DECOMPOSITION OF A METRIC SPACE BY CAPACITORS

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We dedicate this paper to Louis Nirenberg on occasion of his 70th birthday, whose contribution to differential equations has had tremendous infuence, also on this paper

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1. Introduction

The main subject of this work is to develop a method of decomposition of a metric measure space into subsets possessing certain uniformity.

Let X be a metric space with a distance d. Denote by $\mathcal{A}$ an algebra of sets generated by all open balls, and let X be equipped with a measure μ defined on $\mathcal{A}$. Let us note that $\mathcal{A}$ is not necessarily a σ-algebra.

We assume also that *a capacity* is defined on X : any couple (F, G) of $\mathcal{A}$-sets $F \subset G \subset X$ is referred to as *a capacitor* and is assigned a non-negative number $\operatorname{cap}(F, G)$. Of course, the capacity should satisfy certain axioms - see the next section for the exact definition and examples. In particular, the Wiener capacity will satisfy it.

Let us denote by $\mathbb{B}_r^x = \{y \in X : d(x, y) < r\}$ a ball of radius r centred at the point $x \in X$. For any ball $B = \mathbb{B}_r^x$, we denote by $\widetilde{B}$ the ball $\mathbb{B}_{2r}^x$ of the double radius.

We make the following assumptions about the structure of X :

(A0): the total measure μX is finite, let $\mu X = M < \infty$;

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(A1): any ball $\mathbb{B}_r^x$ is covered by at most ν other balls of radius $r/50$ where ν does not depend on r, x;

(A2): for any ball $B \subset X$ we have

$$\mathrm{cap}(B, \widetilde{B}) \leq \gamma$$

where the constant γ does not depend on B.

(A3): there are numbers $\alpha > 0$ and $\rho > \frac{5}{4}\delta$ (the number $\delta \geq 0$ is one which is involved in the axiomatic properties of capacity below) such that for *any* $x \in X$

$$\mu\mathbb{B}_{\rho/5}^x < \alpha$$

and for *some* $x \in X$

$$\mu\mathbb{B}_{\rho}^x \geq \alpha.$$

We say that the capacitors (F_1, G_1) and (F_2, G_2) do not intersect if the sets G_1 and G_2 do not intersect.

Our main result is the following theorem.

Theorem 1.1. *Let a metric space X equipped with a measure μ and with a capacity* cap *satisfy the hypotheses (A0)-(A3). Let k be the smallest integer satisfying the inequality*

$$k \geq c\frac{M}{\nu\alpha}, \tag{1.1}$$

(where $c = \frac{1}{60}$). Then it is possible to find at least k capacitors (F_i, G_i), $i = 1, 2, ...k$ so that the following assertions hold:

(X1): *the capacitors (F_i, G_i), $i = 1, 2, ...k$, do not intersect pairwise;*

(X2): *for any $i = 1, 2, ... k$*

$$\mu F_i \geq \alpha\ ; \tag{1.2}$$

(X3): *for any $i = 1, 2, ... k$*

$$\mu(G_i \backslash F_i) \leq 6\nu\alpha\ . \tag{1.3}$$

(X4): *for any $i = 1, 2, ... k$*

$$\mathrm{cap}(F_i, G_i) \leq 22\nu\gamma\ ; \tag{1.4}$$

Of course, the number of disjoint sets F_i satisfying (1.2), cannot be larger than $\frac{M}{\alpha}$ so the desired lower bound (1.1) for k is optimal up to a constant factor c/ν. Let us mention also that all constants $\frac{1}{60}$, $22, 6$ etc. are technical and are in no case optimal.

The method of proof is based on the arguments on N.Korevaar [3] which he applied to obtain the upper bounds of the higher eigenvalues of the Laplace operator of the conformal metric on the Riemannian manifolds (see also [6] and [4] for the two-dimensional case). Indeed, each capacitor provided by Theorem 1.1, can be used to produce a test function for the eigenvalue problem. We have modified the proof of N.Korevaar so that it works in a rather abstract setting. The advantage of having an abstract setting will be clear from the applications we will show below: alongside with the case of Laplace-Beltrami operator on Riemannian manifolds, we are able to estimate from above the eigenvalues of elliptic operators on graphs and to derive lower bounds for the number of cycles on a graph.

Let us note that the geometric properties of manifolds or graphs which enter the final estimates are basically those involved in (A1) and (A2):

- the constant ν which can be also derived from the doubling volume property;
- and the constant γ - the upper bound of certain capacitors - which can also be derived from the volume growth function of the balls.

The condition (A0) holds automatically in compact/finite cases which our results are mainly aimed at. The condition (A4) puts practically no additional restriction - it is just a way of choosing a parameter α (the value of ρ is never required). We will show that for the case of manifolds, α may take any value from $(0, M)$ whereas on graphs $\alpha \in (m, M)$ where m is the maximal measure of a vertex.

Let X be a finite connected non-complete graph, let $\mu_0(x)$ be a combinatorial measure on X (=the degree of a vertex x) and let $\mu(x)$ be another measure on X. Let us define an operator on functions on X as

$$L = \frac{\mu_0(x)}{\mu(x)}\Delta \tag{1.5}$$

where Δ is a combinatorial Laplace operator on X. Let $m = \min_x \mu(x)$ and $M = \mu(X)$. Then we claim that the eigenvalues λ_k of the operator L satisfy the inequality

$$\lambda_k \le C\frac{k}{M}$$

provided

$$1 \le k < c\frac{M}{m}$$

where the constants C and c are expressed via the parameters ν and γ from the properties $(A1)$ and (A2)[1], and depend only on the combinatorial structure of X (and do not depend on μ).

Let us assume further that any vertex $x \in X$ has at least K adjacent edges. Let $\mathcal{C}_k$ denote the total number of closed paths on X of k edges each (self-intersections are allowed). There is a simple well-known connection between $\mathcal{C}_k$ and the adjacency matrix A of the graph:

$$\mathcal{C}_k \ge \frac{1}{k}\operatorname{tr} A^k.$$

By exploiting upper the bounds of eigenvalues of Δ, we obtain the following lower bound of $\mathcal{C}_k$: for any even $k > 1$

$$\mathcal{C}_k \ge \frac{NK^{k+1}}{4000\gamma\nu^2 k^2}$$

where ν and γ are the constants from the conditions (A1) and (A2) respectively. Let us note for comparison that if any vertex has exactly K adjacent edges then

$$\mathcal{C}_k \le \frac{NK^k}{2}.$$

The structure of this paper is the following. In Section 2 we introduce the abstract notion of capacity and explain it on Riemannian manifolds. We also show how Theorem 1.1 works on manifolds to give the upper bounds of the eigenvalues of the Laplace operator obtained originally in [6] and [3].

In Section 3 we introduce discrete elliptic operators on graphs and apply Theorem 1.1 in that setting to obtain the upper estimates of their eigenvalues. In Section 4 we apply those estimates to prove a lower bound for a number of cycles of an even length on a graph.

Section 5-7 are devoted to the proof of Theorem 1.1.

2. Capacity and examples on Riemannian manifolds

2.1. Abstract definition of capacity. As was mentioned already, any couple (F, G) of sets $F, G \in \mathcal{A}$ such that $F \subset G$ will be referred to as *a capacitor.* We denote by Ω^r the r-neighbourhood of the set $\Omega \subset X$ namely

$$\Omega^r = \{x \in X \mid \; d(x, \Omega) < r\}.$$

A capacity $\operatorname{cap}(F, G)$ is a real-valued function on the set of all capacitors satisfying the following natural hypotheses:

- **(C0):** $\operatorname{cap}(F, G) \ge 0$ (the value $+\infty$ is admitted);
- **(C1):** there is a number $\delta \ge 0$ such that for any $r > \delta$ and for all $\mathcal{A}$-sets E, F, G such that $E^r \subset F \subset G$ we have

$$\operatorname{cap}(F, G) = \operatorname{cap}(F \backslash E, G)$$

  (which means that the essentially interior points of F do not affect the capacity - the latter is rather a function of $G \backslash F$);
- **(C2):** if $F_1 \subset F_2$ and $G_1 \supset G_2$ then

$$\operatorname{cap}(F_1, G_1) \le \operatorname{cap}(F_2, G_2);$$

- **(C3):** for any capacitor (F, G)

$$\operatorname{cap}(F, G) = \operatorname{cap}(X \backslash G, X \backslash F);$$

- **(C4):** for any two capacitors (F_1, G_1) and (F_2, G_2)

$$\operatorname{cap}(F_1 \cup F_2, G_1 \cup G_2) \le \operatorname{cap}(F_1, G_1) + \operatorname{cap}(F_2, G_2)$$

[1]See Sections 3-4 below for explanation how to define capacity and operators on a graph

The properties (C3) and (C4) imply the following one which will be the most useful:

(C5): for any two capacitors (F_1, G_1) and (F_2, G_2) we have

$$\operatorname{cap}(F_1 \backslash G_2, G_1 \backslash F_2) \leq \operatorname{cap}(F_1, G_1) + \operatorname{cap}(F_2, G_2)\ .$$

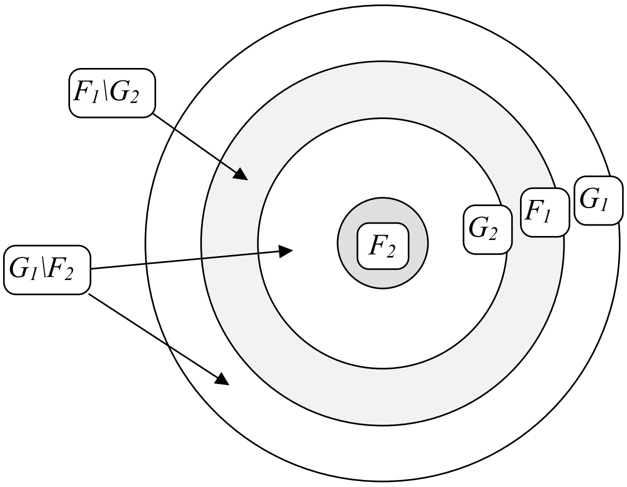

FIGURE 1. Illustration of property (C5)

Indeed, we have by (C3) and (C4)

$$\begin{aligned}
&\operatorname{cap}(F_1 \backslash G_2, G_1 \backslash F_2) \stackrel{(C3)}{=} \operatorname{cap}\left(X \backslash (G_1 \backslash F_2), X \backslash (F_1 \backslash G_2)\right) \\
&= \operatorname{cap}\left((X \backslash G_1) \cup F_2, (X \backslash F_1) \cup G_2\right) \\
&\stackrel{(C4)}{\leq} \operatorname{cap}\left(X \backslash G_1, X \backslash F_1\right) + \operatorname{cap}\left(F_2, G_2\right) \\
&\stackrel{(C3)}{=} \operatorname{cap}(F_1, G_1) + \operatorname{cap}(F_2, G_2)\ .
\end{aligned}$$

2.2. Capacity on a Riemannian manifold. All hypotheses (C0)-(C4) hold for a standard variational definition of a capacity on a Riemannian manifold. Indeed, let X be a Riemannian manifold with a Riemannian measure μ_0 (which is not necessarily to be μ - we will discuss this later on). Let us define for some $p > 0$ a capacity

$$\operatorname{cap}_p(F, G) = \inf_{f \in \mathcal{F}(F,G)} \int_X |\nabla f|^p \, d\mu_0 \tag{2.1}$$

where $\mathcal{F}(F, G)$ is a set of all real-valued Lipschitz functions $f(x)$ on X such that $0 \leq f(x) \leq 1$ and $f|_F = 1,\ f|_{X \backslash G} = 0$.

The condition (C0) is obviously true, (C2) follows from $\mathcal{F}(F_1, G_1) \supset \mathcal{F}(F_2, G_2)$, and (C3) follows from the fact that if $f \in \mathcal{F}(F, G)$ then $1 - f \in \mathcal{F}(X \backslash G, X \backslash F)$ with the same integral (2.1).

To prove (C1) (with $\delta = 0$) let us note that for any $f \in \mathcal{F}(F \backslash E, G)$ the function

$$w(x) = \begin{cases} f(x), & x \in X \backslash F \\ 1, & x \in F \end{cases}$$

belong to $\mathcal{F}(F, G)$ and

$$|\nabla w| \leq |\nabla f|$$

whence it follows

$$\operatorname{cap}_p(F, G) \leq \operatorname{cap}_p(F \backslash E, G)\,.$$

The inequality to the opposite direction follows from (C2).

Finally, (C4) follows from the fact that if $f \in \mathcal{F}(F_1, G_1)$, $g \in \mathcal{F}(F_2, G_2)$ then the function $w = \max(f, g) \in \mathcal{F}(F_1 \cup F_2, G_1 \cup G_2)$ and

$$|\nabla w| \leq \max(|\nabla f|, |\nabla g|)$$

whence

$$\int_X |\nabla w|^p \, d\mu_0 \leq \int_X |\nabla f|^p \, d\mu_0 + \int_X |\nabla g|^p \, d\mu_0.$$

See [5] for more details on various definitions and properties of capacities.

2.3. Theorem 1.1 on Riemannian manifolds. We introduce a distance d on a Riemannian manifold X as the Riemannian distance. We could also define a measure μ as the Riemannian measure μ_0 but for our purposes we need to divorce μ from the Riemannian structure and admit that μ is *some* measure (for example, a Riemannian measure of another metric). Let us take also $\operatorname{cap} \equiv \operatorname{cap}_p$ with some $p > 0$ and check the hypotheses (A1)-(A3).

The hypothesis (A1) depends only of the distance function (no capacity or measure is involved) and holds, for example, on a compact Riemannian manifold X with some ν depending on the geometry of the manifold. Another case when (A1) is true is a complete manifold of a non-negative Ricci curvature - then it is not necessarily compact, and ν depends only on the dimension of X.

The hypothesis (A1) can be also deduced from the doubling volume property

$$\mu_0 \mathbb{B}_{2r}^x \leq D \, \mu_0 \mathbb{B}_r^x \tag{2.2}$$

should it hold with a constant D independent of $x \in X$ and $r > 0$. The number ν depends then only on D (see [2]). The doubling volume property (2.2) is true for both above cases of compact manifolds and non-negatively curved manifolds.

The hypothesis (A2) depends on the distance and capacity and is valid on the manifold X provided the following holds:

- for any ball $\mathbb{B}_r^x$ and some positive n, C

$$\mu_0 \mathbb{B}_r^x \leq C r^n \tag{2.3}$$

- $\operatorname{cap} \equiv \operatorname{cap}_p$ and $p \leq n$; moreover, if $p < n$ then, in addition, X must be a compact.

The condition (2.3) holds, for example, if $\dim X = n$ and if X is compact or if X is complete and has a non-negative Ricci curvature.

Let us show how to derive (A2) from the two hypotheses above. Let $B = \mathbb{B}_r^x$ and let us take a function

$$f(\cdot) = \max\left(1, \ 2 - \frac{d(x, \cdot)}{r}\right)$$

so that $f|_B = 1$, $f|_{X \setminus \widetilde{B}} = 0$ and f is linear in radius in $\widetilde{B} \setminus B$. Therefore $f \in \mathcal{F}(B, \widetilde{B})$ and $|\nabla f| \leq r^{-1}$ whence we obtain

$$\operatorname{cap}_p(B, \widetilde{B}) \leq \frac{1}{r^p} \mu_0(\widetilde{B}) \leq \frac{1}{r^p} C \, (2r)^n = 2^n C r^{n-p}.$$

If $n = p$ then we get exactly (A2) with $\gamma = 2^n C$. In this case, it does not matter whether X is compact or not. If $n > p$ and X is compact then r can be always reduced to $\operatorname{diam} X$ so we take $\gamma = 2^n C (\operatorname{diam} X)^{n-p}$.

The hypothesis (A3) depends on the distance d and on the measure μ. The number ρ itself will never be needed for applications whereas the range of possible values of α is very important. If ρ is chosen already then α can be taken arbitrarily to satisfy

$$\sup_{x \in X} \mu \mathbb{B}_{\rho/5}^x < \alpha < \sup_{x \in X} \mu \mathbb{B}_\rho^x. \tag{2.4}$$

For the case when the measure μ is absolutely continuous with respect to the Riemannian measure μ_0 with a positive density, ρ can be taken as follows

$$0 < \rho < R := \inf_x \sup_y d(x, y), \tag{2.5}$$

and α ranges the entire interval $(0, M)$.

To show that, let us introduce a function $V(x) \equiv \mu \mathbb{B}_{\rho/5}^x$ which is obviously continuous on X. If X is compact then $V(x)$ attains its supremum on X. If X is not compact then $V(x)$ attains its supremum as well, due to $V(x) \to 0$ as $x \to \infty$ because of finiteness of the total measure of X.

Let the supremum of $V(x)$ attains at a point x, then we have due to (2.5) a strict inequality $\mu\mathbb{B}^x_{\rho/5} < \mu\mathbb{B}^x_\rho$, and α can be chosen from

$$\mu\mathbb{B}^x_{\rho/5} < \alpha < \mu\mathbb{B}^x_\rho \tag{2.6}$$

which implies obviously (2.4).

Let us show that by varying ρ as in (2.5) and α as in (2.6) we can get any value $\alpha \in (0, M)$. Indeed, if $\rho \to R$ then $\sup_{x\in X} \mu\mathbb{B}^x_{\rho/5} \to M$ whereas if $\rho \to 0$ then $\sup_{x\in X} \mu\mathbb{B}^x_\rho \to 0$. To substantiate the latter, let us assume from the contrary that there is a sequence x_i, ρ_i such that $\rho_i \to 0$ and

$$\mu\mathbb{B}^{x_i}_{\rho_i} > \varepsilon > 0. \tag{2.7}$$

Then either there is a convergent subsequence of x_i, and, thus, $\mu\mathbb{B}^{x_i}_{\rho_i} \geq \varepsilon$ which contradicts the absolute continuity of μ, or there is a subsequence of x_i diverging to ∞ which together with (2.7) contradicts the finiteness of $\mu X = M$.

We conclude, that all conditions (A0)-(A3) hold provided X is either a compact Riemannian manifold or a complete non-compact manifold of a non-negative Ricci curvature. In both cases the distance d is the geodesic distance, the capacity $\mathrm{cap} \equiv \mathrm{cap}_p$ with $p \in (0, n]$ (where $n = \dim X$) for the compact case and with $p = n$ for the non-compact case, and the measure μ is absolutely continuous with respect to the Riemannian measure with a positive density. The number α may take any value within the interval $(0, M)$.

2.4. Upper bounds of eigenvalues of Laplace operator. Let us show how to apply Theorem 1.1 to obtain upper bounds for the eigenvalues of the Laplace operator on a n-dimensional Riemannian manifold X. We mainly follow the argument of Korevaar [3].

Let g_0 denote a Riemannian metric of X, $d(\cdot,\cdot)$ be the geodesic distance of g_0, and $\mathrm{cap} = \mathrm{cap}_n$. Let us suppose that the hypotheses (A1) and (A2) hold for these d and cap. Let g be another Riemannian metric on X conformal to g_0 and let μ be the Riemannian measure of the metric g. Of course, we have to explicitly assume (A0) - finiteness of the measure μ. Moreover, let us also assume that the Laplace operator Δ of the metric g has in $L^2(X, \mu)$ a discrete non-negative spectrum $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq ....$ (note, that $\lambda_1 = 0$ because $1 \in L^2(X, \mu)$ is the first eigenfunction). For example, it is always the case when X is compact.

As follows from a general spectral theory, if one wants to prove that

$$\lambda_k \leq A$$

for some A then it suffices to construct a set of k functions ϕ_i on X which are Lipschitz, have non-intersecting supports and for all $i = 1, 2, ...k$

$$\frac{\int |\nabla\phi_i|^2 \, d\mu}{\int \phi_i^2 d\mu} \leq A. \tag{2.8}$$

Given an integer k, let us take $\alpha = c\frac{M}{\nu k} < M$ (which will discharge (A3)), and find by Theorem 1.1 k capacitors (F_i, G_i) satisfying all conditions (X1)-(X4). For any capacitor (F_i, G_i) constructed in this way, let us take a function $\phi_i \in \mathcal{F}(F_i, G_i)$ which almost optimizes the capacity, say,

$$\int |\nabla_0\phi_i|^n \, d\mu_0 \leq \mathrm{cap}(F_i, G_i) + \varepsilon$$

with a small enough $\varepsilon > 0$ (here ∇_0 and μ_0 are the gradient and the measure of the metric g_0). Since $n = \dim X$ and the metrics g and g_0 are conformal then we have

$$\int |\nabla_0\phi_i|^n \, d\mu_0 = \int |\nabla\phi_i|^n \, d\mu \, .$$

Therefore, we obtain by Hölder inequality

$$\begin{aligned}\int |\nabla\phi_i|^2\,d\mu &= \int\limits_{G_i\setminus F_i} |\nabla\phi_i|^2\,d\mu \\ &\leq \left(\int |\nabla\phi_i|^n\,d\mu\right)^{\frac{2}{n}} (\mu G_i\setminus F_i)^{\frac{n-2}{n}} \\ &= \left(\int |\nabla_0\phi_i|^n\,d\mu_0\right)^{\frac{2}{n}} (\mu G_i\setminus F_i)^{\frac{n-2}{n}} \\ &\leq (\operatorname{cap}(F_i,G_i)+\varepsilon)^{\frac{2}{n}} (\mu G_i\setminus F_i)^{\frac{n-2}{n}}\ .\end{aligned}$$

On the other hand

$$\int \phi_i^2 d\mu \geq \int\limits_{F_i} \phi_i^2 d\mu = \mu F_i$$

whence we get

$$\frac{\int |\nabla\phi_i|^2\,d\mu}{\int \phi_i^2 d\mu} \leq \frac{(\operatorname{cap}(F_i,G_i)+\varepsilon)^{\frac{2}{n}} (\mu\,(G_i\setminus F_i))^{\frac{n-2}{n}}}{\mu F_i}\ .$$

Each term on the right hand side can be estimated by Theorem 1.1 which yields

$$\begin{aligned}\frac{\int |\nabla\phi_i|^2\,d\mu}{\int \phi_i^2 d\mu} &\leq \frac{(2\nu\gamma+\varepsilon)^{\frac{2}{n}} (2\nu\alpha)^{\frac{n-2}{n}}}{\alpha} \\ &= \frac{\text{const}_{\,\nu,\gamma}}{\alpha^{2/n}} = \text{const}_{\,\nu,\gamma}\left(\frac{k}{M}\right)^{\frac{2}{n}}\end{aligned}$$

whence it follows

$$\lambda_k \leq \text{const}_{\,\nu,\gamma}\left(\frac{k}{M}\right)^{\frac{2}{n}}\ . \tag{2.9}$$

For 2-dimensional manifolds, this inequality was proved by Yang and Yau [6], and for dimension $n>2$ by Korevaar [3]. A remarkable feature of (2.9) is that it depends on the metric g only via its volume M. The other geometric properties are hidden in the constant $\text{const}_{\nu,\gamma}$ which depends however only on the background metric g_0.

3. Elliptic operators on graphs and their eigenvalues

3.1. Capacity on a graph. Let X be a connected locally finite graph with the set of edges E. We define a (combinatorial) distance $d(x,y)$ between the vertices $x,y\in X$ as the smallest number of edges in a path connecting x and y. Let μ be so far an arbitrary measure on X. We will write $\mu(x)$ for the measure of a single point set $\{x\}$.

To define a capacity, let any edge $\xi\in E$ be assigned two positive weights - a resistance $\iota(\xi)$ and a length (measure) $\sigma(\xi)$. We orient arbitrarily every edge ξ and define $\langle x,\xi\rangle$ as $+1$ if the vertex $x\in X$ is the end of ξ, -1 if x is the beginning of ξ, and 0 otherwise. Let us also denote by ξ_- and ξ_+ the beginning and the end of ξ respectively. We denote by $\overline{xy}$ an edge which has the beginning at the vertex x and the end at y. Also, we write $x\sim y$ if the vertices x and y are connected by an edge, and $\xi\sim x$ if the point x is either the beginning or the end of the edge ξ.

Given a function $f(x)$ on X, we define its gradient ∇f as a functions on E as follows:

$$\nabla f(\xi) := \frac{f(\xi_+)-f(\xi_-)}{\iota(\xi)} = \sum_{x\in X} f(x)\,\langle x,\xi\rangle\,\iota^{-1}(\xi)\ . \tag{3.1}$$

Let us define a capacity of a capacitor (F,G) on X for any $p>0$:

$$\operatorname{cap}_p(F,G) = \inf_{f\in\mathcal{F}(F,G)} \sum_{\xi\in E} |\nabla f(\xi)|^p\,\sigma(\xi). \tag{3.2}$$

Here $F\subset G$ are subsets of X, and $\mathcal{F}(F,G)$ consists of functions $f(x)$ on X such that $f|_F=1$, $f|_{X\setminus G}=0$.

Let us verify the axioms (C0)-(C4). Axiom (C0) is obvious. Axiom (C2) follows from the fact that $\mathcal{F}(F,G)$ is reverse monotone with respect to F and monotone with respect to G. Axiom (C3) is implied by an observation that there is an one-to-one correspondence between $\mathcal{F}(F,G)$ and $\mathcal{F}(X\backslash G, X\backslash F)$ given by $f \longmapsto 1-f$ with the same sum (3.2).

Let us show that (C1) holds with $\delta=1$. It suffices to show that

$$\operatorname{cap}_p(F,G) \le \operatorname{cap}_p(F\backslash E, G) \tag{3.3}$$

since the opposite inequality follows from (C2). Indeed, given $f\in\mathcal{F}(F\backslash E,G)$, let us define

$$g(x)=\begin{cases} f(x), \ x\in X\backslash E \\ 1, \ x\in E\end{cases}.$$

Obviously, $g\in\mathcal{F}(F,G)$ and (3.3) will be implied by

$$|\nabla g(\xi)| \le |\nabla f(\xi)| \tag{3.4}$$

for any edge $\xi\in E$. Let $\xi=\overline{xy}$. To show (3.4) let us consider three cases:

(1) $x\in E$, $y\in E$. We have obviously $g(x)=g(y)=1$ and $\nabla g(\xi)=0$.
(2) $x\in E$, $y\notin E$. Since $r>\delta=1$ then $y\in E^r\subset F$ whence $y\in F\backslash E$ and $g(y)=f(y)=1=g(x)$ with $\nabla g(\xi)=0$ again.
(3) $x\notin E$, $y\notin E$. In this case the functions f and g coincide on the set $\{x,y\}$ whence $\nabla g(\xi)=\nabla f(\xi)$.

Finally, let us verify (C4). If $f\in\mathcal{F}(F_1,G_1)$, $g\in\mathcal{F}(F_2,G_2)$ then the function $h=\max(f,g)$ belongs to $\mathcal{F}(F_1\cup F_2, G_1\cup G_2)$ and satisfies the inequality

$$|\nabla h| \le \max(|\nabla f|, |\nabla g|)$$

which implies (C4).

Indeed, let $\xi\in E$ and $x=\xi_+$, $y=\xi_-$. If h coincides on $\{x,y\}$ either with f or with g then there is nothing to prove. Otherwise we may assume that $h(x)=f(x)\ge g(x)$ and $h(y)=g(y)\ge f(y)$. Consider two cases:

(1) $h(x)\ge h(y)$. Then

$$0\le \nabla h(\xi)=\frac{h(x)-h(y)}{\iota(\xi)}=\frac{f(x)-g(y)}{\iota(\xi)}\le\frac{f(x)-f(y)}{\iota(\xi)}=\nabla f(\xi).$$

(2) $h(x)<h(y)$. Then

$$0\ge \nabla h(\xi)=\frac{h(x)-h(y)}{\iota(\xi)}=\frac{f(x)-g(y)}{\iota(\xi)}\ge\frac{g(x)-g(y)}{\iota(\xi)}=\nabla g(\xi).$$

3.2. Elliptic operator on a graph. Alongside with the gradient defined by (3.1), we introduce the "divergence" which is by definition the adjoint operator to ∇ considered as one from $L^2(X,\mu)$ to $L^2(E,\sigma)$. Let $(\cdot,\cdot)$ denote the inner product in $L^2(X,\mu)$ and $[\cdot,\cdot]$ be the inner product in $L^2(E,\sigma)$. For arbitrary functions $f(x)$ and $\phi(\xi)$ defined on X and E respectively and with finite supports, we have

$$\begin{aligned}[][\phi,\nabla f] &= \sum_{\xi\in E}\phi(\xi)\nabla f(\xi)\sigma(\xi)\\ &= \sum_{\xi\in E}\phi(\xi)\sigma(\xi)\sum_{x\in X} f(x)\langle x,\xi\rangle\iota^{-1}(\xi)\\ &= \sum_{x\in X} f(x)\left[\frac{1}{\mu(x)}\sum_{\xi\in E}\phi(\xi)\sigma(\xi)\langle x,\xi\rangle\iota^{-1}(\xi)\right]\mu(x).\end{aligned}$$

Therefore, ∇^* can be defined by

$$\nabla^*\phi(x)=\frac{1}{\mu(x)}\sum_{\xi\in E}\phi(\xi)\langle x,\xi\rangle\sigma(\xi)\iota^{-1}(\xi), \tag{3.5}$$

and we have the "integration-by-parts" formula

$$[\phi,\nabla f]=(\nabla^*\phi, f). \tag{3.6}$$

Next we define a self-adjoint operator $L=-\nabla^*\nabla$ which acts in $L^2(X,\mu)$ and can be considered as a discrete elliptic operator on X. The explicit formula for L is as follows:

$$Lf(x)=\frac{\mu_0(x)}{\mu(x)}\left(\sum_{y\sim x} f(y)p(x,y)-f(x)\right) \tag{3.7}$$

where

$$\mu_0(x) := \sum_{\xi\sim x} \sigma(\xi)\, \iota^{-2}(\xi) \tag{3.8}$$

and

$$p(x,y) := \sigma(\overline{xy})\iota^{-2}(\overline{xy})\mu_0(x)^{-1}. \tag{3.9}$$

Let us mention that for any $x \in X$

$$\sum_{y\sim x} p(x,y) = 1,$$

in particular $L1 = 0$.

If $\mu = \mu_0$ then we have a discrete Laplace operator

$$\Delta f(x) = \sum_{y\sim x} f(y)p(x,y) - f(x) \tag{3.10}$$

associated with the choice of σ and ι (in fact what matters is $\sigma\iota^{-2}$). If $\sigma \equiv \iota \equiv 1$ then (3.10) defines a *combinatorial* Laplace operator. In this case, $\mu_0(x)$ is a degree of the point x - the number of edges adjacent to x.

3.3. Eigenvalues of L. Let us consider the eigenvalues of the operator L. Henceforth we assume that the set X is finite, so the space $L^2(X,\mu)$ is of finite dimension. Let N be the cardinal number of X, then the operator $-L$ is non-negative and has N real eigenvalues $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_N$ counted with their multiplicity. The simple facts about λ_i are (see [1] for a consistent account of spectral properties of graphs):

(1) $\lambda_1 = 0$;
(2) for any $i = 1, 2, \ldots N$

$$0 \le \lambda_i \le 2\sup_x \frac{\mu_0(x)}{\mu(x)}\,; \tag{3.11}$$

(3) the trace of $-L$ is

$$\operatorname{tr}(-L) = \sum_{i=1}^{N} \lambda_i = \sum_{x\in X} \frac{\mu_0(x)}{\mu(x)}\,. \tag{3.12}$$

Indeed, the variational principle says that the maximum and the minimum eigenvalues maximizes and minimizes respectively the Rayleigh quotient

$$\mathcal{R}(f) := \frac{(-Lf,f)}{(f,f)} = \frac{[\nabla f, \nabla f]}{(f,f)}.$$

It gives immediately $\lambda_i \ge 0$ and $\lambda_1 = 0$ since $\nabla 1 = 0$.

The upper bound in (3.11) follows from

$$\begin{aligned}
[\nabla f, \nabla f] &= \sum_{\xi\in E} |\nabla f|^2(\xi)\sigma(\xi) = \sum_{\xi\in E} [f(\xi_+) - f(\xi_-)]^2\, \iota^{-2}(\xi)\sigma(\xi) \\
&\le 2\sum_{\xi\in E} \left[f^2(\xi_+) + f^2(\xi_-)\right] \iota^{-2}(\xi)\sigma(\xi) \\
&= 2\sum_{\xi\in E}\sum_{x\in X} f^2(x)\,\langle x,\xi\rangle^2\, \iota^{-2}(\xi)\sigma(\xi) \\
&\le 2\sup_x \frac{\mu_0(x)}{\mu(x)} \sum_{x\in X} f^2(x)\mu(x) = 2\sup_x \frac{\mu_0(x)}{\mu(x)}\,(f,f)\,.
\end{aligned}$$

To show (3.12), let us introduce the basis $\{\delta^x\}_{x\in X}$ in $L^2(X,\mu)$ where δ^x is a function on X which takes value 1 at x and 0 otherwise. Let $\{\delta_x\}$ be a dual basis i.e. $\delta_x(f) \equiv f(x)$. Then

$$\begin{aligned}
\sum_{i=1}^{N} \lambda_i &= \operatorname{tr}(-L) = \sum_{x\in X} \delta_x(-L\delta^x) = -\sum_{x\in X} L\delta^x(x) \\
&= \sum_{x\in X} \frac{\mu_0(x)}{\mu(x)} \left(-\sum_{y\sim x} \delta^x(y)p(x,y) + \delta^x(x) \right) = \sum_{x\in X} \frac{\mu_0(x)}{\mu(x)}
\end{aligned}$$

3.4. Theorem 1.1 on graphs. The hypotheses (A0)-(A3) of Theorem 1.1 do hold with some constants $\nu, \gamma, \rho, \alpha$ assuming that X is a finite graph.

Let us comment on α, ρ. We claim that if X is not a complete graph[2] then α may take any value from the interval (m, M) where $m = \max_{x \in X} \mu(x)$ and $M = \mu(X)$ as before. To show that, let us introduce *an inner radius* $R := \min_x \max_y d(x, y)$. We have $R > 1$ due to non-completeness. Then for any $\rho \in (\frac{5}{4}, R)$ (where $\frac{5}{4}$ comes from $\frac{5}{4}\delta$, and $\delta = 1$ is the constant from the property (C1)) we have a strict inequality $\mu\mathbb{B}^x_\rho > \mu\mathbb{B}^x_{\rho/5}$ since there is an integer $\rho^* \in (\rho/5, \rho)$ and thus there is a strict inclusion $\mathbb{B}^x_{\rho/5} \subset\subset \mathbb{B}^x_{\rho^*} \subseteq \mathbb{B}^x_\rho$.

Therefore if we denote $V(r) := \max_{x \in X} \mu\mathbb{B}^x_r$ then we have $V(\rho) > V(\rho/5)$ for any $\rho \in (\frac{5}{4}, R)$, and for any such ρ, we can choose α arbitrarily from the interval $(V(\rho/5), V(\rho)]$. Hence, α can take any value from the interval $(V(1/4), V(R)) = (m, M)$ what was claimed.

Let us also mention that for the capacity cap_2, the constant γ can be taken as follows:

$$\gamma = \sup_{x \in X,\, r>0} \frac{\mu_0 \mathbb{B}^x_{2r}}{\lceil r \rceil^2} \tag{3.13}$$

where μ_0 is defined by (3.8) and $\lceil \cdot \rceil$ is the ceiling function. Indeed, by definition, γ is an upper bound of all capacities $\mathrm{cap}_2(\mathbb{B}^x_r, \mathbb{B}^x_{2r})$ over all $x \in X$, $r > 0$. Let us define a trial function $f \in \mathcal{F}(\mathbb{B}^x_r, \mathbb{B}^x_{2r})$

$$f(y) = \begin{cases} 1, & \text{if } y \in \mathbb{B}^x_r \\ 0, & \text{if } y \notin \mathbb{B}^x_{2r} \\ \frac{b - d(x,y)}{b-a}, & \text{otherwise} \end{cases}$$

where $b = \lceil 2r \rceil$ and a is the largest integer smaller than r i.e. $a = r - 1$ if r is integer and $a = \lfloor r \rfloor$ otherwise.

It is easy to see that if an edge ξ has both ends outside $\mathbb{B}^x_{2r}$ then $\nabla f(\xi) = 0$. Otherwise we have

$$|\nabla f(\xi)| \le \frac{1}{(b-a)\,\iota(\xi)}$$

whence

$$\begin{aligned} \mathrm{cap}_2(\mathbb{B}^x_r, \mathbb{B}^x_{2r}) &\le [\nabla f, \nabla f] \le \frac{1}{(b-a)^2} \sum_{\xi \sim \mathbb{B}^x_{2r}} \iota^{-2}(\xi)\sigma(\xi) \\ &\le \frac{1}{(b-a)^2} \sum_{y \in \mathbb{B}^x_{2r}} \mu_0(y) = \frac{\mu_0 \mathbb{B}^x_{2r}}{(b-a)^2} \end{aligned}$$

Finally, we note that since $a < r < 2r \le b$ then $b - a > 2r - r = r$ and moreover, $b - a \ge \lceil r \rceil$ because $b - a$ is an integer..

Let us apply Theorem 1.1 to the setting of a finite graph equipped with the measure $\mu(x)$, the resistance $\iota(\xi)$ and the length $\sigma(\xi)$. The theorem says that there are at least k capacitors (F_i, G_i), $i = 1, 2, ...k$, with the following properties:

- $\mu F_i \ge \alpha$
- $\mathrm{cap}_2(F_i, G_i) \le 22\nu\gamma$

provided the numbers α, ν, k, γ satisfy the following requirements:

- $\alpha \in (m, M)$
- ν is such a number that any ball of radius r on X is covered by at most ν balls of radii $r/50$;
- $k = \lceil \frac{M}{60\alpha\nu} \rceil$
- $\gamma = \sup_{x \in X, r>0} \mathrm{cap}_2(\mathbb{B}^x_r, \mathbb{B}^x_{2r})$ or γ is defined by (3.13).

If f_i is a function which optimizes the capacity of the capacitor (F_i, G_i) , then we have

$$\mathcal{R}(f_i) = \frac{[\nabla f_i, \nabla f_i]}{(f_i, f_i)} = \frac{\mathrm{cap}_2(F_i, G_i)}{(f_i, f_i)} \le \frac{\mathrm{cap}_2(F_i, G_i)}{\mu F_i} \le 22\frac{\nu\gamma}{\alpha}.$$

Since the cardinal of the set $\{f_i\}$ is at least k, and the functions f_i, f_j, $i \ne j$, are orthogonal in $L^2(X, \mu)$ (because the different capacitors do not intersect) then we obtain that

$$\lambda_k \le 22\frac{\nu\gamma}{\alpha}$$

[2]A graph is complete if any two vertices are connected by an edge

Let us express α via k. Given an integer k, we take $\alpha = \frac{M}{60\nu k}$. To satisfy the restriction $\alpha \in (m, M)$, it suffices to have

$$1 \le k < \frac{M}{60\nu m} \tag{3.14}$$

Therefore, for those k we have

$$\lambda_k \le 1320\nu^2\gamma\frac{k}{M}. \tag{3.15}$$

Let us underline that the constant ν reflects only the combinatorial structure of X and does not depend on capacities and measures. The constant γ depends only on the capacity and is independent of the measure μ. Alternatively, γ may be chosen according to (3.13) and, thus, is determined by the measure μ_0.

Hence, the measure μ takes part in (3.15) only through its total mass M. Another place where μ is essential, is the restriction (3.14) on k. The inequality (3.15) can be considered as a discrete analogue of (2.9).

For the most interesting particular case when $\sigma(\xi) \equiv \iota(\xi) \equiv 1$, we have the following statement.

Theorem 3.1. *Let X be a finite non-complete graph, let $\mu(x)$ be a measure on the vertex set which defines an operator*

$$L = \frac{\mu_0(x)}{\mu(x)}\Delta \tag{3.16}$$

where $\mu_0(x)$ is the degree of a vertex x, and Δ is a combinatorial Laplace operator. Let $m = \min_x \mu(x)$ and $M = \mu(X)$.

Then the eigenvalues λ_k of the operator L satisfy the inequality

$$\lambda_k \le C\frac{k}{M}$$

provided

$$1 \le k < c\frac{M}{m}$$

where the constants $C = 1320\nu^2\gamma$ and $c = \frac{1}{60\nu}$ depend only on the combinatorial structure of X (and do not depend on μ).

3.5. Mappings of graphs. We will show here an application of Theorem 3.1. Let we have two finite graphs X and Y, and a mapping $I : Y \to X$ which is onto. We say that I is *a contraction* if for any points $p, q \in Y$,

$$d_X(I(p), I(q)) \le d_Y(p, q) \tag{3.17}$$

where d_X and d_Y are combinatorial distances on X and Y respectively.

It is simple that (3.17) holds for all $p, q \in Y$ if and only if it holds for all neighbouring $p, q \in Y$ (indeed, connect p and q by the shortest path and apply (3.17) to any consecutive pair of the vertices on this path). In other words, the condition (3.17) means that any neighbouring points $p, q \in Y$ are sent by I either to neighbouring points on X, or to the same point on X.

We say that D is a degree of the mapping I if any point $x \in X$ has at most D points $y \in Y$ such that $I(y) = x$.

Let us introduce conductance and measure on the edge set of X in the most simple way: $\sigma \equiv \iota \equiv 1$. Respectively, let μ_0 be a combinatorial measure on X (that is $\mu_0(x)$ is a number of edges adjacent to x), and $\text{cap} \equiv \text{cap}_2$. Let μ denote a combinatorial measure on Y. Then we can transfer the measure μ to X by taking

$$\mu(x) := \mu\left(I^{-1}(x)\right).$$

Hence, the graph X is endowed by measure μ, distance $d = d_X$ and capacity cap, and we can apply Theorem 3.1 to estimate the eigenvalues of the operator L defined by (3.16). Let us show that there is a direct relation between the eigenvalues of L on X and those of the combinatorial Laplace operator Δ on Y.

Proposition 3.2. *If $I : Y \to X$ is onto and contraction then for any k, $1 \le k \le \operatorname{card} X$, we have*

$$\lambda_k(\Delta) \le D^2\lambda_k(L) \tag{3.18}$$

Proof. Let $f_1, f_2, ...f_N$ be eigenfunctions of L which are orthonormal in $L^2(X,\mu)$, and let $g_1, g_2, ...g_N$ be their pullbacks to Y by I, that is $g_i = f_i \circ I$. Then g_i are obviously orthonormal in $L^2(Y,\mu)$, too (this makes use of the fact that I is onto). The inequality (3.18) will follow from

$$[g_i, g_i]_Y \le D^2 [f_i, f_i]_X \tag{3.19}$$

where

$$[f,f]_X := \sum_{\xi \in E_X} \left(f(\xi_+) - f(\xi_-)\right)^2 \tag{3.20}$$

and

$$[g,g]_Y = \sum_{\eta \in E_Y} \left(g(\eta_+) - g(\eta_-)\right)^2 \tag{3.21}$$

(we will suppress the subscript i in the sequel).

To show (3.19), let us fix an edge $\eta \in E_Y$ and note that by the contraction property, the points $I(\eta_+)$ and $I(\eta_-)$ either coincide or are connected by an edge ξ in X so we can write $I(\eta) = \xi$. In the first case, the corresponding term in (3.21) vanishes. In the second case,

$$|f(\xi_+) - f(\xi_-)| = |g(\eta_+) - g(\eta_-)|$$

We are left to show that for any $\xi \in E_X$, there is at most D^2 edges $\eta \in E_Y$ with $I(\eta) = \xi$. Indeed, since each point ξ_+, ξ_- has at most D sources in Y, then the number of possible edges between them does not exceed D^2. ∎

Hence, we conclude by Theorem 3.1 that for $1 \le k < c_\nu \frac{M}{m}$, we have the following upper bound for the eigenvalues of the combinatorial Laplace operator on Y

$$\lambda_k(\Delta) \le C_{\nu,\gamma} \frac{D^2 k}{M}. \tag{3.22}$$

It is important to mention that the upper bound (3.22) involves the structure of Y only via the degree D and M. The other parameters - ν and γ - reflect geometry of the graph X.

4. Number of cycles

We show here an application of the estimate (3.15) to a combinatorial problem of estimating a number of cycles on a graph. Throughout this section, X will be a finite graph. Let us denote by N the cardinal of X, and for any vertex $x \in X$ we denote by N_x the number of edges adjacent to x. We assume that for all x

$$N_x \ge K \tag{4.1}$$

with a positive integer K. This number will enter the final estimate of the of number of cycles.

Let us denote by $\mathcal{C}_k$ the number of all closed paths (cycles) of k edges (it is allowed to have all sorts of self-intersections). Out aim is to provide the upper and lower estimates for $\mathcal{C}_k$.

4.1. Number of cycles and the trace of a combinatorial Laplace operator.

To match the setting of the previous section, let us take $\iota(\xi) = \sigma(\xi) = 1$ for any edge $\xi \in E$. Therefore, for any $x \in X$

$$\mu_0(x) = \sum_{\xi \sim x} \sigma(\xi)\, \iota^{-2}(\xi) = N_x$$

and let us define $\mu(x) \equiv \mu_0(x) = N_x$. Then the operator (3.10) acquires the form

$$\Delta f(x) = \frac{1}{N_x} \sum_{y \sim x} f(y) - f(x)$$

and is called a combinatorial Laplace operator.

Let us identify the operator Δ with its matrix in the orthonormal basis $\left\{\widetilde{\delta^x}\right\}$ where $\widetilde{\delta^x} := \frac{\delta^x}{\sqrt{N_x}}$ so that

$$\Delta_{xy} = \left(\Delta\widetilde{\delta^x}, \widetilde{\delta^y}\right) = -\left[\nabla\widetilde{\delta^x}, \nabla\widetilde{\delta^y}\right]$$

whence

$$\Delta_{xy} = \begin{cases} -1, & \text{if } x = y \\ \frac{1}{\sqrt{N_x N_y}}, & \text{if } x \sim y \\ 0, & \text{otherwise} \end{cases}. \tag{4.2}$$

Let us put

$$A = K\left(I + \Delta\right)$$

where I is the identity matrix $N \times N$, or in other words, the $N \times N$ matrix A has the elements

$$A_{xy} = \begin{cases} \frac{K}{\sqrt{N_x N_y}}, & \text{if } x \sim y \\ 0, & \text{otherwise} \end{cases} . \tag{4.3}$$

Let us note that the eigenvalues α_i of the matrix A are expressed via λ_i (eigenvalues of $-\Delta$) as

$$\alpha_i = K\left(1 - \lambda_i\right). \tag{4.4}$$

On the other hand, the number of cycles can be estimated as follows

$$\mathcal{C}_k \geq \frac{1}{k} \operatorname{tr} A^k. \tag{4.5}$$

Indeed, it is important that every element A_{xy} is either 0 or is at most 1 when $x \sim y$, due to (4.1). Therefore, we obtain from the rule of multiplication of matrices that $\left(A^k\right)_{xy}$ does not exceed the number of all paths of k edges connecting x and y. Hence, $\left(A^k\right)_{xx}$ does not exceed the number of all k-paths which start and end at x. Since each closed k-path has at most k different vertices and

$$\sum_x \left(A^k\right)_{xx} = \operatorname{tr} A^k$$

then we obtain (4.5).

Finally, since $\operatorname{tr} A^k = \alpha_1^k + \alpha_2^k + ... + \alpha_N^k$ then we obtain from (4.4) and (4.5)

$$\mathcal{C}_k \geq \frac{K^k}{k} \sum_{i=1}^{N} \left(1 - \lambda_i\right)^k. \tag{4.6}$$

Let us also mention that for a K-regular graph (i.e. when all $N_x = K$.) one gets also an upper bound

$$\mathcal{C}_k \leq \frac{K^k}{2} \sum_{i=1}^{N} \left(1 - \lambda_i\right)^k . \tag{4.7}$$

Indeed, for a K-regular graph, $\left(A^k\right)_{xx}$ is exactly equal to the number of all k-paths which start and end at x. Since each closed k-path has at least 2 vertices then we obtain (4.7).

4.2. Estimates of $\mathcal{C}_k$. For the Laplace operator we have from (3.11) $0 \leq \lambda_i \leq 2$ and $|1 - \lambda_i| \leq 1$. If graph X is K-regular then we derive from (4.7)

$$\mathcal{C}_k \leq \frac{NK^k}{2}. \tag{4.8}$$

We will show that there is a similar lower bound of $\mathcal{C}_k$ (without assuming K-regularity) implied by the upper bound (3.15) of λ_i.

Theorem 4.1. *Let X be a connected non-complete graph with N vertices each of them having at least K adjacent edges. Then for any even $k > 1$ we have*

$$\mathcal{C}_k \geq \frac{NK^{k+1}}{4000\gamma\nu^2 k^2} \tag{4.9}$$

where ν and γ are the constants from the conditions (A1) and (A2) respectively.

Proof. A trivial lower bound for even k

$$\mathcal{C}_k \geq \frac{K^k}{k} \tag{4.10}$$

follows obviously from (4.6) if one uses $\lambda_1 = 0$ and $\left(1 - \lambda_i\right)^k \geq 0$. In particular, (4.10) holds on a complete graph which is not covered by Theorem 4.1. However, if N is very large (fix the other parameters K, γ, ν) then (4.9) is substantially sharper and, moreover, is comparable to the upper bound (4.8).

To obtain (4.9), we will estimate $\left(1 - \lambda_i\right)^k$ from below using the upper bounds for λ_i. Let us rewrite (3.15) as follows

$$\lambda_i \leq ai \tag{4.11}$$

where

$$a = \frac{1320\nu^2\gamma}{NK},$$

provided

$$1 \le i < i_0 \equiv \frac{N}{60\nu}$$

If $a(k+1) \ge 1$ then the right hand side of (4.9) is non-positive, and there is nothing to prove.

Let us assume that $a(k+1) < 1$ and denote by i_1 the maximal integer i for which $ai \le 1$. Then $i_1 \ge k+1 > 2$ and

$$\frac{i_1}{i_0} \le \frac{1}{ai_0} = \frac{K}{22\nu\gamma} < 1$$

because from the definition $\nu \ge K$ and $\gamma \ge \operatorname{cap}(\mathbb{B}^x_{0.5}, \mathbb{B}^x_1) = [\delta^x, \delta^x] \ge K$. Hence, for any $i \le i_1$ we have (4.11). We deduce from (4.6) for any even k

$$\mathcal{C}_k \ge \frac{K^k}{k} \sum_{i=1}^{N} (1-\lambda_i)^k \ge \frac{K^k}{k} \left(1 + \sum_{i=2}^{i_1} (1-ai)^k\right)$$

Let us estimate the sum on the right hand side:

$$\sum_{i=2}^{i_1} (1-ai)^k \ge \int_2^{1/a} (1-at)^k dt = \frac{1}{a}\frac{(1-2a)^{k+1}}{k+1} \ge \frac{1}{a}\frac{1-2a(k+1)}{k+1} = \frac{1}{a(k+1)} - 2$$

whence

$$\mathcal{C}_k \ge \frac{K^k}{k}\left(\frac{1}{a(k+1)} - 1\right).$$

Finally, we sum up this inequality with (4.10), apply $k+1 \le 3k/2$ and obtain (4.9). ∎

5. Decomposition of the space by balls

The rest of the paper is devoted to the proof Theorem 1.1. We apply the construction which was introduced by Korevaar [3] in the setting of Riemannian manifolds. The purpose of it is to locate places on X where the measure μ is concentrated at most.

Let us denote by $\mathbb{A}^x_{r,R}$ an annulus $\mathbb{B}^x_R \setminus \mathbb{B}^x_r$. For any annulus $A = \mathbb{A}^x_{r,R}$ we denote by $\widetilde{A}$ the "double" annulus $\mathbb{A}^x_{r/2,2R}$. For any set $\Omega \subset X$, we denote by Ω^r the r-neighbourhood of Ω, namely, $\Omega^r = \{y \in X : d(y,\Omega) < r\}$. For any ball B, we denote by $\widetilde{B}$ the concentric ball of the double radius.

Given two balls in X, we say that they *overlap* if the distance between their centres is smaller than the sum of their radii, and they are *disjoint* otherwise. In a Euclidean space overlapping in the above sense is equivalent to intersecting and disjointness is equivalent to non-intersecting but in a general metric space it is not necessarily true: whereas disjointness implies always non-intersecting, overlapping may occur with an empty intersection.

For any integer j let us consider a family Σ_j of all balls of radius $\rho_j \equiv 5^j\rho$ on X having measure μ at least α. By the hypothesis (A3), the class Σ_{-1} is empty whereas Σ_j is non-empty for $j \ge 0$. We shall construct inductively a triple of sets $\chi_j, \theta_j, \beta_j$ so that

- **(B1):** χ_j is a discrete set of points of X increasing on j;
- **(B2):** θ_j is a set of ρ_j-balls with centres at the points of χ_{j-1};
- **(B3):** β_j is a set of ρ_j-balls with centres at the points of $\chi_j \setminus \chi_{j-1}$;
- **(B4):** moreover, the double balls from β_j are mutually disjoint and are disjoint with any ball from θ_j.

We construct these sets inductively. As the inductive basis we put that any of the sets χ_j, θ_j, β_j is empty for a negative j. Let us describe the inductive step from $j-1$ to j.

The set θ_j is defined as in 5 since χ_{j-1} is known by the inductive hypothesis. Next we choose a maximal set of balls from Σ_j (i.e. balls of radius $\rho_j = 5^j\rho$ with the measure at least α), say, $B_1, B_2, ...$, such that their double $\widetilde{B}_i$ are disjoint and, moreover, each $\widetilde{B}_i$ is disjoint with any ball from θ_j.

The set β_j consists by definition of all the balls $B_1, B_2, ...$. We admit that β_j may be empty. Let us also note that the word "maximal" means that if we take any additional ball B from Σ_j, then its double $\widetilde{B}$ will overlap with either a ball from θ_j or with one of the balls $\widetilde{B}_i$. Existence of the maximal family

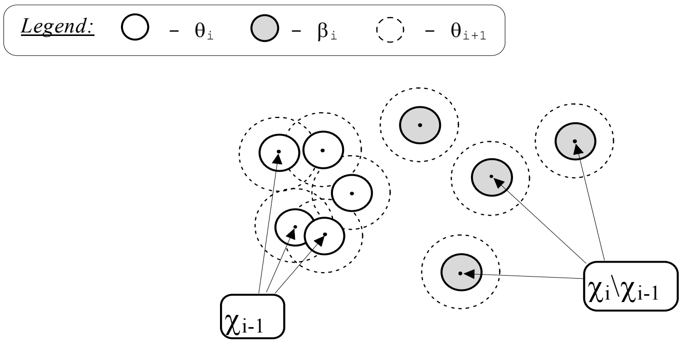

FIGURE 2. Balls of θ_j and β_j

$\{B_j\}$ follows from the fact that any disjoint set of balls from Σ_j contains at most $\frac{M}{\alpha}$ balls because any ball from Σ_j has a measure at least α. If we take a family with the maximum number of balls then it will be maximal in the above sense.

Finally, χ_j consists of χ_{j-1} plus all centres of β_j (see Fig. 2).

For example, set β_0 is a maximal set of ρ-balls of a measure at least α such that their double are mutually disjoint. Set χ_0 consists of the centres of β_0, and θ_0 is empty. Set θ_1 consists of 5ρ-balls concentric to those of β_0. To find β_1, we proceed as in the inductive step, and so on.

Lemma 5.1. *A union of all balls from θ_j covers a union of all balls from Σ_{j-1}.*

Proof. Indeed, let some ball $B \in \Sigma_{j-1}$ is not covered by θ_j. By definition, θ_j consists of thickened by factor 5 balls from θ_{j-1} and β_{j-1}. If they do not cover B then the distance between the centre of B and each of their centres is at least $4\rho_{j-1}$, and the double ball of B must be disjoint with the double of any ball from $\theta_{j-1} \cup \beta_{j-1}$.

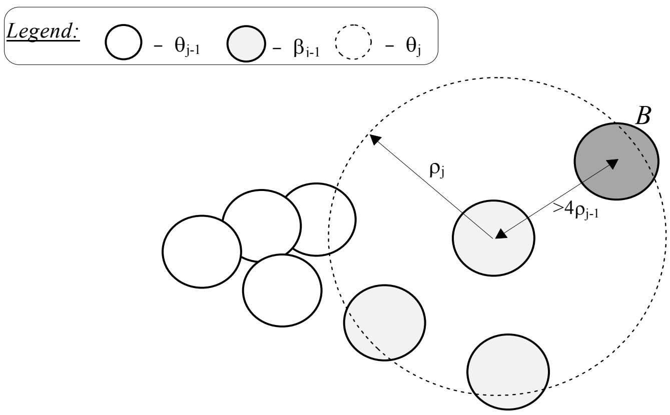

FIGURE 3. Ball B must belong to β_{j-1}

Because of maximality of β_{j-1}, the ball B must belong to β_{j-1} which contradicts the assumption that it is not covered by θ_j. ∎

Corollary 5.2. *The union of all balls from all θ_j coincides with the entire space X. The total number of all balls in all β_j is finite.*

Indeed, a union of all ρ_{j-1}-ball over all j with a fix centre coincides with X since $\rho_{j-1} \to \infty$ as $j \to \infty$. Therefore, the union of all balls from Σ_{j-1} is also X . By Lemma 5.1 so is the union of all balls from all θ_j.

To prove the second statement, let us note that all ball from β_j are disjoint across all j, and the measure of any ball from β_j is at least α. Therefore, the number of balls in all β_j does not exceed $\frac{M}{\alpha}$ and, thus, is finite. ∎

As follows from above the total number of points in $\bigcup_j \chi_j$ (that are centres of all balls β_j) is finite, and number of balls in each θ_j is finite, too. Of course, if the diameter of X is finite, then the union of all balls from θ_j coincides with X already for some finite j so the number of all balls in all θ_j is also finite. But if the diameter of X is infinite then it cannot be covered by a finite number of balls which means that in this case the total number of all balls in all θ_j is infinite.

Lemma 5.3. *For any ball $B \in \beta_j$ we have $\mu(B) \geq \alpha$ and*

$$\mu(\widetilde{B}) \leq \nu\alpha \tag{5.1}$$

Indeed, inequality $\mu(B) \geq \alpha$ is a part of the definition of β_j. Let us prove (5.1). By the hypothesis (A1), the ball $\widetilde{B}$ of radius $2\rho_j$ can be covered by at most ν balls of radius ρ_{j-1}. Let us remove the phantoms from this covering i.e. assume that each of those ρ_{j-1}-balls does intersect $\widetilde{B}$. Due to our construction, the ball $\widetilde{B}$ does not intersect θ_j.

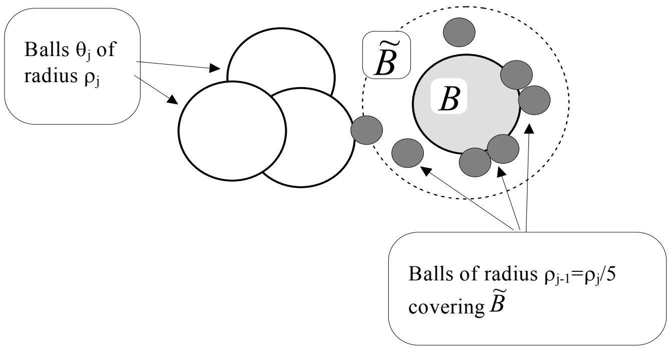

FIGURE 4. Any ρ_{j-1}-ball intersecting $\widetilde{B}$ has measure $< \alpha$

Therefore, none of the chosen balls of radius ρ_{j-1} can lie inside the union of ball from θ_j which implies by Lemma 5.1 that none of those ρ_{j-1}-balls is from Σ_{j-1}. It means that the measure of each of those ball is smaller than α which implies (5.1). ∎

We shall use the families of balls constructed above to produce enough number of capacitors satisfying (X1)-(X4). Here we consider two simple cases when the desired family of capacitors can be found easily while postponing a general case to the next sections. Let us denote by β the number of balls in the union $\cup\beta_j$ of all sets β_j. The first simple case will be when the number β is large enough whereas the second simple case will be the opposite one when $\beta = 1$. A general case which will be treated in the next sections can be regarded as a kind of interpolation between those two extreme situations.

The first simple case is when β satisfies (1.1) i.e.

$$\beta \geq c\frac{M}{\nu\alpha}.$$

If $B_1, B_2, ...B_\beta$ denote all the balls from $\cup\beta_j$ then their double $\widetilde{B}_i$ are disjoint across all β_j which obviously follows from the properties 5-5. On the other hand, by Lemma 5.3 we have $\mu(B_i) \geq \alpha$ and $\mu(\widetilde{B}_i) \leq \nu\alpha$. Since we have by (A2) that $\mathrm{cap}(B_i, \widetilde{B}_i) \leq \gamma$ then the family of capacitors $(B_i, \widetilde{B}_i)$ satisfies all the conditions (X1)-(X4).

Let us now consider the second simple case when

$$\beta = 1 < c\frac{M}{\nu\alpha}.$$

Let emphasize that logically the following argument is not necessary and will never be referred to later. We consider this case solely for the purpose to simplify understanding the next sections.

In this case, β_0 consists of a single ball $B_0 \equiv \mathbb{B}_\rho^{x_0}$, each next β_j , $j \geq 1$, is empty, and each θ_j consists of a single ball $B_j \equiv \mathbb{B}_{\rho_j}^{x_0}$ concentric to that of β_0. Let us consider the annuli

$$A_j \equiv B_j \backslash B_{j-1}$$

for any $j \geq 1$ and $A_0 \equiv B_0$. As follows from (A1), A_j can be covered by at most ν balls of the radius ρ_{j-1}. If $j \geq 1$ then each of the ρ_{j-1}-balls which intersects A_j does not lie in θ_{j-1}(indeed, θ_{j-1}consists of a single ball B_{j-1}) and therefore has a measure at most α. Hence we conclude that

$$\nu_j \equiv \mu(A_j) \leq \nu\alpha.$$

By Lemma 5.3, the same inequality applies to the measure $\nu_0 \equiv \mu(A_0)$ where $A_0 \equiv B_0$. On the other hand, we have

$$\nu_0 + \nu_1 + \nu_2 + ... = M.$$

Let us split the series $\nu_0 + \nu_1 + \nu_2 + ...$ in the following way (without changing the order):

$$\underbrace{(\nu_0 + ... + \nu_{i_1 - 1})}_{\geq \alpha} + \nu_{i_1} + \underbrace{(\nu_{i_1+1} + ... + \nu_{i_2 - 1})}_{\geq \alpha} + \nu_{i_2} + \underbrace{(\nu_{i_2+1} + ... + \nu_{i_3 - 1})}_{\geq \alpha} + \nu_{i_3} + ... \tag{5.2}$$

where each group in brackets is chosen so that the sum of that group is as least α (except for possibly the last bracket). Moreover, let us do that in the optimal way in the sense that any bracket contains the minimal possible number of terms. For example, the very first group contains the only term ν_0 since $\nu_0 \geq \alpha$ so that $i_1 = 1$ but we prefer to represent that group in a general way, too.

Now we define F_n, $n \geq 1$, to be a union of those annuli A_j whose measures form the n-th bracket in (5.2). Namely, F_1 is simply the ball B_0, and for $n \geq 1$ the set F_n is a union of the successive annuli $A_{i_{n-1}+1}, ..., A_{i_n - 1}$. In other words, F_n is the annulus with the interior radius $\rho_{i_{n-1}}$ and with exterior radius $\rho_{i_n - 1}$ or $F_n = B_{i_n - 1} \backslash B_{i_{n-1}}$. By our construction, all F_n except for possibly the last one have measure at least α that conforms the condition (X2). Let us notice that there is a substantial gap between any pair of the consecutive sets F_n and F_{n+1}. Indeed, the exterior radius of the former one is equal $\rho_{i_n - 1}$ whereas the interior radius of the latter is ρ_{i_n} which is by factor 5 larger than $\rho_{i_n - 1}$. Therefore, the double annuli $\widetilde{F}_n$ and $\widetilde{F}_{n+1}$ do not intersect, all sets $\widetilde{F}_n$ are disjoint, and we can take $G_n = \widetilde{F}_n$.

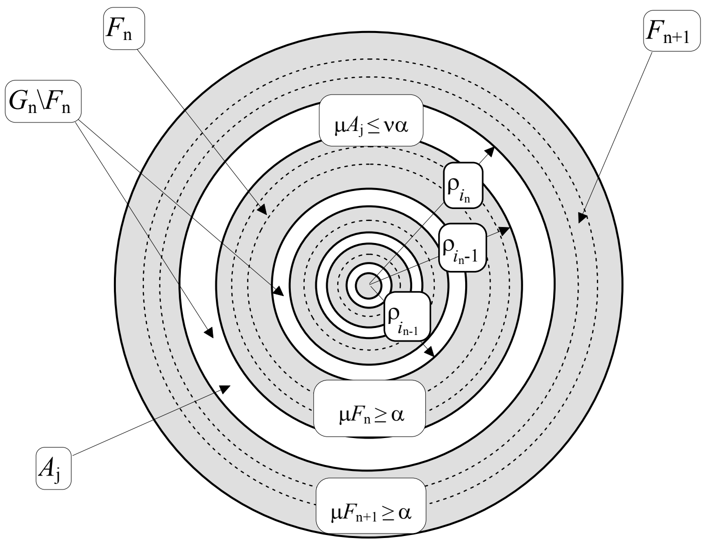

FIGURE 5. Sets F_n

Next, let us estimate measure of the difference $G_n \backslash F_n$. Since this difference lies in the union of two annuli $A_{i_{n-1}}$ and A_{i_n} whose measures $\nu_{i_{n-1}}$ and ν_{i_n} do not exceed $\nu\alpha$ then we have

$$\mu\left(G_n \backslash F_n\right) \leq 2\nu\alpha$$

which gives us (X3).

To prove the condition (X4) let us denote for simplicity by H_j (where H is for "half") the ball centred at x_0 of the radius $\frac{1}{2}\rho_j$ so that $G_n = \widetilde{B}_{i_n - 1} \backslash H_{i_{n-1}}$ and $F_n = B_{i_n - 1} \backslash \widetilde{H}_{i_{n-1}}$. Then we have by (C5) and

(A2)

$$\begin{aligned}\operatorname{cap}(F_n, G_n) &= \operatorname{cap}(B_{i_n-1}\backslash\widetilde{H}_{i_{n-1}}, \widetilde{B}_{i_n-1}\backslash H_{i_{n-1}}) \\ &\le \operatorname{cap}(B_{i_n-1}, \widetilde{B}_{i_n-1}) + \operatorname{cap}(H_{i_{n-1}}, \widetilde{H}_{i_{n-1}}) \\ &\le 2\gamma\,.\end{aligned}$$

Thus, we have all the conditions (X1)-(X4).

Finally, we have to show that the number of the sets F_n is large enough. Let us estimate from above the sum of all terms in any bracket in (5.2). Indeed, any bracket possesses the property of minimality which means that the sum of all its summands but the last one is less than α. Since the last summand (as any other) is at most $\nu\alpha$ then the total sum in any bracket does not exceed $(\nu+1)\alpha$. Any single term between the consecutive brackets is at most $\nu\alpha$. Therefore, if the number of brackets is k then the number of the intermediate single terms is $\le k$, and the total sum in (5.2) is less than or equal to $(2\nu+1)\,\alpha k$. On the other hand, the sum is equal to M whence we get

$$k \ge \frac{M}{(2\nu+1)\,\alpha}.$$

All sets F_n except for the last one have measure at least α. Therefore, the number of sets F_n satisfying all the conditions (X1)-(X4) is at least

$$\frac{M}{(2\nu+1)\,\alpha} - 1 \ge \frac{M}{(2\nu+1)\,\alpha} - c\frac{M}{\nu\alpha} \ge c\frac{M}{\nu\alpha}$$

provided c is small enough, for example, $c \le \frac{1}{6}$.

6. A tree of components

We say that a set of balls in X is *connected* if any two balls from this set can be connected by a sequence of balls from this set so that any two consecutive balls in this sequence overlap. For example, in $\mathbb{R}^n$ the set of balls is connected if their union is a connected set in a topological sense. Any connected set of balls in X will be also referred to as a *component*. We say that two components overlap if one of balls of the first component overlaps with a ball of the second component. Otherwise, the components are disjoint. Obviously, any (finite or countable) set of balls in X will split into maximal connected subsets which will be called components of this set. For the sake of simplicity of notations, we normally do not distinguish between a set of balls and their union as subsets in X unless it may cause a confusion.

For any integer j, let us look at components of each set of balls β_j and θ_j. By the construction, any component of β_j consists of a single ball which does overlap any component of θ_j. Each of those component will be treated as a "vertex" in a graph $\mathbb{G}$. Inside the graph, we shall distinguish vertices which came from different j. Let us denote by $\mathbb{G}_j$ the set of all components of β_j and θ_j. Thus, the graph $\mathbb{G}$ consists of the levels $\mathbb{G}_j$.We also distinguish between β-components and θ-components with the obvious meaning of these terms.

Let us construct edges of the graph $\mathbb{G}$. Any component at level j lies inside a component of the next level $j+1$ because at the next step of our construction, any ball thickens by a factor 5 so that the component will not split but may merge with another thickened component. So, we connect by an edge any j-level component with a $(j+1)$-level component which it belongs to. We refer to the former component as a child and to the latter one as a parent.

Hence, the graph $\mathbb{G}$ has a structure of a (inverse) tree (see Fig. 6). Indeed, each vertex of $\mathbb{G}$ may have several children while it may have at most one parent. Let us mention also that those of vertices of $\mathbb{G}$ who have no children are exactly β-components.

Let us mention that for a negative j the set $\mathbb{G}_j$ is empty. If j is sufficiently large and positive then $\mathbb{G}_j$ consists of a single θ-component. Indeed, since the total number of all β-components is finite by Corollary 5.2 then for large enough j, $\mathbb{G}_j$ has no β-components. In particular, any constituent ball at level $j+1$ is obtained from a concentric ball at level j by thickening by a factor 5. Obviously, for large enough j all balls at level j will merge into a single component.

Let K be one of the components at level j, in other words, $K \in \mathbb{G}_j$. Let us denote by K^* the union of all children components of K. For example, if K is a β-component then K^* is empty. If K is a θ-component then K is a union of some balls of the radius ρ_j , and K^* is a union of their concentric balls of the radius ρ_{j-1} (see Fig. 7).

The main idea of the proof of Theorem 1.1 is to estimate from above the measure $\mu(K\backslash K^*)$ and the capacity $\operatorname{cap}(K^*, K)$ by using a combinatorial structure of the graph $\mathbb{G}$. Let us denote by $\mathcal{N}_s(K)$ (where

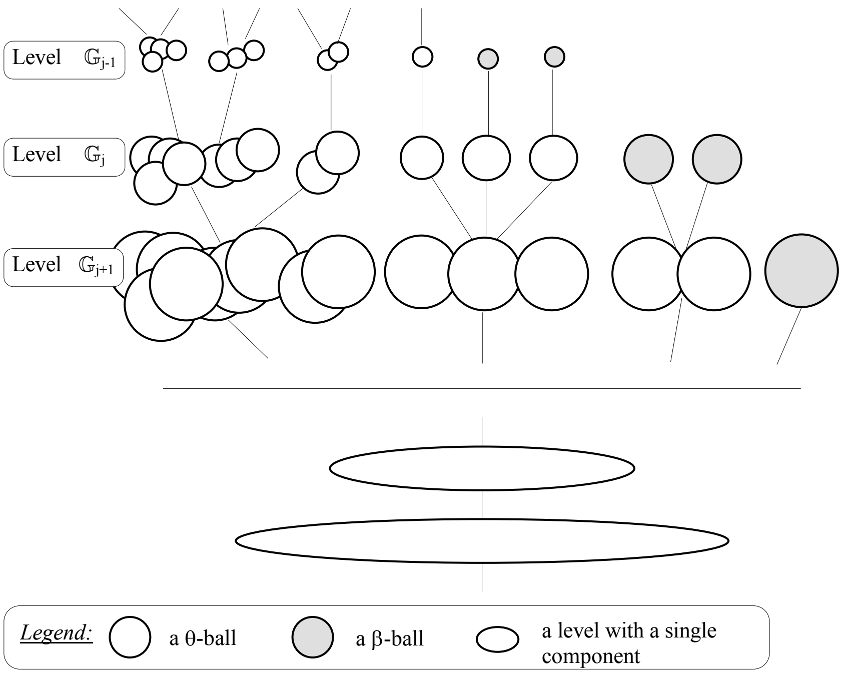

FIGURE 6. Tree of components β and θ

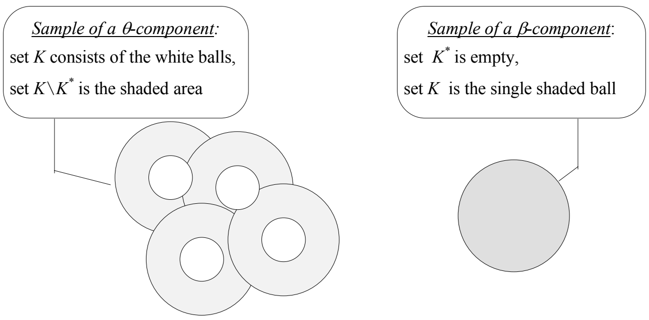

FIGURE 7. Sets K and K^*

s is a non-negative integer) the number of components at the level $j-s$ (where j is the level of K) which are grandchildren of K. Let us denote by $\mathcal{N}_s^\theta(K)$ the number of $\theta-$components at the level $j-s$ which are grandchildren of K (see Fig. 8).

We follow a convention that K is a grandchild of itself, so $\mathcal{N}_0(K)=1$ and $\mathcal{N}_0^\theta(K)=0$ or 1 depending on whether K is a β-component or not.

Let us denote

$$\mathcal{N}(K) \equiv \mathcal{N}_1(K) - \mathcal{N}_0^\theta(K) + \frac{\mathcal{N}_2(K) - \mathcal{N}_1^\theta(K)}{5} + \frac{\mathcal{N}_3(K) - \mathcal{N}_2^\theta(K)}{5^2} + ... \tag{6.1}$$

This is a non-negative number since every term on the right hand side of (6.1) is non-negative. Indeed, each θ-component which is counted in $\mathcal{N}_s^\theta(K)$, has at least one child at the previous level $j-s-1=j-(s+1)$. Since the number of K's grandchildren at the level $j-(s+1)$ is $\mathcal{N}_{s+1}(K)$ then $\mathcal{N}_{s+1}(K) \geq \mathcal{N}_s^\theta(K)$.

Lemma 6.1. *We have for any component $K \in \mathbb{G}$*

$$\mu(K \backslash K^*) \leq \nu\alpha(2\mathcal{N}(K)+1) \tag{6.2}$$

and

$$\operatorname{cap}(K^*, K) \leq \nu\gamma(10\mathcal{N}(K)+1). \tag{6.3}$$

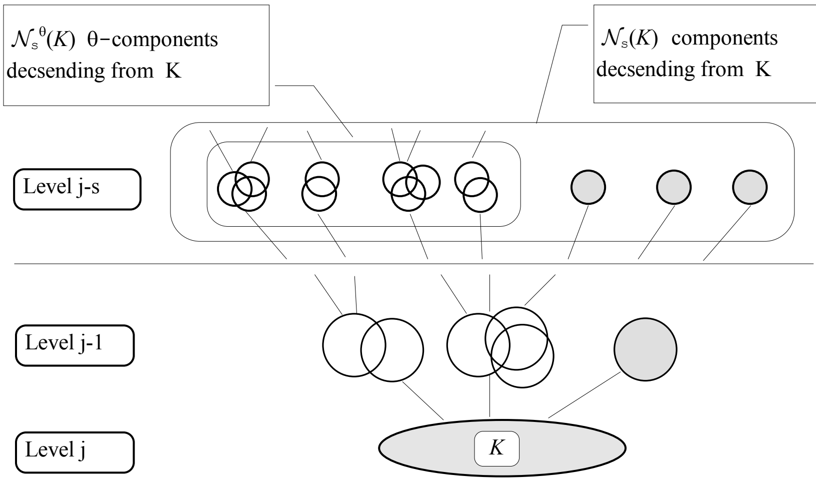

FIGURE 8. Numbers $\mathcal{N}_s$ and $\mathcal{N}_s^\theta$

Proof. If K is a β-component, then K is a ball, say $K = \mathbb{B}^x_{\rho_j}$. Since in this case K^* is empty, by (6.1) we have $\mathcal{N}(K) = 0$ the inequality (6.2) is true by Lemma 5.3. The inequality (6.3) follows from $\operatorname{cap}(K^*, K) \le \operatorname{cap}(\mathbb{B}^x_{\rho_j/2}, \mathbb{B}^x_{\rho_j}) \le \gamma$ by (A2) (however, we will never need (6.3) with an empty K^*).

Now let K be a θ-component at level $j \ge 1$. Let us take $\mathcal{N}_1(K)$ children of K, denote them $K_1, K_2, K_3, \ldots$, and recall that each of them is a connected set of some balls of radius ρ_{j-1}. Let us denote by V the set of all centres of those balls. In particular, K is a union of all ρ_j-balls with the centres at V.

We split the further proof into several steps.

6.1. Introducing a graph structure on the set V. Let us form another graph whose set of vertices would be V. We introduce edges in that graph by using an inductive procedure. At the first step, we connect two points from V by an (abstract) edge if and only if they belong to different components K_i and the distance between them is smaller than $2\rho_j$. In other words, the ρ_j-balls centred at those points must overlap.

Let us identify for a moments vertices from V which belong to the same K_i, and denote that auxiliary graph by V^* (see Fig. 9). We claim that this graph with the edges as above is connected. Indeed, if V^* splits into two disjoint parts then the sets of ρ_i-balls centred at one part would not overlap that of the other. In other words, the set K would not been connected. Now we allow to remove some edges from V^* provided it will not break its connectedness. Let us apply the following elementary fact.

Lemma 6.2. *If a connected graph has k vertices then it is possible to remove all its edges except for some $k-1$ edges so that the remaining graph is still connected.*

Since V^* consists of $\mathcal{N}_1(K)$ vertices then it is possible to leave as little as $\mathcal{N}_1(K) - 1$ edges, and the graph will still be connected. Let us transplant those $\mathcal{N}_1(K) - 1$ edges back to the graph V and forget about V^*. Finally, let us assign to each of the remaining edges a (abstract) length equal to the distance between its ends in X. By our construction, each length does not exceed $2\rho_j$.That finishes our first step.

Let us state the combined result of the above construction: given a component K at level j, we can connect some of the centres of its constituent balls by the abstract edges so that the length of any edge is at most $2\rho_j$, and the number of edges is $\mathcal{N}_1(K) - \mathcal{N}_0^\theta(K)$. Indeed, if K is a θ-component, then $\mathcal{N}_0^\theta(K) = 1$ and the number of edges is $\mathcal{N}_1(K) - 1$ as was shown above. If K is a β-component then both $\mathcal{N}_1(K)$ and $\mathcal{N}_0^\theta(K)$ are vanishing, and we get no edges.

Now let us take each of the components K_i and repeat the procedure above with K_i instead of K. For each K_i, we produce $\mathcal{N}_1(K_i) - \mathcal{N}_0^\theta(K_i)$ edges of the length at most $2\rho_{j-1}$. The total number of the edges of "the second generation" is equal to

$$\sum_i \left(\mathcal{N}_1(K_i) - \mathcal{N}_0^\theta(K_i)\right) = \mathcal{N}_2(K) - \mathcal{N}_1^\theta(K)\ .$$

We proceed further with the same construction for the children of K_i and so on until we exhaust all the grandchildren of K. At the step s, we add $\mathcal{N}_s(K) - \mathcal{N}_{s-1}^\theta(K)$ edges to the graph V each of the length at most $2\rho_{j-s+1}$. At each step, we have connectedness of a graph which is obtained from V by identifying

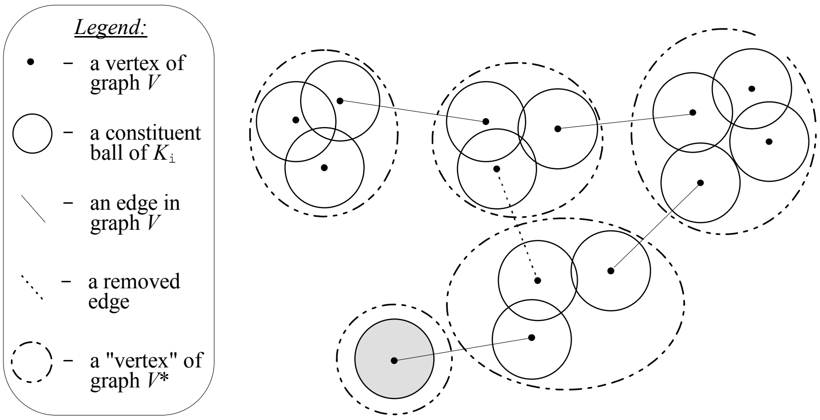

FIGURE 9. Graphs V and V^*

the vertices belonging to the same component of the level $j - s$. At the last step, each vertex can be identified only with itself, so the graph V equipped with the edges of all generations becomes connected (see Fig. 10).

Let us estimate the total length of the edges of V. By the construction, it does not exceed

$$\sum_{s>0} 2\rho_{j-s+1} \left(\mathcal{N}_s(K) - \mathcal{N}^\theta_{s-1}(K)\right) = 2 \cdot 5^j \rho \sum_{s>0} \frac{\mathcal{N}_s(K) - \mathcal{N}^\theta_{s-1}(K)}{5^{s-1}} = 2\rho_j \mathcal{N}(K)\,.$$

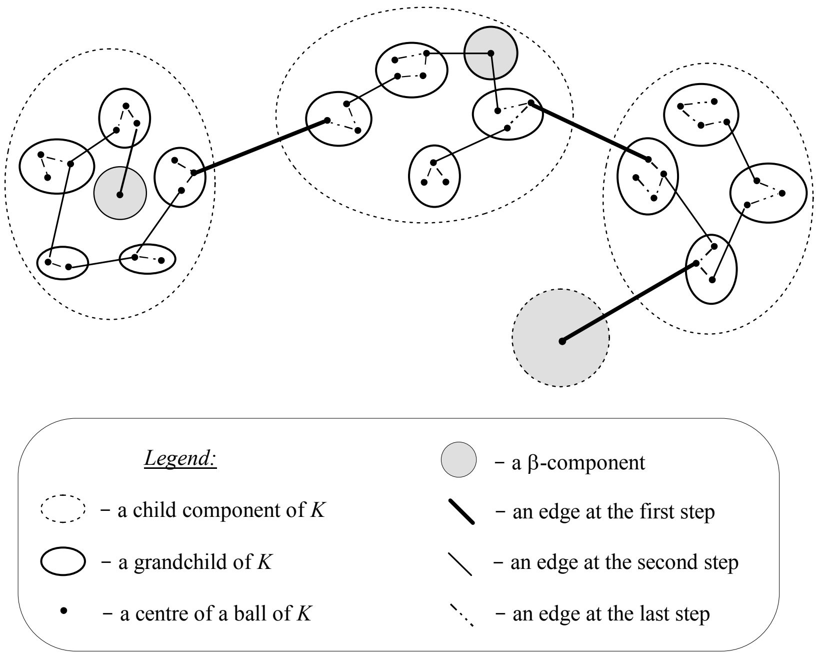

FIGURE 10. Edges in graph V

6.2. Covering of a graph by a family of balls. Next we need the following lemma.

Lemma 6.3. *Let W be a finite connected graph whose edges are equipped by a length. Let the total length of all edges be $\leq L$. We claim that for any $R > 0$, there is at most $1 + \frac{L}{R}$ vertices of W such that any other vertex can be connected to one of them by a path of a length less than R.*

Proof. We introduce a distance between vertices of W as the shortest length of a path connecting two vertices. Then W is a metric space, and the statement is that W can be covered by at most $1 + \frac{L}{R}$ open balls of radius R each.

If $L < R$ then any ball in W of radius R covers all the vertices. Since $1 < 1 + \frac{L}{R}$ then there is nothing to prove in this case. If $L \geq R$ then we shall undertake an inductive procedure with respect to $\left\lfloor \frac{L}{R} \right\rfloor$. Namely, we want to reduce the question to a subgraph $W' \subset W$ whose edges have the total length at most $L' = L - R$ so that $\left\lfloor \frac{L'}{R} \right\rfloor = \left\lfloor \frac{L}{R} \right\rfloor - 1$. By the inductive hypothesis, we can claim that the vertices of W' are covered by at most $1+\frac{L'}{R} = \frac{L}{R}$ balls of radius R. If the rest $W \backslash W'$ is covered by a single ball of radius R then we have done.

Before we can find W', let us do some preliminary modifications of W. First of all, we get rid of superfluous edges: remove any edge if after its removal the graph remains connected. The total length of edges may only decrease and will be at most L. Let W have already a minimal set of edges. Then it contains no cycles: otherwise one of the edges of a cycles could be removed.

Next, let us make from the graph W an oriented tree i.e. introduce a direction on any edge of W so that each vertex has at most 1 incoming edge. Indeed, take any vertex $a \in W$ and direct all adjacent edges to look outward from a. Let $b_1, b_2, ...$ be the ends of those edges. Any of the vertices b_i has already an incoming edge, so all other edges of b_i are directed outward. Next we repeat this procedure with their ends and proceed until we orient all edges. Since the graph contains no cycles then it will never happen that we come across a vertex that has been considered before. Therefore, each vertex has at most one incoming edge; moreover, each vertex but a has exactly one incoming edge. In other words, we can regard W as an oriented tree with the root at a.

The subgraph W' will be obtained from W by erasing some vertices of W and removing all edges adjacent to the erased vertices. To find out what should be erased let us associate with any vertex $x \in W \backslash \{a\}$ a subgraph $W_x \subset W$ which consists of the vertex x, of all children of x (i.e. the ends of edges outcoming from x), of all grandchildren of x of all generations, and of the unique parent of x. We assume that if W_x contains two vertices then W_x contains also the edge between them should it exists in W.

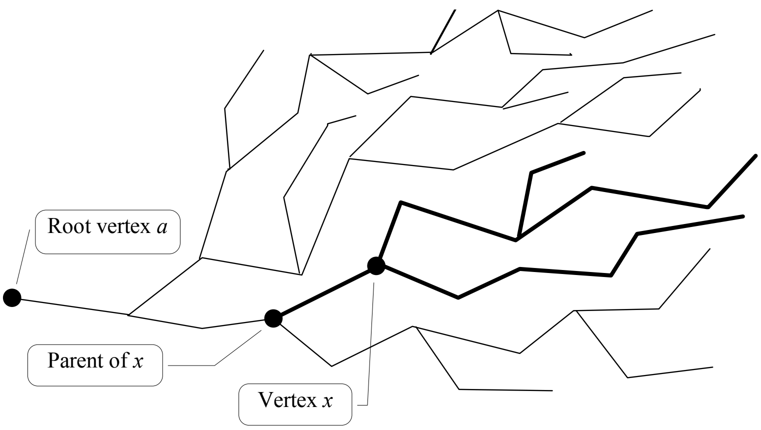

FIGURE 11. Graph W and subgraph W_x (bold)

For any vertex $x \in W \backslash \{a\}$, we define $h(x)$ as the maximal length of a path which starts at the parent of x and lives in W_x so $h(x)$ can be regarded as a "height" of W_x. Let us first consider a simple case when $\max h(x) < R$ over all x. Let us choose x to be a neighbour of a such that the longest path in W which necessarily starts at a goes through x. Since $h(x)$ is the length of that path then the longest path in W has the length $< R$ and, therefore, the entire graph W is covered by a single ball of radius R centred at a. In this case, we have nothing to do.

Now we consider the main case when for some $x \neq a$ we have $h(x) \geq R$. Out of all $x \neq a$ with $h(x) \geq R$, we choose a vertex x at which $h(x)$ attains the minimum. Let $y_1, y_2, ...$ be the children of x. Since $h(y_i) < h(x)$ then by the choice of x we have $h(y_i) < R$. In particular, any path emanating from x has the length $< R$. It implies that the ball of radius R centred at x covers W_x except for possibly the parent of x.

Hence, a subgraph W' can be obtained from W by erasing all vertices of W_x except for the parent of x. As we have shown the erased vertices are covered by one ball of radius R. On the other hand, the total length of the erased edges is at least $h(x) \geq R$ whence it follows that the total length of all edges of W' does not exceed $L - R$. ∎

6.3. Estimating $\mu(K\backslash K^*)$ and $\operatorname{cap}(K^*,K)$. Let us proceed with the proof of Lemma 6.1. To that end, let us apply Lemma 6.3 to the graph V with $R=\rho_j$. We find that there is at most $2\mathcal{N}(K)+1$ points $x_1,x_2,...$ in V such that the ρ_j-balls on the graph V with the centres $x_1,x_2,...$ cover all points of V. Since the distance on graph V is bounded below by the distance in X then the balls $\mathbb{B}_{\rho_j}^{x_i}$ in X also cover the entire V. Moreover, the component K is a union of all ρ_j-balls centred at the points of V whence the double balls $\mathbb{B}_{2\rho_j}^{x_i}$ not only cover the vertices of V but also cover the entire component K.

Now let us replace by hypothesis (A1) each ball $\mathbb{B}_{2\rho_j}^{x_i}$ by at most ν balls of radius $\rho_{j-2}=(2\rho_j)/50$ that cover the ball $\mathbb{B}_{2\rho_j}^{x_i}$. We obtain at most $\nu\,(2\mathcal{N}(K)+1)$ balls of radius ρ_{j-2} that cover K. Let us remove all those balls of this family that lie either entirely inside $\overset{o}{K}$ or entirely outside K. The set of the remaining ρ_{j-2}-balls still covers $K\backslash K^*$, and each of them does intersect $K\backslash K^*$.

Let us note that K does not intersect any other component of level j and, thus, it intersects only those components at level $j-1$ that are its children. In particular, the difference $K\backslash K^*$ does not intersect any ball from θ_{j-1} (and β_{j-1}). Therefore, if a ρ_{j-2}-ball intersects $K\backslash K^*$ then it is not covered by θ_{j-1}. By Lemma 5.1, such a ball does not belong to Σ_{j-2} which implies that its measure is smaller than α. Since $K\backslash K^*$ is covered by at most $\nu\,(2\mathcal{N}(K)+1)$ of those balls then we conclude that

$$\mu(K\backslash K^*)\leq\nu\alpha\,(2\mathcal{N}(K)+1)$$

what was to be proved.

The capacity $\operatorname{cap}(K^*,K)$ is estimated in a similar way. Let us denote by K_i $(i=1,2,...)$ the children components of K, so that $K^*=\bigcup_i K_i$. We claim that

$$\operatorname{cap}(K^*,K)=\operatorname{cap}\left(\bigcup_i K_i,K\right)\overset{(C1)}{=}\operatorname{cap}\left(\bigcup_i(K_i\backslash K_i^*),K\right)$$

where we refer to the axiom (C1) of the capacity with $F=\bigcup_i K_i$, $E=\bigcup_i K_i^*$ and $G=K$. To justify applicability of (C1), we have to show that $E^r\subset F$ for some $r>\delta$. Since the component K_i^* is at the level $j-2$, it is a union of balls of radii ρ_{j-2} whereas K_i is a union of the concentric balls of radii ρ_{j-1}. Therefore, we can take

$$r=\rho_{j-1}-\rho_{j-2}=4\cdot5^{j-2}\rho\geq\frac{4}{5}\rho>\delta$$

because according to (A3) we have $\rho>\frac{5}{4}\delta$.

Let us apply the above argument to any K_i to get that $K_i\backslash K_i^*$ can be covered by at most $\nu\,(2\mathcal{N}(K_i)+1)$balls of radius ρ_{j-3}. If B is one of those balls then $\widetilde{B}\subset K$ and by (C2) and (A2)

$$\operatorname{cap}(B,K)\leq\operatorname{cap}(B,\widetilde{B})\leq\gamma\,.$$

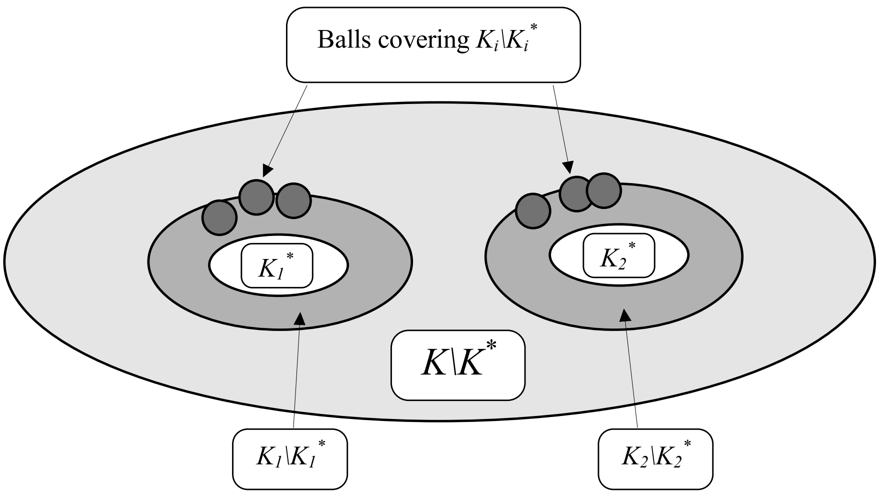

FIGURE 12. Component K and its children K_1,K_2 $(K^*=K_1\cup K_2)$

Therefore,

$$\begin{aligned} \operatorname{cap}(K^*, K) &\le \gamma \sum_i \nu\left(2\mathcal{N}(K_i)+1\right) \\ &= 2\nu\gamma \sum_i \mathcal{N}(K_i) + \nu\gamma\mathcal{N}_1(K) \\ &= 2\nu\gamma \sum_i \sum_{s=1}^{\infty} \frac{\mathcal{N}_s(K_i) - \mathcal{N}^{\theta}_{s-1}(K_i)}{5^{s-1}} + \nu\gamma\mathcal{N}_1(K) \\ &= 2\nu\gamma \sum_{s=1}^{\infty} \frac{\mathcal{N}_{s+1}(K) - \mathcal{N}^{\theta}_{s}(K)}{5^{s-1}} + \nu\gamma\mathcal{N}_1(K) \\ &= 10\nu\gamma \sum_{s=2}^{\infty} \frac{\mathcal{N}_{s}(K) - \mathcal{N}^{\theta}_{s-1}(K)}{5^{s-1}} + \nu\gamma\mathcal{N}_1(K) \\ &= 10\nu\gamma\left(\mathcal{N}(K) - \left(\mathcal{N}_1(K) - \mathcal{N}^{\theta}_0(K)\right)\right) + \nu\gamma\mathcal{N}_1(K) \\ &\le \nu\gamma\left(10\mathcal{N}(K)+1\right) \end{aligned}$$

where we have used that the number of children of K is $\mathcal{N}_1(K) \ge 1$ whereas $\mathcal{N}^{\theta}_0(K) = 1$. Thus, we have finished the proof of Lemma 6.1. ∎

7. Chains of the component tree

We recall that our aim is to construct enough number of capacitors (F_i, G_i) satisfying (X1)-(X4). Let us first show how we can extract a sample capacitor satisfying (X2)-(X4) from a tree $\mathbb{G}$ constructed above. Afterwards, we will show that the number of them is large enough.

Given a component $K \in \mathbb{G}$, let us introduce notations

$$\widehat{\mu}(K) = \mu(K \backslash K^*)$$

and

$$\widehat{c}(K) = \operatorname{cap}(K^*, K)$$

where $K^* \subset X$ is obtained by subtracting from K of a union of all its children. The quantities $\widehat{\mu}(K)$ and $\widehat{c}(K)$ are estimated from above in terms of $\mathcal{N}(K)$ by Lemma 6.1.

Suppose that we have a *chain* in the graph $\mathbb{G}$ i.e. a sequence of components $K_1, K_2, ...K_s$ such that K_{i+1} is a parent of K_i. Let K_0 denote a union of all children of K_1, and K_{-1} be a union of all children of K_0 (K_0 and/or K_{-1} may be empty). Let us also denote by K_{s+1} the (unique) parent of K_s. We refer to K_0 as the child of the chain and to K_{s+1} as the parent of the chain. A set of components $K_0, K_1, ...K_s, K_{s+1}$ is called an *extension* of the chain $K_1, K_2, ...K_s$.

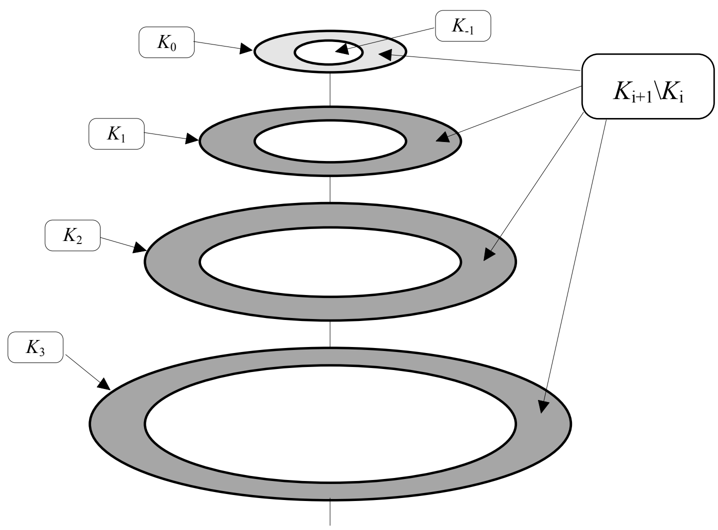

Figure 13. Chain of components

Given a chain, we associate with it a capacitor (F, G) where $F = K_s \backslash K_0$ and $G = K_{s+1} \backslash K_{-1}$. The conditions (X2)-(X4) will follow from the following hypotheses:

(a): any component $K_1, K_2, ...K_s$ has at most one child (which implies that any component but K_1 has exactly one child) and the parent K_{s+1} has a unique child.

(b): $\sum_{i=1}^{s} \widehat{\mu}(K_i) \geq \alpha$

(c): $\widehat{\mu}(K_{s+1}) \leq 3\nu\alpha$ and $\widehat{\mu}(K_0) \leq 3\nu\alpha$

(d): $\widehat{c}(K_{s+1}) \leq 11\nu\gamma$ and $\widehat{c}(K_0) \leq 11\nu\gamma$

The condition (a) implies that $\widehat{\mu}(K_i) = \mu(K_i) - \mu(K_{i-1})$ whence it follows

$$\sum_{i=1}^{s} \widehat{\mu}(K_i) = \mu(K_s \backslash K_0) = \mu(F) .$$

Therefore, (b) is equivalent to the condition (X2) for the capacitor (F, G). The inequalities of (c) together with $G \backslash F = (K_{s+1} \backslash K_s) \cup (K_0 \backslash K_{-1})$ imply (X3). Finally, by (A3)

$$\operatorname{cap}(F, G) \leq \operatorname{cap}(K_s, K_{s+1}) + \operatorname{cap}(K_{-1}, K_0) = \widehat{c}(K_{s+1}) + \widehat{c}(K_0) ,$$

so (X4) follows from (d).

If $\mathcal{C}$ denotes a chain (a set of components $K_1, K_2, ...K_s$) then we write

$$\widehat{\mu}(\mathcal{C}) \equiv \sum_{i=1}^{s} \widehat{\mu}(K_i)$$

$$\widehat{c}(\mathcal{C}) \equiv \operatorname{cap}(K_s \backslash K_0, K_{s+1} \backslash K_{-1}) = \operatorname{cap}(F, G)$$

and call $\widehat{\mu}(\mathcal{C})$ by a measure of the chain, and $\widehat{c}(\mathcal{C})$ by a capacity of the chain. These definitions do not assume that the chain satisfies the conditions above.

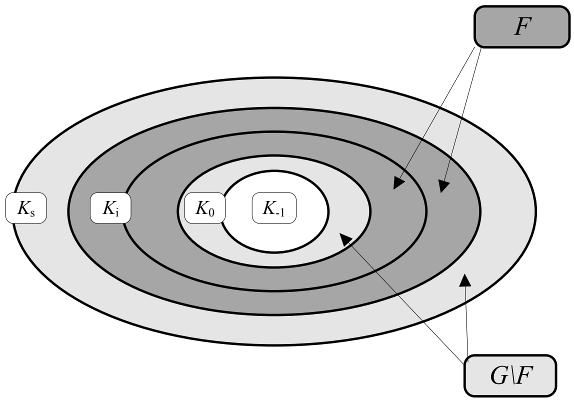

Figure 14. Capacitor (F, G)

Any chain satisfying all conditions (a)-(d) will be a *good* chain. We say that two chains are disjoint if their capacitors are disjoint as in (X1). Next we intend to extract from the graph $\mathbb{G}$ a large enough number k of good chains which are also mutually disjoint. Whenever the number k satisfies (1.1) then the proof is finished. If the total number β of all β-components satisfies $\beta \geq c\frac{M}{\nu\alpha}$ then this case was considered in section 5. From now we assume that

$$\beta < c\frac{M}{\nu\alpha}. \tag{7.1}$$

We would like to partition the graph $\mathbb{G}$ into chains satisfying initially only the condition (b). We first find a level $\mathbb{G}_j$ which possesses a single component, and this component has a measure at least $M/2$ where M is the total measure of X. Indeed, as was remarked after the proof of Corollary 5.2, for large enough j any $\mathbb{G}_j$ has only one component which is a union of ρ_j-balls with the same for all large

j centres. Obviously, the measure of the component at level j tends to M as $j \to \infty$. In fact, if X has a finite diameter, then this component covers the entire X already for a finite j, but in general we can only claim that when $j \to \infty$, then its measure becomes arbitrarily close to M. In particular, for some j_0 the measure of the unique component at level j_0 is greater than $M/2$. Let us forget about the levels $\mathbb{G}_j$ with $j > j_0 + 1$ and pretend that the graph $\mathbb{G}$ terminates at level $j_0 + 1$.

First, we find a chain which starts at the level j_0 and which has a measure at least α. Indeed, we take an arbitrary path on $\mathbb{G}$ starting at level j_0 and proceed until the measure of the chain along this path becomes for the first time greater than or equal to α. That will occur because the path can terminate only at a β-component, and any β-component contributes at least α toward the measure of a chain. Let us mark all the components of the constructed chain as engaged. Let us take a non-engaged component at the highest available level $\leq j_0$, and construct a chain of a measure $\geq \alpha$ starting from that component. Any path which emanates from that component will not intersect any engaged component because any component has at most one parent. Each component of the second chain becomes also engaged. We repeat this construction until all components of $\mathbb{G}$ up to the level j_0 are engaged. Let us note that any β-component may serve as a chain of a single element.

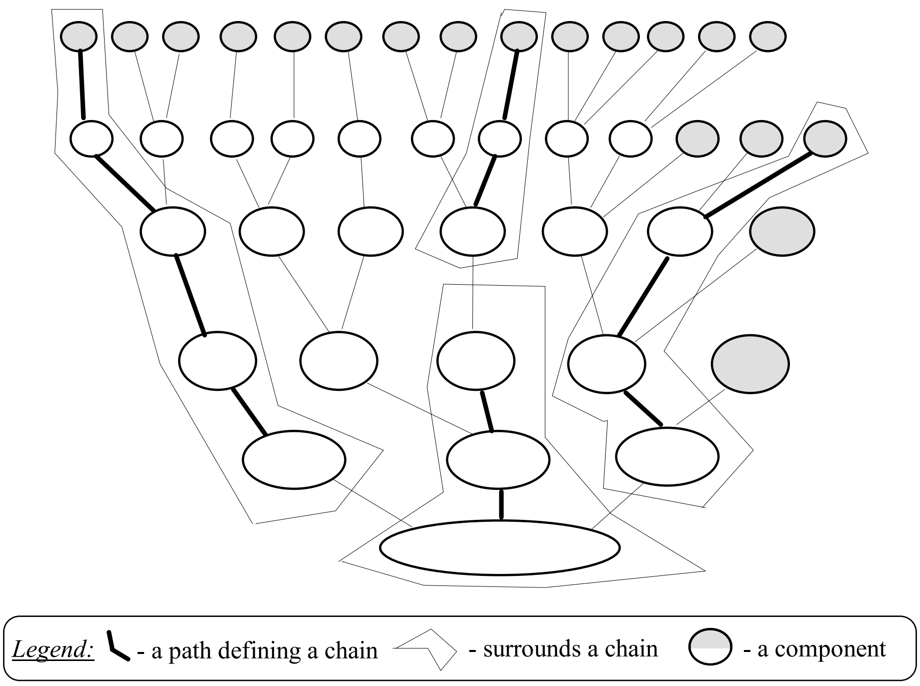

FIGURE 15. Selecting chains on graph $\mathbb{G}$

Let us emphasize that we construct all the chain in an optimal way in the sense that the number of elements in any chain cannot be reduced without breaking the condition (b). In particular, if we denote one of the constructed chain by $\mathcal{C}$ and if K is its youngest component then

$$\widehat{\mu}(\mathcal{C}) \leq \widehat{\mu}(K) + \alpha \, . \tag{7.2}$$

On the other hand, the total sum $\sum \widehat{\mu}(\mathcal{C})$ over all chains $\mathcal{C}$ coincides with the total sum $\sum \widehat{\mu}(K)$ over all components $K \in \mathbb{G}$ (up to the level j_0) which is equal to the measure of the component at level j_0 i.e. is at least $M/2$:

$$\sum_{\mathcal{C}} \widehat{\mu}(\mathcal{C}) = \sum_{K} \widehat{\mu}(K) \geq \frac{M}{2} \, . \tag{7.3}$$

We have by (6.2) for any component K

$$\widehat{\mu}(K) \leq 2\nu\alpha\mathcal{N}(K) + \nu\alpha.$$

(where $\mathcal{N}(K)$ is defined by (6.1)) whence we have that for any chain $\mathcal{C}$

$$\widehat{\mu}(\mathcal{C}) \leq 2\nu\alpha\mathcal{N}(K) + (\nu + 1)\alpha \tag{7.4}$$

provided K is the youngest component of $\mathcal{C}$.

Next we need the following lemma.

Lemma 7.1. *We have*

$$\sum_K \mathcal{N}(K) = \frac{5}{4}(\beta - 1)$$

where the sum is taken over all components $K \in \mathbb{G}$, and $\mathcal{N}(K)$ is defined by (6.1).

Proof. Let us denote by $\mathbb{G}_j^\theta$ and $\mathbb{G}_j^\beta$ respectively the sets of θ-components and β-components at level j. Let $n_j = \operatorname{card}\mathbb{G}_j$, $n_j^\theta = \operatorname{card}\mathbb{G}_j^\theta$ and $n_j^\beta = \operatorname{card}\mathbb{G}_j^\beta$ so that $n_j = n_j^\beta + n_j^\theta$. It follows from these definitions that for any positive integer s

$$\sum_{K\in\mathbb{G}_j} \left(\mathcal{N}_s(K) - \mathcal{N}_{s-1}^\theta(K)\right) = n_{j-s} - n_{j-s+1}^\theta = n_{j-s} - n_{j-s+1} + n_{j-s+1}^\beta\,. \tag{7.5}$$

If we sum up (7.5) over all j we obtain

$$\sum_{K\in\mathbb{G}} \left(\mathcal{N}_s(K) - \mathcal{N}_{s-1}^\theta(K)\right) = \sum_j (n_{j-s} - n_{j-s+1}) + \sum_j n_{j-s+1}^\beta\,.$$

Summation in each of the sums on the right hand side is taken over all integers j so by changing $j - s$ to j we see that the sums do not depend on s, and we can put $s = 1$ in each of them. Let us recall that for negative j we have $n_j = 0$ whereas for a large enough positive j we have $n_j = 1$. Therefore, the first sum is equals to

$$\sum_j (n_{j-1} - n_j) = 0 - 1 = -1$$

The second sum is equal to $\sum_j n_j^\beta = \beta$ whence

$$\sum_{K\in\mathbb{G}} \left(\mathcal{N}_s(K) - \mathcal{N}_{s-1}^\theta(K)\right) = \beta - 1. \tag{7.6}$$

Finally, we recall definition (6.1) of $\mathcal{N}(K)$ and obtain

$$\begin{aligned}\sum_{K\in\mathbb{G}} \mathcal{N}(K) &= \sum_{K\in\mathbb{G}}\sum_{s=1}^{\infty} \frac{\mathcal{N}_s(K) - \mathcal{N}_{s-1}^\theta(K)}{5^{s-1}} \\ &= \sum_{s=1}^{\infty}\sum_{K\in\mathbb{G}} \frac{\mathcal{N}_s(K) - \mathcal{N}_{s-1}^\theta(K)}{5^{s-1}} = \sum_{s=1}^{\infty} \frac{\beta - 1}{5^{s-1}} = \frac{5}{4}(\beta - 1)\end{aligned}$$

what was to be proved. ∎

Let us denote by N the total number of all chains. Then by (7.4) and by Lemma 7.1 we have

$$\sum_{\mathcal{C}} \widehat{\mu}(\mathcal{C}) \le 2\nu\alpha \sum_K \mathcal{N}(K) + (\nu + 1)\alpha N < \frac{5}{2}\nu\alpha\beta + (\nu + 1)\alpha N\,.$$

Combinig with (7.3) we obtain

$$\operatorname{card}\{\mathcal{C}\} = N \ge \frac{M - 5\nu\alpha\beta}{2(\nu + 1)\alpha}\,. \tag{7.7}$$

Next we will subtract from all chains those which are bad (=not good) in the sense above. Let us first estimate from above the number of chains which do not satisfy (a). To this end, let us define for any component K the number

$$f(K) = (\mathcal{N}_1(K) - 1)_+ = \mathcal{N}_1(K) - \mathcal{N}_0^\theta(K)$$

(the second equality is an obvious identity). By (7.6) we have

$$\sum_{K\in\mathbb{G}} f(K) = \beta - 1\,. \tag{7.8}$$

In particular, it implies that

$$\operatorname{card}\{K : f(K) \ge 1\} < \beta\,.$$

But the condition $f(K) \geq 1$ is equivalent to the fact that K has more than one child. Therefore, the total number of components with more than one child is less than β, and the number of chains having a component with more than one child is also smaller than β.

We have also to subtract those chains whose parents have more than one child. If P is a component having more than one child then it may serve as a parent to at most $\mathcal{N}_1(P) = f(P) + 1 \leq 2f(P)$ chains. By (7.8) the sum $\sum f(P)$ over all possible parents is less than β. Thus, the number of chain that may have such a parent is less than 2β.

Combining together the above results, we see that the number of chains satisfying the conditions (a) and (b) is at least as much as

$$\frac{M - 5\nu\alpha\beta}{2(\nu+1)\alpha} - 3\beta\,. \tag{7.9}$$

Now we subtract from this number the chains that break (c) or (d). The number of chains (satisfying (a)) whose parent has a measure $\widehat{\mu}$ greater than $3\nu\alpha$ does not exceed the total number of all components K with $\widehat{\mu}(K) > 3\nu\alpha$. Since by Lemma 6.1

$$\widehat{\mu}(K) \leq \nu\alpha\left(2\mathcal{N}(K) + 1\right)$$

then $\widehat{\mu}(K) > 3\nu\alpha$ may happen only if $\mathcal{N}(K) > 1$. But as follows from Lemma 7.1, the number of all components with $\mathcal{N}(K) > 1$ is smaller than $\frac{5}{4}\beta$. We conclude that the number of chains (satisfying (a)) whose parent has a measure $\widehat{\mu}$ greater than $3\nu\alpha$ is smaller than $\frac{5}{4}\beta$. The same applies to the child of a chain whence we have that the number of chains satisfying (a),(b) and (c) is at least

$$\frac{M - 5\nu\alpha\beta}{2(\nu+1)\alpha} - 5.5\beta\,. \tag{7.10}$$

Next, we have to remove all chains for which (d) is not true. The number of chains satisfying (a) but not satisfying (d) does not exceed two times the number of all components K such that

$$\widehat{c}(K) > 11\nu\gamma. \tag{7.11}$$

We have by (6.3)

$$\widehat{c}(K) \leq 10\nu\gamma\mathcal{N}(K) + \nu\gamma$$

whence (7.11) may occur only when $\mathcal{N}(K) > 1$, and the number of such components is smaller than $\frac{5}{4}\beta$. Combining with (7.10) we deduce that the number of good chains is at least

$$\frac{M - 5\nu\alpha\beta}{2(\nu+1)\alpha} - 8\beta\,.$$

Since we have assumed that β is bounded above as (7.1) then the number of good chains is bounded below by

$$\begin{aligned} & \frac{(1-5c)M}{2(\nu+1)\alpha} - 8c\frac{M}{\alpha\nu} \\ = & \left(\frac{(1-5c)\nu}{2(\nu+1)} - 8c\right)\frac{M}{\nu\alpha} \\ > & \left(\frac{1}{4} - \frac{37}{4}c\right)\frac{M}{\nu\alpha} \\ \geq & \ 5c\frac{M}{\alpha\nu} \end{aligned}$$

provided $c \leq \frac{1}{57}$ (here we applied $\frac{\nu}{\nu+1} \geq \frac{1}{2}$).

Let us forget about bad chains and speak only about good chains. We have to extract from all (=good) chains a subset of chains which would give rise to mutually disjoint capacitors. Let us extend any chain by adding to it its child and its parent. Obviously, if two extended chains do not intersect as sets of the graph $\mathbb{G}$ then their capacitors do not intersect either. Although by construction any two chains do not intersect in $\mathbb{G}$, their extension may do. Let $K_0, K_1, K_2 \ldots. K_{s+1}$ be an extended chain (we follow the previous notations). Either the child K_0 or the parent K_{s+1} may belong to another extended chain. We claim that each K_0 and K_{s+1} may belong to at most 2 other extended chains. Indeed, K_0 may be a component (no child no parent) to at most one other chain. It may not be a child to another chain since it has only one parent K_1. Finally, K_0 may be the parent again to at most 1 other chain since any parent of a (good) chain has only one child. Similarly, K_{s+1} may belong to at most 2 other extended chains.

Therefore, each extended chain may intersect at most 4 other extended chains. Since the total number of extended chains is at least $5c\frac{M}{\alpha\nu}$ then we can extract out of them at least $c\frac{M}{\alpha\nu}$ non-intersecting extended chains and, thus, finish the proof of Theorem 1.1.

Remark 7.1. *As one sees from the proof, we have used the properties (A1) and (A2) only for balls of radius $r \geq c\rho$ with an absolute constant $c > 0$, for example $c = \frac{1}{125}$. Therefore, one may want to require from the first that (A1) and (A2) hold only for such balls. This may have applications on non-compact manifolds, and we intend to return to this elsewhere.*

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