) and \(\dim H_p = 0\) for all \(p \geq 1\). In particular, \(\chi = 1\).

Proof of Theorem 4.3. Without loss of generality assume that \(bc\) is an edge. Since the number of connected components of the graphs \((V, E)\) and \((V', E')\) is obviously the same, the identity (4.7) for \(p = 0\) follows from Proposition 3.2.
For $p \geq 1$ consider the long exact sequence (4.1), that is,
\[ \ldots \leftarrow H_p(\Omega/\Omega') \leftarrow H_p(\Omega) \leftarrow H_p(\Omega') \leftarrow H_{p+1}(\Omega/\Omega') \leftarrow \ldots, \]
which implies the identity
\[ \dim H_p(\Omega) = \dim H_p(\Omega') \quad \text{for } p \geq 1, \]
if we prove that
\[ \dim H_p(\Omega/\Omega') = 0 \quad \text{for } p \geq 1. \tag{4.8} \]
The condition (4.8) means that
\[ \ker \partial|_{\Omega_p/\Omega'_p} \subset \text{Im} \partial|_{\Omega_{p+1}/\Omega'_{p+1}} \]
that is, if
\[ v \in \Omega_p \quad \text{and} \quad \partial v = 0 \mod \Omega'_{p-1} \tag{4.9} \]
then there exists $\omega \in \Omega_{p+1}$ such that
\[ \partial \omega = v \mod \Omega'_p. \tag{4.10} \]
In fact, it suffices to have $\omega \in A_{p+1}$ because then the identity (4.10) implies $\partial \omega \in \Omega_p$ and, hence, $\omega \in \Omega_{p+1}$. 

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Consider first the case $p = 1$. Every 1-path $v \in \Omega_1$ has the form

$$v = v^{ab}e_{ab} + v^{ac}e_{ac} + v'$$

where $v' \in A'_1 = \Omega'_1$. Since $(\partial v)^a = 0$ and

$$(\partial v)^a = \sum_k (v^{ka} - v^{ak}) = -(v^{ab} + v^{ac}),$$

it follows that

$$v^{ab} + v^{ac} = 0,$$

whence

$$v = v^{ab} (e_{ab} - e_{ac}) \text{ mod } \Omega'_1.$$

For 2-form $\omega = v^{ab}e_{abc}$ we have

$$\partial \omega = v^{ab} (e_{bc} - e_{ac} + e_{ab}) = v \text{ mod } \Omega'_1,$$

which finishes the proof of (4.8) in the case $p = 1$.

Consider now the case $p = 2$. For any $v \in \Omega_2$ and any vertex $j \neq a, b, c$, we have

$$(\partial v)^{aj} = \sum_{k \in V} (v^{kaj} - v^{akj} + v^{ajk}) = -(v^{abj} + v^{acj}),$$
because there are no incoming edges at $a$ and only two outcoming edges $ab$ and $ac$. By (4.9) we have $(\partial v)^a_j = 0$ whence

$$v^{ab}j + v^{ac}j = 0. \quad (4.11)$$

Denote by $J$ the set of vertices $j$ such that either $j = c$ or both $bj$ and $cj$ are edges:

![Diagram](此处插入一个简单的示意图，展示顶点a, b, c及它们之间的边关系)

We claim that

$$j \in V \setminus J \Rightarrow v^{ab}j = v^{ac}j = 0.$$

If $j = a$ or $b$ then this is trivial. Otherwise, $j \neq a, b, c$ and either $bj$ or $cj$ is not an edge. If $bj$ is not an edge then $v^{ab}j = 0$ whence by (4.11) also $v^{ac}j = 0$, and the same is valid if $cj$ is not an edge.

It follows that $v$ can be represented in the form

$$v = \sum_{j \in J} v^{ab}j e_{abj} + \sum_{j \in J} v^{ac}j e_{acj} + v'$$

$$= \sum_{j \in J} v^{abj} (e_{abj} - e_{acj}) + v', \quad (4.12)$$
where $v' \in \mathcal{A}'_2$. In the last line we have used (4.11) for $j \neq c$ and $e_{acj} = 0$ for $j = c$.

For any $j \in J$, we have
\[
\partial (e_{abj} - e_{acj}) = (e_{bj} - e_{aj} + e_{ab}) - (e_{cj} - e_{aj} + e_{ac}) = e_{bj} - e_{cj} + e_{ab} - e_{ac} \in \mathcal{A}_1.
\]

Since $\partial v \in \mathcal{A}'_1$, it follows from (4.12) that $\partial v' \in \mathcal{A}_1$. Since $v' \in \mathcal{A}'_2$, we conclude that $\partial v' \in \mathcal{A}'_1$ whence $v' \in \Omega'_2$. Therefore,
\[
v = \sum_{j \in J} v^{abj} (e_{abj} - e_{acj}) \mod \Omega'_2. \quad (4.13)
\]

Since $(\partial v)^{ab} = 0$ and
\[
(\partial v)^{ab} = \sum_{j \in V} (v^{jab} - v^{ajb} + v^{abj}) = \sum_{j \in J} v^{abj},
\]

it follows that
\[
\sum_{j \in J} v^{abj} = 0. \quad (4.14)
\]
Consider the 3-path
\[ \omega = \sum_{j \in J} v^{abj} e_{abcj}. \]

For any \( j \in J \setminus \{c\} \) we have \( e_{abcj} \in E_3 \) whereas for \( j = c \) we have \( e_{abcj} = 0 \). Hence, \( \omega \in A_3 \). Since

\[ \partial e_{abcj} = e_{bcj} - e_{acj} + e_{abj} - e_{abc} \]

and \( e_{bcj} \in \Omega'_2 \), it follows from (4.13) and (4.14) that

\[ \partial \omega = \sum_{j \in J} v^{abj} (e_{abj} - e_{acj} - e_{abc}) = v \mod \Omega'_2, \]

which finishes the proof in the case \( p = 2 \).

Consider the case \( p \geq 3 \). Any \( p \)-path \( v \in \Omega_p \) has the form

\[ v = \sum_{\gamma \in E'_{p-2}} v^{ab\gamma} e_{ab\gamma} + \sum_{\gamma \in E'_{p-2}} v^{ac\gamma} e_{ac\gamma} + v', \quad (4.15) \]
where \( v' \in \mathcal{A}_p' \). Using product of paths and (1.14), we obtain

\[
\partial e_{ab\gamma} = \partial (e_{ab}e_\gamma) = (\partial e_{ab})e_\gamma + e_{ab}\partial e_\gamma \\
= (e_b - e_a)e_\gamma + e_{ab}\partial e_\gamma \\
= e_{b\gamma} - e_{a\gamma} + e_{ab}\partial e_\gamma
\]

and a similar formula for \( \partial e_{ac\gamma} \), whence it follows that

\[
\partial v = \sum_{\gamma \in E'_{p-2}} \left( v^{ab\gamma}e_{b\gamma} + v^{ac\gamma}e_{c\gamma} - (v^{ab\gamma} + v^{ac\gamma})e_{a\gamma} \right) + e_{ab} \sum_{\gamma \in E'_{p-2}} v^{ab\gamma}\partial e_\gamma + e_{ac} \sum_{\gamma \in E'_{p-2}} v^{ac\gamma}\partial e_\gamma + \partial v'.
\] (4.16)

Let \( \gamma = \gamma_0...\gamma_{p-2} \) where \( \gamma_i \in V' \). We claim that if \( \gamma_0 \neq c \) then

\[
v^{ab\gamma} + v^{ac\gamma} = 0.
\] (4.18)

If \( \gamma_0 = a \) or \( b \) then we have trivially \( v^{ab\gamma} = v^{ac\gamma} = 0 \). Otherwise let us look at the component \( e_{a\gamma} \) in (4.16)-(4.17). Since it occurs only once, namely in the last term of (4.16), while \( (\partial v)^{a\gamma} = 0 \), we obtain (4.18). Note that
if $\gamma_0 = c$ then $v^{ac\gamma} = 0$ but $v^{ab\gamma}$ may be non-zero, so that (4.18) may not be valid.

Denote by $\Gamma$ the set of paths $\gamma \in E'_{p-2}$ such that either $\gamma_0 = c$ or both $b\gamma$ and $c\gamma$ are in $E'_{p-1}$:

It follows from (4.18) that if $\gamma \in E'_{p-2} \setminus \Gamma$ then both $v^{ab\gamma}$ and $v^{ac\gamma}$ vanish. Indeed, since $\gamma_0 \neq c$, we have (4.18). Since $b\gamma$ or $c\gamma$ is not in $E'_{p-1}$, one of the terms $v^{ab\gamma}, v^{ac\gamma}$ vanish, whence the second term also vanish by (4.18). Hence, the summation in (4.15) can be restricted to $\gamma \in \Gamma$:

$$v = \sum_{\gamma \in \Gamma} v^{ab\gamma}e_{ab\gamma} + \sum_{\gamma \in \Gamma} v^{ac\gamma}e_{ac\gamma} + v'$$

$$= \sum_{\gamma \in \Gamma} v^{ab\gamma}(e_{ab\gamma} - e_{ac\gamma}) + v', \quad (4.19)$$
where in the second line we have used (4.18) for $\gamma_0 \neq c$ and $e_{ac}\gamma = 0$ for $\gamma_0 = c$. Set

$$u = \sum_{\gamma \in \Gamma} v^{ab\gamma} e_{\gamma},$$

(4.20)

so that we can rewrite (4.19) in the form

$$v = (e_{ab} - e_{ac}) u + v'$$

(4.21)

whence

$$\partial v = (e_b - e_c) u + (e_{ab} - e_{ac}) \partial u + \partial v'.$$

(4.22)

Let us show that $\partial u = 0$. Indeed, since the $(p - 1)$-paths $\partial v$, $e_b u$, $e_c u$, and $\partial v'$ are in $\mathcal{R}'_{p-1}$, it follows from (4.22) that also

$$(e_{ab} - e_{ac}) \partial u \in \mathcal{R}'_{p-1}.$$

We have the identity

$$(e_{ab} - e_{ac}) \partial u = \sum_{i_0, \ldots, i_{p-3} \in V'} (\partial u)^{i_0 \ldots i_{p-3}} (e_{abi_0 \ldots i_{p-3}} - e_{aci_0 \ldots i_{p-3}}).$$
If $i_0 \neq b$ then $e_{abi_0...i_{p-3}} \notin R'_{p-1}$ so that the coefficient $(\partial u)^{i_0...i_{p-3}}$ must vanish. If $i_0 = b$ then $i_0 \neq c$, $e_{aci_0...i_{p-3}} \notin R'_{p-1}$ and again $(\partial u)^{i_0...i_{p-3}} = 0$. Hence, we conclude that $\partial u = 0$, which was claimed.

It follows that

$$\partial v = (e_b - e_c) u + \partial v'.$$

Since $\partial v, e_b u, e_c u \in A'_{p-1}$, it follows that $\partial v' \in A'_{p-1}$ whence $v' \in \Omega'_{p-1}$. Substituting this into (4.21), we obtain

$$v = (e_{ab} - e_{ac}) u \mod \Omega'_p. \quad (4.23)$$

Consider a $(p + 1)$-path $\omega = e_{abc} u$. Since

$$\partial (e_{abc} u) = (\partial e_{abc}) u - e_{abc} \partial u = (e_{bc} - e_{ac} + e_{ab}) u,$$

we have

$$\partial (e_{abc} u) = e_{bc} u + v \mod \Omega'_p. \quad (4.24)$$

We are left to show that $e_{bc} u \in \Omega'_p$. That $e_{bc} u \in A'_p$ follows from the definition of $\Gamma$ and (4.20). Next we have

$$\partial (e_{bc} u) = (e_b - e_c) u + e_{bc} \partial u = e_b u - e_c u \in A'_{p-1},$$
which implies that $e_{bc}u \in \Omega'_p$. From (4.24) we conclude that

$$\partial (e_{abc}u) = v \mod \Omega'_p,$$

which finishes the proof. ■
4.4 Removing a vertex of degree 1 + 1

Recall that a pair $cb$ of distinct vertices on a graph is a *semi-edge* if $cb$ is not an edge but there is a vertex $j$ such that $cjb$ is an edge:

\[
\begin{array}{c}
\bullet b \\
\uparrow & \nearrow \\
\bullet j \\
\bullet c
\end{array}
\]

**Theorem 4.4** Suppose that a graph $(V, E)$ has a vertex $a$ such that there is only one outcoming edge $ab$ from $a$ and only one incoming edge $ca$, where $b \neq c$. Let $V' = V \setminus \{a\}$ and $E' = E \setminus \{ab, ca\}$.

Then the following is true.

(a) For any $p \geq 2$,

\[
\dim H_p(V, E) = \dim H_p(V', E').
\]
(b) If \( cb \) is an edge or a semi-edge in \((V', E')\) then (4.25) is satisfied also for \( p = 0, 1 \), that is, for all \( p \geq 0 \).

(c) If \( cb \) is neither edge nor semi-edge in \((V', E')\), but \( b, c \) belong to the same connected component of \((V', E')\) then

\[
\dim H_1(V, E) = \dim H_1(V', E') + 1
\]

and \( \dim H_0(V, E) = \dim H_0(V', E') \).

(d) If \( b, c \) belong to different connected components of \((V', E')\) then

\[
\dim H_1(V, E) = \dim H_1(V', E')
\]

and \( \dim H_0(V, E) = \dim H_0(V', E') - 1 \).

Consequently, in the case (b), \( \chi(V, E) = \chi(V', E') \), whereas in the cases (c) and (d), \( \chi(V, E) = \chi(V', E') - 1 \).
**Example.** Consider the graphs

\[
(V, E) = a \bullet \quad \downarrow \quad \bullet d \quad \text{and} \quad (V', E') = \downarrow \quad \bullet d
\]

Since \( cb \) is semi-edge in \((V', E')\) we have case (b) so that all homologies of \((V, E)\) and \((V', E')\) are the same. Removing further vertex \(d\) we obtain a digraph \( b \bullet \rightarrow c \bullet \) that will be denoted by \((V'', E'')\). It is a star-like graph with all \( \dim H_p (V'', E'') = 0 \) for \( p \geq 1 \). Since \( cb \) is neither edge nor semi-edge in \((V'', E'')\), but the graph is connected, we conclude by case (c) that

\[
H_p (V', E') = H_p (V'', E'') \quad \text{for} \quad p \geq 2,
\]

and

\[
\dim H_1 (V', E') = \dim H_1 (V'', E'') + 1 = 1.
\]

It follows that \( \dim H_p (V, E) = 0 \) for \( p \geq 2 \) and \( \dim H_1 (V, E) = 1 \).
Example. Consider a digraph (a kind of anti-snake):

We start building this graph with $1 \to 2$. Since 21 is neither edge nor semi-edge, adding a path $2 \to 3 \to 1$ increases $\dim H_1$ by 1 and preserves other homologies. Since 23 is an edge, adding a path $2 \to 4 \to 3$ preserves all homologies. Since 34 is neither edge nor semi-edge, adding a path $3 \to 5 \to 4$ increases $\dim H_1$ by 1 and preserves other homologies. Similarly, adding a path $5 \to 6 \to 4$ preserves all homologies.

One can repeat this pattern arbitrarily many times. By doing so we construct a digraph with a prescribed value of $\dim H_1$ while keeping $\dim H_p = 0$ for all $p \geq 2$. Consequently, the Euler characteristic $\chi$ can take arbitrary negative values.
Proof of Theorem 4.4. Proof of (a). The identity (4.25) for \( p \geq 2 \) will follow if we prove that

\[
\dim H_p (\Omega/\Omega') = 0 \text{ for } p \geq 2. \tag{4.26}
\]

In order to prove (4.26) it suffices to show that

\[
\ker \partial|_{\Omega_p/\Omega'_p} = 0,
\]

which is equivalent to

\[
v \in \Omega_p, \quad \partial v = 0 \mod \Omega'_{p-1} \Rightarrow v = 0 \mod \Omega'_p. \tag{4.27}
\]

By the definition (3.5) of \( \Omega_p \), (4.27) is equivalent to

\[
v \in \mathcal{A}_p \text{ and } \partial v \in \mathcal{A}'_{p-1} \Rightarrow v \in \mathcal{A}'_p. \tag{4.28}
\]

Hence, let us prove (4.28) for all \( p \geq 2 \).

Every elementary allowed \( p \)-path on \((V, E)\) either contains one of the edges \( ab, ca \) or is allowed in \((V', E')\). Let us show that, for any \( v \) as in (4.28), its components \( v^{\cdots ab\cdots} \) and \( v^{\cdots ca\cdots} \) vanish, which will imply that
\( v \in A_p'. \) Any such component can be written in the form \( v^{\alpha ab\beta} \) or \( v^{\gamma ca\beta} \) where \( \alpha, \beta, \gamma \) are some paths. Consider the following cases. For further applications, in the Cases 1,2 we assume only that \( v \in \Omega_p \) (whereas in the Case 3 \( v \) is as in (4.28)).

**Case 1.** Let us consider first the component \( v^{\alpha ab\beta} \) where \( \beta \) is non-empty. If \( \alpha ab\beta \) is not allowed in \( (V, E) \) then \( v^{\alpha ab\beta} = 0 \) by definition. Let \( \alpha ab\beta \) be allowed in \( (V, E) \). The path \( \alpha a\beta \) is not allowed because the only outcoming edge from \( a \) is \( ab \). Since \( \partial v \in A_{p-1} \), we have

\[
(\partial v)^{\alpha a\beta} = 0.
\]

Let us show that

\[
(\partial v)^{\alpha a\beta} = \pm v^{\alpha ab\beta}, \tag{4.29}
\]

which will imply \( v^{\alpha ab\beta} = 0 \). Indeed, by (1.6) \( (\partial v)^{\alpha a\beta} \) is the sum of the terms \( \pm v^\omega \) where \( \omega \) is a \( p \)-path that is obtained from \( \alpha a\beta \) by inserting one vertex. Since there is no edge from \( a \) to \( \beta \), the only way \( \omega \) can be allowed is when \( \omega = \alpha ab\beta \). Since for any other \( \omega \) we have \( v^\omega = 0 \), we obtain (4.29), which implies that \( v^{\alpha ab\beta} = 0 \).

**Case 2.** In the same way one proves that \( v^{\gamma ca\beta} = 0 \) provided \( \gamma \) is non-empty, using the fact that the only incoming edge in \( a \) is \( ca \).
Case 3. Consider now an arbitrary component $v^{\alpha ab \beta}$. If $\beta$ is non-empty then $v^{\alpha ab \beta} = 0$ by Case 1. Let $\beta$ be empty. Then $\alpha$ must have the form $\alpha = \gamma c$ so that $v^{\alpha ab \beta} = v^{\gamma cab}$. If $\gamma$ is non-empty then $v^{\gamma cab} = 0$ by Case 2. Finally, let $\gamma$ be also empty so that $v^{\alpha ab \beta} = v^{cab}$ (which is only possible if $p = 2$). Since $\partial v \in A_1'$, we have

$$(\partial v)^{ab} = 0.$$ 

On the other hand,

$$(\partial v)^{ab} = \sum_{i \in V} v^{iab} - v^{aib} + v^{abi}.$$ 

Here all the terms of the form $v^{iab}$ vanish, except possibly for $v^{cab}$, because $ia$ is not an edge unless $i = c$. All the terms $v^{aib}$ vanish because $ai$ is not an edge. All the terms $v^{abi}$ vanish by Case 1. Hence, we obtain

$$(\partial v)^{ab} = v^{cab}$$

whence $v^{cab} = 0$ follows, thus finishing the proof of the part (a).
Proof of \((b), (c), (d)\). If \(b, c\) belong to the same connected component of \((V', E')\) then the number of connected components of \((V, E)\) and that of \((V', E')\) are the same, so that

\[
\dim H_0(\Omega) = \dim H_0(\Omega'), \quad (4.30)
\]

whereas if \(b, c\) belong to different connected components of \((V', E')\) then after joining them by \(a\) the number of connected components reduces by 1, so that

\[
\dim H_0(\Omega) = \dim H_0(\Omega') - 1. \quad (4.31)
\]

To handle \(H_1\) we use the long exact sequence (4.1) that by (4.26) has the form

\[
0 \leftarrow H_0(\Omega/\Omega') \leftarrow H_0(\Omega) \leftarrow H_0(\Omega') \leftarrow H_1(\Omega/\Omega') \leftarrow H_1(\Omega) \leftarrow H_1(\Omega') \leftarrow 0. \quad (4.32)
\]

Since we know already the relation between \(H_0(\Omega')\) and \(H_0(\Omega)\), to obtain the relation between \(H_1(\Omega')\) and \(H_1(\Omega)\) we need to compute \(\dim H_0(\Omega/\Omega')\) and \(\dim H_1(\Omega/\Omega')\) from the quotient complex \(\Omega/\Omega'\). Observe that

\[
\Omega_0 = \Omega_0' + \text{span} \{e_a\}, \quad \Omega_1 = \Omega_1' + \text{span} \{e_{ab}, e_{ca}\} \quad (4.33)
\]
so that the quotient complex $\Omega/\Omega'$ has the form

$$0 \leftarrow \text{span} \{e_a\} \leftarrow \text{span} \{e_{ab}, e_{ca}\} \leftarrow \Omega_2/\Omega'_2 \leftarrow \cdots$$

We need to determine $\text{Im} \partial|_{\Omega_1/\Omega'_1}$, $\ker \partial|_{\Omega_1/\Omega'_1}$, $\text{Im} \partial|_{\Omega_2/\Omega'_2}$. Since

$$\partial e_{ab} = e_b - e_a = -e_a \mod \Omega'_0,$$

it follows that

$$\text{Im} \partial|_{\Omega_1/\Omega'_1} = \Omega_0/\Omega'_0,$$

whence

$$\dim H_0 (\Omega/\Omega') = 0. \quad (4.34)$$

For any scalars $k, l \in \mathbb{K}$, we have

$$\partial(ke_{ab} + le_{ca}) = (l - k)e_a \mod \Omega'_0,$$

so that $\partial(ke_{ab} + le_{ca}) = 0$ if and only if $k = l$, that is

$$\ker \partial|_{\Omega_1/\Omega'_1} = \text{span}(e_{ab} + e_{ca}) \mod \Omega'_1. \quad (4.35)$$
Let us now compute $\text{Im} \partial|_{\Omega_2/\Omega'_2}$. For any $v \in \Omega_2$ we have by the above Cases 1,2 that $v^{abi} = v^{jca} = 0$, which implies that $v$ has the form

$$v = v' + v^{cab}e_{cab}, \quad (4.36)$$

where $v' \in A'_2$. It follows that

$$\partial v = \partial v' + v^{cab} (e_{ab} - e_{cb} + e_{ca}). \quad (4.37)$$

Since all 1-paths $\partial v$, $e_{ab}$ and $e_{ca}$ belong to $A_1$, it follows that $\partial v' - v^{cab}e_{cb} \in A_1$ whence also $\partial v' - v^{cab}e_{cb} \in A'_1$. Therefore,

$$\partial v = v^{cab} (e_{ab} + e_{ca}) \mod \Omega'_1. \quad (4.38)$$

Next consider two cases.

(i) Let $\Omega_2$ contain an element $v$ with $v^{cab} \neq 0$. Then by (4.38)

$$\text{Im} \partial|_{\Omega_2/\Omega'_2} = \text{span} (e_{ab} + e_{ca}) \mod \Omega'_1, \quad (4.39)$$

which together with (4.35) implies

$$\dim H_1 (\Omega/\Omega') = 0. \quad (4.40)$$
Substituting (4.34) and (4.40) into the exact sequence (4.32), we obtain that the identity
\[ \dim H_p(\Omega') = \dim H_p(\Omega) \]
holds for all \( p \geq 0 \).

(ii) Assume that \( v^{cab} = 0 \) for all \( v \in \Omega_2 \). Then by (4.38)
\[ \text{Im}\, \partial|_{\Omega_2/\Omega_2'} = 0, \]
which together with (4.35) implies
\[ \dim H_1(\Omega/\Omega') = 1. \quad (4.41) \]

Using again the exact sequence (4.32), that is,
\[ 0 \leftarrow H_0(\Omega) \leftarrow H_0(\Omega') \leftarrow H_1(\Omega/\Omega') \leftarrow H_1(\Omega) \leftarrow H_1(\Omega') \leftarrow 0, \]
we obtain by (2.5) and (4.41)
\[ \dim H_1(\Omega') - \dim H_1(\Omega) + 1 - \dim H^0(\Omega') + \dim H^0(\Omega) = 0 \quad (4.42) \]
Let us now specify when (i) or (ii) occur. Assume first that \( cb \) is an edge:

![Diagram showing edge \( cb \) between vertices \( a \) and \( b \)]

Then

\[
\partial e_{cab} = e_{ab} - e_{cb} + e_{ca} \in A_1,
\]

whence it follows that \( e_{cab} \in \Omega_2 \). Hence, we have the case (i) with \( v = e_{cab} \).

Assume now that \( cb \) is not an edge. Denote by \( J \) the set of vertices \( j \in V' \) such that the 2-path \( cjb \) is allowed in \((V', E')\):

![Diagram showing set \( J \) and vertices \( j \)]

Assume first that \( J \) is non-empty, that is, \( cb \) is a semi-edge, and set

\[
v = e_{cab} - \frac{1}{|J|} \sum_{j \in J} e_{cjb},
\]
where $|J|$ is the number of elements in $J$. It is clear that $v \in \mathcal{A}_2$. We have

$$
\partial v = (e_{ab} - e_{cb} + e_{ca}) - \frac{1}{|J|} \sum_{j \in J} (e_{jb} - e_{cb} + e_{cj})
= (e_{ab} + e_{ca}) - \frac{1}{|J|} \sum_{j \in J} (e_{jb} + e_{cj}),
$$

where the term $e_{cb}$ has cancelled out. It follows from (4.43) that $\partial v \in \mathcal{A}_1$ whence $v \in \Omega_2$, and we obtain again the case $(i)$. This finishes the proof of $(b)$.

Let us show that if $J = \emptyset$ (that is, when $cb$ is neither edge nor semi-edge) then we have the case $(ii)$. Any 2-path $v \in \Omega_2$ has the form (4.36) and $\partial v$ is given by (4.37). It follows that

$$
(\partial v)^{cb} = (\partial v')^{cb} - v^{cab}.
$$

Since $\partial v \in \mathcal{A}_1$ and $cb$ is not an edge, we have $(\partial v)^{cb} = 0$. We have by (1.6)

$$
(\partial v')^{cb} = \sum_{j \in V'} (v')^{jcb} - (v')^{cjb} + (v')^{cbj},
$$

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which implies that \((\partial v')^{cb} = 0\) as no elementary 2-path of the form \(jcb, cj b, cb j\) is allowed in \((V', E')\), whereas \(v' \in A'_2\). It follows that \(v^{cab} = 0\) so that we have the case \((ii)\).

If in addition \(b, c\) belong to the same connected component of \((V', E')\) then we have (4.30), that is,

\[
\dim H^0 (\Omega) = \dim H^0 (\Omega').
\]

Substituting into (4.42), we obtain

\[
\dim H_1 (\Omega) = \dim H_1 (\Omega') + 1.
\]

which proves part \((c)\).

If \(b, c\) belong to different components of \((V', E')\) then we have by (4.31)

\[
\dim H^0 (\Omega) = \dim H^0 (\Omega') - 1,
\]

whence by (4.42)

\[
\dim H_1 (\Omega) = \dim H_1 (\Omega'),
\]

which finishes the proof of part \((d)\).

Finally, the identities for the Euler characteristic follows easily from the relations between \(\dim H_p (\Omega)\) and \(\dim H_p (\Omega')\).
4.5 Suspension

Let a digraph \((V, E)\) have a subgraph \((V', E')\) such that \(V \setminus V' = \{a, b\}\) and \(E \setminus E' = \{ia, ib, i \in V'\}\):

\[
\begin{array}{c}
\bullet^a \\
\uparrow \quad \uparrow \\
\rightarrow \quad \rightarrow \\
V' \\
\rightarrow \quad \rightarrow \\
\downarrow \quad \downarrow \\
\bullet^b \\
\end{array}
\]

The digraph \((V, E)\) is called a suspension of \((V', E')\) and is denoted by \(\text{Sus} (V', E')\). Similarly, if \(a\) and \(b\) have outcoming edges then \((V, E)\) is an inverse suspension of \((V', E')\).

The next theorem determines the homologies of a suspension.

**Theorem 4.5** If \((V, E)\) is a suspension (or inverse suspension) of \((V', E')\) then, for any \(p \geq 1\),

\[
H_p (V, E) \cong \tilde{H}_{p-1} (V', E').
\] (4.44)
Here $\tilde{H}_p$ is a reduced homology: $\tilde{H}_p = H_p$ for $p \geq 1$ and $\tilde{H}_0 \cong H_0/\text{const}$.

Denoting the digraph $(V', E')$ by $G$, we can write the identity (4.44) in the functorial form as follows:

$$H_p (\text{Sus } G) = \tilde{H}_{p-1} (G).$$

It follows that $\chi (\text{Sus } G) = 2 - \chi (G)$. 
Example. Consider the digraph $G = (V, E)$ as follows:

Clearly, $G = \text{Sus } G'$ where $G'$ is the subgraph with vertices $\{2, 3, 4, 5\}$. Also, $G' = \text{Sus } G''$ where $G''$ is a subgraph with vertices $\{4, 5\}$. Since $\dim H_0 (G'') = 2$ and $\dim H_p (G'') = 0$ for $p \geq 1$, we obtain by (4.44)

\[
\begin{align*}
\dim H_0 (G') &= 1, \quad \dim H_1 (G') = 1, \quad \dim H_p (G') = 0 \text{ for } p \geq 2, \\
\dim H_0 (G) &= 1, \quad \dim H_1 (G) = 0, \quad \dim H_2 (G) = 1, \quad \dim H_p (G) = 0 \text{ for } p \geq 3.
\end{align*}
\]

Consequently, $\chi (G) = 2$. 

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In the digraph $G$ we have

$$\dim \Omega_0 = |V| = 6 \quad \text{and} \quad \dim \Omega_1 = |E| = 12$$

and

$$\mathcal{A}_2 = \text{span} \{e_{024}, e_{025}, e_{034}, e_{035}, e_{124}, e_{125}, e_{134}, e_{135}\}.$$ 

The set of semi-edges is empty, whence $\dim \Omega_2 = \dim \mathcal{A}_2 = 8$ and, hence, $\Omega_2 = \mathcal{A}_2$. Alternatively, one can see that because all the 2-paths spanning $\mathcal{A}_2$ are triangles so that all they are $\partial$-invariant. Also, there are no allowed 3-paths, so that $\mathcal{A}_3 = \{0\}$ whence $\dim \Omega_p = 0$ for all $p \geq 3$.

A spanning element in $H^2(G)$ is

$$v = e_{024} - e_{025} - e_{034} + e_{035} - e_{124} + e_{125} + e_{134} - e_{135}$$

as $v \neq 0$ and $\partial v = 0$. 

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**Example.** Let $S$ be any cycle graph that is neither triangle nor square. We regards $S$ as a circle. Define $S_n$ inductively by $S_1 = S$ and $S_{n+1} = S S S_n$. Then $S_n$ can be regarded as $n$-dimensional sphere. Here is an example of $S_2$:

![Diagram of S_2]

Since $\chi(S) = 0$ by Proposition 3.7, it follows that $\chi(S_n) = 0$ if $n$ is odd and $\chi(S_n) = 2$ if $n$ is even. Theorem 4.5 also implies that $\dim H_n(S_n) = \dim H_1(S) = 1$, which gives an example of a non-trivial $H_n$ with an arbitrary $n$. 
Proof of Theorem 4.5. For any $p \geq 0$ consider a linear mapping

$$
\tau : \mathcal{A}_p' \to \mathcal{A}_{p+1},
$$
defined by

$$
\tau v = v (e_a - e_b).
$$

(4.45)

Since every vertex from $V'$ is connected to $a$ and $b$, the path $\tau v$ is indeed allowed. By the product rule (1.14) we have

$$
\partial (\tau v) = (\partial v) (e_b - e_a) + (-1)^{p+1} v \partial (e_a - e_b) = \tau \partial v
$$

so that the operators $\partial$ and $\tau$ commute. It follows that

$$
\tau (\Omega_p') \subset \Omega_{p+1}.
$$

Indeed, if $v \in \Omega_p'$ then

$$
v \in \mathcal{A}_p' \quad \text{and} \quad \partial v \in \mathcal{A}_{p-1}'
$$

whence

$$
\tau v \in \mathcal{A}_{p+1} \quad \text{and} \quad \partial (\tau v) = \tau (\partial v) \in \mathcal{A}_p,
$$
whence $\tau v \in \Omega_{p+1}$. Hence, we have the commutative following diagram for all $p \geq 1$:

$$
\begin{array}{ccc}
\Omega'_{p-1} & \xleftarrow{\partial} & \Omega'_{p} \\
\downarrow{\tau} & & \downarrow{\tau} \\
\Omega_{p} & \xleftarrow{\partial} & \Omega_{p+1}
\end{array}
$$

(4.46)

Let us extend it to the case $p = 0$. Set $\Omega'_{-1} = \mathbb{K}$ as in the case of reduced homology. The operator $\tau : \mathbb{K} \to \Omega_0$ is also defined by (4.45), which now amounts to $\tau 1 = e_a - e_b$. The operator $\partial$ should be replaced by $\tilde{\partial} : \Omega'_0 \to \mathbb{K}$ where $\tilde{\partial}e_i = 1$ (this is the same operator $\tilde{\partial}$ that is used in the reduced homologies and in the product rule). The above argument, based on the product rule, remains valid. Hence, the diagram (4.46) remains commutative also for $p = 0$, where it takes the form

$$
\begin{array}{ccc}
\mathbb{K} & \xleftarrow{\tilde{\partial}} & \Omega'_0 \\
\downarrow{\tau} & & \downarrow{\tau} \\
\Omega_0 & \xleftarrow{\partial} & \Omega_1
\end{array}
$$

Consider the digraph $(V'', E'')$ that is obtained by adding to $(V', E')$ the vertex $a$ and all the edges $ia$ with $i \in V'$, that is, $(V'', E'')$ is a cone.
over \((V', E')\):

Let us mark by a double dash " all the notation related to this digraph. For any \(p \geq 0\), define a linear mapping \(\rho : \mathcal{A}_p \rightarrow \mathcal{A}_p''\) by

\[
\rho e_{i_0...i_p} = \begin{cases} 
  e_{i_0...i_p}, & \text{if } i_p \neq b \\
  e_{i_0...i_{p-1}a}, & \text{if } i_p = b.
\end{cases}
\tag{4.47}
\]

Clearly, \(\rho\) is surjective. Let us show that \(\rho\) commutes with \(\partial\). If \(v = e_{i_0...i_p}\) with \(i_p \neq b\) then \(\rho v = v\) and \(\rho (\partial v) = \partial v\) so that \(\rho (\partial v) = \partial (\rho v)\). If \(i_p = b\) then, setting \(u = e_{i_0...i_{p-1}}\), we obtain \(\rho v = ue_a\) and

\[
\partial (\rho v) = (\partial u) e_a + (-1)^p u.
\]
On the other hand, we have

$$\partial v = (\partial u) e_a + (-1)^p u$$

whence it follows that

$$\rho (\partial v) = (\partial u) e_b + (-1)^p u,$$

which proves that $\rho (\partial v) = \partial (\rho v)$.

It follows that $\rho$ maps $\Omega_p$ to $\Omega''_p$ and the following diagram is commutative for any $p \geq 0$:

\[
\begin{array}{ccc}
\Omega_p & \xleftarrow{\partial} & \Omega_{p+1} \\
\downarrow{}^\rho & & \downarrow{}^\rho \\
\Omega''_p & \xleftarrow{\partial} & \Omega''_{p+1}
\end{array}
\]

(4.48)

We will merge the diagrams (4.48) and (4.46), and for that we need to verify that the following sequence is exact for all $p \geq -1$:

$$0 \longrightarrow \Omega'_p \longrightarrow \Omega_{p+1} \xrightarrow{\rho} \Omega''_{p+1} \longrightarrow 0.$$  

(4.49)

Since $\tau$ is injective and $\rho$ is surjective, it suffices to show that $\text{Im} \tau = \ker \rho$. We have

$$\tau e_{i_0...i_p} = e_{i_0...i_pa} - e_{i_0...i_pb}$$
so that \( \text{Im} \, \tau \) consists of all \( p \)-paths of the form

\[
\sum_{i_0, \ldots, i_p \in V'} c^{i_0 \ldots i_p} \left( e_{i_0 \ldots i_p a} - e_{i_0 \ldots i_p b} \right)
\]  

(4.50)

with arbitrary coefficients \( c^{i_0 \ldots i_p} \). Observe that, for any \( u \in \Omega_{p+1} \),

\[
\rho u = \sum_{i_0, \ldots, i_{p+1} \in V'} u^{i_0 \ldots i_{p+1}} e_{i_0 \ldots i_{p+1}} + \sum_{i_0, \ldots, i_p \in V'} \left( u^{i_0 \ldots i_p a} + u^{i_0 \ldots i_p b} \right) e^{i_0 \ldots i_p a}.
\]  

(4.51)

Then the equation \( \rho u = 0 \) that amounts to the system

\[
\begin{cases}
  u^{i_0 \ldots i_{p+1}} = 0, & \text{for all } i_0 \ldots i_{p+1} \in V', \\
  u^{i_0 \ldots i_p a} + u^{i_0 \ldots i_p b} = 0, & \text{for all } i_0 \ldots i_p \in V',
\end{cases}
\]  

(4.52)

that is, to the identity

\[
u = \sum_{i_0, \ldots, i_p \in V'} u^{i_0 \ldots i_p a} \left( e_{i_0 \ldots i_p a} - e_{i_0 \ldots i_p b} \right).
\]  

(4.53)

Comparing with (4.50) we see that \( \text{Im} \, \tau = \ker \rho \).
Hence, we have constructed the following commutative diagram where the rows are chain complexes and the columns are exact:

\[
\begin{array}{ccccccccc}
0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \cdots & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \leftarrow & \Omega'_{-1} & \leftarrow & \partial & \Omega'_{0} & \leftarrow & \partial & \Omega'_{1} & \leftarrow & \partial & \Omega'_{p-1} & \leftarrow & \partial & \Omega'_{p} & \leftarrow & \partial & \cdots \\
\downarrow & \tau & & \downarrow & \tau & & \downarrow & \tau & & \downarrow & \tau & & \downarrow & \tau & & \downarrow & \tau & & \downarrow & \tau \\
0 & \leftarrow & \Omega_{0} & \leftarrow & \partial & \Omega_{1} & \leftarrow & \partial & \Omega_{2} & \leftarrow & \partial & \Omega_{p} & \leftarrow & \partial & \Omega_{p+1} & \leftarrow & \partial & \cdots \\
\downarrow & \rho & & \downarrow & \rho & & \downarrow & \rho & & \downarrow & \rho & & \downarrow & \rho & & \downarrow & \rho & & \downarrow & \rho \\
0 & \leftarrow & \Omega''_{0} & \leftarrow & \partial & \Omega''_{1} & \leftarrow & \partial & \Omega''_{2} & \leftarrow & \partial & \Omega''_{p} & \leftarrow & \partial & \Omega''_{p+1} & \leftarrow & \partial & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \cdots & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \cdots \\
\end{array}
\]

(4.54)

The homologies of the first chain complex in (4.54) are the reduced homologies \( \widetilde{H} (\Omega') \), while the second and the third complexes yield the homologies \( H (\Omega) \) and \( H (\Omega'') \) respectively. By (2.17) we obtain a long exact sequence

\[
0 \leftarrow \cdots \leftarrow H_p (\Omega'') \leftarrow H_p (\Omega) \leftarrow \widetilde{H}_{p-1} (\Omega') \leftarrow H_{p+1} (\Omega'') \leftarrow \cdots 
\]
Since \((V'', E'')\) is a star-like, we have by Theorem 3.4 \(H_p (\Omega'') = \{0\}\) for any \(p \geq 1\), whence it follows that

\[
\dim H_p (\Omega) = \dim \tilde{H}_{p-1} (\Omega'),
\]

which was to be proved. ■