Analysis on manifolds and volume growth

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Setup

Let M be always a Riemannian manifold that is geodesically complete and non-compact. Let $d(x,y)$ denote the geodesic distance on M and μ be the Riemannian measure. Consider geodesic balls

$$B(x,r) = \{y \in M : d(x,y) < r\},$$

that are necessarily precompact, and their volumes:

$$V(x,r) = \mu(B(x,r)).$$

In this lecture we collect some old and new results relating the rate growth of $V(x,r)$ as $r \to \infty$ to the properties of certain PDEs on M.

Recall that the Laplace operator Δ on M is given in the local coordinates $x_1, ..., x_n$ as follows:

$$\Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left(\sqrt{\det g} g^{ij} \frac{\partial}{\partial x_j} \right)$$

where $g = (g_{ij})$ is the Riemannian metric tensor and $(g^{ij}) = (g_{ij})^{-1}$.

Parabolicity

A function $u \in C^2(M)$ is called superharmonic if $\Delta u \leq 0$. Manifold M is called *parabolic* if any positive superharmonic function on M is constant, and *non-parabolic* otherwise.

Equivalent characterizations of the parabolicity:

- there exists no positive fundamental solution of $-\Delta$;
- $\int^{\infty} p_t(x,y)\,dt = \infty$ for all/some $x, y \in M$, where $p_t(x,y)$ is the heat kernel of Δ;
- the capacity of any compact set is zero;
- Brownian motion on M is recurrent.

Theorem of Polya (1921): $\mathbb{R}^n$ is parabolic for $n \leq 2$ and non-parabolic for $n > 2$.

Let us fix a reference point x_0 and denote $V(r) = V(x_0, r)$.

Theorem 1 *(Cheng-Yau, '75) If for all large enough r*

$$V(r) \leq Cr^2 \tag{1}$$

then M is parabolic.

Theorem 2 *(AG '83, Karp '82, Varopoulos '83) If*

$$\int^{\infty} \frac{r\,dr}{V(r)} = \infty \tag{2}$$

then M is parabolic.

For example, (2) is satisfied if

$$V(r) \leq C r^2 \log r.$$

The condition (2) is sharp: if $f(r)$ is a smooth convex function such that $f'(r) > 0$ and

$$\int^{\infty} \frac{r\,dr}{f(r)} < \infty,$$

then there is a non-parabolic manifold such that $V(r) = f(r)$ for large r.

Stochastic completeness

Manifold M is called *stochastically complete* if for all $x \in M$ and $t > 0$

$$\int_M p_t(x,y)\, d\mu(y) = 1.$$

Equivalent characterizations of the stochastic completeness:

- Lifetime of Brownian motion on M is ∞ almost surely.
- For some/any $\lambda > 0$, any bounded solution u to $\Delta u - \lambda u = 0$ on M is identical zero.
- For some/any $T \in (0, \infty]$, the Cauchy problem

  $$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } M \times (0,T) \\ u\big|_{t=0} = 0 \end{cases} \tag{3}$$

  has the only bounded solution $u \equiv 0$.

Theorem 3 *(AG '86) If*

$$\int^{\infty} \frac{r\,dr}{\log V(r)} = \infty \tag{4}$$

then M is stochastically complete.

In particular, M is stochastically complete provided

$$V(r) \leq \exp\left(Cr^2\right). \tag{5}$$

(Davies '92, Karp and Li '83, Takeda '89).

The condition (4) is sharp: if $f(r)$ is a smooth convex function such that $f'(r) > 0$ and

$$\int^{\infty} \frac{r dr}{f(r)} < \infty$$

then there exists a stochastically incomplete manifold with $V(r) = \exp(f(r))$.

Liouville properties

Theorem 4 *(S.T.Yau '78) If u is a harmonic function on M and $u \in L^p(M,\mu)$ with $1 < p < \infty$ then $u \equiv \mathrm{const}$.*

In other words, any geodesically complete manifold satisfies *L^p-Liouville property* if $1 < p < \infty$. For $p = 1$ and $p = \infty$ this is not true: there are manifolds with non-trivial L^1 harmonic function as well as those with non-trivial L^∞ harmonic functions.

Theorem 5 *(AG '89) Assume that*

$$\int^{\infty} \frac{r\,dr}{\log V(r)} = \infty.$$

Then any positive superharmonic function $u \in L^1(M,\mu)$ is identical constant.

Open Questions:. *Find "good" conditions to ensure that*

(a) any harmonic function $u \in L^1$ is identical constant;

(b) any harmonic function $u \in L^\infty$identical constant.

Bounded solutions of Schrödinger equations

Let $Q(x)$ be a nonnegative continuous function on M, $Q \not\equiv 0$. Consider the equation

$$\Delta u - Qu = 0 \tag{6}$$

and ask if (6) has a non-trivial bounded solution (equivalently: a positive bounded solution).

Set $|x| = d(x, x_0)$ and denote

$$q(r) = \inf_{|x|=r} Q(x) \quad \text{and} \quad F(r) = \int_0^{r/2} \sqrt{q(t)}dt.$$

Theorem 6 *(AG, '90) If for all large enough r*

$$V(r) \le Cr^2 \exp\left(CF(r)^2\right) \tag{7}$$

then (6) has no bounded solution except for $u \equiv 0$.

Example. Let $Q \equiv 1$. Then (6) has no bounded solution (except for zero) if and only if M is stochastically complete. We have $q \equiv 1$, $F(r) = r/2$, and (7) becomes $V(r) \le \exp(Cr^2)$, which coincides with the condition (5) for the stochastic completeness.

Example. Let Q have compact support. In this case (6) has no bounded solution (except for zero) if and only if M is parabolic. Since $q(r) = 0$ for large enough r, we obtain that $F(r) = \text{const}$ for large r, and (7) becomes $V(r) \leq Cr^2$, which coincides with the condition (1) for the parabolicity.

Example. Assume that, for large $|x|$

$$Q(x) \geq \frac{c}{|x|^2 \log |x|}.$$

Then

$$F(r) \geq \int_2^{r/2} \frac{c}{t\sqrt{\log t}} dt \simeq \sqrt{\log r}$$

so that (7) is satisfied provided

$$V(r) \leq Cr^N.$$

In this case (6) has no bounded solution except for zero.

On the other hand, if in $\mathbb{R}^n$

$$Q(x) \leq \frac{C}{|x|^2 \log^{1+\varepsilon} |x|}$$

then (6) has a positive solution in $\mathbb{R}^n$.

Escape rate

Let $\{X_t\}_{t\geq 0}$ be Brownian motion on M, that is the diffusion process generated by Δ, whose transition density is $p_t(x,y)$.

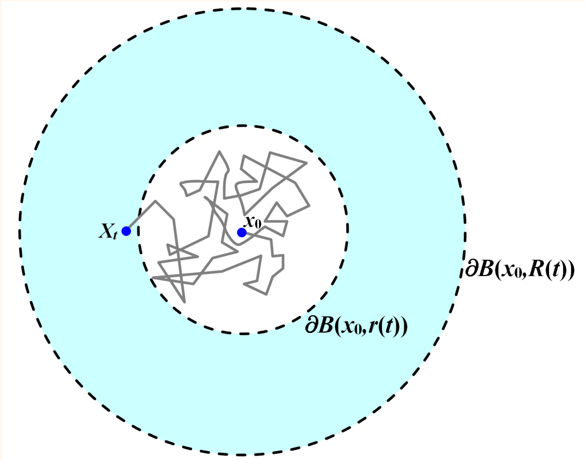
An increasing function $R(t)$ is called *an upper rate function* if, with probability 1, we have $|X_t| < R(t)$ for all t large enough.

Similarly, an increasing function $r(t)$ is called a *lower rate function* if, with probability 1, we have $|X_t| > r(t)$ for all t large enough.

Hence, for large enough t,
X_t is contained in the annulus

$$B(x_0, R(t)) \setminus B(x_0, r(t))$$

almost surely.



For example, in $\mathbb{R}^n$ the following upper rate function is known:

$$R(t) = \sqrt{(4+\varepsilon)\, t \log\log t} \quad \text{(Khinchin's law of iterated } \log\text{)}.$$

If $n > 2$ and $r(t)/\sqrt{t}$ is decreasing then $r(t)$ is a lower rate function in $\mathbb{R}^n$ if and only if

$$\int^{\infty} \left(\frac{r(t)}{\sqrt{t}}\right)^{n-2} \frac{dt}{t} < \infty \tag{8}$$

(Dvoretzky and Erdös, '50). Here is an example of such a function: $r(t) = \frac{C\sqrt{t}}{\log^{\frac{1+\varepsilon}{n-2}} t}$.

Theorem 7 *(AG and M.Kelbert '98, AG '99) Assume that, for all r large enough,*

$$V(r) \leq Cr^N, \tag{9}$$

with some $N, C > 0$. Then the following function is an upper rate function:

$$R(t) = \sqrt{2Nt \log t}. \tag{10}$$

Under assumption (9), the upper rate function (10) is optimal (AG, Kelbert '00).

Similar result for simple random walks on graphs: Hardy–Littlewood (1914, for $\mathbb{Z}$) and M.Barlow–E.Perkins '89.

Theorem 8 *(AG and E.P. Hsu '09) Let M be a Cartan-Hadamard manifold and assume that*

$$\int^{\infty} \frac{rdr}{\log V(r)} = \infty. \tag{11}$$

Define a function $\varphi(t)$ as follows:

$$t = \int_{r_0}^{\varphi(t)} \frac{rdr}{\log V(r)}.$$

Then $R(t) = \varphi(Ct)$ is an upper rate function.

Example. If $V(r) = Cr^N$ then

$$t \simeq \frac{\varphi^2(t)}{\log \varphi(t)}$$

whence $R(t) \simeq \varphi(t) \simeq \sqrt{t \log t}$ that matches (10).

Example. If $V(r) = \exp(r^\alpha)$ where $0 < \alpha < 2$ then

$$t \simeq \varphi(t)^{2-\alpha}$$

whence $R(t) = Ct^{\frac{1}{2-\alpha}}$.

Example. If $V(r) = \exp(r^2)$ then

$$t \simeq \log \varphi(t)$$

whence $R(t) = \exp(Ct)$.

Theorem 9 *(E.P. Hsu and G.Qin '10) On any complete manifold M, satisfying (11), define function $\varphi(t)$ as follows:*

$$t = \int_{r_0}^{\varphi(t)} \frac{r\,dr}{\log V(r) + \log\log r}.$$

Then $R(t) = C\varphi(Ct)$ is an upper rate function.

Example. Let $V(r) \leq C \log r$. Then

$$t \simeq \frac{\varphi^2(t)}{\log\log \varphi(t)}$$

and we obtain an upper rate function

$$R(t) = C\sqrt{t \log\log t}.$$

Theorem 10 *(AG '99) Assume that a relative Faber-Krahn inequality holds on M (for example, this is the case when $Ricci_M \geq 0$). Assume that*

$$\int^{\infty} \frac{rdr}{V(r)} < \infty$$

so that M is non-parabolic and X_t is transient. Denote

$$\gamma(r) := \left(\int_r^{\infty} \frac{sds}{V(s)} \right)^{-1} . \tag{12}$$

Let $r(t)$ be an increasing positive function on $(0, \infty)$ such that

$$\int^{\infty} \frac{\gamma(r(t))}{V(\sqrt{t})} dt < \infty. \tag{13}$$

Then $r(t)$ is a lower rate function for Brownian motion on M.

Example. Let $V(x,r) \simeq r^N$ for large r and some $N > 2$. We obtain from (12) $\gamma(t) \simeq t^{N-2}$, and (13) amounts to

$$\int^{\infty} \frac{r^{N-2}(t)dt}{t^{N/2}} < \infty ,$$

which coincides with the Dvoretzky–Erdös condition (8).

Semilinear PDE

Consider on M the inequality

$$\Delta u + u^{\sigma} \leq 0 \tag{14}$$

and ask if it has a non-negative solution u except for $u \equiv 0$. Here $\sigma > 1$ is a given parameter. Any solution of (14) is superharmonic. Hence, if M is parabolic then u must be identical zero. In particular, this is the case if $V(r) \leq Cr^2$.

Theorem 11 *(AG and Y.Sun '14) Assume that, for all large r,*

$$V(r) \leq Cr^p \log^q r, \tag{15}$$

where

$$p = \frac{2\sigma}{\sigma - 1}, \quad q = \frac{1}{\sigma - 1}. \tag{16}$$

Then any nonnegative solution of (14) is identical zero.

The values of the exponents p and q in (16) are sharp: if either $p > \frac{2\sigma}{\sigma-1}$ or $p = \frac{2\sigma}{\sigma-1}$ and $q > \frac{1}{\sigma-1}$ then there is a manifold satisfying (15) where the inequality (14) has a positive solution.

Equivalent reformulation: if, for some $\alpha > 2$

$$V(r) \le Cr^{\alpha} \log^{\frac{\alpha-2}{2}} r, \tag{17}$$

then, for any $\sigma \le \frac{\alpha}{\alpha-2}$, any nonnegative solution of (14) is identical zero.

For example, in $\mathbb{R}^n$ with $n > 2$ (17) holds with $\alpha = n$ which implies that, for $\sigma \le \frac{n}{n-2}$, any nonnegative solution of (14) is identical zero.

It is known that, for any $\sigma > \frac{n}{n-2}$, (14) has a positive solution in $\mathbb{R}^n$ (Mitidieri and Pohozaev, '98).

Conjecture. *If*

$$\int^{\infty} \frac{r^{2\sigma-1} dr}{V(r)^{\sigma-1}} = \infty \tag{18}$$

then any nonnegative solution of (14) is identical zero.

In particular, the function (17) satisfies (18) with $\sigma = \frac{\alpha}{\alpha-2}$.

Similar results for the semilinear heat equation: Yuhua Sun.

Heat kernel lower bounds

Theorem 12 *(Coulhon and AG, '97) Assume that, for all* $r \geq r_0 > 0$,

$$V(r) \leq Cr^{\alpha}, \tag{19}$$

for some $C, \alpha > 0$*. Then, for all large enough* t*,*

$$p_t(x_0, x_0) \geq \frac{1/4}{V(\sqrt{Kt \log t})}, \tag{20}$$

where $K = K(x_0, r_0, C, \alpha) > 0$*. Consequently,*

$$p_t(x_0, x_0) \geq \frac{c}{(t \log t)^{\alpha/2}} \tag{21}$$

The best possible lower bound would be

$$p_t(x_0, x_0) \geq \frac{c}{V(\sqrt{t})}$$

that is valid on manifolds of non-negative Ricci curvature (Li and Yau '85). However, under the hypothesis (19), the lower bound (20) is optimal (AG and Kelbert, '00).

Theorem 13 *(Coulhon and AG, '97) Assume that the function $V(r)$ is doubling, that is,*

$$V(2r) \leq CV(r),$$

and that, for all $t > 0$,

$$p_t(x_0, x_0) \leq \frac{C}{V\left(\sqrt{t}\right)}.$$

Then, for all $t > 0$,

$$p_t(x_0, x_0) \geq \frac{c}{V\left(\sqrt{t}\right)}.$$

Recurrence revisited

For any $\alpha \in (0,2)$, the operator $(-\Delta)^{\alpha/2}$ is a generator of a jump process on M that is called *the α-process.* It is a natural generalization of the symmetric stable Levy process of index α in $\mathbb{R}^d$. By a general semigroup theory, the Green function $G^{(\alpha)}(x,y)$ of $(-\Delta)^{\alpha/2}$ is given by

$$G^{(\alpha)}(x,y) = \int_0^\infty t^{\alpha/2-1} p(t,x,y)dt,$$

and the recurrence of the α-process is equivalent to $G^{(\alpha)} \equiv \infty$, that is, to

$$\int^\infty t^{\alpha/2-1} p(t,x,x)dt = \infty. \tag{22}$$

Theorem 14 *(AG '99) If for all $r \geq r_0 > 0$*

$$V(r) \leq Cr^\alpha, \tag{23}$$

then the α-process is recurrent.

Indeed, by Theorem 12 we have $p_t(x_0,x_0) \geq \frac{c}{t^{\alpha/2}\log^{\alpha/2} t}$. Substituting into (22) we see that the integral diverges.

Heat kernel upper bounds

Theorem 15 *(Barlow, Coulhon, AG '01) Let M be a manifold of bounded geometry. Assume that, for all $x \in M$ and $r \geq r_0 > 0$*

$$V(x,r) \geq cr^N$$

Then, for all $x \in M$ and large enough t,

$$p_t(x,x) \leq Ct^{-\frac{N}{N+1}}. \tag{24}$$

For any $N \geq 1$, there exists an example of a manifold with $V(x,r) \simeq r^N$ and

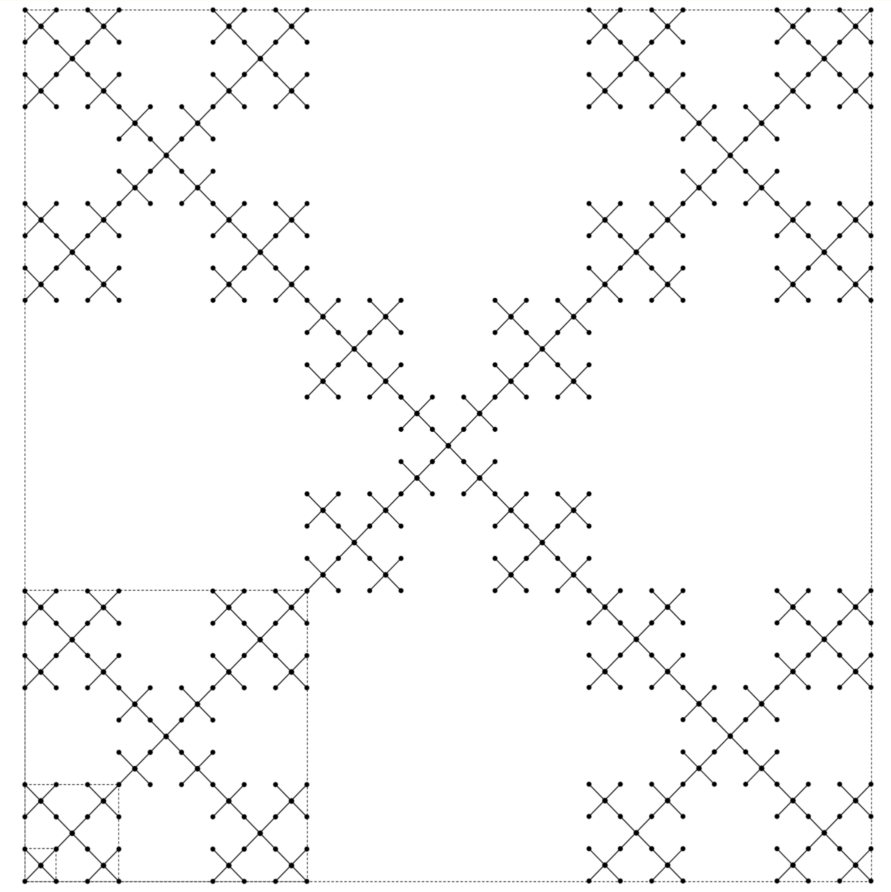
$$p_t(x,x) \simeq t^{-\frac{N}{N+1}}.$$

Indeed, take any fractal graph where the volume function is r^α and the on-diagonal decay of the heat kernel is given by $t^{-\alpha/\beta}$. It is known that such a graph exists for any α, β satisfying

$$2 \leq \beta \leq \alpha + 1$$

(Barlow '04). Choose $\alpha = N$ and $\beta = N + 1$ and then inflate the graph into a manifold.

Here is one of such graphs, the *Vicsek tree*:



For this fractal

$\alpha = \frac{\log 5}{\log 3}$

and

$\beta = \alpha + 1 = \frac{\log 15}{\log 3}$

Biparabolic manifolds

A function $u \in C^4(M)$ is called *bi-superharmonic* if $\Delta u \leq 0$ and $\Delta^2 u \geq 0$.

For example, let M be nonparabolic and consider the Green operator

$$Gf = \int_0^\infty g(x,y)\, f(y)\, d\mu(y)$$

where $g(x,y) = \int_0^\infty p_t(x,y)\, dt$ is the Green function. Then $u = Gf$ is bi-superharmonic if u is finite and f is non-negative and superharmonic.

Another example of bi-superharmonic functions in a precompact domain $\Omega \subset M$: if f is a non-negative continuous function on $\partial\Omega$ then

$$u(x) = \mathbb{E}_x\left(\tau_\Omega f\left(X_{\tau_\Omega}\right)\right)$$

solves the following boundary value problem

$$\begin{cases} \Delta^2 u = 0 \ \text{ in } \Omega \\ -\Delta u|_{\partial\Omega} = f, \\ u|_{\partial\Omega} = 0, \end{cases}$$

and, hence, is bi-superharmonic in Ω.

A manifold M is called *biparabolic*, if any positive bi-superharmonic function on M is harmonic, that is $\Delta u = 0$.

Note that the notion of parabolicity also admits a similar equivalent definition: M is parabolic if and only if any positive superharmonic function on M is harmonic.

One can show that $\mathbb{R}^n$ is biparabolic if and only if $n \leq 4$. For example, if $n > 4$ then $u(x) = |x|^{-(n-4)}$ is bi-superharmonic but not harmonic.

Theorem 16 *(Faraji and AG, to appear) Manifold M is biparabolic provided*

$$V(r) \leq C \frac{r^4}{\log r} \quad \textit{for large } r. \tag{25}$$

The condition (25) is not far from optimal in the following sense: for any $\beta > 1$ there exists a manifold M with

$$V(r) \leq C\, r^4 \log^\beta r$$

that is not biparabolic.

Conjecture. *If*

$$V(r) \leq C r^4 \log r \quad \textit{or} \quad \textit{even} \quad \int^\infty \frac{r^3 dr}{V(r)} = \infty,$$

then M is biparabolic.

Recall that M is parabolic if and only if $G \equiv \infty$ that is, $Gf \equiv \infty$ for any non-zero $f \geq 0$.

Lemma 17 *The following conditions are equivalent:*
(i) M is biparabolic.
(ii) $G^2 \equiv \infty$ (that is, $G^2 f \equiv \infty$ for any non-zero $f \geq 0$)

Proof of Theorem 16. Assuming (25), we prove that $G^2 f \equiv \infty$ for any non-negative non-zero function f. It is easy to compute that

$$G^2 f(x) = \int_0^\infty t P_t f(x)\, dt = \int_0^\infty \int_M t p_t(x,y) f(y) d\mu(y) dt.$$

Fix an arbitrary $x \in M$ and choose $R > 0$ so big that the ball $B(x_0, R)$ contains both supp f and x. By the local Harnack inequality, we have, for all $x, y \in B(x_0, R)$ and $t > 2R^2$

$$p_t(x,y) \geq c p_{t-R^2}(x_0, x_0) \geq c p_t(x_0, x_0),$$

where $c = c(x_0, R) > 0$. Hence, we obtain, for large enough t_0,

$$G^2 f(x) \geq \int_{t_0}^\infty \int_{B(x_0,R)} t p_t(x,y) f(y) d\mu(y) dt \geq c ||f||_{L^1} \int_{t_0}^\infty t p_t(x_0, x_0) dt.$$

By Theorem 12, we have, for large t,

$$p_t\left(x_0, x_0\right) \geq \frac{1/4}{V\left(\sqrt{Kt\log t}\right)} \geq \frac{c}{v\left(\sqrt{t\log t}\right)},$$

where

$$v(r) = \frac{r^4}{\log r}.$$

For large t we have

$$v(\sqrt{t\log t}) \simeq t^2 \log t,$$

whence

$$\int_{t_0}^{\infty} tp_t(x_0, x_0)dt \geq c\int_{t_0}^{\infty} \frac{tdt}{v\left(\sqrt{t\log t}\right)} \simeq \int_{t_0}^{\infty} \frac{dt}{t\log t} = \infty.$$

We conclude that

$$G^2 f\left(x\right) = \infty,$$

which was to be proved. ■

Now let us construct for any $\beta > 1$ an example of a manifold M that is not biparabolic and satisfies

$$V(r) \leq Cr^4 \log^\beta r.$$

Fix $n \geq 2$ and consider a smooth manifold $M = \mathbb{R} \times \mathbb{S}^{n-1}$, where any point $x \in M$ is represented in the polar form as (r, θ) where $r \in \mathbb{R}$ and $\theta \in \mathbb{S}^{n-1}$.

Define the Riemannian metric g on M by

$$\mathrm{g} = dr^2 + \psi^2(r) d\theta^2, \tag{26}$$

where $d\theta^2$ is the standard Riemannian metric on $\mathbb{S}^{n-1}$ and $\psi(r)$ is a smooth positive function on $\mathbb{R}$. Define the area function $S(r)$, $r \in \mathbb{R}$, by

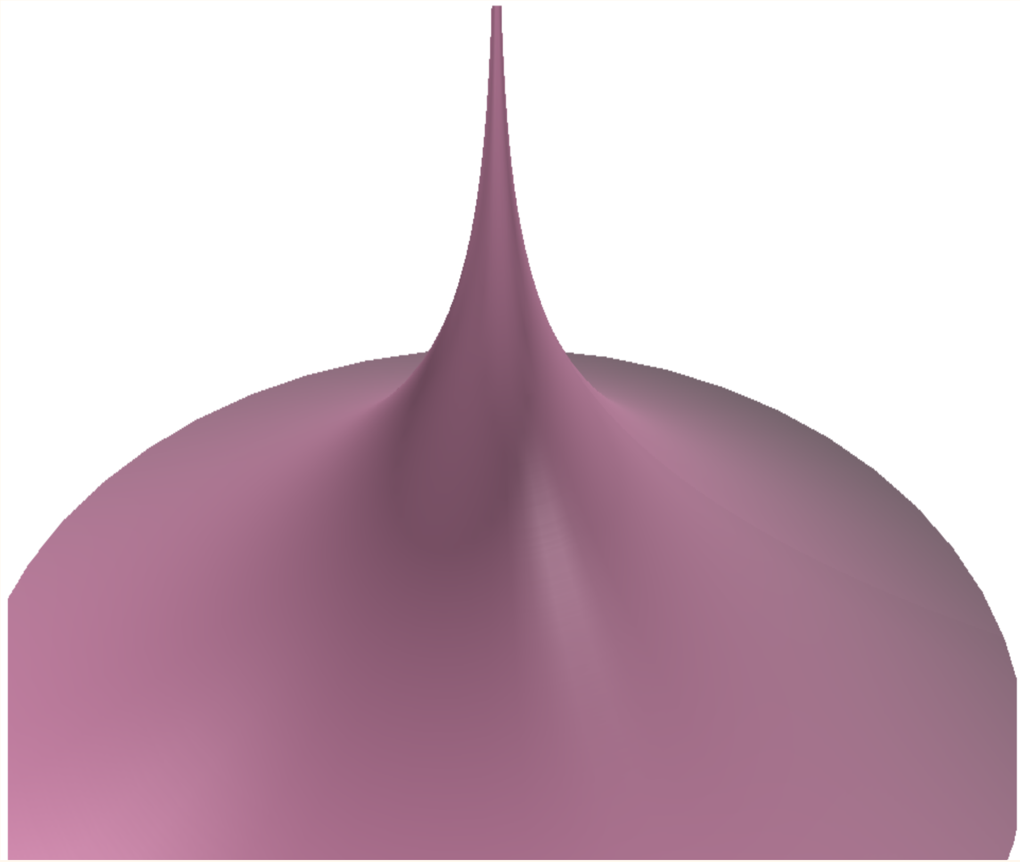
$$S(r) = \omega_n \psi(r)^{n-1},$$

where ω_n is the volume of $\mathbb{S}^{n-1}$. We choose $S(r)$ as follows:

$$S(r) = \begin{cases} e^{-r^\alpha}, & r > 2, \\ |r|^3 \log^\beta |r|, & r < -2, \end{cases} \tag{27}$$

where α, β are arbitrary real numbers such that

$$\boxed{\alpha > 2 \quad \text{and} \quad \beta > 1.} \tag{28}$$



Proposition 18 *Under the hypotheses* (27) *and* (28)*, the manifold* M *is not biparabolic, and the volume growth function of* M *satisfies*

$$V(r) \leq Cr^4 \log^\beta r. \tag{29}$$

Proof. Fix a reference point $x_0 = (0,0)$. The volume estimate (29) follows from

$$V(r) \simeq \int_{-r}^{r} S(t)\, dt.$$

In order to prove that M is not biparabolic, it suffices to construct a positive harmonic function h on M such that the function

$$u := Gh$$

is finite at least at one point. Indeed, in this case we have $u \in C^\infty(M)$ and $\Delta u = -h$. Hence, $\Delta u < 0$ and

$$\Delta^2 u = \Delta h = 0$$

so that u is bi-superharmonic, but not harmonic; hence, M is not biparabolic.

Choose h as follows:

$$h(r) = \int_{-\infty}^{r} \frac{dt}{S(t)}. \tag{30}$$

It is finite by (27) and harmonic on M because it depends only on r and

$$\Delta h = \frac{\partial^2 h}{\partial r^2} + \frac{S'(r)}{S(r)} \frac{\partial h}{\partial r} = \frac{1}{S(r)} \frac{\partial}{\partial r} \left(S(r) \frac{\partial h}{\partial r} \right) = 0.$$

Then one proves that, for any $x = (r, \theta)$,

$$g(x_0, x) \simeq \begin{cases} h(r), & \text{if } r < -2, \\ 1, & \text{if } r > 2. \end{cases}$$

We have

$$Gh(x_0) = \int_M g(x_0, x) h(x) d\mu(x) \simeq 1 + \int_{-\infty}^{-2} h^2(r) S(r) dr + \int_2^{\infty} h(r) S(r) dr.$$

For $r < -2$ we have

$$S(r) = |r|^3 \log^\beta |r| \quad \text{and} \quad h(r) \simeq \int_{-\infty}^{r} \frac{dt}{|t|^3 \log^\beta |t|} \simeq \frac{1}{|r|^2 \log^\beta |r|}.$$

Since $\beta > 1$, we obtain

$$\int_{-\infty}^{-2} h^2(r)\, S(r)\, dr \simeq \int_{-\infty}^{-2} \frac{1}{|r| \log^{\beta} |r|} dr < \infty.$$

For $r > 2$

$$S(r) = e^{-r^{\alpha}} \quad \text{and} \quad h(r) \simeq \int_0^r e^{t^{\alpha}} dt \simeq \frac{e^{r^{\alpha}}}{r^{\alpha-1}}.$$

Since $\alpha > 2$, we have

$$\int_2^{+\infty} h(r)\, S(r)\, dr \simeq \int_2^{\infty} \frac{dr}{r^{\alpha-1}} < \infty.$$

Hence, $Gh(x_0) < \infty$, which was to be proved. ■