Negative eigenvalues of Schrödinger operators

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1 Upper estimate in $\mathbb{R}^n$, $n \geq 3$

1.1 Introduction and statement

Given a non-negative L^1_{loc} function $V(x)$ on $\mathbb{R}^n$, consider the Schrödinger type operator

$$H_V = -\Delta - V$$

where $\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$ is the classical Laplace operator. More precisely, H_V is defined as a form sum of $-\Delta$ and $-V$, so that, under certain assumptions about V, the operator H_V is self-adjoint in $L^2(\mathbb{R}^n)$.

Denote by $\operatorname{Neg}(H_V)$ the number of negative eigenvalues of H_V (counted with multiplicity), assuming that its spectrum in $(-\infty, 0)$ is discrete. For example, the latter is the case when $V(x) \to 0$ as $x \to \infty$. We are are interested in obtaining estimates of $\operatorname{Neg}(H_V)$ in terms of the potential V.

Suppose that $-V$ is an attractive potential field in quantum mechanics. Then H_V is the Hamiltonian of a particle that moves in this field, and the negative eigenvalues of H_V correspond to so called *bound states* of the particle, that is, the negative energy levels E_k that are inside a potential well.

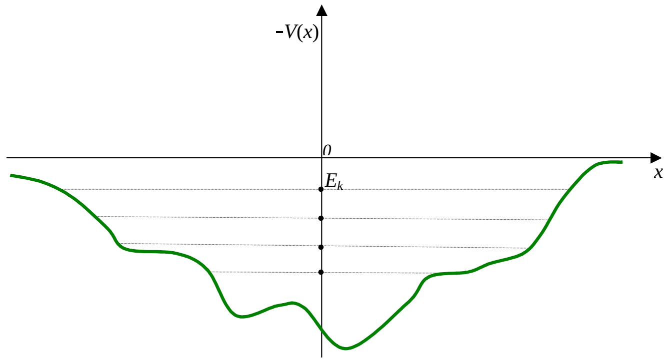

Figure 1:

Hence, $\operatorname{Neg}(H_V)$ determines the number of bound states of the system. In particular, if $-V$ is the potential field of an electron in an atom, then $\operatorname{Neg}(H_V)$ is the maximal number of possible electron orbits in the atom.

Estimates of $\operatorname{Neg}(H_V)$, especially upper bounds, are of paramount importance for quantum mechanics.

We start with a famous theorem of Cwikel-Lieb-Rozenblum.

Theorem 1 *Assume $n \geq 3$ and $V \in L^{n/2}(\mathbb{R}^n)$. Then H_V can be defined as a self-adjoint operator with the domain in $W^{1,2}(\mathbb{R}^n)$, its negative spectrum is discrete, and the following estimate is true*

$$\operatorname{Neg}(H_V) \leq C_n \int_{\mathbb{R}^n} V(x)^{n/2}\, dx. \tag{1}$$

This estimate was proved independently by the above named authors in 1972-1977. Later Lieb used (1) to prove the stability of the matter in the framework of quantum mechanics.

The estimate (1) implies that, for a large parameter α,

$$\operatorname{Neg}(\alpha V) = O\left(\alpha^{n/2}\right) \quad \text{as } \alpha \to \infty. \tag{2}$$

This is a so called semi-classical asymptotic (that corresponds to letting $\hbar \to 0$), and it is expected from another consideration that $\operatorname{Neg}(\alpha V)$ should behave as $\alpha^{n/2}$, at least for a reasonable class of potentials.

1.2 Counting function

Before the proof of Theorem 1, let us give an exact definition of the operator H_V and its counting function. Given a potential V in $\mathbb{R}^n$, that is, a non-negative function from $L^1_{loc}(\mathbb{R}^n)$, define the bilinear energy form by

$$\mathcal{E}_V(f,g) = \int_{\mathbb{R}^n} \nabla f \cdot \nabla g dx - \int_{\mathbb{R}^n} V f g dx$$

for all $f, g \in \mathcal{D} := C_0^\infty(\mathbb{R}^n)$, and the corresponding quadratic form $\mathcal{E}_V(f) := \mathcal{E}_V(f,f)$.

For any open set $\Omega \subset \mathbb{R}^n$, we consider a restriction of $\mathcal{E}_V$ to $\mathcal{D}_\Omega := C_0^\infty(\Omega)$. The form $(\mathcal{E}_V, \mathcal{D}_\Omega)$ is called *closable* in $L^2(\Omega)$ if

1. it is semi-bounded below, that is, for some constant $K \geq 0$,

$$\mathcal{E}_V(f) \geq -K \|f\|_2^2 \quad \text{for all } f \in \mathcal{D}_\Omega; \tag{3}$$

2. and, for any sequence $\{f_n\} \subset \mathcal{D}_\Omega$,

$$\|f_n\|_2 \to 0 \text{ and } \mathcal{E}_V(f_n - f_m) \to 0 \implies \mathcal{E}_V(f_n) \to 0.$$

Here $\|\cdot\|_2$ is the L^2-norm with respect to the Lebesgue measure.

A closable form $(\mathcal{E}_V, \mathcal{D}_\Omega)$ has a unique extension to a subspace $\mathcal{F}_{V,\Omega}$ of $L^2(\Omega)$ so that $\mathcal{F}_{V,\Omega}$ is a Hilbert space with respect to the inner product

$$(f,g)_\mathcal{E} := \mathcal{E}_V(f,g) + (K+1)(f,g), \tag{4}$$

(that is, $(\mathcal{E}_V, \mathcal{F}_{V,\Omega})$ is *closed*) and $\mathcal{D}_\Omega$ is dense in $\mathcal{F}_{V,\Omega}$.

Being a closed symmetric form, $(\mathcal{E}_V, \mathcal{F}_{V,\Omega})$ has the *generator* $H_{V,\Omega}$ that can be defined as an (unbounded) operator in $L^2(\Omega)$ with a maximal possible domain $\operatorname{dom}(H_{V,\Omega}) \subset \mathcal{F}_{V,\Omega}$ such that

$$\mathcal{E}_V(f,g) = (H_{V,\Omega} f, g) \quad \forall f \in \operatorname{dom}(H_{V,\Omega}) \text{ and } g \in \mathcal{F}_{V,\Omega}. \tag{5}$$

In fact, $H_{V,\Omega}$ is a self-adjoint operator in $L^2(\Omega)$.

For example, for $f,g\in\mathcal{D}_\Omega$ we have

$$\mathcal{E}_V(f,g)=\int_\Omega \nabla f\cdot\nabla g dx-\int_\Omega Vfgdx=\int_\Omega(-\Delta f-Vf)\,gdx$$

so that

$$H_{V,\Omega}f=-\Delta f-Vf.$$

The operator $H_{V,\Omega}$ is called the Friedrichs extension of $-\Delta-V$.

Since $H_{V,\Omega}$ is self-adjoint, the spectrum of $H_{V,\Omega}$ is real. It follows from (3) that the spectrum of $H_{V,\Omega}$ is semi-bounded below. The *counting function* $\mathcal{N}_\lambda$ of $H_{V,\Omega}$ is defined by

$$\mathcal{N}_\lambda(H_{V,\Omega})=\dim\operatorname{Im}\mathbf{1}_{(-\infty,\lambda)}(H_{V,\Omega}), \tag{6}$$

where $\mathbf{1}_{(-\infty,\lambda)}(H_{V,\Omega})$ is the spectral projector of $H_{V,\Omega}$ of the interval $(-\infty,\lambda)$. For example, if the spectrum of $H_{V,\Omega}$ is discrete and $\{\varphi_k\}$ is an orthonormal basis of eigenfunctions with eigenvalues $\{\lambda_k\}$ then $\mathbf{1}_{(-\infty,\lambda)}(H_{V,\Omega})$ is the projection on the subspace of $L^2(\Omega)$ spanned by all φ_k with $\lambda_k<\lambda$. It follows that $\mathcal{N}_\lambda(H_{V,\Omega})$ is the number of eigenvalues $\lambda_k<\lambda$ counted with multiplicity. The definition (6) has advantage that it makes sense for any spectrum.

Lemma 2 *The following identity is true for all real* λ*:*

$$\mathcal{N}_\lambda(H_{V,\Omega})=\sup\left\{\dim\mathcal{V}:\mathcal{V}\prec\mathcal{D}_\Omega \text{ and } \mathcal{E}_V(f)<\lambda\|f\|_2^2\quad\forall f\in\mathcal{V}\setminus\{0\}\right\}, \tag{7}$$

where $\mathcal{V}\prec\mathcal{D}_\Omega$ *means that* $\mathcal{V}$ *is a subspace of* $\mathcal{D}_\Omega$*. In fact, it suffices to restrict* sup *to finite dimensional subspaces* $\mathcal{V}$*.*

For example, if the spectrum of $H_{V,\Omega}$ is discrete and $\{\varphi_k\}$ is an orthonormal basis of eigenfunctions with eigenvalues $\{\lambda_k\}$ then the condition $\mathcal{E}_V(f)<\lambda\|f\|_2^2$ is satisfied exactly for $f=\varphi_k$ provided $\lambda_k<\lambda$, because

$$\mathcal{E}_V(\varphi_k)=(H_{V,\Omega}\varphi_k,\varphi_k)=\lambda_k(\varphi_k,\varphi_k)<\lambda\|\varphi_k\|_2^2.$$

The optimal space $\mathcal{V}$ in (8) is spanned by all $\{\varphi_k\}$ with $\lambda_k<\lambda$, and its dimension is equal to $\mathcal{N}_\lambda(H_{V,\Omega})$.

In the case if $H_{V,\Omega}$ is not defined as a self-adjoint operator, we still use (7) as the definition of $\mathcal{N}_\lambda(H_{V,\Omega})$

There is also a version of counting function with non-strict inequality:

$$\mathcal{N}^*_\lambda(H_{V,\Omega})=\dim\operatorname{Im}\mathbf{1}_{(-\infty,\lambda]}(H_{V,\Omega}).$$

Then the following identity is true:

$$\mathcal{N}^*_\lambda(H_{V,\Omega})=\sup\{\dim\mathcal{V}:\mathcal{V}\prec\mathcal{F}_{V,\Omega}\ \text{ and } \mathcal{E}_V[f]\le\lambda\mu[f]\quad\forall f\in\mathcal{V}\}. \tag{8}$$

1.3 Reduction to operator $\frac{1}{V}\Delta$

In the most part of the proof of Theorem 1, we will assume that $V > 0$ and, moreover, $\frac{1}{V} \in L^{\infty}_{loc}(\mathbb{R}^n)$. This can be achieved as follows. Choose some positive function $U \in C^{\infty}(\mathbb{R}^n) \cap L^{n/2}$ and consider the potential $V_\varepsilon = V + \varepsilon U$ for $\varepsilon > 0$. For this potential we have $\frac{1}{V_\varepsilon} \in L^{\infty}_{loc}$. On the other hand, we have by monotonicity

$$\mathcal{N}_0(H_V) \leq \mathcal{N}_0(H_{V_\varepsilon}).$$

Hence, any estimate of $\mathcal{N}_0(H_{V_\varepsilon})$ will translate to that of $\mathcal{N}_0(H_V)$ by letting $\varepsilon \to 0$. For example, if we know already that

$$\mathcal{N}_0(H_{V_\varepsilon}) \leq C_n \int_{\mathbb{R}^n} V_\varepsilon^{n/2} dx,$$

then we obtain (1) by letting $\varepsilon \to 0$.

Set $H_V \equiv H_{V,\mathbb{R}^n}$. Our aim is to prove the upper bound (1), that is,

$$\mathcal{N}_0(H_V) \leq C_n \int_{\mathbb{R}^n} V^{n/2} dx. \tag{9}$$

In fact, the same argument works also for the number $\mathcal{N}_0^*(H_V)$ of non-positive eigenvalues.

For $\lambda = 0$ the identity (7) becomes

$$\mathcal{N}_0(H_{V,\Omega}) = \sup\{\dim \mathcal{V} : \mathcal{V} \prec \mathcal{D}_\Omega \text{ and } \mathcal{E}_V(f) < 0 \ \forall f \in \mathcal{V} \setminus \{0\}\}. \tag{10}$$

The condition $\mathcal{E}_V(f) < 0$ here is equivalent to

$$\int_\Omega |\nabla f|^2 dx - \int_\Omega V f^2 dx < 0 \tag{11}$$

for all non-zero $f \in \mathcal{V}$ where $\mathcal{V}$ is a subspace of $\mathcal{D}_\Omega$.

We will interpret this inequality in terms of the counting function of another operator. Consider a new measure μ defined by

$$d\mu = V(x)\, dx$$

and the energy form

$$\mathcal{E}(f) = \int_{\mathbb{R}^n} |\nabla f|^2 dx$$

for $f \in \mathcal{D}_\Omega$. Then (11) can be rewritten in the form $\mathcal{E}(f) < \|f\|^2_{2,\mu}$ so that

$$\mathcal{N}_0(H_{V,\Omega}) = \sup\left\{\dim \mathcal{V} : \mathcal{V} \prec \mathcal{D}_\Omega \text{ and } \mathcal{E}(f) < \|f\|^2_{2,\mu} \ \forall f \in \mathcal{V} \setminus \{0\}\right\}. \tag{12}$$

The right hand side here is the counting function of another operator. Indeed, denoted by $\mathcal{L}_{V,\Omega}$ the generator of the energy form $(\mathcal{E}, \mathcal{D}_\Omega)$ in $L^2(\Omega, \mu)$. Assuming that $\frac{1}{V} \in L^{\infty}_{loc}$, this form can be shown to be closable, so that its generator $\mathcal{L}_{V,\Omega}$ is a self-adjoint operator in $L^2(\Omega,\mu)$. Note also that this operator is positive definite because so is $\mathcal{E}$.

By definition, we have, for all $f,g\in \operatorname{dom}(\mathcal{L}_{V,\Omega})$,

$$\mathcal{E}(f,g)=(\mathcal{L}_{V,\Omega}f,g)_\mu .$$

In particular, for $f,g\in\mathcal{D}_\Omega$ this implies

$$-\int_\Omega(\Delta f)\,g dx=\int_\Omega\nabla f\cdot\nabla g\ dx=\int_\Omega(\mathcal{L}_{V,\Omega}f)\,gV dx,$$

whence $\mathcal{L}_{V,\Omega}f=-\frac{1}{V}\Delta f$ that is, $\mathcal{L}_{V,\Omega}=-\frac{1}{V}\Delta$.

The counting function $\mathcal{N}_\lambda(\mathcal{L}_{V,\Omega})$ of the operator $\mathcal{L}_{V,\Omega}$ is defined exactly as for $H_{V,\Omega}$. Lemma 2 for this operator means that

$$\mathcal{N}_\lambda(\mathcal{L}_{V,\Omega})=\sup\left\{\dim\mathcal{V}:\mathcal{V}\prec\mathcal{D}_\Omega\text{ and }\mathcal{E}(f)<\lambda\|f\|_{2,\mu}^2\quad\forall f\in\mathcal{V}\setminus\{0\}\right\}.\tag{13}$$

For $\lambda=1$ the right hand side of (13) coincides with that of (12), which implies

$$\mathcal{N}_0(H_{V,\Omega})=\mathcal{N}_1(\mathcal{L}_{V,\Omega}).\tag{14}$$

In particular, for the case $\Omega=\mathbb{R}^n$, we have $\mathcal{N}_0(H_V)=\mathcal{N}_1(\mathcal{L}_V)$. The identity (14) is called *Birman-Schwinger principle.*

Informally the identity (14) reflects the equivalence of the inequalities $-\Delta-V\le 0$ and $-\frac{1}{V}\Delta\le 1$ that are understood in the sense of quadratic forms.

1.4 Case of small V

Here we illustrate the usage of (14) by proving a particular case of Theorem 1 as follows.

Lemma 3 *If $n\ge 3$ then there is a constant $c_n>0$ such that*

$$\int_{\mathbb{R}^n}V^{n/2}dx<c_n\Rightarrow\mathcal{N}_0(H_V)=0.$$

Proof. As was explained above, we can assume without loss of generality, that $\frac{1}{V}\in L^\infty_{loc}$. By (14) we need to prove that the spectrum of $\mathcal{L}_V$ below 1 is empty, that is,

$$\inf\operatorname{spec}\mathcal{L}_V\ge 1.$$

This is equivalent to the claim that the operator $\mathcal{L}_V$ in $L^2(\mathbb{R}^n,\mu)$ is invertible and

$$\left\|\mathcal{L}_V^{-1}\right\|\le 1.$$

The inverse operator is defined by

$$\mathcal{L}_V^{-1}f=u\quad\Leftrightarrow\quad\mathcal{L}_Vu=f,$$

where $f \in L^2(\mathbb{R}^n, \mu)$ and $u \in \operatorname{dom}(\mathcal{L}_V)$. Hence, it suffices to prove that

$$\mathcal{L}_V u = f \quad \Rightarrow \|u\|_{2,\mu} \le \|f\|_{2,\mu}.$$

Multiplying $\mathcal{L}_V u = f$ by u and integrating against μ, we obtain

$$\mathcal{E}(u) = (\mathcal{L}_V u, u)_\mu = (f, u)_\mu$$

that is,

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx = \int_{\mathbb{R}^n} u f d\mu.$$

By Sobolev inequality, we have

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \ge c_n \left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}}.$$

Note that this is the only place where $n > 2$ is used.

Using the Hölder inequality and the above lines, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} u^2 V dx &\le \left(\int_{\mathbb{R}^n} |u|^{2\frac{n}{n-2}} \, dx \right)^{\frac{n-2}{n}} \left(\int_{\mathbb{R}^n} V^{\frac{n}{2}} dx \right)^{\frac{2}{n}} \\ &\le c_n^{-1} \left(\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \right) \left(\int_{\mathbb{R}^n} V^{\frac{n}{2}} dx \right)^{\frac{2}{n}} \\ &= c_n^{-1} \left(\int_{\mathbb{R}^n} u f d\mu \right) \left(\int_{\mathbb{R}^n} V^{\frac{n}{2}} dx \right)^{\frac{2}{n}} \\ &\le c_n^{-1} \left(\int_{\mathbb{R}^n} f^2 d\mu \right)^{1/2} \left(\int_{\mathbb{R}^n} u^2 d\mu \right)^{1/2} \left(\int_{\mathbb{R}^n} V^{\frac{n}{2}} dx \right)^{\frac{2}{n}} \end{aligned} \tag{15}$$

whence

$$\|u\|_{2,\mu} \le c_n^{-1} \left(\int_{\mathbb{R}^n} V^{\frac{n}{2}} dx \right)^{\frac{2}{n}} \|f\|_{2,\mu}.$$

Clearly, if $\int_{\mathbb{R}^n} V^{\frac{n}{2}} dx$ small enough then $\|u\|_{2,\mu} \le \|f\|_{2,\mu}$, which was to be proved. ■

The argument in the proof of Lemma 3 allows to prove another part of Theorem 1.

Lemma 4 *If $V \in L^{n/2}(\mathbb{R}^n)$ then the form $(\mathcal{E}_V, W^{1,2})$ is closed in $L^2(\mathbb{R}^n)$. Consequently, the operator H_V is defined as a self-adjoint operator in $L^2(\mathbb{R}^n)$ and its domain is a subspace of $W^{1,2}(\mathbb{R}^n)$.*

Proof. It follows from the hypothesis that, for any $\varepsilon > 0$, V can be split to a sum of two potentials $V = V_1 + V_2$ where

$$\|V_1\|_{n/2} \le \varepsilon \quad \text{and} \quad V_2 \in L^\infty.$$

It follows from (15) that

$$\mathcal{E}(u) \geq c_n \left(\int_{\mathbb{R}^n} V_1^{n/2} dx\right)^{-2/n} \int_{\mathbb{R}^n} u^2 V_1 dx \geq c_n \varepsilon^{-1} \int_{\mathbb{R}^n} u^2 V_1 dx.$$

Choosing ε sufficiently small, we obtain $c_n \varepsilon^{-1} \geq 2$ whence

$$\begin{aligned} \int_{\mathbb{R}^n} u^2 V dx &= \int_{\mathbb{R}^n} u^2 V_1 dx + \int_{\mathbb{R}^n} u^2 V_2 dx \\ &\leq \frac{1}{2}\mathcal{E}(u) + K \|u\|_2^2, \end{aligned} \tag{16}$$

where $K = \|V_2\|_{L^\infty}$. Since the form $(\mathcal{E}, W^{1,2})$ is closed and $\int_{\mathbb{R}^n} u^2 V dx$ satisfies the domination condition (16), it follows by the KLMN-theorem from the theory of quadratic forms, that the form $(\mathcal{E}_V, W^{1,2})$ is also closed. Hence, its generator H_V is well-defined as a self-adjoint semi-bounded below operator, whose domain is a subspace of $W^{1,2}$. ∎

It remains to prove the main part of Theorem 1: the estimate (1) or, equivalently, (9). As it was explained above, we can assume that $\frac{1}{V} \in L^\infty_{loc}$. Moreover, let us show that it suffices to treat the case $V \in C^\infty$. Indeed, consider a sequence $\{V_k\}_{k=1}^\infty$ of smooth positive functions V_k being mollifications of V, so that $V_k \overset{L^{n/2}}{\longrightarrow} V$. Using the universal inequality

$$\mathcal{N}_0(U+V) \leq \mathcal{N}_0(2U) + \mathcal{N}_0(2V),$$

that follows from (12), we obtain

$$\begin{aligned} \mathcal{N}_0(V) &\leq \mathcal{N}_0(V_k + |V - V_k|) \\ &\leq \mathcal{N}_0(2V_k) + \mathcal{N}_0(2|V - V_k|). \end{aligned} \tag{17}$$

Choose k large enough so that $\|V - V_k\|_{n/2}$ is small enough. Then by Lemma 3

$$\mathcal{N}_0(2|V - V_k|) = 0.$$

Assuming that (9) is proved for smooth potentials, we have

$$\mathcal{N}_0(2V_k) \leq \text{const} \int_{\mathbb{R}^n} V_k^{n/2} dx.$$

Substituting into (17) and passing to the limit as $k \to \infty$, we obtain (9).

1.5 Proof of Theorem 1

The proof below is due to Li and Yau '83 but it is presented here from somewhat different angle. As it was explained above, we can assume from now on that $V \in C^\infty$ and $V > 0$.

In a precompact domain Ω the operator $\mathcal{L}_{V,\Omega}$ has discrete positive spectrum. Denote its eigenvalues by $\lambda_k(\Omega)$, where $k = 1, 2, ...$, so that the sequence $\{\lambda_k(\Omega)\}$ is increasing, and each eigenvalue is counted with multiplicity. The main part of the proof of Theorem 1 is contained in the following statement.

Theorem 5 (AG, Yau 2003) *Assume that there is a Radon measure ν in $\mathbb{R}^n$ and $\alpha > 0$ such that, for all precompact open sets Ω,*

$$\lambda_1(\Omega) \geq \nu(\Omega)^{-\alpha}. \tag{18}$$

Then, for any positive integer k and any precompact open set Ω,

$$\lambda_k(\Omega) \geq c\left(\frac{k}{\nu(\Omega)}\right)^{\alpha}, \tag{19}$$

where $c = c(\alpha) > 0$.

For example, if $V = 1$ then $\mathcal{L}_{V,\Omega}$ is the Laplace operator $-\Delta$ with the Dirichlet boundary condition on $\partial\Omega$. The hypothesis (18) is satisfies if ν is a multiple of the Lebesgue measure as by the Faber-Krahn inequality

$$\lambda_1(\Omega) \geq c_n (\operatorname{vol}\Omega)^{-2/n}.$$

Then (19) becomes

$$\lambda_k(\Omega) \geq c_n'\left(\frac{k}{\operatorname{vol}\Omega}\right)^{2/n},$$

that is also known to be true. Moreover, it matches the Weyl's asymptotic formula $\lambda_k(\Omega) \sim \widetilde{c}_n\left(\frac{k}{\operatorname{vol}\Omega}\right)^{2/n}$ as $k \to \infty$.

The point of Theorem 5 is that V in the definition of $\mathcal{L}_{V,\Omega}$ can be arbitrary and measure ν can be arbitrary. By the way, there is no restriction of the dimension n in Theorem 5. Moreover, exactly in this form it is true on any Riemannian manifold instead of $\mathbb{R}^n$.

Proof of Theorem 1 using Theorem 5. Let us use the variational principle:

$$\lambda_1(\Omega) = \inf_{u\in\mathcal{D}_\Omega}\frac{(\mathcal{L}_{V,\Omega}u,u)_\mu}{(u,u)_\mu} = \inf_{u\in\mathcal{D}_\Omega}\frac{\mathcal{E}(u)}{(u,u)_\mu}.$$

Using again the Sobolev inequality

$$\int_\Omega |\nabla u|^2\,dx \geq c_n\left(\int_\Omega |u|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}}$$

and the Hölder inequality

$$(u,u)_\mu = \int_\Omega u^2 V\,dx \leq \left(\int_\Omega |u|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}}\left(\int_\Omega V^{n/2}dx\right)^{\frac{2}{n}},$$

we obtain

$$\frac{\mathcal{E}(u)}{(u,u)_\mu} \geq c_n\left(\int_\Omega V^{n/2}dx\right)^{-\frac{2}{n}}.$$

Hence, setting $d\nu = c_n^{-n/2}V^{n/2}dx$ and minimizing in u, we obtain

$$\lambda_1(\Omega) \geq \nu(\Omega)^{-2/n}.$$

By Theorem 5, we conclude that

$$\lambda_k(\Omega) \geq c\left(\frac{k}{\nu(\Omega)}\right)^{2/n}. \tag{20}$$

We need to estimate the counting function

$$\mathcal{N}_1(\mathcal{L}_{V,\Omega}) = \#\{k : \lambda_k(\Omega) < 1\}.$$

By (20), $\lambda_k(\Omega) < 1$ implies $k \leq C\nu(\Omega)$ whence also

$$\mathcal{N}_1(\mathcal{L}_{V,\Omega}) \leq C\nu(\Omega) = C\int_\Omega V^{n/2}dx.$$

It follows by (14) that also

$$\mathcal{N}_0(H_{V,\Omega}) \leq C\int_\Omega V^{n/2}dx \leq C\int_{\mathbb{R}^n} V^{n/2}dx. \tag{21}$$

We are left to pass from $H_{V,\Omega}$ to $H_{V,\mathbb{R}^n}$. Recall that

$$\mathcal{N}_0(H_{V,\mathbb{R}^n}) = \sup\{\dim\mathcal{V} : \mathcal{V} \prec \mathcal{D}_{\mathbb{R}^n},\ \ \mathcal{E}_V(f) < 0\ \forall f \in \mathcal{V}\setminus\{0\}\},$$

where $\mathcal{V}$ is a finite-dimensional subspace of $\mathcal{D}_{\mathbb{R}^n}$. For any such $\mathcal{V}$ there exists a precompact open set Ω containing $\operatorname{supp} f$ for all $f \in \mathcal{V}$ (for it suffices to have $\operatorname{supp} f \subset \mathcal{V}$ for the elements of a basis of $\mathcal{V}$). Hence, $\mathcal{V} \prec \mathcal{D}_\Omega$ and by (21)

$$\dim\mathcal{V} \leq C\int_{\mathbb{R}^n} V^{n/2}dx,$$

whence the same estimate for $\mathcal{N}_0(H_{V,\mathbb{R}^n})$ follows. ■

Nash inequality

For the proof of Theorem 5 we need a Nash type inequality.

Lemma 6 *Assume that* (18) *holds, that is, for all precompact open sets* Ω*,*

$$\lambda_1(\Omega) \geq \nu(\Omega)^{-\alpha}.$$

Then, for all such Ω *and non-negative* $f \in \mathcal{D}_\Omega$*,*

$$\mathcal{E}(f) \geq c\left(\int_\Omega f^2 d\mu\right)^{1+\alpha}\left(\int_\Omega f d\mu \int_\Omega f d\nu\right)^{-\alpha}, \tag{22}$$

where $c = 2^{-2\alpha-1}$*.*

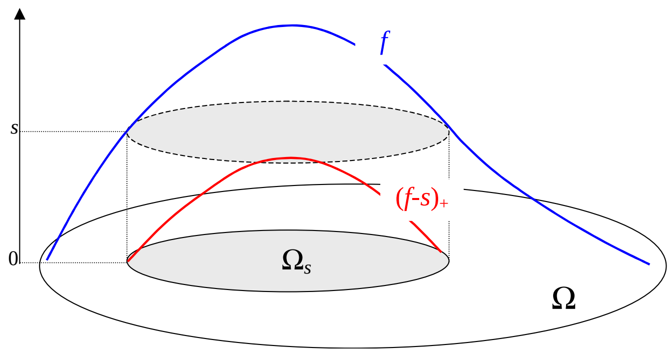

Figure 2:

Remark. If $V \equiv 1$ then both μ and ν are Lebesgue measures, $\alpha = 2/n$, and (22) becomes

$$\mathcal{E}(f) \geq \|f\|_2^{2+4/n} \|f\|_1^{-4/n},$$

which is a classical Nash inequality.

Proof. Fix $s > 0$ and observe that

$$\mathcal{E}\left((f-s)_+\right) \leq \mathcal{E}(f). \tag{23}$$

Set

$$\Omega_s := \{x \in \Omega : f(x) > s\}$$

and note that supp $(f-s)_+ \subset \overline{\Omega}_s \subset \Omega$.

It follows from the variational property of $\lambda_1(\Omega_s)$ and from (23), that

$$\int_\Omega (f-s)_+^2 \, d\mu = \int_{\Omega_s} (f-s)_+^2 \, d\mu \leq \frac{\mathcal{E}\left((f-s)_+\right)}{\lambda_1(\Omega_s)} \leq \frac{\mathcal{E}(f)}{\lambda_1(\Omega_s)}. \tag{24}$$

Since

$$\nu(\Omega_s) \leq \frac{1}{s} \int_\Omega f d\nu$$

we obtain by hypothesis

$$\frac{1}{\lambda_1(\Omega_s)} \leq \nu(\Omega_s)^\alpha \leq s^{-\alpha} \left(\int_\Omega f d\nu\right)^a.$$

Substituting into (24) and using

$$f^2 - 2sf \leq (f-s)_+^2,$$

we obtain

$$\int_\Omega f^2 d\mu - 2s \int_\Omega f d\mu \leq s^{-\alpha} \left(\int_\Omega f d\nu\right)^\alpha \mathcal{E}(f). \tag{25}$$

Let us choose s from the condition

$$2s \int_\Omega f d\mu = \frac{1}{2} \int_\Omega f^2 d\mu.$$

With this s we obtain

$$\frac{1}{2}\int_{\Omega} f^2 d\mu \le \left(\frac{1}{4}\frac{\int_{\Omega} f^2 d\mu}{\int_{\Omega} f d\mu}\right)^{-\alpha}\left(\int_{\Omega} f d\nu\right)^{\alpha}\mathcal{E}(f)$$

whence

$$\left(\int_{\Omega} f^2 d\mu\right)^{1+\alpha} \le 2^{2\alpha+1}\left(\int_{\Omega} f d\mu\right)^{\alpha}\left(\int_{\Omega} f d\nu\right)^{\alpha}\mathcal{E}(f),$$

and (22) follows. ■

1.6 Proof of Theorem 5

In the proof we work with the heat semigroup $\{P_t\}_{t\ge 0}$ of the operator $\mathcal{L}_{V,\Omega}$, that is defined by

$$P_t^{\Omega} = e^{-t\mathcal{L}_{V,\Omega}}.$$

Since $\mathcal{L}_{V,\Omega}$ is a self-adjoint non-negative definite operator in $L^2(\Omega,\mu)$, the operator P_t^{Ω} is bounded self-adjoint operator in $L^2(\Omega,\mu)$ for any $t \ge 0$. In fact, it is an integral operator:

$$P_t^{\Omega} f(x) = \int_{\Omega} p_t^{\Omega}(x,y) f(y) d\mu(y)$$

where $p_t^{\Omega}(x,y)$ is the *heat kernel* of $\mathcal{L}_{V,\Omega}$. We will use the following general properties of the heat kernel:

1. $p_t(x,y)$ is a smooth function of $x, y \in \mathbb{R}^n$ and $t > 0$.

2. positivity: $p_t(x,y) \ge 0$;

3. the symmetry: $p_t^{\Omega}(x,y) = p_t^{\Omega}(y,x)$;

4. the semigroup identity

$$\int_{\Omega} p_t^{\Omega}(x,z)\, p_s^{\Omega}(z,y)\, d\mu(z) = p_{t+s}^{\Omega}(x,y);$$

5. the total mass inequality:

$$\int_{\Omega} p_t^{\Omega}(x,y)\, d\mu(y) \le 1.$$

The last step before the actual proof of Theorem 5 is the following lemma.

Lemma 7 *If (18) holds then, for any precompact open set $\Omega \subset \mathbb{R}^n$,*

$$\int_{\Omega} p_t^{\Omega}(x,x)\, d\mu(x) \le \frac{C\nu(\Omega)}{t^{1/\alpha}}. \tag{26}$$

where $C = C(\alpha)$.

Proof. Fix $\Omega \subset \mathbb{R}^n$, $s > 0$ and consider the function

$$u_t(x,y) = p^{\Omega}_{t+s}(x,y).$$

By the semigroup identity we have

$$u_t(x,y) = \int_\Omega p^{\Omega}_t(y,z)\, p^{\Omega}_s(z,x)\, d\mu(z) = P^{\Omega}_t f(y)$$

where $f = p^{\Omega}_s(\cdot, x)$ (considering x as fixed). Since $f \in L^2(\Omega,\mu)$, it follows that $u_t(x,\cdot)$ lies in $\operatorname{dom}(\mathcal{L}_{V,\Omega})$. The Nash inequality (22) extends easily to such functions, so that we obtain

$$\int_\Omega u_t^2 d\mu \le C \left(\int_\Omega u_t d\mu \int_\Omega u_t d\nu \right)^{\frac{\alpha}{\alpha+1}} \mathcal{E}(u_t)^{\frac{1}{\alpha+1}},$$

where $C = C(\alpha)$ and integration is done with respect to y. Set

$$v_t(x) := \int_\Omega u_t^2(x,y)\, d\mu(y) = \int_\Omega p^{\Omega}_{t+s}(x,y) p^{\Omega}_{t+s}(y,x)\, d\mu(y) = p^{\Omega}_{2(t+s)}(x,x).$$

Using

$$\int_\Omega u_t d\mu = \int_\Omega p^{\Omega}_{t+s}(x,y)\, d\mu(y) \le 1 \tag{27}$$

and

$$\mathcal{E}(u_t) = (\mathcal{L}_{V,\Omega} u_t, u_t)_\mu = -\left(\frac{d}{dt} u_t, u_t \right)_\mu = -\frac{1}{2}\frac{d}{dt}(u_t,u_t)_\mu = -\frac{1}{2}\frac{d}{dt} v_t(x),$$

we obtain

$$v_t(x) \le C \left(\int_\Omega u_t d\nu \right)^{\frac{\alpha}{\alpha+1}} \left(-\frac{d}{dt} v_t(x) \right)^{\frac{1}{\alpha+1}}. \tag{28}$$

Integrating (28) against $d\mu(x)$ and using the Hölder inequality

$$\int F^{\frac{\alpha}{\alpha+1}} G^{\frac{1}{\alpha+1}} d\mu \le \left[\int F d\mu \right]^{\frac{\alpha}{\alpha+1}} \left[\int G d\mu \right]^{\frac{1}{\alpha+1}},$$

we obtain

$$\begin{aligned} \int_\Omega v_t(x)\, d\mu(x) &\le C \int \underbrace{\left[\int u_t d\nu(y) \right]^{\frac{\alpha}{\alpha+1}}}_{F} \underbrace{\left[-\frac{\partial v_t}{\partial t} \right]^{\frac{1}{\alpha+1}}}_{G} d\mu(x) \\ &\le C \left[\int \int u_t d\nu(y)\, d\mu(x) \right]^{\frac{\alpha}{\alpha+1}} \left[- \int \frac{\partial v_t}{\partial t} d\mu(x) \right]^{\frac{1}{\alpha+1}}. \end{aligned}$$

Observe that (27) implies

$$\int \int u_t d\nu(y)\, d\mu(x) = \int \left(\int u_t(x,y)\, d\mu(x) \right) d\nu(y) \le \int_\Omega d\nu = \nu(\Omega). \tag{29}$$

Denoting

$$w\left(t\right) := \int_{\Omega} v_t\left(x\right) d\mu\left(x\right) = \int_{\Omega} p^{\Omega}_{2(t+s)}\left(x,x\right)\mu\left(x\right),$$

we obtain from above

$$w(t) \leq C\nu(\Omega)^{\frac{\alpha}{\alpha+1}}\left(-\frac{dw}{dt}\right)^{\frac{1}{\alpha+1}}. \tag{30}$$

Solving this differential inequality by separation of variables, we obtain

$$w(t) \leq \frac{C\nu\left(\Omega\right)}{t^{1/\alpha}}.$$

Finally, choosing $s = t$ we obtain

$$\int_{\Omega} p^{\Omega}_{4t}\left(x,x\right)\mu\left(x\right) \leq \frac{C\nu\left(\Omega\right)}{t^{1/\alpha}},$$

which was to be proved. ■

Proof of Theorem 5. We need to show that

$$\lambda_k\left(\Omega\right) \geq c\left(\frac{k}{\nu\left(\Omega\right)}\right)^{\alpha}.$$

Note that

$$\int_{\Omega} p^{\Omega}_t\left(x,x\right) d\mu\left(x\right) = \operatorname{trace} P^{\Omega}_t.$$

On the other hand, all the eigenvalues of P^{Ω}_t are equal to $e^{-t\lambda_k(\Omega)}$, whence

$$\operatorname{trace} P^{\Omega}_t = \sum_{k=1}^{\infty} e^{-t\lambda_k(\Omega)}.$$

Hence, applying (26), we obtain

$$\sum_{k=1}^{\infty} e^{-t\lambda_k(\Omega)} \leq \frac{C\nu\left(\Omega\right)}{t^{1/\alpha}}.$$

Restricting the summation to the first k terms, we obtain

$$ke^{-t\lambda_k(\Omega)} \leq \frac{C\nu\left(\Omega\right)}{t^{1/\alpha}}$$

whence

$$\lambda_k\left(\Omega\right) \geq \frac{1}{t}\ln\frac{kt^{1/\alpha}}{C\nu\left(\Omega\right)}.$$

Choosing t from the condition

$$\frac{kt^{1/\alpha}}{C\nu\left(\Omega\right)} = e,$$

that is,

$$t = \left(Ce\frac{\nu\left(\Omega\right)}{k}\right)^{\alpha},$$

we obtain

$$\lambda_k\left(\Omega\right) \geq \frac{1}{t} = \left(\frac{1}{Ce}\frac{k}{\nu\left(\Omega\right)}\right)^{\alpha},$$

which finishes the proof of Theorem 5. ■

2 Minimal surfaces

Let M be a two-dimensional manifold immersed in $\mathbb{R}^3$ as an oriented minimal surface. The Riemannian metric on M is induced by the Euclidean structure of $\mathbb{R}^3$. Denote by σ the Riemannian area on M.

For any function $f \in C_0^\infty(M)$ consider a deformation M_f of M given by the mapping $x \mapsto x + f(x)\nu(x)$ where $\nu(x)$ is the unit normal vector field on M that is determined by the orientation. Then

$$\sigma(M_f) = \sigma(M) + \delta\sigma(f) + o(\|f\|)$$

where $\delta\sigma(f)$ is the first variation of the area functional, that is given by

$$\delta\sigma(f) = \int_M f\mathcal{H}d\sigma,$$

where $\mathcal{H} = \mathcal{H}(x)$ is the mean curvature of M. By definition, M is called a minimal surface, if the first variation $\delta\sigma(f)$ vanishes for all f, which is equivalent to $\mathcal{H} \equiv 0$ on M. Assuming that M is minimal, there is the following nice formula for the second variation:

$$\delta^2\sigma(f) = \int_M (|\nabla f|^2 + 2Kf^2)d\sigma\,, \tag{31}$$

where $K = K(x)$ is the Gauss curvature of M at the point $x \in M$. Note that since $\mathcal{H} \equiv 0$, we have $K \leq 0$. If $\delta^2\sigma(f) \geq 0$ for all f then the minimal surface M is called *stable*. In particular, all area minimizers are stable.

However, in general a minimal surface is not necessarily stable. By definition, the *stability index* $\operatorname{ind}(M)$ of the minimal surface is the Morse index of the $\delta^2\sigma$, that us,

$$\operatorname{ind}(M) = \sup\left\{\dim \mathcal{V} : \mathcal{V} \prec C_0^\infty(M) \text{ s.t. } \delta^2\sigma(f) < 0 \text{ for all } f \in \mathcal{V} \setminus \{0\}\right\}.$$

In other words, we have

$$\operatorname{ind}(M) = \mathcal{N}_0(H_V)\,,$$

where $V = -2K$ and

$$H_V = -\Delta - V = -\Delta + 2K$$

and Δ is the Laplace-Beltrami operator on M. The operator H_V is called the stability operator of M.

It turns out that for the stability operator the upper bound of Theorem 1 is satisfied.

Theorem 8 (AG, Yau 2003) *For any immersed oriented minimal surface M in $\mathbb{R}^3$, we have*

$$\operatorname{ind}(M) \leq C \int_M |K|\, d\sigma, \tag{32}$$

where C is an absolute constant.

The proof goes in the same way as the one of Theorem 1 using Theorem 5. Using specific properties of Gauss curvature, we first prove for the operator $\mathcal{L}_{V,\Omega} = -\frac{1}{V}\Delta$ in $\Omega \subset M$ the eigenvalue estimate

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-1},$$

where $d\mu = |K|\, d\sigma$. By Theorem 5 this implies

$$\lambda_k(\Omega) \geq c' \frac{k}{\mu(\Omega)}$$

and then as in the proof of Theorem 1,

$$\mathcal{N}_0(H_V) \leq C\mu(M)$$

that is (32).

In the case of geodesically complete M the estimate (32) was proved in Tysk in 1987 (a better value of the constant C is due to Micallef 2001). In 1985 Fischer-Colbrie proved, also for complete minimal surfaces, that the finiteness of ind(M) is equivalent to the finiteness of the total curvature of M.

3 Lower estimates in $\mathbb{R}^2$

Here we are concerned with $\mathcal{N}_0(H_V)$ in $\mathbb{R}^2$.

3.1 A counterexample to the upper bound

In the case $n = 2$, the estimate (1) of Theorem 1 becomes

$$\mathcal{N}_0(H_V) \leq C \int_{\mathbb{R}^2} V(x)\, dx,$$

which however is *wrong*. To see that, consider in $\mathbb{R}^2$ the potential

$$V(x) = \frac{1}{|x|^2 \ln^2 |x|} \quad \text{if } |x| > e$$

and $V(x) = 0$ if $|x| \leq e$. For this V we have

$$\int_{\mathbb{R}^2} V(x)\, dx < \infty,$$

whereas $\operatorname{Neg}(H_V) = \infty$. Indeed, consider the function

$$f(x) = \sqrt{\ln |x|} \sin\left(\frac{1}{2} \ln \ln |x|\right)$$

that satisfies in the region $\{|x| > e\}$ the differential equation

$$\Delta f + \frac{1}{2} V(x) f = 0.$$

For any positive integer k, function f has constant sign in the ring

$$\Omega_k := \left\{ x \in \mathbb{R}^2 : \pi k < \frac{1}{2} \ln \ln |x| < \pi (k+1) \right\},$$

and vanishes on $\partial \Omega_k$. For each function $f_k = f \mathbf{1}_{\Omega_k}$ we have

$$\begin{aligned}
\mathcal{E}_V (f_k) &= \int_{\Omega_k} |\nabla f_k|^2 \, dx - \int_{\Omega_k} V f_k^2 dx \\
&= -\int_{\Omega_k} f_k \Delta f_k dx - \int_{\Omega_k} V f_k^2 dx \\
&= -\frac{1}{2} \int_{\Omega_k} V f_k^2 dx < 0.
\end{aligned}$$

The same inequality holds for linear combination of functions f_k since the intersection of their supports has measure 0.

Hence, the space $\mathcal{V} = \operatorname{span}\{f_k\}$ has infinite dimension and $\mathcal{E}_V (f) < 0$ for all non-zero $f \in \mathcal{V}$, which implies $\mathcal{N}_0 (H_V) = \infty$.

In fact, one can show that no upper bound of the form

$$\mathcal{N}_0 (H_V) \le \int_{\mathbb{R}^2} V(x) \, W(x) \, dx$$

can be true, no matter how we choose a weight $W(x)$.

3.2 Lower bound of $\mathcal{N}_0 (H_V)$

It turns out that in the case $n = 2$, instead of an upper bound, a lower bound in (1) is true.

Theorem 9 (AG, Netrusov, Yau, 2004) *For any non-negative potential V in $\mathbb{R}^2$,*

$$\mathcal{N}_0 (H_V) \ge c \int_{\mathbb{R}^2} V(x) \, dx \tag{33}$$

with some absolute constant $c > 0$.

Let us describe an approach to the proof. Since

$$\mathcal{N}_0 (H_V) = \sup \left\{ \dim \mathcal{V} : \mathcal{V} \prec \mathcal{D}_{\mathbb{R}^2} \text{ and } \mathcal{E}_V (f) < 0 \, \forall f \in \mathcal{V} \setminus \{0\} \right\},$$

it suffices to construct a subspace $\mathcal{V}$ of $\mathcal{D}_{\mathbb{R}^2}$ such that $\mathcal{E}_V$ is negative on $\mathcal{V}$ and

$$\dim \mathcal{V} \ge c \int_{\mathbb{R}^2} V(x) \, dx.$$

We will construct $\mathcal{V}$ as $\operatorname{span}\{f_k\}$ where $\{f_k\}_{k=1}^N$ is a sequence of functions with disjoint compact supports such that $\mathcal{E}_V (f_k) < 0$. Then $\mathcal{E}_V (f) < 0$ will be true for any non-zero function $f \in \operatorname{span}\{f_k\}$, and $\dim \mathcal{V} = N$. Hence, it suffices to construct a sequence $\{f_k\}_{k=1}^{N}$ of functions with compact disjoint supports such that, for any $k = 1, ..., N$,

$$\int_{\mathbb{R}^2} |\nabla f_k|^2 \, dx < \int_{\mathbb{R}^2} V f_k^2 dx,$$

and

$$N \geq c \int_{\mathbb{R}^2} V(x) \, dx.$$

Each function f_k will be constructed as follows. Fix two reals $0 < r < R$ and consider the annulus

$$A = \left\{ x \in \mathbb{R}^2 : r < |x| < R \right\}$$

and denote by $2A$ the annulus

$$2A = \left\{ x \in \mathbb{R}^2 : \frac{1}{2} r < |x| < 2R \right\}.$$

Consider the following function

$$f(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin 2A, \\ \frac{1}{\ln 2} \ln \frac{2|x|}{r}, & \frac{r}{2} \leq |x| \leq r, \\ \frac{1}{\ln 2} \ln \frac{2R}{|x|}, & R \leq |x| \leq 2R. \end{cases}$$

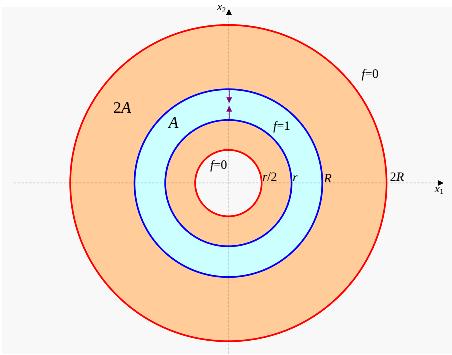

Figure 3:

This function f is harmonic in each of the four domains, whence we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla f|^2 \, dx &= \int_{\left\{\frac{r}{2} \leq |x| \leq r\right\}} |\nabla f|^2 \, dx + \int_{\{R \leq |x| \leq 2R\}} |\nabla f|^2 \, dx \\ &= \int_{\partial\left\{\frac{r}{2} \leq |x| \leq r\right\}} f \frac{\partial f}{\partial \nu} dl + \int_{\partial\{R \leq |x| \leq 2R\}} f \frac{\partial f}{\partial \nu} dl \\ &= f'(r) \, 2\pi r - f'(R) \, 2\pi R \\ &= \frac{1}{(\ln 2) \, r} 2\pi r + \frac{1}{(\ln 2) \, R} 2\pi R \\ &= \frac{4\pi}{\ln 2} < 20. \end{aligned}$$

Suppose that we have a sequence of annuli $\{A_k\}_{k=1}^N$, with different centers and different radii, but such that the sequence $\{2A_k\}_{k=1}^N$ is disjoint. Then, defining f_k for each pair $(A_k, 2A_k)$ as above, we obtain a sequence of functions with disjoint supports and with

$$\int_{\mathbb{R}^2} |\nabla f_k|^2 \, dx < 20.$$

Note that

$$\int_{\mathbb{R}^2} V f_k^2 dx \geq \int_{A_k} V dx.$$

Hence, the condition $\int_{\mathbb{R}^2} |\nabla f_k|^2 \, dx < \int_{\mathbb{R}^2} V f_k^2 dx$ will be satisfied if

$$\int_{A_k} V dx \geq 20.$$

Consider again measure μ given by $d\mu = V dx$ and restate our problem as follows: construct N annuli $\{A_k\}_{k=1}^N$ such that

(i) $\{2A_k\}_{k=1}^N$ are disjoint,

(ii) $\mu(A_k) \geq 20$ for each k,

(iii) and $N \geq c\mu(\mathbb{R}^2)$.

Of course, if $\mu(\mathbb{R}^2) < 20$ then such a sequence cannot be constructed. In this case we argue differently. Choose some $0 < r < R$ and consider the function

$$f(x) = \begin{cases} 1, & |x| \leq r \\ 0, & x \geq R, \\ \frac{1}{\ln \frac{R}{r}} \ln \frac{R}{|x|}, & r \leq |x| \leq R. \end{cases}$$

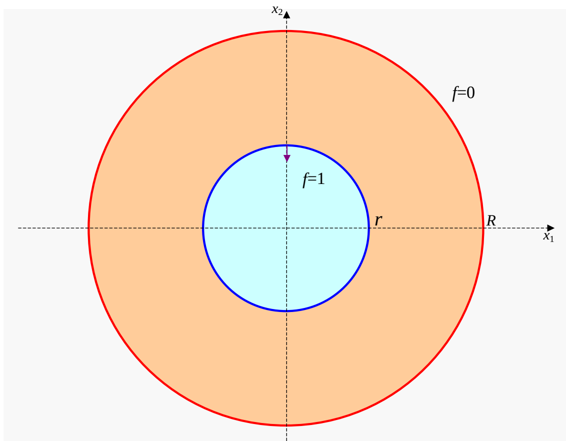

Figure 4:

For this function

$$\int_{\mathbb{R}^2} |\nabla f|^2 \, dx = -f'(r) \, 2\pi r = \frac{2\pi}{\ln \frac{R}{r}}$$

while

$$\int_{\mathbb{R}^2} V f^2 dx \geq \int_{\{|x| \leq r\}} V dx.$$

Taking r and $\frac{R}{r}$ large enough, we obtain $\int_{\mathbb{R}^2} |\nabla f|^2 \, dx < \int_{\mathbb{R}^2} V f^2 dx$ whence $\mathcal{N}_0(H_V) \geq 1$. If $\mu(\mathbb{R}^2) = \int_{\mathbb{R}^2} V dx$ is bounded by some constant, say 20, then we obtain $\mathcal{N}_0(H_V) \geq c\mu(\mathbb{R}^2)$ just by taking c small enough.

Hence, in the main part we can assume that $\mu(\mathbb{R}^2)$ is large enough. In this case, the sequence of annuli satisfying (i)-(iii) can be always constructed. In fact, the positive answer is given by the following abstract theorem.

Theorem 10 *Let (X, d) be a metric space and μ is a non-atomic Borel measure on X. Assume that the following properties are satisfied.*

1. *All metric balls $B(x,r) = \{y \in X : d(x,y) < r\}$ are precompact.*
2. *There exists a constant M such that, for any ball $B(x,r)$ there is a family of at most M balls of radii $r/2$ that cover $B(x,r)$.*

Then there is a constant $c = c(M) > 0$ such that, for any $0 < v < \mu(X)$ there exists at least $c\frac{\mu(X)}{v}$ annuli $\{A_k\}$ such that

(i) *$\{2A_k\}$ are disjoint*

(ii) *and $\mu(A_k) \geq v$ for any k.*

Of course, $\mathbb{R}^2$ satisfies all the hypotheses of Theorem 10. Taking $v = 20$ we obtain that if $\mu(\mathbb{R}^2) > 20$ then there exists at least $c'\mu(\mathbb{R}^2)$ annuli satisfying (i) and (ii), which finishes the proof of Theorem 9.

We leave Theorem 10 without proof, only mentioning that it can be regarded as a sophisticated version of the ball covering argument. Note also that annuli in the statement cannot be replaced by balls.

4 Eigenvalues on $\mathbb{S}^2$

Let us show one more application of Theorem 10.

Theorem 11 *Let λ_k, $k = 1, 2, ...,$ be the k-th smallest eigenvalue of the Laplace-Beltrami operator Δ on $(\mathbb{S}^2, g)$, where g is an arbitrary Riemannian metric on $\mathbb{S}^2$. Then, for any k,*

$$\lambda_k \leq C\frac{k-1}{\mu(\mathbb{S}^2)}, \tag{34}$$

where C is a universal constant and μ is the Riemannian volume of the metric g.

In fact, this theorem holds also for any closed Riemann surface, where the constant C depends also on the genus of the surface. However, the general case follows from the estimate for $\mathbb{S}^2$.

Note that $\lambda_1 = 0$ so that the case $k = 1$ is trivial. For $k = 2$ Theorem 11 was proved by Hersch in 1970 for the sphere and then for any Riemann surface by Yang and Yau in 1980. For a general k, Yau stated (34) as a conjecture, which was proved by Korevaar in 1993.

The main point of (34) that the constant C does not depend on the Riemannian metric g. The metric enters (34) only through the total area $\mu(\mathbb{S}^2)$. This is essentially two-dimensional phenomenon as such estimates do not hold in higher dimensions.

Let us show how Theorem 11 can be obtained from Theorem 10. Consider the counting function for Δ on $(\mathbb{S}^2, g)$:

$$\mathcal{N}_\lambda = \#\{j \geq 1 : \lambda_j < \lambda\}.$$

Note that $\lambda_k < \lambda$ will follow from $\mathcal{N}_\lambda \geq k$. We will prove that, for all $\lambda > 0$,

$$\mathcal{N}_\lambda \geq C^{-1}\mu\left(\mathbb{S}^2\right)\lambda. \tag{35}$$

If (35) is already proved, then choosing here $\lambda = C\frac{k}{\mu(\mathbb{S}^2)}$, where $k \geq 2$, we obtain $\mathcal{N}_\lambda \geq k$ and, hence,

$$\lambda_k < \lambda = C\frac{k}{\mu\left(\mathbb{S}^2\right)} \leq 2C\frac{k-1}{\mu\left(\mathbb{S}^2\right)},$$

which proves (34).

Let us prove (35) for any $\lambda > 0$. The counting function admits variational characterization

$$\mathcal{N}_\lambda = \sup\left\{\dim \mathcal{V} : \mathcal{V} \prec D_{\mathbb{S}^2},\ \mathcal{E}(f) < \lambda \|f\|_2^2\ \forall f \in \mathcal{V} \setminus \{0\}\right\}$$

where

$$\mathcal{E}(f) = \int_{\mathbb{S}^2} |\nabla f|_g^2\, d\mu \quad \text{and} \quad \|f\|_2^2 = \int_{\mathbb{S}^2} f^2 d\mu.$$

Hence, it suffices to construct at least $N = C^{-1}\mu\left(\mathbb{S}^2\right)\lambda$ functions f with disjoint supports and with $\mathcal{E}(f) < \lambda \|f\|_2^2$.

If λ is small enough, namely, if $C^{-1}\mu\left(\mathbb{R}^2\right)\lambda \leq 1$ then we need to construct only one function, and it always exists: $f \equiv 1$. Hence, we can assume that $\lambda > \frac{C}{\mu(\mathbb{S}^2)}$.

Any metric g on $\mathbb{S}^2$ is conformally equivalent to the canonical metric g_0 on $\mathbb{S}^2$. Denote by μ_0 the canonical Riemannian measure on $\mathbb{S}^2$. Note that the energy is a conformal invariant:

$$\mathcal{E}(f) = \int_{\mathbb{S}^2} |\nabla f|_g^2\, d\mu = \int_{\mathbb{S}^2} |\nabla f|_{g_0}^2\, d\mu_0.$$

Let d be the geodesic distance on $\left(\mathbb{S}^2, g_0\right)$. As in $\mathbb{R}^2$, one can show that, for any annulus A on $\mathbb{S}^2$ (with respect to d) one can construct a test function f supported in $2A$ and such that $f|_A = 1$ and $\mathcal{E}(f) < K$ where K is some constant. On the other hand,

$$\|f\|_2^2 \geq \int_A f^2 d\mu = \mu(A),$$

so that $\mathcal{E}(f) < \lambda \|f\|_2^2$ will follow from $K \leq \lambda\mu(A)$. Hence, we need to construct at least $N = C^{-1}\mu\left(\mathbb{S}^2\right)\lambda$ annuli A_k on $\mathbb{S}^2$ so that $2A_k$ are disjoint and

$$\mu(A_k) \geq \frac{K}{\lambda}.$$

Let us emphasize that measure μ is defined by the metric g, whereas the annuli are defined using the distance function of g_0.

Let us apply Theorem 10 to the metric space $(\mathbb{S}^2, d)$ with measure μ. Set $v := \frac{K}{\lambda} < C^{-1}K\mu(\mathbb{S}^2)$. Choosing $C > K$, we have $v < \mu(\mathbb{S}^2)$ so that Theorem 10 can be applied. Hence, we obtain at least $c\frac{\mu(\mathbb{S}^2)}{v} = \frac{c}{K}\mu(\mathbb{S}^2)\lambda$ annuli A_k with disjoint $2A_k$ and with

$$\mu(A_k) \geq v = \frac{K}{\lambda},$$

which finishes the proof of (35) with $C = \frac{K}{c}$.

5 Upper estimate in $\mathbb{R}^2$

5.1 Statement of the result

Consider a tiling of $\mathbb{R}^2$ into a sequence of annuli $\{U_n\}_{n\in\mathbb{Z}}$ defined by

$$U_n \overset{n\leq 0}{=} \{e^{-2^{|n|}} < |x| < e^{-2^{|n|-1}}\}, \quad U_0 = \{e^{-1} < |x| < e\}, \quad U_n \overset{n\geq 0}{=} \{e^{2^{n-1}} < |x| < e^{2^n}\}$$

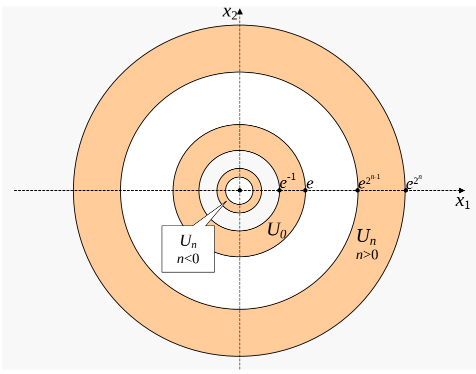

Figure 5:

Given a potential (=a non-negative L^1_{loc}-function) $V(x)$ on $\mathbb{R}^2$ and $p > 1$, define for any $n \in \mathbb{Z}$ the following quantities:

$$A_n = \int_{U_n} V(x)(1 + |\ln|x||)\,dx\,, \qquad B_n = \left(\int_{\{e^n<|x|<e^{n+1}\}} V^p(x)|x|^{2(p-1)}\,dx\right)^{1/p} \tag{36}$$

The main result of this section is the following theorem.

Theorem 12 (AG, N.Nadirashvili, 2012) *For any potential V in $\mathbb{R}^2$ and for any $p > 1$, we have*

$$\operatorname{Neg}(V) \leq 1 + C\sum_{\{n\in\mathbb{Z}:A_n>c\}}\sqrt{A_n} + C\sum_{\{n\in\mathbb{Z}:B_n>c\}}B_n, \tag{37}$$

where C, c are positive constants depending only on p.

The additive term 1 in (37) reflects a special feature of $\mathbb{R}^2$: for any non-zero potential V, there is at least 1 negative eigenvalue of H_V, no matter how small are the sums in (37), as it was shown in the course of the proof of Theorem 9.

Let us compare (37) with previously known upper bounds. A simpler (and coarser) version of (37) is

$$\operatorname{Neg}(V) \leq 1 + C \int_{\mathbb{R}^2} V(x)(1 + |\ln|x||)\, dx + C \sum_{n \in \mathbb{Z}} B_n. \tag{38}$$

Indeed, if $A_n > c$ then $\sqrt{A_n} \leq c^{-1/2} A_n$ so that the first sum in (37) can be replaced by $\sum_{n \in \mathbb{Z}} A_n$ thus yielding (38).

The estimate (38) was obtained by Solomyak in 1994. In fact, he proved a better version:

$$\operatorname{Neg}(V) \leq 1 + C \|A\|_{1,\infty} + C \sum_{n \in \mathbb{Z}} B_n, \tag{39}$$

where A denotes the whole sequence $\{A_n\}_{n \in \mathbb{Z}}$ and $\|A\|_{1,\infty}$ is the weak l^1-norm (the Lorentz norm) given by

$$\|A\|_{1,\infty} = \sup_{s > 0} s \# \{n : A_n > s\}.$$

Clearly, $\|A\|_{1,\infty} \leq \|A\|_1$ so that (39) is better than (38).

However, (39) also follows from (37) using an observation that

$$\|A\|_{1,\infty} \leq \sup_{s > 0} s^{1/2} \sum_{\{A_n > s\}} \sqrt{A_n} \leq 4 \|A\|_{1,\infty}.$$

In particular, we have

$$\sum_{\{A_n > c\}} \sqrt{A_n} \leq 4 c^{-1/2} \|A\|_{1,\infty},$$

so that (37) implies (39). As we will see below, our estimate (37) provides for certain potentials strictly better results than (39).

In the case when $V(x)$ is a radial function, that is, $V(x) = V(|x|)$, the following estimate was proved by physicists Chadan, Khuri, Martin, Wu in 2003:

$$\operatorname{Neg}(V) \leq 1 + \int_{\mathbb{R}^2} V(x)(1 + |\ln|x||)\, dx. \tag{40}$$

Although this estimate is better than (38), we will see that our main estimate (37) gives for certain potentials strictly better estimates than (40).

Another upper estimate for a general potential in $\mathbb{R}^2$ was obtained by Molchanov and Vainberg in 2010:

$$\operatorname{Neg}(V) \leq 1 + C \int_{\mathbb{R}^2} V(x) \ln \langle x \rangle\, dx + C \int_{\mathbb{R}^2} V(x) \ln\left(2 + V(x) \langle x \rangle^2\right) dx, \tag{41}$$

where $\langle x \rangle = e + |x|$. However, due to the logarithmic term in the second integral, this estimate never implies the linear semi-classical asymptotic

$$\operatorname{Neg}(\alpha V) = O(\alpha) \quad \text{as } \alpha \to \infty, \tag{42}$$

that is expected to be true for "nice" potentials. Observe that the Solomyak estimates (38) and (39) are linear in V so that they imply (42) whenever the right hand side is finite.

Our estimate (37) gives both linear asymptotic (42) for "nice" potentials and non-linear asymptotics for some other potentials. Let us emphasize two main novelties in our estimate (37): using the square root of A_n instead of linear expressions, and the restriction of the both sums in (37) to the values $A_n > c$ and $B_n > c$, respectively, which allows to obtain significantly better results.

The reason for the terms $\sqrt{A_n}$ in (37) can be explained as follows. Different parts of the potential V contributes differently to $\operatorname{Neg}(V)$. The high values of V that are concentrated on relatively small areas, contribute to $\operatorname{Neg}(V)$ via the terms B_n, while the low values of V scattered over large areas, contribute via the terms A_n. Since we integrate V over annuli, the long range effect of V becomes similar to that of an one-dimensional potential. In $\mathbb{R}^1$ one expects $\operatorname{Neg}(\alpha V) \simeq \sqrt{\alpha}$ as $\alpha \to \infty$ which explains the appearance of the square root in (37).

By the way, the following estimate of $\operatorname{Neg}(V)$ in $\mathbb{R}^1_+$ was proved by Solomyak:

$$\operatorname{Neg}(V) \leq 1 + C \sum_{n=0}^{\infty} \sqrt{a_n} \tag{43}$$

where

$$a_n = \int_{I_n} V(x)(1 + |x|)\, dx$$

and $I_n = [2^{n-1}, 2^n]$ if $n > 0$ and $I_0 = [0, 1]$. Clearly, the sum $\sum \sqrt{a_n}$ here resembles $\sum \sqrt{A_n}$ in (37), which is not a coincidence. In fact, our method allows to improve (43) by restricting the sum to those n for which $a_n > c$.

Returning to (38), one can apply a suitable Hölder inequality to combine the both terms of (38) in one as follows. Assume that $\mathcal{W}(r)$ is a positive monotone increasing function on $(0, +\infty)$ that satisfies the following Dini type condition both at 0 and at ∞:

$$\int_0^{\infty} \frac{r\,|\ln r|^{\frac{p}{p-1}}\, dr}{\mathcal{W}(r)^{\frac{1}{p-1}}} < \infty. \tag{44}$$

Then

$$\operatorname{Neg}(V) \leq 1 + C \left(\int_{\mathbb{R}^2} V^p(x)\, \mathcal{W}(|x|)\, dx \right)^{1/p}, \tag{45}$$

where the constant C depends on p and $\mathcal{W}$. Here is an example of a weight function $\mathcal{W}(r)$ that satisfies (44):

$$\mathcal{W}(r) = r^{2(p-1)} \langle \ln r \rangle^{2p-1} \ln^{p-1+\varepsilon} \langle \ln r \rangle, \tag{46}$$

where $\varepsilon > 0$. In particular, for $p = 2$, we obtain the following estimate:

$$\operatorname{Neg}(V) \leq 1 + C \left(\int_{\mathbb{R}^2} V^2(x)\, |x|^2 \langle \ln |x| \rangle^3 \ln^{1+\varepsilon} \langle \ln |x| \rangle dx \right)^{1/2}. \tag{47}$$

5.2 Examples

Example 1. Assume that, for all $x \in \mathbb{R}^2$,

$$V(x) \leq \frac{\alpha}{|x|^2}$$

for a small enough positive constant α. Then, for all $n \in \mathbb{Z}$,

$$B_n \leq \alpha \left(\int_{\{e^n < |x| < e^{n+1}\}} \frac{1}{|x|^2} dx \right)^{1/p} \simeq \alpha$$

so that $B_n < c$ and the last sum in (37) is void, whence we obtain

$$\operatorname{Neg}(V) \leq 1 + C \int_{\mathbb{R}^2} V(x) \left(1 + |\ln |x||\right) dx. \tag{48}$$

The estimate (48) in this case follows also from the estimate (41) of Molchanov and Vainberg.

Example 2. Assume that a potential V satisfies the following condition: for some constant K and all $n \in \mathbb{Z}$,

$$\sup_{\{e^n < |x| < e^{n+1}\}} V \leq K \inf_{\{e^n < |x| < e^{n+1}\}} V. \tag{49}$$

For such potential we have

$$B_n \simeq \int_{\{e^n < |x| < e^{n+1}\}} V dx, \tag{50}$$

so that (38) implies

$$\operatorname{Neg}(V) \leq 1 + C \int_{\mathbb{R}^2} V(x) \left(1 + |\ln |x||\right) dx + C' \int_{\mathbb{R}^2} V(x)\, dx,$$

where the constant C' depends also on K. Of course, the second term here can be absorbed by the first one thus yielding (48) with $C = C(K)$.

The estimate (48) in this case can be obtained from the estimate (40) of Chadan, Khuri, Martin, Wu by comparing V with a radial potential.

Example 3. Let

$$V(x) = \frac{\alpha}{|x|^2 \left(1 + \ln^2 |x|\right)},$$

where $\alpha > 0$ is small enough. Then as in the first example $B_n < c$, while A_n can be computed as follows: for $n > 0$

$$A_n \simeq \int_{e^{2^{n-1}}}^{e^{2^n}} \frac{\alpha}{r^2 \ln^2 r} (\ln r)\, r dr = \alpha \int_{e^{2^{n-1}}}^{e^{2^n}} d \ln \ln r \simeq \alpha, \tag{51}$$

and the same for $n \leq 0$, so that $A_n < c$ for all n. Hence, the both sums in (37) are void, and we obtain

$$\operatorname{Neg}(V) = 1.$$

This result cannot be obtained by any of the previously known estimates. Indeed, in the estimates of Chadan, Khuri, Martin, Wu and of Molchanov, Vainberg one has $\int_{\mathbb{R}^2} V(x)(1+|\ln|x||)\,dx = \infty$, and in the estimate (39) of Solomyak one has $\|A\|_{1,\infty} = \infty$. As will be shown below, if $\alpha > 1/4$ then $\mathrm{Neg}(V)$ can be ∞. Hence, $\mathrm{Neg}(V)$ exhibits a non-linear behavior with respect to the parameter α, which cannot be captured by linear estimates.

Example 4. Assume that $V(x)$ is locally bounded and

$$V(x) = o\left(\frac{1}{|x|^2 \ln^2 |x|}\right) \quad \text{as } x \to \infty.$$

Similarly to the above computation we see that $A_n \to 0$ and $B_n \to 0$ as $n \to \infty$, which implies that the both sums in (37) are finite and, hence,

$$\mathrm{Neg}(V) < \infty.$$

This result is also new. Note that in this case the integral $\int_{\mathbb{R}^2} V(x)(1+|\ln|x||)\,dx$ may be divergent; moreover, the norm $\|A\|_{1,\infty}$ can also be ∞ as one can see in the next example.

Example 5. Choose $q > 0$ and set

$$V(x) = \frac{1}{|x|^2 \ln^2 |x| (\ln\ln|x|)^q} \quad \text{for } |x| > e^2 \tag{52}$$

and $V(x) = 0$ for $|x| \le e^2$. For $n \ge 2$ we have

$$A_n \simeq \int_{e^{2^{n-1}}}^{e^{2^n}} \frac{1}{r^2 \ln^2 r (\ln\ln r)^q} (\ln r)\, r dr = \int_{e^{2^{n-1}}}^{e^{2^n}} \frac{d\ln\ln r}{(\ln\ln r)^q} \simeq \frac{1}{n^q},$$

and, by (50),

$$B_n \simeq \int_{e^n}^{e^{n+1}} \frac{1}{r^2 \ln^2 r (\ln\ln r)^q} r dr = \int_{e^n}^{e^{n+1}} \frac{d\ln r}{\ln^2 r (\ln\ln r)^q} \simeq \frac{1}{n^2 \ln^q n}.$$

Let α be a large real parameter. Then

$$A_n(\alpha V) \simeq \frac{\alpha}{n^q}, \tag{53}$$

and the condition $A_n(\alpha V) > c$ is satisfied for $n \le C\alpha^{1/q}$, whence we obtain

$$\sum_{\{A_n(\alpha V) > c\}} \sqrt{A_n(\alpha V)} \le C \sum_{n=1}^{\lceil C\alpha^{1/q} \rceil} \sqrt{\frac{\alpha}{n^q}} \simeq C\sqrt{\alpha}\left(\alpha^{1/q}\right)^{1-q/2} = C\alpha^{1/q}.$$

It is clear that $\sum_n B_n(\alpha V) \simeq \alpha$. Hence, we obtain from (37)

$$\mathrm{Neg}(\alpha V) \le C\left(\alpha^{1/q} + \alpha\right). \tag{54}$$

If $q \geq 1$ then the leading term here is α. Combining this with (33), we obtain

$$\operatorname{Neg}(\alpha V) \simeq \alpha \quad \text{as} \quad \alpha \to \infty.$$

If $q > 1$ then this follows also from (40) and (39); if $q = 1$ then only the estimate (39) of Solomyak gives the same result as in this case $A_n \simeq \frac{1}{n}$ and $\|A\|_{1,\infty} < \infty$.

If $q < 1$ then the leading term in (54) is $\alpha^{1/q}$ so that

$$\operatorname{Neg}(\alpha V) \leq C\alpha^{1/q}.$$

As was shown by Birman and Laptev, in this case, indeed, $\operatorname{Neg}(\alpha V) \simeq \alpha^{1/q}$ as $\alpha \to \infty$. Observe that in this case $\|A\|_{1,\infty} = \infty$, and neither of the estimates previous estimates (38), (40), (39), (41) yields even the finiteness of $\operatorname{Neg}(\alpha V)$, leaving alone the correct rate of growth in α.

Example 6. Let V be a potential in $\mathbb{R}^2$ such that

$$\sum_{n\in\mathbb{Z}} \sqrt{A_n(V)} + \sum_{n\in\mathbb{Z}} B_n(V) < \infty. \tag{55}$$

Applying (37) to αV, we obtain

$$\operatorname{Neg}(\alpha V) \leq 1 + C\alpha^{1/2} \sum_{n\in\mathbb{Z}} \sqrt{A_n(V)} + \alpha \sum_{n\in\mathbb{Z}} B_n(V).$$

Combining with the lower bound (33) and letting $\alpha \to \infty$, we see that

$$c\alpha \int_{\mathbb{R}^2} V dx \leq \operatorname{Neg}(\alpha V) \leq \alpha \sum_{n\in\mathbb{Z}} B_n(V) + o(\alpha),$$

in particular,

$$\operatorname{Neg}(\alpha V) \simeq \alpha \quad \text{as } \alpha \to \infty.$$

Furthermore, if V satisfies the condition (49) then, using (50), we obtain a more precise estimate

$$\operatorname{Neg}(\alpha V) \simeq \alpha \int_{\mathbb{R}^2} V(x)\, dx \quad \text{as } \alpha \to \infty. \tag{56}$$

For example, (55) is satisfied for the potential (52) of Example 5 with $q > 2$, as it follows from (53). By a more sophisticated argument, one can show that (56) holds also for $q > 1$.

Example 7. Set $R = e^{2^m}$ where m is a large integer, choose $\alpha > \frac{1}{4}$ and consider the following potential on $\mathbb{R}^2$

$$V(x) = \frac{\alpha}{|x|^2 \ln^2 |x|} \quad \text{if } \ e < |x| < R$$

and $V(x) = 0$ otherwise. Computing A_n as in (51) we obtain $A_n \simeq \alpha$ for any $1 \leq n \leq m$, whence it follows that

$$\sum_{n\in\mathbb{Z}} \sqrt{A_n} = \sum_{n=1}^{m} \sqrt{A_n} \simeq \sqrt{\alpha}m \simeq \sqrt{\alpha} \ln \ln R.$$

Also, we obtain by (50) $B_n \simeq \frac{a}{n^2}$, for $1 \leq n < 2^m$, whence

$$\sum_{n\in\mathbb{Z}} B_n(V) \simeq \sum_{n=1}^{2^m-1} \frac{\alpha}{n^2} \simeq \alpha.$$

By (37) we obtain

$$\operatorname{Neg}(V) \leq C\sqrt{\alpha} \ln\ln R + C\alpha. \tag{57}$$

Observe that both (39) and (40) give in this case a weaker estimate

$$\operatorname{Neg}(V) \leq C\alpha \ln\ln R.$$

Let us estimate $\operatorname{Neg}(V)$ from below. Considering the function

$$f(x) = \sqrt{\ln|x|} \sin\left(\sqrt{\alpha - \frac{1}{4}} \ln\ln|x|\right)$$

that satisfies in the region $\Omega = \{e < |x| < R\}$ the differential equation $\Delta f + V(x) f = 0$, and counting the number N of rings

$$\Omega_k := \left\{x \in \mathbb{R}^2 : \pi k < \sqrt{\alpha - \frac{1}{4}} \ln\ln|x| < \pi(k+1)\right\}$$

in Ω, we obtain

$$\operatorname{Neg}(V) \geq N \simeq \sqrt{\alpha} \ln\ln R$$

(assuming that $\alpha >> \frac{1}{4}$). On the other hand, (33) yields $\operatorname{Neg}(V) \geq c\alpha$. Combining these two estimates, we obtain the lower bound

$$\operatorname{Neg}(V) \geq c\left(\sqrt{\alpha} \ln\ln R + \alpha\right),$$

that matches the upper bound (57).

5.3 The energy form revisited

We consider a somewhat different energy form than in $\mathbb{R}^n$, $n \geq 3$. For any open set $\Omega \subset \mathbb{R}^2$, consider a function space

$$\mathcal{F}_{V,\Omega} = \left\{f \in L^2_{loc}\left(\overline{\Omega}\right) : \int_\Omega |\nabla f|^2 dx < \infty, \int_\Omega V f^2 dx < \infty\right\}$$

and the quadratic form on $\mathcal{F}_{V,\Omega}$:

$$\mathcal{E}_{V,\Omega}(f) = \int_\Omega |\nabla f|^2 dx - \int_\Omega V f^2 dx. \tag{58}$$

We will use the following quantity:

$$\operatorname{Neg}(V,\Omega) := \sup\left\{\dim \mathcal{V} : \mathcal{V} \prec \mathcal{F}_{V,\Omega} : \mathcal{E}_{V,\Omega}(f) \leq 0 \text{ for all } f \in \mathcal{V}\right\}. \tag{59}$$

Clearly, we have $\mathcal{N}_0(H_V) \leq \text{Neg}(V, \mathbb{R}^2)$, but in $\mathbb{R}^2$ we do not loose much when we estimate a larger quantity Neg instead of $\mathcal{N}_0$. (Observe that $\mathcal{F}_{V,\mathbb{R}^2}$ contains $f = \text{const}$ and $\mathcal{E}(f) \leq 0$ so that $\text{Neg}(V, \mathbb{R}^2) \geq 1$, but as we know, $\mathcal{N}_0(H_V) \geq 1$). Theorem 12 contains the estimate of $\text{Neg}(V) = \text{Neg}(V, \mathbb{R}^2)$.

For bounded domains with smooth boundary, $\text{Neg}(V, \Omega)$ is equal to the number of non-positive eigenvalues of the *Neumann* problem in Ω for $-\Delta - V$.

A useful feature of $\text{Neg}(V, \Omega)$ is subadditivity with respect to Ω. We say that a sequence $\{\Omega_k\}$ of open sets $\Omega_k \subset \mathbb{R}^2$ is a *partition* of Ω if all the sets Ω_k are disjoint, $\Omega_k \subset \Omega$, and $\overline{\Omega} \setminus \bigcup_k \Omega_k$ has measure 0.

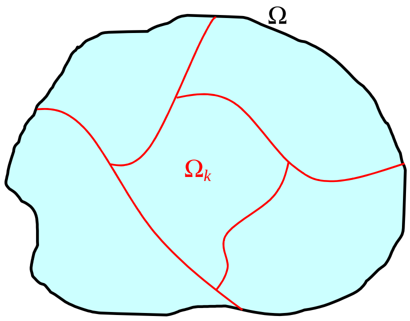

Figure 6:

Lemma 13 *If $\{\Omega_k\}$ is a partition of Ω, then*

$$\text{Neg}(V, \Omega) \leq \sum_k \text{Neg}(V, \Omega_k). \tag{60}$$

The idea of the proof is the same as in the classical Weyl's argument: adding additional Neumann boundaries inside Ω increases the space of test functions and, hence, the number of non-negative eigenvalues.

5.4 One negative eigenvalue in a disc

Denote by D_r the open disk of radius r in $\mathbb{R}^2$, that is, $D_r = \{x \in \mathbb{R}^2 : |x| < r\}$, and set $D_1 \equiv D$.

Lemma 14 *For any $p > 1$ there is $\varepsilon > 0$ such that, for any potential V in D,*

$$\|V\|_{L^p(D)} \leq \varepsilon \Rightarrow \text{Neg}(V, D) = 1.$$

Sketch of proof. Since always $\text{Neg}(V, D) \geq 1$, we need only to prove that $\text{Neg}(V, D) \leq 1$. We will prove that if $u \in \mathcal{F}_{V,D}$ then

$$u \perp 1 \text{ in } L^2(D) \quad \text{and} \quad \mathcal{E}_{V,D}(u) \leq 0 \quad \Rightarrow \quad u = 0,$$

which will imply that $\text{Neg}(V, D) \leq 1$.

Extend $u \in \mathcal{F}_{V,D}$ to $\mathbb{R}^2$ using the inversion $\Phi(x) = \frac{x}{|x|^2}$: for any $x \notin D$, set $u(x) = u(\Phi(x))$. By conformal invariance of energy, we have

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx = 2\int_D |\nabla u|^2 dx \le 2\int_D V u^2 dx.$$

Choose a cutoff function φ such that $\varphi|_{D_2} \equiv 1$, $\varphi|_{\mathbb{R}^2\setminus D_3} = 0$ and set $u^* = u\varphi$. Then it follows that

$$\int_{D_4} |\nabla u^*|^2 dx \le C\int_D V u^2 dx,$$

with some absolute constant C. Since $u\perp 1$, one uses in the proof the Poincaré inequality in D in the form $\|u\|_{L^2} \le C\|\nabla u\|_{L^2}$.

Next, we have by Hölder inequality

$$\int_D V u^2 dx \le \left(\int_D V^p dx\right)^{1/p}\left(\int_D |u|^{\frac{2p}{p-1}} dx\right)^{1-1/p},$$

and by Sobolev inequality

$$\left(\int_D |u|^{\frac{2p}{p-1}} dx\right)^{1-1/p} \le \left(\int_{D_4} |u^*|^{\frac{2p}{p-1}} dx\right)^{1-1/p} \le C\int_{D_4} |\nabla u^*|^2 dx.$$

Combining the above three lines, we obtain

$$\int_{D_4} |\nabla u^*|^2 dx \le C\left(\int_D V^p dx\right)^{1/p}\int_{D_4} |\nabla u^*|^2 dx. \tag{61}$$

Assuming that $\|V\|_{L^p(D)}$ is small enough, we see that (61) is only possible if $u^* = \text{const}$. Since $u\perp 1$ in $L^2(D)$, it follows that $u \equiv 0$. ■

Corollary 15 *Let Ω be a domain in $\mathbb{R}^2$ that is bilipschitz equivalent to D_r. Then*

$$\int_\Omega V^p dx \le c r^{2-2p} \Rightarrow \operatorname{Neg}(V,\Omega) = 1. \tag{62}$$

where $c > 0$ depends on p and on the Lipschitz constant of the mapping between D_r and Ω.

Proof. Indeed, if $\Omega = D_r$ then (62) follows from Lemma 14 by scaling transformation. For a general Ω one shows that $\operatorname{Neg}(V,\Omega) \le \operatorname{Neg}(CV^*, D_r)$ where V^* is the pull-back of V under the bilipschitz mapping $L : D_r \to \Omega$ where the constant C depends on the Lipschitz constant. ■

5.5 Negative eigenvalues in a square

Denote by Q the unit square in $\mathbb{R}^2$.

Lemma 16 *For any $p > 1$ and for any potential V in Q,*

$$\operatorname{Neg}(V, Q) \leq 1 + C \|V\|_{L^p(Q)}, \tag{63}$$

where C depends only on p.

Proof. It suffices to construct a partition $\mathcal{P}$ of Q into a family of N disjoint subsets such that

1. $\operatorname{Neg}(V, \Omega) = 1$ for any $\Omega \in \mathcal{P}$;
2. $N \leq 1 + C \|V\|_{L^p(Q)}$.

Indeed, if such a partition exists then we obtain by Lemma 13

$$\operatorname{Neg}(V, Q) \leq \sum_{\Omega \in \mathcal{P}} \operatorname{Neg}(V, \Omega) = N, \tag{64}$$

and (63) follows from the above bound of N.

The elements of a partition will be of two shapes: it is either a square of the side length $0 < l \leq 1$ or a *step*, that is, a set of the form $\Omega = A \setminus B$ where A is a square of the side length l, and B is a square of the side length $\leq l/2$ that is attached to one of corners of A.

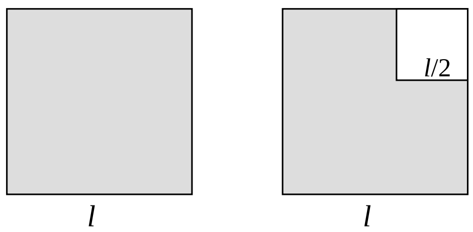

Figure 7: A square and a step of size l

In the both cases we refer to l as the size of Ω. By Corollary 15, the condition 1 for such a set Ω will follow from

$$\int_\Omega V^p dx \leq c l^{2-2p}. \tag{65}$$

Apart from the shape, we will distinguish also the *type* of a set $\Omega \in \mathcal{P}$ of size l as follows: we say that

- Ω is of a large type, if
$$\int_\Omega V^p dx > c l^{2-2p},$$

- Ω is of a medium type if

$$c' l^{2-2p} < \int_\Omega V^p dx \le c l^{2-2p}, \tag{66}$$

- and Ω is of small type if

$$\int_\Omega V^p dx \le c' l^{2-2p}. \tag{67}$$

Here c is the constant from (65) and $c' \in (0, c)$ will be chosen below. In particular, if Ω is of small or medium type then $\mathrm{Neg}\,(V, \Omega) = 1$.

The construction of the partition $\mathcal{P}$ will be done by induction. At each step $i \ge 1$ of induction we will have a partition $\mathcal{P}^{(i)}$ of Q such that

1. each $\Omega \in \mathcal{P}^{(i)}$ is either a square or a step;
2. If $\Omega \in \mathcal{P}^{(i)}$ is a step then Ω is of a medium type.

At step 1 we have just one set: $\mathcal{P}^{(1)} = \{Q\}$. At any step $i \ge 1$, partition $\mathcal{P}^{(i+1)}$ is obtained from $\mathcal{P}^{(i)}$ as follows. If $\Omega \in \mathcal{P}^{(i)}$ is small or medium then Ω becomes one of the elements of the partition $\mathcal{P}^{(i+1)}$. If $\Omega \in \mathcal{P}^{(i)}$ is large, then it is a square, and it will be further partitioned into a few sets that will become elements of $\mathcal{P}^{(i+1)}$. Denoting by l the side length of the square Ω, let us first split Ω into four equal squares $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ of side length $l/2$ and consider the following cases.

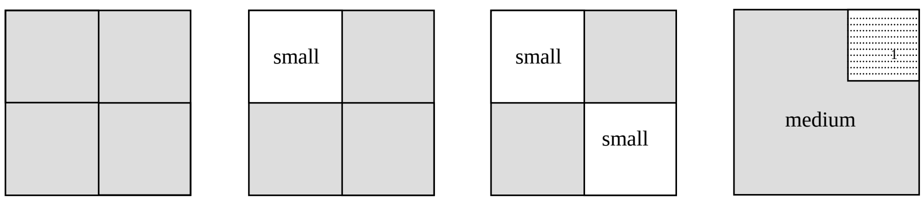

Figure 8: Various possibilities of partitioning of a square Ω (the shaded shapes are of medium or large type, the hatched shape Ω_1 can be of any type)

Case 1. If among $\Omega_1, ..., \Omega_4$ the number of small squares is at most 2, then all sets $\Omega_1, ..., \Omega_4$ become elements of $\mathcal{P}^{(i+1)}$.

Case 2. If among $\Omega_1, ..., \Omega_4$ there are exactly 3 small squares, say, $\Omega_2, \Omega_3, \Omega_4$, then we have

$$\int_{\Omega \setminus \Omega_1} V^p dx = \int_{\Omega_2 \cup \Omega_3 \cup \Omega_4} V^p dx \le 3c' \left(\frac{l}{2}\right)^{2-2p} = 3c' 2^{2p-2} l^{2-2p} < c l^{2-2p},$$

where we choose c' to satisfy $3c' 2^{2p-2} < c$. On the other hand, we have

$$\int_\Omega V^p dx > c l^{2-2p}.$$

Therefore, by reducing the size of Ω_1 (but keeping Ω_1 attached to the corner of Ω) one can achieve the equality

$$\int_{\Omega\setminus\Omega_1} V^p dx = cl^{2-2p}.$$

Hence, we obtain a partition of Ω into two sets Ω_1 and $\Omega\setminus\overline{\Omega_1}$, where the step $\Omega\setminus\overline{\Omega_1}$ is of medium type, while the square Ω_1 can be of any type. Both Ω_1 and $\Omega\setminus\overline{\Omega_1}$ become elements of $\mathcal{P}^{(i+1)}$.

Case 3. Let us show that all 4 squares $\Omega_1, ..., \Omega_4$ cannot be small. Indeed, in this case we would have

$$\int_\Omega V^p dx = \sum_{k=1}^{4} \int_{\Omega_k} V^p dx \le 4c' \left(\frac{l}{2}\right)^{2-2p} = \left(4c' 2^{2p-2}\right) l^{2-2p}.$$

Let us choose c' so small that $4c'2^{2p-2} < c$. Then the above estimate contradicts the assumption that Ω is of large type.

As we see from construction, at each step i only large squares get partitioned further, and the size of the large type squares in $\mathcal{P}^{(i+1)}$ reduces at least by a factor 2. If the size of a square is small enough then it is necessarily of small type, because the right hand side of (67) goes to ∞ as $l \to 0$. Hence, the process will stop after finitely many steps. After sufficiently many steps we obtain a partition $\mathcal{P}$ where all the elements are either of small or medium types. In particular, we have $\operatorname{Neg}(V, \Omega) = 1$ for any $\Omega \in \mathcal{P}$.

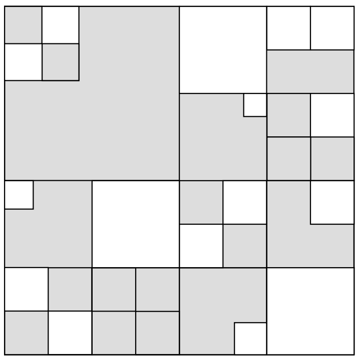

Figure 9: An example of a final partition $\mathcal{P}$. The shaded shapes are of medium type, the white squares are of small type.

Let N be a number of elements of $\mathcal{P}$. We need to show that

$$N \le 1 + C \|V\|_{L^p(Q)}. \tag{68}$$

At each step of construction, denote by L the number of large elements, by M the number of medium elements, and by S the number of small elements. Let us show that the quantity $2L+3M-S$ is non-decreasing during the construction. Indeed, at each step we split one large square Ω, so that by removing this square, L decreases by 1. However, we add new elements of partitions, which contribute to the quantity $2L+3M-S$ as follows.

1. If Ω is split into $s \le 2$ small and $4-s$ medium/large squares as in Case 1, then the value of $2L+3M-S$ has the increment at least
$$-2+2(4-s)-s = 6-3s \ge 0.$$

2. If Ω is split into 1 square and 1 step as in Case 2, then one obtains at least 1 medium set and at most 1 small, so that $2L+3M-S$ has the increment at least
$$-2+3-1=0.$$

(Luckily, Case 3 cannot occur. In that case, we would have 4 new small squares so that L and M would not have increased, whereas S would have increased at least by 3, so that no quantity of the type C_1L+C_2M-S would have been monotone increasing).

Since for the partition $\mathcal{P}^{(1)}$ we have $2L+3M-S \ge -1$, this inequality will remain true at all steps of construction and, in particular, it is satisfied for the final partition $\mathcal{P}$. For the final partition we have $L=0$, whence it follows that $S \le 1+3M$ and, hence,
$$N = S+M \le 1+4M. \tag{69}$$

Let us estimate M. Let $\Omega_1, ..., \Omega_M$ be the medium type elements of $\mathcal{P}$ and let l_k be the size of Ω_k. Each Ω_k contains a square $\Omega'_k \subset \Omega_k$ of the size $l_k/2$, and all the squares $\{\Omega'_k\}_{k=1}^M$ are disjoint, which implies that
$$\sum_{k=1}^M l_k^2 \le 4. \tag{70}$$

Using the Hölder inequality and (70), we obtain
$$M = \sum_{k=1}^M l_k^{\frac{2}{p'}} l_k^{-\frac{2}{p'}} \le \left(\sum_{k=1}^M l_k^2\right)^{1/p'} \left(\sum_{k=1}^M l_k^{-\frac{2p}{p'}}\right)^{1/p} \le 4^{1/p'} \left(\sum_{k=1}^M l_k^{2-2p}\right)^{1/p}.$$

Since by (66) $c' l_k^{2-2p} < \int_{\Omega_k} V^p dx$, it follows that
$$M \le C\left(\sum_{k=1}^M \int_{\Omega_k} V^p dx\right)^{1/p} \le C\left(\int_Q V^p dx\right)^{1/p}.$$

Combining this with $N \le 1+4M$, we obtain $N \le 1 + C\,\|V\|_{L^p(Q)}$, thus finishing the proof. ■

Corollary 17 *Let Ω be a domain in $\mathbb{R}^2$ that is bilipschitz equivalent to D. Then*

$$\operatorname{Neg}(V,\Omega) \leq 1 + C\left(\int_{\Omega} V^p dx\right)^{1/p},$$

where $C > 0$ depends on p and on the Lipschitz constant of the mapping between D and Ω.

5.6 One negative eigenvalue in $\mathbb{R}^2$

Now we would like to obtain conditions for $\operatorname{Neg}(V,\mathbb{R}^2) = 1$ in terms of some weighted L^1-norms. The method that we have used in the case $n \geq 3$ (Lemma 3) was based on the operator $\mathcal{L}_V = -\frac{1}{V}\Delta$ and estimating of $\left\|\mathcal{L}_V^{-1}\right\|$ in $L^2(\mathbb{R}^n, Vdx)$.

The hidden reason why it was possible is the existence of the positive Green function $g(x,y) = \frac{c_n}{|x-y|^{n-2}}$ of $-\Delta$. In fact, the operator $\mathcal{L}_V^{-1}$ is given by

$$\mathcal{L}_V^{-1} f = \int_{\mathbb{R}^n} g(x,y) f(y) V(y) dy.$$

The application of the Sobolev in the proof of Lemma 3 can be replaced by a direct estimate of the norm of this integral operator in $L^2(\mathbb{R}^n, Vdx)$. In fact, the classical proof of the Sobolev inequality uses this approach.

One of the difficulties in $\mathbb{R}^2$ is the absence of a positive Green function of the Laplace operator. To overcome this difficulty, we introduce an auxiliary potential $V_0 \in C_0^\infty(\mathbb{R}^2)$, such that $V_0 \not\equiv 0$ and $V_0 \geq 0$.

Lemma 18 (AG, 2006) *Operator $H_0 = -\Delta + V_0$ has a positive Green function $g(x,y)$ that admits the following estimate*

$$g(x,y) \simeq \ln\langle x\rangle \wedge \ln\langle y\rangle + \ln_+ \frac{1}{|x-y|}, \tag{71}$$

where $\langle x\rangle := e + |x|$ and $\wedge$ means $\min$.

By Lemma 14 there exists V_0 such that $\operatorname{Neg}(V_0,\mathbb{R}^2) = 1$. Fix such V_0 and, hence, the Green function $g(x,y)$ of H_0 for what follows.

For a given potential V, define as measure μ by $d\mu = Vdx$ and consider the integral operator G_V defined by

$$G_V f(x) = \int_{\mathbb{R}^2} g(x,y) f(y) d\mu(y).$$

Denote by $\|G_V\|$ the norm of G_V in the space $L^2(\mathbb{R}^2,\mu)$.

Lemma 19 *If $\|G_V\| \leq \frac{1}{2}$ then* $\operatorname{Neg}(V,\mathbb{R}^2) = 1$.

Sketch of the proof. The idea is that the operator G_V is the inverse of the operator $\frac{1}{V}H_0$ in $L^2(\mu)$ so that $\|G_V\| \leq \frac{1}{2}$ implies that the spectrum of $\frac{1}{V}H_0$ is confined in $[2,\infty)$. This implies that $H_0 \geq 2V$ in the sense of quadratic forms, that is,

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} V_0 u^2 dx \geq 2\int_{\mathbb{R}^2} V u^2 dx$$

for all $u \in \mathcal{F}_V$. If $\mathcal{V}$ is a subspace of $\mathcal{F}_V$ where $\mathcal{E}_V \leq 0$ then for any $u \in \mathcal{V}$

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx \leq \int_{\mathbb{R}^2} V u^2 dx.$$

Combining the two lines, we obtain

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx \leq \int_{\mathbb{R}^2} V_0 u^2 dx,$$

that is, $\mathcal{E}_{V_0}(u) \leq 0$. Taking $\sup \dim \mathcal{V}$ we obtain

$$\operatorname{Neg}\left(V,\mathbb{R}^2\right) \leq \operatorname{Neg}\left(V_0,\mathbb{R}^2\right) = 1.$$

■

The next step is estimating the norm $\|G_V\|$ in terms of V. Since $g(x,y)$ is symmetric in x,y, we have a simple estimate

$$\|G_V\| \leq \sup_y \int_{\mathbb{R}^2} g(x,y)\, d\mu(x),$$

which together with Lemma 18 leads to

$$\|G_V\| \leq C\int_{\mathbb{R}^2} \ln\langle x\rangle\, d\mu(x) + C\sup_{y\in\mathbb{R}^2}\int_{\mathbb{R}^2} \ln_+ \frac{1}{|x-y|}\, d\mu(x).$$

However, $\|G_V\|$ admits a better estimate, as will be explained below.

5.7 Transformation to a strip

It will be more convenient to estimate first $\operatorname{Neg}(V,S)$ where S is a strip in $\mathbb{R}^2$ defined by

$$S = \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R},\ 0 < x_2 < \pi\right\}.$$

The strip S is the image of $\mathbb{R}^2_+$ under the conformal mapping $z \mapsto \ln z$:

Let $\gamma(x,y)$ be the push-forward of the Green function $g(x,y)$ under this mapping, that is,

$$\gamma(x,y) = g\left(e^x, e^y\right).$$

Using the estimate (72) of g, it is possible to show that

$$\gamma(x,y) \leq C\langle x_1\rangle \wedge \langle y_1\rangle + C\ln_+ \frac{1}{|x-y|}. \tag{72}$$

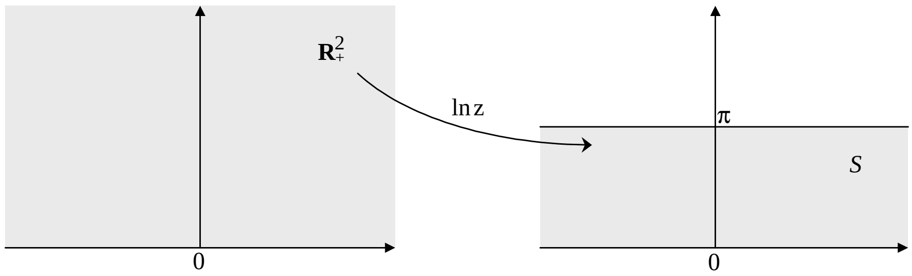

Figure 10:

For example, x_1 arises from $\ln|e^x| = \ln|e^{x_1+ix_2}| = \ln e^{x_1} = x_1$.

Consider also the corresponding integral operator

$$\Gamma_V f(x) = \int_S \gamma(x,y) f(y)\, d\mu(y), \tag{73}$$

where measure μ is defined as above by $d\mu = V(x)\,dx$. Denote by $\|\Gamma_V\|$ the norm of Γ_V in $L^2(S,\mu)$. Lemma 19 implies the following.

Lemma 20 $\|\Gamma_V\| \le \frac{1}{8} \Rightarrow \mathrm{Neg}(V,S) = 1.$

The main point in the proof is that the holomorphic mappings are conformal and, hence, preserve the Dirichlet integral.

5.8 Estimating $\|\Gamma_V\|$

For any $n \in \mathbb{Z}$ set

$Q_n = S \cap \{n < x_1 < n+1\}$,
$S_n = S \cap \left\{-2^{|n|} < x_1 < -2^{|n|-1}\right\}$ for $n < 0$,
$S_0 = S \cap \{-1 < x_1 < 1\}$,
$S_n = S \cap \{2^{n-1} < x_1 < 2^n\}$ for $n > 0$,

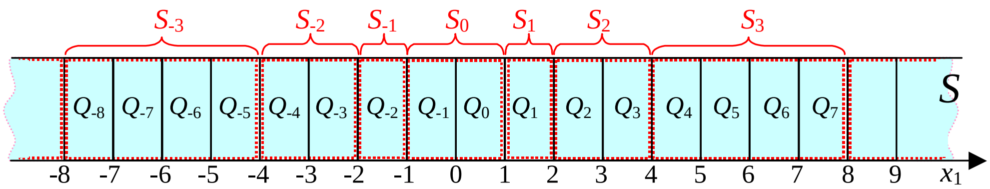

Figure 11:

$$a_n(V) = \int_{S_n} (1+|x_1|)\, V(x)\, dx \simeq 2^{|n|} \int_{S_n} V(x)\, dx \tag{74}$$

$$b_n(V) = \left(\int_{Q_n} V^p(x)\, dx\right)^{1/p}. \tag{75}$$

Lemma 21 *The operator Γ_V admits the following norm estimate in $L^2(S,\mu)$:*

$$\|\Gamma_V\| \le C \sup_{n\in\mathbb{Z}} a_n(V) + C \sup_{n\in Z} b_n(V). \tag{76}$$

Approach to the proof. Note that by (72)

$$\begin{aligned} |\Gamma_V f(x)| \le\ & C\int_S (1+|x_1|\wedge|y_1|)\,|f(y)|\,V(y)\,dy \\ & + C\int_S \ln_+ \frac{1}{|x-y|} f(y)\,|V(y)|\,dy. \end{aligned} \tag{77}$$

The second integral operator can be estimated by the Hölder inequality:

$$\int_S \ln_+ \frac{1}{|x-y|} V(y)\,dy \le \left(\int_{B(x,1)} \left(\ln_+ \frac{1}{|x-y|}\right)^{p'} dy\right)^{1/p'} \left(\int_{B(x,1)\cap S} V^p(y)\,dy\right)^{1/p}.$$

The first integral here is equal to a finite constant depending only on p, but independent of x. The second integral is bounded by $C\sup_n b_n(V)$.

It is much more subtle to estimate the norm of the first integral operator in (77) via $C\sup_{n\in\mathbb{Z}} a_n(V)$. This problem is reduced to an one dimensional problem by integrating in the direction x_2. Then we apply a certain weighted Hardy inequality. We skip the details as the argument is quite lengthy. ■

Corollary 22 *There is a constant $c>0$ such that*

$$\sup_n a_n(V) \le c \quad \text{and} \quad \sup_n b_n(V) \le c \quad \Rightarrow \quad \operatorname{Neg}(V,S) = 1.$$

Proof. Assuming that the constant c here is small enough, we obtain from (76) that $\|\Gamma_V\| \le \frac{1}{8}$, whence by Lemma 20 $\operatorname{Neg}(V,S)=1$. ■

5.9 Rectangles

For all $\alpha \in [-\infty,+\infty)$, $\beta \in (-\infty,+\infty]$ such that $\alpha<\beta$, denote by $P_{\alpha,\beta}$ the rectangle

$$P_{\alpha,\beta} = \left\{(x_1,x_2)\in\mathbb{R}^2 : \alpha < x_1 < \beta, \quad 0 < x_2 < \pi\right\}.$$

Note that $P_{\alpha,\beta} \subset S$.

Lemma 23 *For any potential V in a rectangle $P_{\alpha,\beta}$ with the length $\beta-\alpha\ge 1$, we have*

$$\operatorname{Neg}(V, P_{\alpha,\beta}) \le \operatorname{Neg}(17V, S),$$

where V is extended to S by setting $V=0$ outside $P_{\alpha,\beta}$.

Sketch of the proof. It suffices to show that any function $u \in \mathcal{F}_{V,P}$ can be extended to $\mathcal{F}_{V,S}$ so that

$$\int_S |\nabla u|^2 \, dx \leq 17 \int_P |\nabla u|^2 \, dx. \tag{78}$$

Attach to P from each side one rectangle, say P' from the left and P'' from the right, each having the length $4(\beta - \alpha)$ (to ensure that the latter is $> \pi$). Extend function u to P' by applying four times symmetries in the vertical sides, so that

$$\int_{P'} |\nabla u|^2 \, dx = 4 \int_P |\nabla u|^2 \, dx.$$

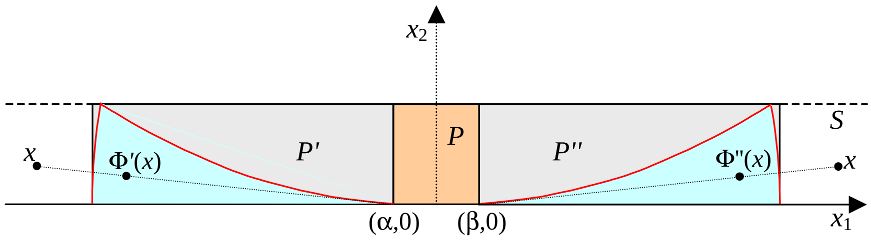

Figure 12: Extension of function u from P to S.

Then slightly reduce P' by taking intersections with the circle of radii $\beta - \alpha$ centered at $(\alpha, 0)$. Now we extend u from P' to the left by using the inversion Φ' at the point $(\alpha, 0)$ in the aforementioned circle. By the conformal invariance of the Dirichlet integral, we have

$$\int_{S \cap \{x_1 < \alpha\}} |\nabla u|^2 \leq 8 \int_P |\nabla u|^2 \, dx.$$

Extending u in the same way to the right of P, we obtain (78). ■

5.10 Sparse potentials

We say that a potential V in S is *sparse* if

$$\sup_n b_n(V) < c_0,$$

where c_0 is a small enough positive constant, depending only on p. It follows from Corollary 22 that, for a sparse potential,

$$\sup_n a_n(V) \leq c \ \Rightarrow \operatorname{Neg}(V, S) = 1.$$

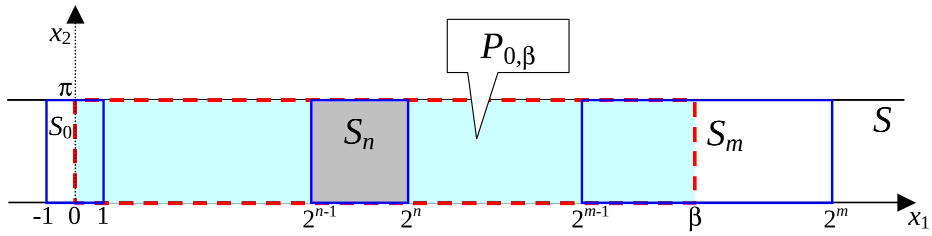

Figure 13: Rectangle $P_{0,\beta}$ is covered by the sequence $S_n, 0 \leq n \leq m$

Corollary 24 *Let V be a sparse potential in $P_{\alpha,\beta}$ where $\beta - \alpha \geq 1$. Then*

$$(\beta - \alpha) \int_{P_{\alpha,\beta}} V(x)\, dx \leq c \quad \Rightarrow \quad \mathrm{Neg}(V, P_{\alpha,\beta}) = 1. \tag{79}$$

Proof. Take $\alpha = 0$ so that $\beta \geq 1$. Let m be a non-negative integer such that $2^{m-1} < \beta \leq 2^m$.

Then $a_n(V) = 0$ for $n < 0$ and for $n \geq m+1$. For $0 \leq n \leq m$

$$a_n(V) \leq 2^{n+1} \int_{S_n} V(x)\, dx \leq 2^{m+1} \int_{P_{0,\beta}} V(x)\, dx \leq 4\beta \int_{P_{0,\beta}} V(x)\, dx, \tag{80}$$

so that $a_n(17V)$ are small enough for all $n \in \mathbb{Z}$. By Corollary 22 $\mathrm{Neg}(17V, S) = 1$, and by Lemma 23 $\mathrm{Neg}(V, P_{0,\beta}) = 1$. ■

Lemma 25 *Let V be a sparse potential in $P_{\alpha,\beta}$ where $\beta - \alpha \geq 1$. Then*

$$\mathrm{Neg}(V, P_{\alpha,\beta}) \leq 1 + C\left((\beta - \alpha) \int_{P_{\alpha,\beta}} V(x)\, dx\right)^{1/2}. \tag{81}$$

In particular, for a sparse potential in S_n,

$$\mathrm{Neg}(V, S_n) \leq 1 + C\sqrt{a_n(V)}. \tag{82}$$

Proof. Without loss of generality set $\alpha = 0$. Set also

$$J = \int_{P_{0,\beta}} V(x)\, dx$$

and recall that, by Corollary 24, if $\beta J \leq c$ for sufficiently small c then $\mathrm{Neg}(V, P_{0,\beta}) = 1$. Hence, in this case (81) is trivially satisfied, and we assume in the sequel that $\beta J > c$.

Due to Lemma 23, it suffices to prove that

$$\mathrm{Neg}(V, S) \leq C(\beta J)^{1/2}.$$

Consider a sequence of reals $\{r_k\}_{k=0}^N$ such that

$$0 = r_0 < r_1 < ... < r_{N-1} < \beta \leq r_N$$

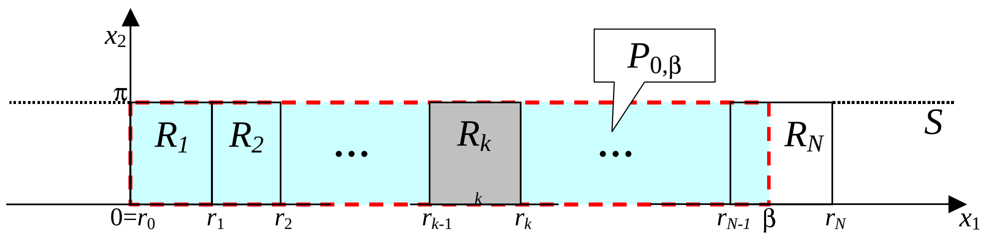

Figure 14: The sequence $\{R_k\}_{k=1}^N$ of rectangles covering $P_{0,\beta}$

and the corresponding sequence of rectangles

$$R_k := P_{r_{k-1},r_k} = \{(x_1, x_2) : r_{k-1} < x_1 < r_k, \quad 0 < x_2 < \pi\}$$

where $k = 1, ..., N$, that covers $P_{0,\beta}$.

Denote $l_k = r_k - r_{k-1}$ and $J_k = \int_{R_k} V(x)\, dx$. By Corollary 24,

$$l_k \geq 1 \text{ and } \ l_k J_k \leq c \quad \Rightarrow \ \operatorname{Neg}(V, R_k) = 1 \tag{83}$$

Let us construct the sequence $\{r_k\}_{k=0}^N$ to satisfy (83) for all $k = 1, ..., N$. If r_{k-1} is already defined and $r_{k-1} < \beta$ then choose $r_k > r_{k-1}$ to satisfy the identity

$$l_k J_k = c. \tag{84}$$

If such r_k does not exist then set $r_k = \beta + 1$; in this case, we have $l_k J_k < c$. Let us show that in the both cases $l_k = r_k - r_{k-1} \geq 1$. Indeed, if $l_k < 1$ then $r_k < \beta + 1$ so that (84) is satisfied. By Hölder inequality, (84) and $l_k < 1$, we obtain

$$\left(\int_{R_k} V^p dx\right)^{1/p} \geq \frac{1}{(\pi l_k)^{1/p'}} \int_{R_k} V dx = \frac{c}{(\pi l_k)^{1/p'}\, l_k} \geq \frac{c}{\pi^{1/p'}},$$

which contradicts the assumption that V is sparse. Hence, $l_k \geq 1$.

As soon as we reach $r_k \geq \beta$ we stop the process and set $N = k$. Since always $l_k \geq 1$, the process will indeed stop in a finite number of steps.

We obtain a partition of S into N rectangles $R_1, ..., R_N$ and two half-strips: $S \cap \{x_1 < 0\}$ and $S \cap \{x_1 > r_N\}$, and in the both half-strips we have $V \equiv 0$. In each R_k we have $\operatorname{Neg}(V, R_k) = 1$ whence it follows that

$$\operatorname{Neg}(V, S) \leq 2 + \sum_{k=1}^{N} \operatorname{Neg}(V, R_k) = N + 2.$$

Let us estimate N from above. In each R_k with $k \leq N - 1$ we have by (84) $\frac{1}{J_k} = \frac{1}{c} l_k$. Therefore, we have

$$N - 1 = \sum_{k=1}^{N-1} \frac{1}{\sqrt{J_k}} \sqrt{J_k} \leq \left(\frac{1}{c} \sum_{k=1}^{N-1} l_k\right)^{1/2} \left(\sum_{k=1}^{N-1} J_k\right)^{1/2} \leq \left(\tfrac{1}{c}\beta\right)^{1/2} J^{1/2}.$$

Using also $3 \leq 3\left(\frac{1}{c}\beta J\right)^{1/2}$, we obtain $N+2 \leq 4\left(\frac{1}{c}\beta J\right)^{1/2}$, which finishes the proof of (81).

The estimate (82) follows trivially from (81) and (74) as S_n is a rectangle $P_{\alpha,\beta}$ with the length $1 \leq \beta - \alpha \leq 2^{|n|+1}$. ■

Lemma 26 *For any sparse potential in the strip S,*

$$\operatorname{Neg}(V,S) \leq 1 + C \sum_{\{n: a_n(V) > c\}} \sqrt{a_n(V)}, \tag{85}$$

for some constant $C, c > 0$ depending only on p.

Proof. Let us enumerate in the increasing order those values n where $a_n(V) > c$. So, we obtain an increasing sequence $\{n_i\}$, finite or infinite, such that $a_{n_i}(V) > c$ for any index i. The difference $S \setminus \bigcup_i S_{n_i}$ can be partitions into a sequence $\{T_j\}$ of rectangles, where each rectangle T_j either fills the gap in S between successive rectangles S_{n_i} or T_j may be a half-strip that fills the gap between S_{n_i} and $+\infty$ or $-\infty$.

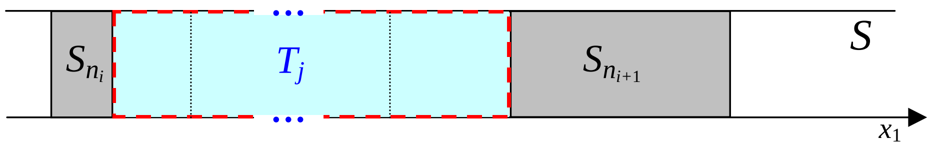

Figure 15: Partitioning of the strip S into rectangles S_{n_i} and T_j

By construction, each T_j is a union of some rectangles S_k with $a_k(V) \leq c$. It follows from Corollary 22 that $\operatorname{Neg}(V, T_j) = 1$. Since by construction

$$\#\{T_j\} \leq 1 + \#\{S_{n_i}\},$$

it follows that

$$\begin{aligned} \operatorname{Neg}(V,S) &\leq \sum_j \operatorname{Neg}(V,T_i) + \sum_i \operatorname{Neg}(V,S_{n_i}) \\ &\leq 1 + \#\{S_{n_i}\} + \sum_i \operatorname{Neg}(V,S_{n_i}) \\ &\leq 1 + 2\sum_i \operatorname{Neg}(V,S_{n_i}). \end{aligned}$$

In each S_{n_i} we have by (82) and $a_{n_i}(V) > c$ that

$$\operatorname{Neg}(V,S_{n_i}) \leq C\sqrt{a_{n_i}(V)}.$$

Substituting into the previous estimate, we obtain (85). ■

5.11 Arbitrary potentials in a strip

We use notation $a_n(V)$ and $b_n(V)$ defined by (74) and (75).

Theorem 27 *For any $p > 1$ and for any potential V in the strip S, we have*

$$\operatorname{Neg}(V, S) \leq 1 + C \sum_{\{n \in \mathbb{Z}: a_n(V) > c\}} \sqrt{a_n(V)} + C \sum_{\{n \in \mathbb{Z}: b_n(V) > c\}} b_n(V), \tag{86}$$

where the positive constants C, c depend only on p.

Proof. Let $\{n_i\}$ be a sequence of all $n \in \mathbb{Z}$ for which $b_n(V) > c$. Let $\{T_j\}$ be rectangles that fill the gaps in S between successive Q_{n_i} or between Q_{n_i} and $\pm\infty$.

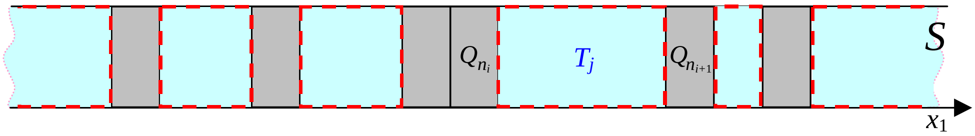

Figure 16: Partitioning of the strip S into rectangles Q_{n_i} and T_j

If the sequence $\{n_i\}$ is empty then V is sparse, and (86) follows from Lemma 26. Assume that $\{n_i\}$ is non-empty.

Consider the potentials $V' = V\mathbf{1}_{\cup T_j}$ and $V'' = V\mathbf{1}_{\cup Q_{n_i}}$. Since $V = V' + V''$, we have

$$\operatorname{Neg}(V, S) \leq \operatorname{Neg}(2V', S) + \operatorname{Neg}(2V'', S).$$

The potential $2V'$ is sparse by construction, whence by Lemma 26

$$\operatorname{Neg}(2V', S) \leq 1 + C \sum_{\{n: a_n(V') > c\}} \sqrt{a_n(V')}. \tag{87}$$

By Lemma 13 and Lemma 16, we obtain

$$\begin{aligned} \operatorname{Neg}(2V'', S) &\leq \sum_j \operatorname{Neg}(2V'', T_j) + \sum_i \operatorname{Neg}(2V'', Q_{n_i}) \\ &= \#\{T_j\} + \sum_i \left(1 + C \|2V''\|_{L^p(Q_{n_i})}\right) \\ &= \#\{T_j\} + \#\{Q_{n_i}\} + 2C \sum_i b_{n_i}(V). \end{aligned}$$

By construction we have $\#\{T_j\} \leq 1 + \#\{Q_{n_i}\}$. By the choice of n_i, we have $1 < c^{-1} b_{n_i}(V)$, whence

$$\#\{T_j\} + \#\{Q_{n_i}\} \leq 1 + 2\#\{Q_{n_i}\} \leq 1 + 2c^{-1} \sum_i b_{n_i}(V) \leq 3c^{-1} \sum_i b_{n_i}(V)$$

Combining these estimates together, we obtain

$$\operatorname{Neg}(2V'', S) \le C' \sum_i b_{n_i}(V) = C' \sum_{\{n: b_n(V) > c\}} b_n(V) \tag{88}$$

Adding up (87) and (88) yields

$$\operatorname{Neg}(V, S) \le 1 + C \sum_{\{n: a_n(V') > c\}} \sqrt{a_n(V')} + C \sum_{\{n: b_n(V) > c\}} b_n(V). \tag{89}$$

Since $V' \le V$, (89) implies (86), which finishes the proof. ∎

Remark. In fact, we have proved a slightly better inequality (89) than (86).

5.12 Proof of Theorem 12

Let us prove the main Theorem 12, that is, for any potential V in $\mathbb{R}^2$,

$$\operatorname{Neg}(V) \le 1 + C \sum_{\{n \in \mathbb{Z}: A_n > c\}} \sqrt{A_n} + C \sum_{\{n \in \mathbb{Z}: B_n > c\}} B_n, \tag{90}$$

where

$$A_n(V) = \int_{U_n} V(x)(1 + |\ln |x||)\, dx, \quad B_n(V) = \left(\int_{W_n} V^p(x) |x|^{2(p-1)} dx \right)^{1/p},$$

$$U_n = \begin{cases} \{e^{2^{n-1}} < |x| < e^{2^n}\}, & n \ge 1, \\ \{e^{-1} < |x| < e\}, & n = 0, \\ \{e^{-2^{|n|}} < |x| < e^{-2^{|n|-1}}\}, & n \le -1, \end{cases}$$

and

$$W_n = \left\{ e^n < |x| < e^{n+1} \right\}.$$

Consider an open set $\Omega = \mathbb{R}^2 \setminus L$ where $L = \{\, x_1 \ge 0, x_2 = 0\}$ and the mapping $\Psi : \Omega \to \widetilde{S}$ where $\Psi(z) = \ln z$ and

$$\widetilde{S} = \left\{ (y_1, y_2) \in \mathbb{R}^2 : \quad 0 < y_2 < 2\pi \right\}.$$

Using the inverse mapping $\Phi = \Psi^{-1}$, define a potential $\widetilde{V}$ on $\widetilde{S}$ by $\widetilde{V}(y) = V(\Phi(y)) |J_\Phi(y)|$ where J_Φ is the Jacobian of Φ. It is possible to prove that

$$\operatorname{Neg}\left(V, \mathbb{R}^2\right) \le \operatorname{Neg}(V, \Omega) = \operatorname{Neg}(\widetilde{V}, \widetilde{S}). \tag{91}$$

Since the strips $\widetilde{S}$ and S are bilipschitz equivalent, Theorem 27 holds also for $\widetilde{S}$, that is,

$$\operatorname{Neg}(\widetilde{V}, \widetilde{S}) \le 1 + C \sum_{\{n: a_n > c\}} \sqrt{a_n} + C \sum_{\{n: b_n(V) > c\}} b_n, \tag{92}$$

where

$$a_n = \int_{S_n} (1 + |y_1|)\, \widetilde{V}(y)\, dy, \quad b_n = \left(\int_{Q_n} \widetilde{V}^p dy \right)^{1/p},$$

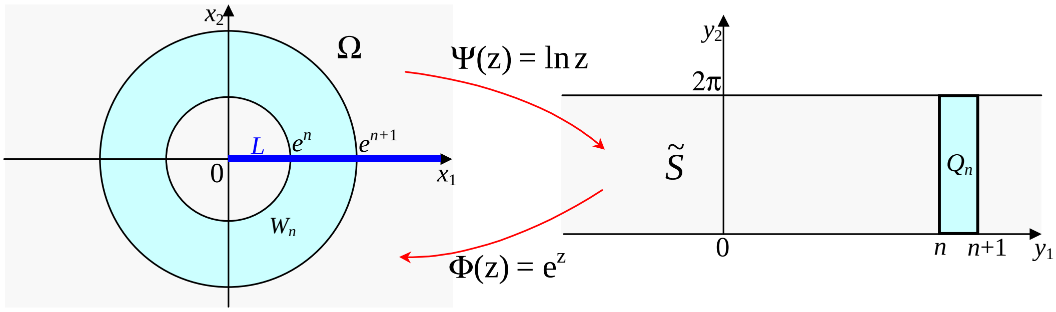

Figure 17: Conformal mapping $\Psi : \Omega \to \widetilde{S}$

and

$$Q_n = \Psi\left(W_n \setminus L\right), \qquad S_n = \Psi\left(U_n \setminus L\right).$$

Since $J_\Psi = \frac{1}{|x|^2}$, we obtain, using the change $y = \Psi\left(x\right)$,

$$\begin{aligned} b_n^p &= \int_{Q_n} \widetilde{V}^p\left(y\right) dy = \int_{W_n} V^p\left(x\right) \left|J_\Phi\left(y\right)\right|^p \left|J_\Psi\left(x\right)\right| dx \\ &= \int_{W_n} V^p\left(x\right) \left|J_\Psi\left(x\right)\right|^{1-p} dx \\ &= \int_{W_n} V^p\left(x\right) |x|^{2(p-1)} dx = B_n^p. \end{aligned}$$

Similarly, computing a_n and observing that

$$y_1 = \operatorname{Re}\Psi\left(x\right) = \operatorname{Re}\ln x = \ln|x|,$$

we obtain

$$\begin{aligned} a_n &= \int_{S_n} \widetilde{V}\left(y\right)\left(1 + |y_1|\right) dy = \int_{U_n} V\left(x\right) \left|J_\Phi\left(y\right)\right| \left(1 + \left|\ln|x|\right|\right) \left|J_\Psi\left(x\right)\right| dx \\ &= \int_{U_n} V\left(x\right)\left(1 + \left|\ln|x|\right|\right) dx = A_n. \end{aligned}$$

Combining together (91), (92), we obtain (90).