

LOWER ESTIMATES OF HEAT KERNELS FOR NON-LOCAL DIRICHLET FORMS ON METRIC MEASURE SPACES

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ABSTRACT. We are concerned with heat kernel estimates for a non-local Dirichlet form on an Ahlfors regular metric measure space. We use an analytic approach to obtain the full *lower* stable-like estimate of the heat kernel from the *near diagonal lower* estimate. Combining with other known results, we obtain certain equivalent conditions for two-sided stable-like estimates of the heat kernel. The results can be simplified in certain cases, for example, when the Dirichlet form admits an effective resistance.

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1. INTRODUCTION

In recent years there has been a growing interest in heat kernel estimates for *non-local* Dirichlet forms in various settings, see, for example [4], [8], [9], [10], [11], [12], [13], [23], [27] and the references therein. Let (M, d, μ) be a metric measure space and $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$. In many cases of interest, if $(\mathcal{E}, \mathcal{F})$ is of jump type, the heat kernel $p_t(x, y)$ of the associated heat semigroup admits the following two-sided estimate:

$$p_t(x, y) \asymp c \min \left(t^{-\alpha/\beta}, \frac{t}{d(x, y)^{\alpha+\beta}} \right), \quad (1.1)$$

where α is the Hausdorff dimension of the underlying metric space, β is a positive parameter called the *index*, and the symbol \asymp means that both \leq and \geq hold but with different values of a positive constant c . For instance, the heat kernel of the fractional Laplacian $(-\Delta)^{\beta/2}$ on \mathbb{R}^n for any $\beta \in (0, 2)$ admits the estimate (1.1) with $\alpha = n$. Note that $(-\Delta)^{\beta/2}$

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is the generator of a symmetric stable process of index β . Hence, we refer to (1.1) as a *stable-like* estimate.

An important problem in this area is to provide equivalent conditions for the stable-like estimate (1.1) in reasonable terms, in particular, in terms of the jump kernel of the Dirichlet form. In the case $\beta < 2$ this problem was solved by Chen and Kumagai in [12], [13]¹ by showing that, under the standing assumption of α -regularity of (M, d, μ) , the heat kernel estimate (1.1) is equivalent to the following estimate of the jump kernel of the Dirichlet form: $J(x, y) \asymp Cd(x, y)^{-(\alpha+\beta)}$. However, on fractal spaces the index β in (1.1) can actually be larger than 2. The present paper is a part of the project of the authors of creating tools for obtaining stable-like estimate (1.1) for arbitrary $\beta > 0$ including the case $\beta > 2$.

Let us mention for comparison that the theory for obtaining heat kernel bounds for *local* Dirichlet forms on fractal-like spaces has reached by now a certain maturity. It is known that typically the heat kernel of a diffusion on such spaces satisfies *sub-Gaussian* estimate

$$p_t(x, y) \asymp \frac{c_1}{t^{\alpha/\beta}} \exp\left(-c_2 \left(\frac{d^\beta(x, y)}{t}\right)^{\frac{1}{\beta-1}}\right), \quad (1.2)$$

see, for example, [2], [3], [7], [26], [28]. Here α is as above the Hausdorff dimension and β is the *walk dimension* of the underlying space that is an invariant of (M, d) . Various equivalent conditions for (1.2) have been obtained in [5], [18], [24], [25].

Returning to jump type Dirichlet forms, let us mention that the upper bound

$$p_t(x, y) \leq C \min\left(t^{-\alpha/\beta}, \frac{t}{d(x, y)^{\alpha+\beta}}\right) \quad (1.3)$$

for arbitrary β was addressed in [23]. However, a task of obtaining the lower estimate

$$p_t(x, y) \geq c \min\left(t^{-\alpha/\beta}, \frac{t}{d(x, y)^{\alpha+\beta}}\right) \quad (LE)$$

represents a number of challenges. In the both cases of sub-Gaussian and stable-like estimates, the first step towards the lower estimate is a *near-diagonal lower bound*

$$p_t(x, y) \geq ct^{-\alpha/\beta} \quad \text{if } d(x, y) \leq \varepsilon t^{1/\beta}. \quad (NLE)$$

For the local Dirichlet form, (NLE) and a certain *chain condition* (which is a property of the metric) imply rather simply the sub-Gaussian lower bound in (1.2), see [2], [19, Corollary 3.5]. On the contrary, obtaining the stable-like lower bound (LE) from (NLE) is highly non-trivial, which is not surprising because of a “fat” tail of a stable-like heat kernel.

In this paper we present a new method for obtaining (LE) from (NLE) for non-local Dirichlet forms. Methods for obtaining (NLE) will be addressed in a companion paper [16] of the authors, where, hence, the equivalent conditions for the two-sided estimate (1.1) will be obtained.

We are aware of a recent work of Chen, Kumagai and Wang [14] where they provided a method for obtaining the stable-like estimates (1.1) for any $\beta > 0$, even in a more general setting of doubling (rather than regular) measure. The method of [14] uses probabilistic arguments, in particular, Levy system for the derivation of (LE) from (NLE), whereas our approach is purely analytic, based on the parabolic maximum principle.

Our main result – Theorem 2.8, is stated in the next section, after a series of necessary definitions. We give there also an overview of the rest of the paper.

¹Although the results of [12], [13] require some additional assumptions on the metric space (M, d) , this method can be enhanced to work with a general metric structure as it is done in [29].

NOTATION. The letters c, c_i, C, C_i, C', C'' etc denote positive constants whose values are unimportant and may change from one appearance to another. Nevertheless, all constants in conclusions depend only on the parameters in the hypotheses.

We use the expression “ μ -almost all $x, y \in M$ ” as a short hand for “ $\mu \times \mu$ -almost all $(x, y) \in M \times M$ ”.

2. PRELIMINARIES AND MAIN RESULTS

Let (M, d, μ) be a metric measure space, that is, (M, d) is a locally compact and separable metric space, and μ is a Radon measure on M with full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(M, \mu)$. Denote by \mathcal{L} the (non-negative definite) generator of $(\mathcal{E}, \mathcal{F})$, that is a self-adjoint operator in $L^2(M, \mu)$ with $\text{dom}(\mathcal{L}) \subset \mathcal{F}$ such that

$$\mathcal{E}(u, v) = (\mathcal{L}u, v)_{L^2}$$

for all $u \in \text{dom}(\mathcal{L})$ and $v \in \mathcal{F}$. As \mathcal{L} is non-negative definite, it determines the *heat semigroup* $\{P_t\}_{t \geq 0}$, where $P_t = e^{-t\mathcal{L}}$.

For any non-empty open subset $\Omega \subset M$, denote by $C_0(\Omega)$ the space of all continuous functions with compact supports in Ω . Let $\mathcal{F}(\Omega)$ be the closure of $\mathcal{F} \cap C_0(\Omega)$ in \mathcal{F} under $\sqrt{\mathcal{E} + \|\cdot\|_2^2}$ -norm. It is known that the form $(\mathcal{E}, \mathcal{F}(\Omega))$ is a regular Dirichlet form in $L^2(\Omega, \mu)$. Denote by \mathcal{L}_Ω and $\{P_t^\Omega\}_{t \geq 0}$ the generator and the heat semigroup of $(\mathcal{E}, \mathcal{F}(\Omega))$, respectively.

Recall that $(\mathcal{E}, \mathcal{F})$ is called *conservative* if, for any $t > 0$,

$$P_t 1 = 1 \text{ in } M.$$

By a theorem of Beurling and Deny ([15, Theorem 3.2.1]), any regular Dirichlet form admits the following decomposition

$$\mathcal{E}(u, v) = \mathcal{E}^{(L)}(u, v) + \mathcal{E}^{(J)}(u, v) + \mathcal{E}^{(K)}(u, v), \quad (2.1)$$

for all $u, v \in \mathcal{F} \cap C_0(M)$, where $\mathcal{E}^{(L)}$ is the *local* part, $\mathcal{E}^{(K)}$ the *killing* part, and $\mathcal{E}^{(J)}$ is the *jump* part that has the following form

$$\mathcal{E}^{(J)}(u, v) = \int \int_{M \times M \setminus \text{diag}} (u(x) - u(y))(v(x) - v(y)) dj(x, y),$$

where j is a jump measure defined on $M \times M \setminus \text{diag}$.

In this paper we always assume that $\mathcal{E}^{(K)} = 0$ and that the jump measure j has a symmetric density function $J(x, y)$ with respect to $\mu \times \mu$, so that the jump part $\mathcal{E}^{(J)}$ becomes

$$\mathcal{E}^{(J)}(u, v) = \int \int_{M \times M} (u(x) - u(y))(v(x) - v(y)) J(x, y) d\mu(x) d\mu(y). \quad (2.2)$$

In this case (2.2) is satisfied for all $u, v \in \mathcal{F}$.

A family $\{p_t\}_{t > 0}$ of non-negative $\mu \times \mu$ -measurable functions on $M \times M$ is called the *heat kernel* of the form $(\mathcal{E}, \mathcal{F})$ if p_t is the integral kernel of the operator P_t , that is, for any $t > 0$ and for any $f \in L^2(M, \mu)$,

$$P_t f(x) = \int_M p_t(x, y) f(y) d\mu(y) \quad (2.3)$$

for μ -almost all $x \in M$.

We introduce the following notations and conditions to be used in this paper. Fix two positive values α, β . Fix also some value $\bar{R} \in (0, \text{diam}M]$ that will be used for localization of all the hypotheses. For example, \bar{R} can be the diameter of (M, d) but not necessarily.

Denote by $B(x, r)$ the open metric ball in M centered at $x \in M$ and of radius r , that is, $B(x, r) = \{y \in M : d(x, y) < r\}$.

Definition 2.1 (Conditions (V_{\leq}) and (V_{\geq})). Condition (V_{\leq}) means that, for all $x \in M$ and all $r \in (0, \infty)$,

$$\mu(B(x, r)) \leq Cr^\alpha.$$

Condition (V_{\geq}) means that, for all $x \in M$ and all $r \in (0, \bar{R})$,

$$\mu(B(x, r)) \geq C^{-1}r^\alpha. \quad (2.4)$$

Condition (V) means that both (V_{\leq}) and (V_{\geq}) hold.

Definition 2.2 (Conditions (J_{\leq}) and (J_{\geq})). Condition (J_{\leq}) means that, for μ -almost all $x, y \in M$,

$$J(x, y) \leq Cd(x, y)^{-(\alpha+\beta)}.$$

Condition (J_{\geq}) means that

$$J(x, y) \geq C^{-1}d(x, y)^{-(\alpha+\beta)}. \quad (2.5)$$

Condition (J) means that both (J_{\leq}) and (J_{\geq}) hold.

Definition 2.3 (Condition (NLE) – near diagonal lower estimate). The heat kernel $p_t(x, y)$ of $(\mathcal{E}, \mathcal{F})$ exists and there is a constant $\delta' > 0$ such that, for any $t \in (0, \bar{R}^\beta)$ and for μ -almost all $x, y \in M$ with $d(x, y) \leq \delta't^{1/\beta}$,

$$p_t(x, y) \geq ct^{-\alpha/\beta}. \quad (2.6)$$

Definition 2.4 (Condition (DUE) – diagonal upper estimate). The heat kernel $p_t(x, y)$ of $(\mathcal{E}, \mathcal{F})$ exists and satisfies the following estimate

$$p_t(x, y) \leq Ct^{-\alpha/\beta},$$

for any $t \in (0, \bar{R}^\beta)$ and for μ -almost all $x, y \in M$.

Definition 2.5 (Condition (UE) – upper estimate). The heat kernel $p_t(x, y)$ exists and satisfies the following inequality

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}} \right)^{-(\alpha+\beta)},$$

for any $t \in (0, \bar{R}^\beta)$ and for μ -almost all $x, y \in M$.

Note that the following relation is always true:

$$t^{-\alpha/\beta} \left(1 + \frac{d(x, y)}{t^{1/\beta}} \right)^{-(\alpha+\beta)} \simeq \min \left(t^{-\alpha/\beta}, \frac{t}{d(x, y)^{\alpha+\beta}} \right),$$

where \simeq means that the ratio of the both sides is bounded from above and below by positive constants.

Definition 2.6 (Condition (LE) – lower estimate). For any $t \in (0, \bar{R}^\beta)$ and for μ -almost all $x, y \in M$,

$$p_t(x, y) \geq \frac{c}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}} \right)^{-(\alpha+\beta)}.$$

Definition 2.7 (Condition (S) – survival estimate). There exist constants $\varepsilon, \delta > 0$, such that, for any ball $B = B(x, r)$ of radius $r \in (0, \bar{R})$ and for all $t^{1/\beta} \leq \delta r$,

$$P_t^B 1 \geq \varepsilon \quad \mu\text{-a.e. in } \frac{1}{4}B.$$

For any regular Dirichlet form $(\mathcal{E}, \mathcal{F})$, there exists an associated *Hunt process* $(\{X_t\}_{t \geq 0}, \{\mathbb{P}_x\}_{x \in M})$ (cf. [15, Theorem 7.2.1, p.380]). For any open set $\Omega \subset M$, denote by τ_Ω the first exit time of X_t from Ω . Then the following identity is true

$$1 - P_t^B 1(x) = \mathbb{P}_x(\tau_B \leq t)$$

for μ -a.a. $x \in M$. Hence, the survival estimate is equivalent to

$$1 - \mathbb{P}_x(\tau_B \leq t) \geq \varepsilon.$$

The term ‘‘survival’’ refers to the meaning of $1 - \mathbb{P}_x(\tau_B \leq t)$ as the probability of the process X_t to survive in B until time t provided the killing condition is imposed outside B .

The following theorem is the main contribution of this paper.

Theorem 2.8. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$ without killing part and with the jump density J . Then*

$$(NLE) + (V_\geq) + (J_\geq) + (S) \Rightarrow (LE). \tag{2.7}$$

Essentially this theorem provides a way of obtaining of the full lower estimate (LE) of the heat kernel using the near diagonal estimate. As it was already mentioned in Introduction, under the chain condition, (NLE) implies easily the sub-Gaussian lower bound of the heat kernel. The point of Theorem 2.8 is that, taking into account the hypothesis (J_\geq) about the jump kernel (as well a technical condition (S)), one can ensure a much stronger stable-like lower bound (LE) .

Combining the Theorem 2.8 with the previous results of [23], we obtain the following equivalence for two-sided estimates of the heat kernel.

Theorem 2.9. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$ without killing part and with the jump density J . Assume also that (V) is satisfied. Then*

$$(NLE) + (J) + (S) \iff (UE) + (LE). \tag{2.8}$$

We will prove Theorems 2.8, 2.9 in Section 4.

Remark 2.10. In the forthcoming paper [16], we will prove that if the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is of jump type (that is, $\mathcal{E}^{(L)} = 0$, $\mathcal{E}^{(K)} = 0$) then, under the standing assumption (V) ,

$$(J) + (S) \iff (UE) + (LE). \tag{2.9}$$

Hence, in this case (NLE) can be dropped from (2.8). However, the proof of [16] relies on (2.8) and amounts to verifying (NLE) under the other hypotheses, and the latter is extremely involved.

If in addition $\beta < 2$, then (S) also can be dropped from (2.9) thus leading to $(UE) + (LE) \iff (J)$. The latter was also proved by Chen and Kumagai [12], [13], as was already mentioned in Introduction. For an arbitrary $\beta > 0$, we prove in [16] that in (2.9) the condition (S) can be replaced by a more convenient *generalized capacity condition*.

Remark 2.11. If $\bar{R} = \infty$ and $(\mathcal{E}, \mathcal{F})$ is a regular conservative Dirichlet form then Theorem 2.9 can be restated as follows:

$$(NLE) + (V) + (J) + (S) \iff (UE) + (LE),$$

because in this case $(UE) + (LE) \Rightarrow (V)$ by [19, Theorem 3.2].

This rest of the paper is organized as follows. In Section 3, we first prove a comparison theorem (Theorem 3.3) and then derive a comparison inequality (3.14), which is a key technical tool of this paper.

In Section 4, we prove Theorem 4.8 that provides a general way of obtaining off-diagonal lower bounds of the heat kernel using (NLE) and (S) but without (V) and (J) . In particular, this yields the main Theorem 2.8, which then implies also Theorem 2.9. Another consequence of Theorem 4.8 is Theorem 4.13 that deals with a localized version of (NLE) .

In Section 5, we give an example of application of Theorems 2.8 and 2.9 to the case when the effective resistance between points of M is positive; in particular, in this case we have $\beta > \alpha$. We show how (NLE) can be verified in this setting, which leads to equivalent conditions for (1.1) by means of effective resistance, see Theorem 5.5. The argument in this section is similar to those in [6].

3. COMPARISON INEQUALITIES

The main result of this section is a comparison theorem 3.3. This theorem implies an inequality (3.14) below that plays an important role in obtaining the lower bound of the heat kernel.

Let Ω be an open subset of M and I be an open interval in \mathbb{R} . A function $u : I \rightarrow L^2(\Omega)$ is called a *weak solution* to the heat equation

$$\partial_t u + \mathcal{L}_\Omega u = 0 \text{ in } I \times \Omega, \quad (3.1)$$

if, for any $t \in I$, the Fréchet derivative $\partial_t u$ of u exists in $L^2(\Omega)$, the function $u(t)$ belongs to $\mathcal{F}(\Omega)$, and, for any non-negative function $\psi \in \mathcal{F}(\Omega)$,

$$(\partial_t u(t), \psi)_{L^2} + \mathcal{E}(u(t), \psi) = 0. \quad (3.2)$$

A function u is called a weak *subsolution* (resp. *supersolution*) if $\partial_t u + \mathcal{L}_\Omega u \leq 0$ (resp. ≥ 0) weakly in $I \times \Omega$.

It is known that $u = P_t^\Omega f$ is a weak solution to the heat equation in $(0, \infty) \times \Omega$ (see for example [17]). Moreover, $u = P_t^\Omega f$ satisfies the heat equation also in a strong sense: for any $t > 0$, the Fréchet derivative $\partial_t u(t)$ exists in $L^2(\Omega)$, the function $u(t)$ belongs to $\text{dom}(\mathcal{L}_\Omega)$, and $\partial_t u(t) + \mathcal{L}_\Omega u(t) = 0$.

Let A be a subset of Ω . A *cutoff function* of the pair (A, Ω) is any function $\phi \in \mathcal{F}$ such that $0 \leq \phi \leq 1$ μ -a.e. in M , $\phi \equiv 1$ μ -a.e. in A , and $\phi \equiv 0$ μ -a.e. in Ω^c . Denote by $\text{cutoff}(A, \Omega)$ the set of all cutoff functions of (A, Ω) . It is known that if $(\mathcal{E}, \mathcal{F})$ is regular and A is a compact subset of Ω , then the set $\text{cutoff}(A, \Omega)$ is non-empty (see [15, Lemma 1.4.2(ii), p.29]).

We will use the following two previously known results.

Proposition 3.1 ([17, Lemma 4.4]). *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form and let Ω be an open subset of M . For any $u \in \mathcal{F}$, the following two conditions are equivalent:*

- $u_+ \in \mathcal{F}(\Omega)$.
- $u \leq v$ in M for some function $v \in \mathcal{F}(\Omega)$.

Proposition 3.2 (Parabolic maximum principle [21, Proposition 5.2]). *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 . For $T \in (0, \infty]$ and for an open subset Ω of M , let u be a weak subsolution of the heat equation in $(0, T) \times \Omega$ satisfying the following boundary and initial conditions:*

- $u_+(t, \cdot) \in \mathcal{F}(\Omega)$ for any $t \in (0, T)$;
- $u_+(t, \cdot) \xrightarrow{L^2(\Omega)} 0$ as $t \rightarrow 0$.

Then, $u(t, x) \leq 0$ for any $t \in (0, T)$ and for μ -almost all $x \in \Omega$.

The following comparison result is motivated by [22, Theorem 3.1].

Theorem 3.3 (Comparison inequality). *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in L^2 . Let Ω be a precompact open set and K be a compact subset of Ω . Set $V := \Omega \setminus K$. Assume u is a weak subsolution of the heat equation in $(0, T_0) \times V$ for some $T_0 \in (0, \infty]$, satisfying the following two conditions:*

- $u_+(t, \cdot) \in \mathcal{F}(\Omega)$ for any $t \in (0, T_0)$; (3.3)

- $u_+(t, \cdot) \xrightarrow{L^2(V)} 0$ as $t \rightarrow 0$. (3.4)

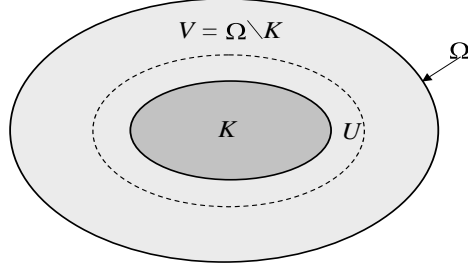


FIGURE 1. Sets Ω , K , V , U

Then for any $t \in (0, T_0)$ and μ -almost every $x \in M$,

$$u(t, x) \leq (1 - P_t^V 1_V(x)) \sup_{0 < s \leq t} \|u_+(s, \cdot)\|_{L^\infty(U)}, \quad (3.5)$$

where U is any open set such that $K \subset U \subset \Omega$ (see Fig. 1).

Proof. Inequality (3.5) is obvious if $x \in V^c = \Omega^c \cup K$. Indeed, (3.5) holds on Ω^c because $u \leq 0$ on Ω^c by (3.3), and (3.5) holds on K because $P_t^V 1_V = 0$ on K and $K \subset U$. Hence, it suffices to prove (3.5) for $x \in V$.

Fix some $T \in (0, T_0)$ and define

$$m := \sup_{0 < s \leq T} \|u_+(s, \cdot)\|_{L^\infty(U)}.$$

It suffices to prove that

$$u(t, \cdot) \leq m(1 - P_t^V 1_V) \text{ a.e. in } V \text{ for all } t \in (0, T), \quad (3.6)$$

whence then (3.5) on V follows by letting $t \rightarrow T-$.

Observe that, for any $t \in (0, T)$, the function $f := (u(t, \cdot) - m)_+$ belongs to $\mathcal{F}(V)$. Indeed, we have $f \in \mathcal{F}(\Omega)$ and $f = 0$ μ -a.e. in U . It follows by [15, Lemma 2.1.4] that, for a quasi-continuous version \tilde{f} of f ,

$$\tilde{f} = 0 \text{ q.e. in } U.$$

On the other hand, the condition $f \in \mathcal{F}(\Omega)$ implies

$$\tilde{f} = 0 \text{ q.e. in } \Omega^c.$$

(see [15, Corollary 2.3.1]). Hence, we have

$$\tilde{f} = 0 \text{ q.e. in } V^c = \Omega^c \cup K \subset \Omega^c \cup U,$$

and $f \in \mathcal{F}(V)$ follows.

Choose a function $\phi \in \text{cutoff}(\Omega, M)$ and set

$$w := u - m\phi.$$

Note that $w \in \mathcal{F}$ and w is a weak subsolution of the heat equation in $(0, T) \times V$ since so are u and $-\phi$ (the cutoff function ϕ is a supersolution of the heat equation in $(0, T) \times V$ because $\phi = 1$ in V – see [22, the last formula on p. 2622]). The initial condition

$$w_+(t, \cdot) \xrightarrow{L^2(V)} 0 \text{ as } t \rightarrow 0$$

follows from $w_+(t, \cdot) \leq u_+(t, \cdot)$ and (3.4). Since $w = u - m$ in Ω , we obtain by the above argument that w satisfies the boundary condition

$$w_+(t, \cdot) \in \mathcal{F}(V) \text{ for any } t \in (0, T). \quad (3.7)$$

By the parabolic maximum principle of Proposition 3.2 we obtain that, for any $t \in (0, T)$,

$$w(t, \cdot) \leq 0 \text{ a.e. in } V \text{ (and hence in } M), \quad (3.8)$$

that is,

$$u(t, \cdot) \leq m\phi \quad \text{a.e. in } M. \quad (3.9)$$

Now let us prove (3.5). Consider the function

$$v = u - m\phi(1 - P_t^V 1_V),$$

where m and ϕ are the same as above. Observe that v is a weak subsolution of the heat equation in $(0, T) \times V$ since u and $P_t^V 1_V$ are weak solutions. The initial condition

$$v_+(t, \cdot) \xrightarrow{L^2(V)} 0 \quad \text{as } t \rightarrow 0$$

follows from $v_+(t, \cdot) \leq u_+(t, \cdot)$ and (3.4). Since by (3.9)

$$v = u - m\phi + m\phi P_t^V 1_V \leq mP_t^V 1_V \quad \text{in } M$$

and $P_t^V 1_V \in \mathcal{F}(V)$, it follows by Proposition 3.1 that v satisfies the boundary condition

$$v_+(t, \cdot) \in \mathcal{F}(V) \quad \text{for any } t \in (0, T).$$

By the maximum principle of Proposition 3.2, we conclude that

$$v(t, \cdot) \leq 0 \quad \text{a.e. in } V \quad \text{for any } t \in (0, T), \quad (3.10)$$

which implies (3.6). \square

Lemma 3.4. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in $L^2(M, \mu)$. Let Ω be any open subset of M and K be a compact subset of Ω . Set $V := \Omega \setminus K$ and let U be any open set such that $K \subset U \subset \Omega$. Then, for any non-negative function $f \in L^\infty(\Omega)$, for any $t \in (0, \infty)$ and μ -a.e. in M ,*

$$P_t^\Omega f - P_t^V f \leq (1 - P_t^V 1_V) \sup_{0 < s \leq t} \|P_s^\Omega f\|_{L^\infty(U)}. \quad (3.11)$$

If in addition $(\mathcal{E}, \mathcal{F})$ is conservative, then, for any $t \in (0, \infty)$ and μ -a.e. in M ,

$$P_t f - P_t^V f \geq (1 - P_t^V 1_V) \inf_{0 < s \leq t} \operatorname{ess\,inf}_U P_s f. \quad (3.12)$$

It follows obviously from (3.12) that

$$P_t f \geq (1 - P_t^V 1_V) \inf_{0 < s \leq t} \operatorname{ess\,inf}_U P_s f. \quad (3.13)$$

Proof. We can assume without loss of generality that U is precompact because the estimates (3.11) and (3.12) become sharper for smaller U . Let us first prove (3.11) for precompact Ω . In this case we have $f \in L^2(\Omega)$. Set $u = P_t^\Omega f - P_t^V f$. Then u is a weak solution of the heat equation in $(0, \infty) \times V$, that obviously satisfies the boundary and initial conditions (3.3), (3.4). By Theorem 3.3 we conclude that, for any $t > 0$ and μ -a.e. in M ,

$$\begin{aligned} P_t^\Omega f - P_t^V f &\leq (1 - P_t^V 1_V) \sup_{0 < s \leq t} \|P_s^\Omega f - P_s^V f\|_{L^\infty(U)} \\ &\leq (1 - P_t^V 1_V) \sup_{0 < s \leq t} \|P_s^\Omega f\|_{L^\infty(U)}. \end{aligned}$$

Let now Ω be arbitrary. Let $\{\Omega_i\}_{i=1}^\infty$ be an exhaustion of Ω by precompact open sets, that is, $\overline{\Omega_i} \subset \Omega_{i+1}$ and $\bigcup_{i=1}^\infty \Omega_i = \Omega$. We can assume that $U \subset \Omega_1$. Set $V_i := \Omega_i \setminus K$. Using (3.11) with Ω, V being replaced by Ω_i, V_i respectively, we obtain that

$$\begin{aligned} P_t^{\Omega_i} f - P_t^{V_i} f &\leq (1 - P_t^{V_i} 1_{V_i}) \sup_{0 < s \leq t} \|P_s^{\Omega_i} f\|_{L^\infty(U)} \\ &\leq (1 - P_t^{V_i} 1_{V_i}) \sup_{0 < s \leq t} \|P_s^\Omega f\|_{L^\infty(U)}. \end{aligned}$$

Passing to the limit as $i \rightarrow \infty$ and using the facts that $P_t^{\Omega_i} f \rightarrow P_t^\Omega f$, $P_t^{V_i} f \rightarrow P_t^V f$ and $P_t^{V_i} 1_{V_i} \rightarrow P_t^V 1_V$ as $i \rightarrow \infty$, we obtain (3.11).

Let us now prove (3.12). Assume without loss of generality, that $f \leq 1$ in M . Applying (3.11) with $\Omega = M$ and with f being replaced by $1 - f$, and using the assumption $P_t 1 = 1$, we obtain

$$\begin{aligned} (1 - P_t^V 1_V) - P_t f + P_t^V f &= P_t(1 - f) - P_t^V(1 - f) \\ &\leq (1 - P_t^V 1_V) \sup_{0 < s \leq t} \|1 - P_s f\|_{L^\infty(U)} \\ &= (1 - P_t^V 1_V) \left(1 - \inf_{0 < s \leq t} \operatorname{ess\,inf}_{z \in U} P_s f(z) \right), \end{aligned}$$

whence (3.12) follows. \square

Corollary 3.5. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular conservative Dirichlet form in L^2 . Let $U, W \subset M$ be any two open sets and K be a compact set such that $K \subset U \subset W$. Then, for any $t > 0$ we have μ -a.e. in M*

$$P_t 1_W \geq (1 - P_t^{K^c} 1_{K^c}) \inf_{0 < s \leq t} \operatorname{ess\,inf}_U P_s^W 1_W. \quad (3.14)$$

Proof. Indeed, it follows from (3.13) with $V = K^c$ and $f = 1_W$ that

$$\begin{aligned} P_t 1_W &\geq (1 - P_t^V 1_V) \inf_{0 < s \leq t} \operatorname{ess\,inf}_U P_s 1_W \\ &\geq (1 - P_t^{K^c} 1_{K^c}) \inf_{0 < s \leq t} \operatorname{ess\,inf}_U P_s^W 1_W, \end{aligned}$$

which was to be proved. \square

4. HEAT KERNEL ESTIMATES

In this section, we will prove Theorems 2.8, 2.9 and 4.13. Before that, we will derive in Theorem 4.8 a lower bound of the heat kernel under conditions (S) and (NLE) for a general jump kernel, which is of an independent interest. The proof employs the key inequality (3.14) obtained above.

4.1. Three Lemmas.

Lemma 4.1. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in $L^2(M, \mu)$. Let Ω be any open subset of M . Let $f \in L^1 \cap L^2(M)$ be non-negative and let $\phi \in \mathcal{F}$ be such that $0 \leq \phi \leq 1$ in M and $\phi = 0$ in Ω . Then, for any $t > 0$,*

$$(1 - P_t^\Omega 1_\Omega, f) \geq - \int_0^t \mathcal{E}(\phi, P_s^\Omega f) ds. \quad (4.1)$$

Remark 4.2. The inequality (4.1) is non-trivial only if the jump part of \mathcal{E} does not vanish, since otherwise $\mathcal{E}(\phi, P_s^\Omega f) = 0$.

Proof. Consider the following function for all $t > 0$:

$$F(t) := (1 - P_t^\Omega 1_\Omega, f).$$

Since $f \geq 0$, we have $F(t) \geq 0$ for any $t > 0$ and, for all $t, s > 0$,

$$F(t + s) - F(s) = (P_s^\Omega 1_\Omega - P_{t+s}^\Omega 1_\Omega, f) = (1 - P_t^\Omega 1_\Omega, P_s^\Omega f) \geq 0.$$

Hence, $F(0) := \lim_{t \rightarrow 0^+} F(t) \geq 0$ and the derivative F' of F exists almost everywhere on $(0, \infty)$. We claim that, for Lebesgue-almost all $s > 0$,

$$F'(s) \geq -\mathcal{E}(\phi, P_s^\Omega f). \quad (4.2)$$

Indeed, by the symmetry of $P_s^\Omega - P_{t+s}^\Omega$, we have

$$F(t + s) - F(s) = (1, P_s^\Omega f - P_{t+s}^\Omega f),$$

and, since $\phi = 0$ in Ω and $0 \leq \phi \leq 1$,

$$\begin{aligned} F(t+s) - F(s) &= (1 - \phi, P_s^\Omega f - P_{t+s}^\Omega f) \\ &\geq (1 - \phi, P_s^\Omega f - P_t P_s^\Omega f) \\ &= (1, P_s^\Omega f - P_t P_s^\Omega f) - (\phi, P_s^\Omega f - P_t P_s^\Omega f) \\ &= (1 - P_t 1, P_s^\Omega f) - (\phi, P_s^\Omega f - P_t P_s^\Omega f) \\ &\geq -(\phi, P_s^\Omega f - P_t P_s^\Omega f). \end{aligned}$$

Since $\phi, P_s^\Omega f \in \mathcal{F}$, dividing this inequality by t and letting $t \rightarrow 0$, we obtain, for Lebesgue-almost all $s > 0$, that

$$F'(s) = \lim_{t \rightarrow 0} \frac{1}{t} (F(t+s) - F(s)) \geq -\lim_{t \rightarrow 0} \frac{1}{t} (\phi, P_s^\Omega f - P_t P_s^\Omega f) = -\mathcal{E}(\phi, P_s^\Omega f),$$

thus proving (4.2). It follows that, for any $t > 0$,

$$F(t) \geq F(t) - F(0) \geq \int_0^t F'(s) ds \geq \int_0^t -\mathcal{E}(\phi, P_s^\Omega f) ds,$$

which finishes the proof. \square

Lemma 4.3. *Let functions $f, g \in \mathcal{F}$ have disjoint supports $F = \text{supp } f$ and $G = \text{supp } g$. Then*

$$\mathcal{E}(f, g) = -2 \int_F \int_G f(x)g(y)J(x, y)d\mu(y)d\mu(x).$$

Proof. Clearly, $\mathcal{E}^{(L)}(f, g) = 0$ and $\mathcal{E}^{(K)}(f, g) = 0$ so that

$$\begin{aligned} \mathcal{E}(f, g) &= \mathcal{E}^{(J)}(f, g) \\ &= \int \int_{M \times M} (f(x) - f(y))(g(x) - g(y))J(x, y)d\mu(y)d\mu(x) \\ &= \int_{F^c \times F} + \int_{F \times F^c} + \int_{F \times F} + \int_{F^c \times F^c} \dots \\ &= 2 \int_F \int_{F^c} (f(x) - f(y))(g(x) - g(y))J(x, y)d\mu(y)d\mu(x) \\ &= -2 \int_F \int_G f(x)g(y)J(x, y)d\mu(y)d\mu(x). \end{aligned}$$

\square

Lemma 4.4. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in $L^2(M, \mu)$ with a jump kernel J . Let x_0, y_0 be two points in M and $r > 0$ be such that the closed balls $\overline{B}(x_0, r)$ and $\overline{B}(y_0, r)$ are disjoint, and $\Omega \subset M$ be an open subset of $B(y_0, r)$. Set $U = B(x_0, r)$ and $V = M \setminus \overline{\Omega}$ (see Fig. 2). Then, for any $t > 0$ and for any non-negative function $f \in L^1 \cap L^2(M)$,*

$$(1 - P_t^V 1_V, f) \geq 2\mu(\Omega) \text{ess inf}_{x \in U, y \in \Omega} J(x, y) \int_0^t (f, P_s^V 1_U) ds. \quad (4.3)$$

If in addition the condition (S) is satisfied, then, for any $t > 0$ such that $t^{1/\beta} \leq \delta r$,

$$1 - P_t^V 1_V \geq 2\epsilon t \cdot \mu(\Omega) \text{ess inf}_{x \in U, y \in \Omega} J(x, y) \quad \mu\text{-a.e. in } B(x_0, r/4), \quad (4.4)$$

where δ, ϵ are the constants from the condition (S).

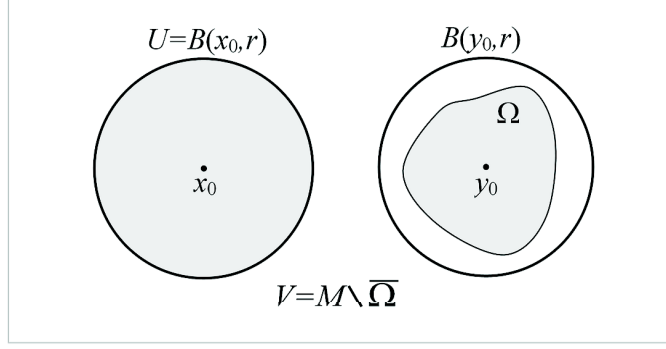


FIGURE 2. Sets U and V

Proof. Let $\phi \in \text{cutoff}(K, \Omega)$ where K is a precompact subset of Ω . Using (4.1) (with $\Omega = V$) and noticing that $\phi = 0$ in V , we obtain

$$(1 - P_t^V 1_V, f) \geq \int_0^t -\mathcal{E}(\phi, P_s^V f) ds. \quad (4.5)$$

Since $P_s^V f$ is supported in $\bar{V} = \Omega^c$ and ϕ is supported in Ω , we obtain by Lemma 4.3

$$\begin{aligned} -\mathcal{E}(\phi, P_s^V f) &= 2 \int_{\bar{V}} \int_{\Omega} P_s^V f(x) \phi(y) J(x, y) d\mu(y) d\mu(x) \\ &\geq 2 \int_U \int_{\Omega} P_s^V f(x) \phi(y) J(x, y) d\mu(y) d\mu(x) \\ &\geq 2 \operatorname{ess\,inf}_{x \in U, y \in \Omega} J(x, y) (P_s^V f, 1_U) \|\phi\|_{L^1} \\ &= 2 \operatorname{ess\,inf}_{x \in U, y \in \Omega} J(x, y) (f, P_s^V 1_U) \|\phi\|_{L^1}. \end{aligned} \quad (4.6)$$

Allowing K to exhaust Ω so that $\|\phi\|_{L^1} \rightarrow \mu(\Omega)$ and substituting this into (4.5), we obtain (4.3).

Let now (S) be satisfied. Assuming that $t^{1/\beta} \leq \delta r$, we obtain by (S) that, for any $0 < s \leq t$,

$$P_s^V 1_U \geq P_s^U 1_U \geq \varepsilon \quad \mu\text{-a.e. in } \frac{1}{4}U$$

whence

$$(f, P_s^V 1_U) \geq \varepsilon \int_{\frac{1}{4}U} f d\mu.$$

Substituting into (4.3), we obtain

$$(1 - P_t^V 1_V, f) \geq 2\mu(\Omega) \operatorname{ess\,inf}_{x \in U, y \in \Omega} J(x, y) \varepsilon t \int_{\frac{1}{4}U} f d\mu,$$

whence (4.4) follows because f is arbitrary. \square

Note that in the setting of Lemma 4.4 we do not assume that balls have finite measure. Hence, in the right hand sides of (4.3) and (4.4) we may have expression of the form $\infty \cdot 0$ that is always set to be 0.

4.2. Condition (S) and conservativeness. The main result of this section is Lemma 4.6. However, we start with existence of cutoff functions as in the next statement. Note that in this section we work with arbitrary regular Dirichlet forms.

For any set $A \subset M$ and any $r > 0$, define its r -neighborhood by

$$A^r := \{y \in M : d(x, y) < r \text{ for some } x \in A\}.$$

Lemma 4.5. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(M, \mu)$. Assume that any metric ball in (M, d) has a finite measure. Assume also that (S) is satisfied. Then, for any bounded measurable set $A \subset M$ and for any $r > 0$, the set cutoff(A, A^r) of cutoff functions of the couple (A, A^r) is non-empty.*

Proof. The idea of the proof follows from [1, Lemma 5.4]. It suffices to prove the claim for small r , say, for $r < \bar{R}$. Let $\lambda = r^{-\beta}$ and $\Omega = A^r$. By the assumption of the finiteness of measure of all balls, we have $1_\Omega \in L^2$. Hence, by [15, Theorem 4.4.1],

$$h := G_\lambda^\Omega 1_\Omega = \int_0^\infty e^{-\lambda t} P_t^\Omega 1_\Omega dt \in \mathcal{F}(\Omega),$$

where G_λ^Ω is the resolvent operator corresponding to $(\mathcal{E}, \mathcal{F}(\Omega))$. We will construct a function $\phi \in \text{cutoff}(A, \Omega)$ by using the function h .

Fix some $x \in A$ and consider the ball $B := B(x, r)$. Since $B \subset \Omega$, we have by (S) that, for all $0 < t \leq (\delta r)^\beta$,

$$P_t^\Omega 1_\Omega \geq P_t^B 1_B \geq \varepsilon \quad \mu\text{-a.e. in } \frac{1}{4}B.$$

Hence, for any $0 \leq f \in L^1 \cap L^2(\frac{1}{4}B)$, we obtain

$$(h, f) \geq \int_0^{(\delta r)^\beta} e^{-\lambda t} (P_t^B 1_B, f) dt \geq \varepsilon \|f\|_{L^1} \int_0^{(\delta r)^\beta} e^{-r^{-\beta} t} dt \geq c^{-1} r^\beta \|f\|_{L^1},$$

where $c = (\varepsilon \delta^\beta e^{-\delta^\beta})^{-1}$. Since A can be covered by at most countable family of the balls like $\frac{1}{4}B$, we obtain that

$$h \geq c^{-1} r^\beta, \quad \mu\text{-a.e. in } A.$$

Finally, we obtain a cutoff function

$$\phi := 1 \wedge \frac{ch}{r^\beta} \in \text{cutoff}(A, \Omega).$$

□

Lemma 4.6. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form without killing part. Assume that any metric ball in (M, d) has a finite measure. Assume also that (S) is satisfied. Then the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is conservative.*

Proof. Set

$$T := (\delta (\bar{R} \wedge 1) / 2)^\beta$$

and let us prove that, for any ball Ω in M of radius $R \geq \bar{R} \wedge 1$ and for any $t \leq T$,

$$P_t 1_\Omega \geq \varepsilon, \quad \mu\text{-a.e. in } \frac{1}{4}\Omega, \quad (4.7)$$

where $\varepsilon > 0$ is the same as in (S). Consequently, we obtain that $P_t 1 \geq \varepsilon$ μ -a.e. on M for all $t \in (0, T)$.

If $R < \bar{R}$ then (4.7) holds just by (S). Assume further that $R \geq \bar{R}$; in particular, in this case $\bar{R} < \infty$. Set $r = \frac{1}{2}\bar{R}$. Then, for any ball $B = B(x, r)$ for all $t^{1/\beta} \leq \delta r$, we have by (S)

$$P_t^B 1_B \geq \varepsilon \quad \mu\text{-a.e. in } \frac{1}{4}B.$$

In particular, this holds for all $t \leq T$. Applying this inequality to any $B(x, r)$ with $x \in \frac{1}{4}\Omega$ and noticing that $B \subset \Omega$ we obtain

$$P_t 1_\Omega \geq P_t^B 1_B \geq \varepsilon \quad \mu\text{-a.e. in } \frac{1}{4}\Omega.$$

Since balls of type $\frac{1}{4}B$ cover all $\frac{1}{4}\Omega$, we obtain (4.7).

The rest of the proof is divided into two cases.

Case 1: $\text{diam}M < \infty$. In this case, M coincides with some ball and, hence, $\mu(M) < \infty$ and $1 \in L^2(M)$. Consequently, we have $P_t 1 \in \mathcal{F}$ for $t > 0$. Since $P_t 1 \geq \varepsilon$ for all $t \in (0, T)$, we obtain

$$1 = 1 \wedge \frac{P_t 1}{\varepsilon} \in \mathcal{F}.$$

Since \mathcal{E} has no killing term, we obtain

$$\mathcal{E}(1, 1) = 0.$$

Therefore, by [15, (iii) \Rightarrow (i) of Theorem 1.6.3, p.58], the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is recurrent and then, by [15, Lemma 1.6.5, p.56], the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is conservative.

Case 2: $\text{diam}M = \infty$.

Fix a point $x_0 \in M$, some $r > 0$ and set $U = B(x_0, r)$. We will prove that, for all $t \in (0, T)$,

$$\frac{P_t 1 - \varepsilon}{1 - \varepsilon} \geq P_t^U 1_U, \quad (4.8)$$

where ε is the constant from (4.7). Indeed, if (4.8) is known already, then passing to the limit as $r \rightarrow \infty$ we will obtain

$$\frac{P_t 1 - \varepsilon}{1 - \varepsilon} \geq P_t 1,$$

which implies that $P_t 1 \geq 1$ and, hence, $P_t 1 = 1$ for all $t \in (0, T)$. By the semigroup property, we conclude that $P_t 1 = 1$ for all $t > 0$.

In order to prove (4.8), choose $R > r$ and set $B_R := B(x_0, R)$ and $\Omega := B_{8R}$. By Lemma 4.5, there is a function $\phi \in \text{cutoff}(B_R, B_{2R})$. Consider the function

$$u := P_t^U 1_U - \frac{P_t 1_\Omega - \varepsilon \phi}{1 - \varepsilon} \quad (4.9)$$

and obtain an upper bound for the L^2 -norm of u_+ . Our argument is motivated by the proof of [21, Propostion 5.2].

We have the following facts.

- (1) By hypothesis, we have $\mu(\Omega) < \infty$ and, hence, $1_U, 1_\Omega \in L^2$. Then the functions $P_t 1_\Omega$ and $P_t^U 1_U$ are weak solutions to the heat equation in $\mathbb{R}_+ \times U$, which implies that, for all $\psi \in \mathcal{F}(U)$,

$$(\partial_t u, \psi) + \mathcal{E}(u, \psi) = \frac{\varepsilon}{1 - \varepsilon} \mathcal{E}(\phi, \psi) \quad (4.10)$$

(cf. (3.2)).

- (2) Since $\phi \leq 1$, we obtain by (4.7) that in $\frac{1}{4}\Omega = B_{2R}$ and, for any $t < T$,

$$P_t 1_\Omega \geq \varepsilon \phi.$$

Since $\phi = 0$ outside B_{2R} , we see that this inequality holds on M . Consequently, $u \leq P_t^U 1_U$ and, by Proposition 3.1, $u_+ \in \mathcal{F}(U)$.

- (3) By the strong continuity of $P_t 1_\Omega$ and $P_t^U 1_U$ in t , we obtain,

$$P_t 1_\Omega \xrightarrow{L^2} 1_\Omega \quad \text{and} \quad P_t^U 1_U \xrightarrow{L^2} 1_U, \quad \text{as } t \rightarrow 0.$$

Hence, by the definition of u , we obtain

$$u(t, \cdot) \xrightarrow{L^2(U)} 0 \quad \text{as } t \rightarrow 0.$$

Therefore, we can apply Lemma 6.1 with $f = \frac{\varepsilon}{1 - \varepsilon} \phi$ and obtain that

$$\|u_+(t, \cdot)\|_{L^2(U)} \leq \frac{2\varepsilon}{1 - \varepsilon} \int_0^t \mathcal{E}(\phi, u_+(s, \cdot)) ds. \quad (4.11)$$

Fix $s > 0$ and denote for simplicity $v = u_+(s, \cdot)$. In order to estimate $\mathcal{E}(\phi, v)$, observe that $\phi = 1$ on B_R while $v \in \mathcal{F}(U)$ and, hence, $\text{supp } v \subset \bar{U}$. Therefore, for the strongly

local part $\mathcal{E}^{(L)}$ of \mathcal{E} , we have $\mathcal{E}^{(L)}(\phi, v) = 0$. For the jump part $\mathcal{E}^{(J)}$ with the jump measure j , we obtain using that $\phi = 1$ on B_R , $v = 0$ on U^c , $v \leq 1$ and $U \subset B_R$, we obtain

$$\begin{aligned} \mathcal{E}^{(J)}(\phi, v) &= \int_{M \times M} (\phi(x) - \phi(y))(v(x) - v(y)) dj(x, y) \\ &= \left(\int_{U^c \times U} + \int_{U \times U^c} \right) (\phi(x) - \phi(y))(v(x) - v(y)) dj(x, y) \\ &= 2 \int_{U \times U^c} (\phi(x) - \phi(y))(v(x) - v(y)) dj(x, y) \\ &= 2 \int_{U \times U^c} (1 - \phi(y))v(x) dj(x, y) \\ &\leq 2 \int_{U \times B_R^c} (1 - \phi(y)) dj(x, y) \leq 2j(U, B_R^c). \end{aligned}$$

Since \mathcal{E} has no killing part, we obtain

$$\mathcal{E}(\phi, u_+(s, \cdot)) \leq 2j(U, B_R^c).$$

Substituting into (4.11), we obtain

$$\|u_+(t, \cdot)\|_{L^2(U)} \leq \frac{4\varepsilon t}{1 - \varepsilon} j(U, B_R^c). \quad (4.12)$$

Let us make sure that the measure $j(U, B_R^c)$ is finite. By Lemma 4.5, there is $\eta \in \text{cutoff}(U, B_R)$. Then we have

$$\begin{aligned} \mathcal{E}(\eta, \eta) &= \mathcal{E}^{(L)}(\eta, \eta) + \mathcal{E}^{(J)}(\eta, \eta) \\ &\geq \int_{U \times B_R^c} (\eta(x) - \eta(y))^2 dj(x, y) \geq j(U, B_R^c), \end{aligned}$$

in particular, $j(U, B_R^c) < \infty$. Hence, we obtain that

$$j(U, B_R^c) \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Substituting into (4.12), using the definition (4.9) of u and Fatou's lemma, we obtain

$$\left\| \left(P_t^U 1_U - \frac{P_t 1 - \varepsilon}{1 - \varepsilon} \right)_+ \right\|_{L^2} = 0,$$

which is equivalent to (4.8). \square

Remark 4.7. The above proof works also in the following situation. Assume that $\text{cutoff}(B_R, B_{2R})$ is non-empty for all $R > 0$ and that (V_{\geq}) and (NLE) hold. The latter two conditions imply trivially (4.7). Hence, applying the above argument, we also obtain that $(\mathcal{E}, \mathcal{F})$ is conservative.

4.3. Lower bounds of the heat kernel. We now estimate the heat kernel from below by using conditions (S) and (NLE) .

Theorem 4.8. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in L^2 with a jump kernel J , and that conditions (S) and (NLE) hold. Then there exist constants $c_1 > 0$ and $0 < c_2 < 1$ such that, for all $t \in (0, \bar{R}^\beta)$ and for μ -almost all $x, y \in M$,*

$$p_t(x, y) \geq \frac{c_1}{t^{\alpha/\beta}} (\{t\mu(B(y, c_2 t^{1/\beta})) \text{ess inf}_{\substack{z \in B(x, c_2 t^{1/\beta}) \\ w \in B(y, c_2 t^{1/\beta})}} J(z, w)\} \wedge 1). \quad (4.13)$$

Proof. Let δ and δ' be the constants from conditions (S) and (NLE) respectively. If $d(x, y) \leq \delta' t^{1/\beta}$, then (4.13) follows trivially from (NLE) since the term in parenthesis on the right-hand side of (4.13) is bounded by 1.

Let us prove (4.13) in the case $d(x, y) > \delta' t^{1/\beta}$. Set

$$\delta_1 = \delta' 2^{-\left(1+\frac{1}{\beta}\right)} \quad \text{and} \quad \sigma = \min\left(\frac{1}{2}, \left(\frac{\delta\delta_1}{5}\right)^\beta\right). \quad (4.14)$$

Fix $t > 0$ and two points $x_0, y_0 \in M$ such that

$$d(x_0, y_0) > \delta' t^{1/\beta}.$$

Set $r = \delta_1 t^{1/\beta}$ and consider the balls $B_{x_0} = B(x_0, r)$, $B_{y_0} = B(y_0, r)$. It follows from (4.14) that $\delta' > 2\delta_1$ and, hence, $d(x_0, y_0) > 2r$. Therefore, the balls $\overline{B_{x_0}}$ and $\overline{B_{y_0}}$ are disjoint. It suffices to prove (4.13) for μ -a.a. $x \in \frac{1}{20}B_{x_0}$ and $y \in \frac{1}{20}B_{y_0}$, because the set

$$\left\{ (x, y) \in M \times M : d(x, y) > \delta' t^{1/\beta} \right\}$$

can be covered by a countable family of the sets of the type $\frac{1}{20}B_{x_0} \times \frac{1}{20}B_{y_0}$ where x_0, y_0 are as above.

By the semigroup identity, we have, for μ -a.a. $x \in B_{x_0}, y \in B_{y_0}$,

$$\begin{aligned} p_t(x, y) &= \int_M p_{\sigma t}(x, z) p_{(1-\sigma)t}(z, y) d\mu(z) \geq \int_{B_{y_0}} p_{\sigma t}(x, z) p_{(1-\sigma)t}(z, y) d\mu(z) \\ &\geq \operatorname{ess\,inf}_{y', z \in B_{y_0}} p_{(1-\sigma)t}(z, y') \int_{B_{y_0}} p_{\sigma t}(x, z) d\mu(z) \\ &= \operatorname{ess\,inf}_{y', z \in B_{y_0}} p_{(1-\sigma)t}(z, y') \cdot P_{\sigma t} 1_{B_{y_0}}(x). \end{aligned} \quad (4.15)$$

We have $d(y', z) < 2r = 2\delta_1 t^{1/\beta}$. Since by (4.14) $\sigma \leq \frac{1}{2}$ and

$$2\delta_1 = \delta' 2^{-1/\beta} \leq \delta' (1-\sigma)^{1/\beta}, \quad (4.16)$$

we obtain

$$d(y', z) < \delta' ((1-\sigma)t)^{1/\beta}.$$

By (NLE), we obtain, for μ -a.a. $y', z \in B_{y_0}$, that

$$p_{(1-\sigma)t}(z, y') \geq c' t^{-\alpha/\beta},$$

where $c' = c(1-\sigma)^{-\alpha/\beta}$. It follows from (4.15) that, for μ -a.a. $x \in B_{x_0}$ and $y \in B_{y_0}$,

$$p_t(x, y) \geq c' t^{-\alpha/\beta} P_{\sigma t} 1_{B_{y_0}}(x). \quad (4.17)$$

We estimate $P_{\sigma t} 1_{B_{y_0}}(x)$ from below by means of the condition (S). Let Ω be a precompact open subset of $\frac{1}{5}B_{y_0}$. Since by Lemma 4.6, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is conservative, we can apply Corollary 3.5. Using (3.14) with $W = B_{y_0}$, $U = \frac{1}{4}B_{y_0}$ and $K = \overline{\Omega}$ and with t being replaced by σt , we obtain, for μ -a.a. $x \in B_{x_0}$,

$$P_{\sigma t} 1_{B_{y_0}}(x) \geq (1 - P_{\sigma t}^{K^c} 1_{K^c}(x)) \inf_{0 < s \leq \sigma t} \operatorname{ess\,inf}_{\frac{1}{4}B_{y_0}} P_s^{B_{y_0}} 1_{B_{y_0}}. \quad (4.18)$$

Since by (4.14)

$$\sigma^{1/\beta} \leq \frac{1}{5} \delta \delta_1 < \delta \delta_1, \quad (4.19)$$

we obtain that, for any $0 < s \leq \sigma t$,

$$s^{1/\beta} \leq (\sigma t)^{1/\beta} < \delta \delta_1 t^{1/\beta} = \delta r.$$

Thus, by condition (S),

$$\inf_{0 < s \leq \sigma t} \operatorname{ess\,inf}_{\frac{1}{4}B_{y_0}} P_s^{B_{y_0}} 1_{B_{y_0}} \geq \varepsilon. \quad (4.20)$$

In order to estimate $1 - P_{\sigma t}^{K^c} 1$ let us apply (4.4) for disjoint balls $\frac{1}{5}B_{x_0}$ and $\frac{1}{5}B_{y_0}$. Set $V = M \setminus \bar{\Omega} = K^c$. Since by (4.19)

$$(\sigma t)^{1/\beta} = \sigma^{1/\beta} \frac{r}{\delta_1} \leq \delta \frac{r}{5},$$

we can apply (4.4) with t and r being replaced by σt and $r/5$. We obtain, for μ -a.a. $x \in \frac{1}{20}B_{x_0}$,

$$1 - P_{\sigma t}^V 1_V(x) \geq 2\varepsilon \sigma t \cdot \mu(\Omega) \operatorname{ess\,inf}_{x' \in \frac{1}{5}B_{x_0}, y' \in \frac{1}{5}B_{y_0}} J(x', y') \quad . \quad (4.21)$$

Combining (4.18), (4.20) and (4.21), we obtain, for μ -a.a. $x \in \frac{1}{20}B_{x_0}$,

$$P_{\sigma t} 1_{B_{y_0}}(x) \geq 2\varepsilon^2 \sigma t \cdot \mu(\Omega) \operatorname{ess\,inf}_{x' \in \frac{1}{5}B_{x_0}, y' \in \frac{1}{5}B_{y_0}} J(x', y').$$

Substituting this into (4.17), we obtain that, for μ -almost all $x \in \frac{1}{20}B_{x_0}$ and $y \in B_{y_0}$,

$$p_t(x, y) \geq c' t^{-\alpha/\beta} \cdot 2\varepsilon^2 \sigma t \cdot \mu(\Omega) \operatorname{ess\,inf}_{x' \in \frac{1}{5}B_{x_0}, y' \in \frac{1}{5}B_{y_0}} J(x', y'), \quad (4.22)$$

which finishes the proof of (4.13) by taking $\Omega \uparrow \frac{1}{5}B_{y_0}$. \square

We are now in a position to prove Theorem 2.8.

Proof of Theorem 2.8. The required estimate (2.7) follows from (4.13) when applying the conditions (V_{\geq}) and (J_{\geq}) . Indeed, by (V_{\geq}) we have

$$\mu(B(y, c_2 t^{1/\beta})) \geq c(c_2 t^{1/\beta})^\alpha = c' t^{\alpha/\beta}.$$

For $z \in B(x, c_2 t^{1/\beta})$ and $w \in B(y, c_2 t^{1/\beta})$, we have

$$\begin{aligned} d(z, w) &\leq d(z, x) + d(x, y) + d(y, w) \leq 2c_2 t^{1/\beta} + d(x, y) \\ &\leq c t^{1/\beta} \left(1 + \frac{d(x, y)}{t^{1/\beta}} \right). \end{aligned}$$

By condition (J_{\geq}) we obtain

$$\begin{aligned} \operatorname{ess\,inf}_{\substack{z \in B(x, c_2 t^{1/\beta}) \\ w \in B(y, c_2 t^{1/\beta})}} J(z, w) &\geq \operatorname{ess\,inf}_{\substack{z \in B(x, c_2 t^{1/\beta}) \\ w \in B(y, c_2 t^{1/\beta})}} c d(z, w)^{-(\alpha+\beta)} \\ &\geq c' t^{-(\alpha+\beta)/\beta} \left(1 + \frac{d(x, y)}{t^{1/\beta}} \right)^{-(\alpha+\beta)}, \end{aligned}$$

whence

$$t \mu(B(y, c_2 t^{1/\beta})) \operatorname{ess\,inf}_{\substack{z \in B(x, c_2 t^{1/\beta}) \\ w \in B(y, c_2 t^{1/\beta})}} J(z, w) \geq c'' \left(1 + \frac{d(x, y)}{t^{1/\beta}} \right)^{-(\alpha+\beta)}.$$

Substituting into (4.13), we obtain

$$p_t(x, y) \geq \frac{c}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}} \right)^{-(\alpha+\beta)},$$

thus proving (LE). \square

4.4. **Proof of Theorem 2.9.** We start the proof of Theorem 2.9 with the following lemma.

Lemma 4.9. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 with a jump kernel J . Then*

$$(LE) \Rightarrow (J_{\geq}) \quad \text{and} \quad (UE) \Rightarrow (J_{\leq}). \quad (4.23)$$

Proof. Let $A, B \subset M$ be any two disjoint compact sets. Since $d(A, B) > 0$, it follows from (LE) , that, for all small enough t and for μ -a.a. $x \in A, y \in B$,

$$p_t(x, y) \geq ctd(x, y)^{-(\alpha+\beta)}.$$

Let $f, g \in \mathcal{F}$ be any two non-negative functions with $\text{supp } f \subset A$ and $\text{supp } g \subset B$. By Lemma 4.3 we have

$$-\mathcal{E}(f, g) = 2 \int_A \int_B f(x)g(y)J(x, y)d\mu(y)d\mu(x).$$

On the other hand, by (LE) ,

$$\begin{aligned} -\mathcal{E}(f, g) &= \lim_{t \rightarrow 0} \frac{1}{t} (P_t f - f, g) = \lim_{t \rightarrow 0} \frac{1}{t} (P_t f, g) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \int_A \int_B p_t(x, y) f(x)g(y) d\mu(y) d\mu(x) \\ &\geq \liminf_{t \rightarrow 0} \frac{1}{t} \int_A \int_B ctd(x, y)^{-(\alpha+\beta)} f(x)g(y) d\mu(y) d\mu(x) \\ &= \int_A \int_B cd(x, y)^{-(\alpha+\beta)} f(x)g(y) d\mu(y) d\mu(x), \end{aligned}$$

which implies

$$\int_{A \times B} f(x)g(y)J(x, y)d\mu(y)d\mu(x) \geq \int_A \int_B \frac{c}{2} d(x, y)^{-(\alpha+\beta)} f(x)g(y) d\mu(y) d\mu(x).$$

Since f, g are arbitrary, the condition (J_{\geq}) follows. The implication $(UE) \Rightarrow (J_{\leq})$ is proved in the same way. \square

The latter implication was also proved in [8, (a) \Rightarrow (c) of Theorem 1.2], however, assuming in addition that $(\mathcal{E}, \mathcal{F})$ was conservative. Here we have dropped this assumption.

Lemma 4.10. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(M, \mu)$. Then,*

$$(NLE) + (UE) + (V) \Rightarrow (S). \quad (4.24)$$

Remark 4.11. Assuming that all metric balls in M are precompact and $(\mathcal{E}, \mathcal{F})$ is conservative, the implication $(UE) + (V_{\leq}) \Rightarrow (S)$ was proved in [19, (3.6), p.2072] and [17, Theorem 3.1, p.96]. The proof below is motivated by [22, Lemma 6.1].

Proof. Let $B := B(x_0, r) \subset M$ be a ball of radius $r \in (0, \bar{R})$. We need to prove that, for some $\varepsilon, \delta \in (0, 1)$ and for all $0 < t \leq (\delta r)^\beta$,

$$P_t^B 1_B \geq \varepsilon \quad \mu\text{-a.e. in } \frac{1}{4}B.$$

By (V_{\leq}) , we have $\mu(B) < \infty$. Hence, by [22, (4.1), p.2626] with $\Omega = M, U = B, K = \overline{\frac{3}{4}B}^c$ and $f = 1_{\frac{1}{2}B}$, we obtain, for all $t > 0$,

$$P_t^B 1_{\frac{1}{2}B}(x) \geq P_t 1_{\frac{1}{2}B}(x) - \sup_{0 < s \leq t} \|P_s 1_{\frac{1}{2}B}\|_{L^\infty(\overline{\frac{3}{4}B}^c)}, \quad \text{for } \mu\text{-a.a. } x \in B. \quad (4.25)$$

We need to estimate the two terms in the right hand side of the above inequality. Note that, for all $t < r^\beta$ and $x \in \frac{1}{4}B$,

$$B(x, \frac{1}{4}t^{1/\beta}) \subset \frac{1}{2}B.$$

By (NLE), we have

$$p_t(x, y) \geq ct^{-\alpha/\beta}$$

for μ -a.e. $x, y \in M$ such that $d(x, y) \leq \delta' t^{1/\beta}$.

Setting $\delta'' = \delta' \wedge \frac{1}{4}$, we obtain, for all $t < r^\beta$ and μ -a.a. $x \in \frac{1}{4}B$ that

$$\begin{aligned} P_t 1_{\frac{1}{2}B}(x) &= \int_{\frac{1}{2}B} p_t(x, y) d\mu(y) \geq \int_{B(x, \delta'' t^{1/\beta})} p_t(x, y) d\mu(y) \\ &\geq ct^{-\alpha/\beta} \mu(B(x, \delta'' t^{1/\beta})) \geq c_0, \end{aligned} \quad (4.26)$$

where the constant $c_0 > 0$ independent of t and x .

On the other hand, we have, for any $y \in \frac{3}{4}B^c$ and $z \in \frac{1}{2}B$,

$$d(y, z) \geq d(x_0, y) - d(x_0, z) \geq \frac{r}{4}.$$

Fix some $\delta \in (0, 1)$ to be specified below. For all $0 < s \leq t \leq (\delta r)^\beta$ and for μ -a.a. y and z as above, we obtain by (UE)

$$p_t(y, z) \leq \frac{Cs}{d(y, z)^{\alpha+\beta}} \leq \frac{Ct}{(r/4)^{\alpha+\beta}} \leq \delta^\beta \frac{C'}{r^\alpha}.$$

Using (V_{\leq}), we obtain for μ -a.a. $y \in \frac{3}{4}B^c$ that

$$P_s 1_{\frac{1}{2}B}(y) = \int_{\frac{1}{2}B} p_s(y, z) d\mu(z) \leq \delta^\beta \frac{C'}{r^\alpha} \mu\left(\frac{1}{2}B\right) \leq C'' \delta^\beta.$$

Choosing δ small enough, we obtain that

$$C'' \delta^\beta \leq \frac{1}{2} c_0,$$

whence it follows that, for all $t \leq (\delta r)^\beta$,

$$\sup_{0 < s \leq t} \|P_s 1_{\frac{1}{2}B}\|_{L^\infty(\frac{3}{4}B^c)} \leq \frac{c_0}{2}.$$

Combining the above inequality, (4.26) and (4.25), we obtain, for all $r \in (0, \bar{R})$ and $t \leq (\delta r)^\beta$,

$$P_t^B 1_{\frac{1}{2}B}(x) \geq c_0 - \frac{c_0}{2} = \frac{c_0}{2}, \quad \text{for } \mu\text{-a.a. } x \in \frac{1}{4}B,$$

which implies (S). □

Proof of Theorem 2.9. The implication

$$(V) + (UE) + (LE) \Rightarrow (NLE) + (J) + (S)$$

follows from Lemmas 4.10 and 4.9 and the trivial implication (LE) \Rightarrow (NLE).

Let us prove the opposite implication in the form

$$(V) + (NLE) + (J) + (S) \Rightarrow (UE) + (LE).$$

The lower bound (LE) here holds by Theorem 2.8. To obtain (UE), we use the result of [23, Cor. 2.7] that says

$$(V) + (J) + (S) \Rightarrow (UE). \quad (4.27)$$

However, the proof in [23] uses the following additional assumptions: $\bar{R} = \infty$; all metric balls in M are precompact; $(\mathcal{E}, \mathcal{F})$ is conservative. The analysis of that proof (that is partly based also on [22]) shows that the same argument goes through without change also if $\bar{R} < \infty$. The precompactness of the balls was used only to ensure the existence of cutoff functions of bounded sets in M , which in our case follows by Lemma 4.5 from (S)+(V). Finally, the conservativeness of $(\mathcal{E}, \mathcal{F})$ follows by Lemma 4.6 also from (S)+(V). Hence, we have (2.9), which finishes the proof. □

4.5. **Condition (LLE).** In this section we remove the technical condition (S) in (2.8) at expense of replacing of (NLE) by a stronger hypothesis (LLE) that was introduced in [20].

Definition 4.12 (Condition (LLE) – localized lower estimate). there exist $c > 0$ and $\delta'' \in (0, 1)$ such that, for any ball $B := B(x_0, r)$ with $r \in (0, \bar{R})$, the heat kernel $p_t^B(x, y)$ exists and satisfies for any $t^{1/\beta} \leq \delta'' r$ the following estimate:

$$p_t^B(x, y) \geq ct^{-\alpha/\beta} \quad \text{for } \mu\text{-a.a. } x, y \in B(x_0, \delta'' t^{1/\beta}).$$

Theorem 4.13. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$ with a jump kernel J . Assume that (M, d, μ) satisfies (V). Then the following equivalences hold:

$$(UE) + (NLE) \iff (LLE) + (J_{\leq}) \tag{4.28}$$

and

$$(UE) + (LE) \iff (LLE) + (J). \tag{4.29}$$

Note that the equivalence (4.28) is a non-local analog of the result in [20, Theorem 4.2] obtained for *local* Dirichlet forms.

Proof. Let us first show that

$$(LLE) + (V_{\geq}) \Rightarrow (S). \tag{4.30}$$

Let $B := B(x_0, r)$ be any ball with radius $r \in (0, \bar{R})$, and let $t > 0$ be such that

$$t^{1/\beta} \leq \frac{1}{4} \delta'' r. \tag{4.31}$$

We need to prove that, for some constant $\varepsilon > 0$,

$$P_t^B 1_B(x) \geq \varepsilon \quad \text{for } \mu\text{-a.a. } x \in \frac{1}{4}B.$$

The ball $\frac{1}{4}B$ is covered by the family of balls $B(z_0, \rho)$ with arbitrary $z_0 \in \frac{1}{4}B$ and with any fixed $\rho > 0$; we will use $\rho = \delta'' t^{1/\beta}$. Hence, it suffices to prove that, for any fixed $z_0 \in \frac{1}{4}B$,

$$P_t^B 1_B(x) \geq \varepsilon \quad \text{for } \mu\text{-a.a. } x \in B(z_0, \rho).$$

Since $B(z_0, r/4) \subset B$, we have, for μ -a.a. $x \in B(z_0, r/4)$,

$$P_t^B 1_B(x) = \int_B p_t^B(x, y) d\mu(y) \geq \int_{B(z_0, r/4)} p_t^{B(z_0, r/4)}(x, y) d\mu(y). \tag{4.32}$$

By (4.31) and (LLE) we have

$$p_t^{B(z_0, r/4)}(x, y) \geq ct^{-\alpha/\beta} \quad \text{for } \mu\text{-a.a. } x, y \in B(z_0, \delta'' t^{1/\beta}).$$

Substituting this estimate to (4.32) and using (V_≥), we obtain that, for μ -a.a. $x \in B(z_0, \delta'' t^{1/\beta})$,

$$P_t^B 1_B(x) \geq ct^{-\alpha/\beta} \mu\left(B\left(z_0, \delta'' t^{1/\beta}\right)\right) \geq \varepsilon,$$

which finishes the proof of (S).

Since by Lemma 4.6, $(\mathcal{E}, \mathcal{F})$ is conservative, combining (4.30) and (4.27) with the following already known result

$$(LLE) \Rightarrow (NLE), \quad \text{by [20, Lemma 4.1]}$$

we obtain

$$(LLE) + (V_{\geq}) + (J_{\leq}) \Rightarrow (UE) + (NLE).$$

The opposite implication is a consequence of (4.23) and the following result:

$$(UE) + (NLE) \Rightarrow (LLE) \quad \text{by [20, Theorem 4.2],}$$

which completes the proof of (4.28).

Finally, in order to prove (4.29), observe that the implication

$$(LLE) + (J) \Rightarrow (UE) + (LE)$$

follows from (4.28), (4.30) and Theorem 2.8, while the opposite implication

$$(UE) + (LE) \Rightarrow (LLE) + (J)$$

follows from (4.28), (4.23). \square

5. SPACES WITH POSITIVE EFFECTIVE RESISTANCE

Given a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ in (M, d, μ) , define the *effective resistance* $R(A, B)$ between two disjoint closed subsets A, B of M by

$$R(A, B)^{-1} = \inf\{\mathcal{E}(u) : u \in \mathcal{F} \cap C_0(M), u|_A = 1 \text{ and } u|_B = 0\}.$$

Note that, for any fixed A , the effective resistance $R(A, B)$ is non-increasing in B . We use also the shortcuts

$$R(x, B) := R(\{x\}, B) \quad \text{and} \quad R(x, y) := R(\{x\}, \{y\}).$$

Fix some constant $\gamma > 0$.

Definition 5.1 (Condition (R_1)). We say that *condition* (R_1) holds if there exists a constant $C > 0$ such that

$$|u(x) - u(y)|^2 \leq Cd(x, y)^\gamma \mathcal{E}(u),$$

for all $u \in \mathcal{F} \cap C(M)$ and for all $x, y \in M$.

Definition 5.2 (Condition (R_2)). We say that *condition* (R_2) holds if there exists a constant $C > 0$ such that

$$R(x, B(x, r)^c) \geq C^{-1}r^\gamma,$$

for all $x \in M$ and $r > 0$.

In this section we assume that $\bar{R} = \infty$ and $\beta > \alpha > 0$. Let us start with the following lemma.

Lemma 5.3. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$ with a jump kernel J and without killing part. Assume that conditions $(V), (R_2)$ are satisfied, and let $\beta = \alpha + \gamma$. Then*

$$(LE) \iff (J_\geq). \tag{5.1}$$

Proof. The implication $(LE) \Rightarrow (J_\geq)$ was proved in Lemma 4.9. To prove the opposite implication

$$(J_\geq) \Rightarrow (LE),$$

we will apply Theorem 2.8. For that, it suffices to verify the conditions (S) and (NLE) .

By [23, Proposition 6.2, p.6419] we have the following implication:

$$(V) + (J_\geq) \Rightarrow (R_1). \tag{5.2}$$

Note also that by [23, Proposition 6.2]), the heat kernel $p_t(x, y)$ is jointly continuous in x, y .

Under the assumption that all metric balls are precompact, the following implication was proved in [23, Theorem 6.13]:

$$(V) + (R_1) + (R_2) \Rightarrow (S). \tag{5.3}$$

In fact, the assumption can be dropped. In the proof of [23, Theorem 6.13], the assumption that all metric balls are precompact is used for the following purposes:

- (I) to ensure the existences of cutoff functions of the pairs $(\frac{1}{2}B, B)$ and (B, M) , that is, $f \in \text{cutoff}(\frac{1}{2}B, B)$ and $\phi \in \text{cutoff}(B, M)$ (where we use the notation from [23, Theorem 6.13]);

(II) to ensure the existence of the Green function for ball $B := B(x_0, r)$ in Section 6.5 of [23].

For (I), in the proof of [23, Theorem 6.13], one can simply set $f := 1_{\frac{1}{2}B}$. As for the existence of the function $\phi \subset \text{cutoff}(B, M)$, one can prove [23, (6.34)] as follows. Note that B can be exhausted by an increasing family of precompact sets $\{\Omega_n\}$ such that $\overline{\Omega_n} \subset \Omega_{n+1}$. Take the function $\phi_n \in \text{cutoff}(\Omega_n, M)$ by regularity of $(\mathcal{E}, \mathcal{F})$. Then, one can prove that [23, (6.34)] holds true for μ -a.a. $x \in \Omega_n$ and hence, for μ -a.a. $x \in B$, since n is arbitrary.

For (II), the precompactness of balls can be replaced by finiteness of volume of balls (which follows from (V_{\leq})) and the main part is to prove that, under condition (R_1) for any fixed bounded open set Ω and any fixed $x \in \Omega$, the variational problem

$$\inf\{\mathcal{E}(u, u) : u \in \mathcal{F}(\Omega), u(x) = 1\}$$

possesses a solution. This can be done as follows. Note that by [23, Lemma 6.1], (R_1) and (V_{\geq}) imply that each $u \in \mathcal{F}$ has a Hölder continuous version. In the rest of the proof, we always use this Hölder continuous version. Since $\text{diam}M \geq \overline{R} = \infty$ and Ω is bounded, we can choose a point $y_0 \in M \setminus \Omega$. For any $u \in \mathcal{F}(\Omega)$ and $z \in \Omega$, by (R_1) , we have

$$|u(z)|^2 = |u(y_0) - u(z)|^2 \leq Cd(z, y_0)^\gamma \mathcal{E}(u, u), \quad (5.4)$$

which implies that the space $\mathcal{F}(\Omega)$ is a Hilbert space under norm $\mathcal{E}(u, u)^{1/2}$. Now, we denote that infimum in the variational problem by c and for any $k \geq 1$, we can choose $u_k \in \mathcal{F}(\Omega)$ such that

$$c \leq \mathcal{E}(u_k, u_k) \leq c + k^{-1}. \quad (5.5)$$

By (5.4) and (R_1) , $\{u_k\}$ is uniformly bounded and equicontinuous, then, by Arzelà-Ascoli theorem, for any compact set $K \subset \Omega$ containing x , there is a uniformly convergent subsequence $\{u_{n_k}\}$ converging to some function $u \in C(K)$. Since $K \subset \Omega$ is arbitrary, we can choose a subsequence (still denoted by $\{u_k\}$) and extend u to a function in $C(\Omega)$, such that u_k converges to u pointwise. Furthermore, by (5.4) and bounded convergence theorem, we have

$$u_k \xrightarrow{L^2} u, \text{ as } k \rightarrow \infty.$$

Hence, since $\mathcal{F}(\Omega)$ is a Hilbert space, by (5.5) and [30, Lemma 2.12], we obtain that $u \in \mathcal{F}(\Omega)$,

$$u_k \rightarrow u \text{ weakly in } \mathcal{F}(\Omega),$$

and

$$\mathcal{E}(u, u) \leq \liminf_{k \rightarrow \infty} \mathcal{E}(u_k, u_k) \leq c.$$

Therefore, the function u is the solution to the above variational problem.

Let us continue the proof of Lemma 5.3. Combining (5.2), (5.3) and [23, Proposition 2.6], we see that all the conditions (R_1) , (S) , (DUE) are satisfied.

Finally, let us verify that

$$(DUE) + (V_{\leq}) + (S) + (R_1) \Rightarrow (NLE). \quad (5.6)$$

Indeed, fix $t > 0$ and a ball $B := B(x, r)$ with $r = \delta^{-1}t^{1/\beta}$. Using (S) and the semigroup property, we obtain that

$$p_t(x, x) \geq c_1 t^{-\alpha/\beta}. \quad (5.7)$$

Using condition (DUE) and the inequality

$$\mathcal{E}(p_t(x, \cdot)) \leq \frac{1}{et} p_t(x, x),$$

(cf. [23, Eq. (6.44)]), we obtain that

$$\mathcal{E}(p_t(x, \cdot)) \leq Ct^{-\alpha/\beta-1}. \quad (5.8)$$

Therefore, it follows from (5.8) and condition (R_1) that, for all $y \in M$

$$|p_t(x, y) - p_t(x, x)|^2 \leq Cd(x, y)^\gamma \mathcal{E}(p_t(x, \cdot)) \leq Cd(x, y)^{\beta-\alpha} t^{-\alpha/\beta-1}.$$

In particular, if $d(x, y) \leq \delta' t^{1/\beta}$ with small enough δ' then we obtain

$$|p_t(x, y) - p_t(x, x)|^2 \leq C \left(\delta' t^{1/\beta} \right)^{\beta-\alpha} t^{-\alpha/\beta-1} = C (\delta')^{\beta-\alpha} t^{-2\alpha/\beta} < \left(\frac{1}{2} c_1 t^{-\alpha/\beta} \right)^2,$$

where c_1 is the same constant as in (5.7). From this and (5.7), we obtain

$$p_t(x, y) \geq p_t(x, x) - \frac{c_1}{2} t^{-\alpha/\beta} \geq \frac{c_1}{2} t^{-\alpha/\beta},$$

for all $t > 0$ and all $x, y \in M$ with $d(x, y) \leq \delta' t^{1/\beta}$, which proves (NLE) . \square

Let us introduce one more condition.

Definition 5.4 (Condition (R_\geq)). We say that *condition* (R_\geq) holds if there exist constants $C, \gamma > 0$ such that for all $x, y \in M$,

$$R(x, y) \geq C^{-1} d(x, y)^\gamma. \quad (5.9)$$

Theorem 5.5. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$ with a jump kernel J and without killing part. Let α, β, γ be three positive numbers such that $\beta = \alpha + \gamma$. Then

$$(UE) + (LE) \iff (V) + (J) + (R_\geq). \quad (5.10)$$

Proof. We first prove the implication:

$$(LE) \Rightarrow \text{"conservativeness"}. \quad (5.11)$$

Indeed, (LE) implies

$$\int_0^\infty p_t(x, y) dt \geq \int_{d(x, y)^\beta}^\infty ct^{-\alpha/\beta} dt = \infty,$$

where we have used the fact that $\beta > \alpha$. Hence, by [15, (1.6.2), p.55], the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is recurrent and furthermore, is conservative by [15, Lemma 1.6.5, p.56]. Then the implication

$$(UE) + (LE) \Rightarrow (V) + (J) + (R_\geq)$$

is a consequence of (5.11) and the following results:

$$\begin{aligned} \text{"conservativeness"} + (UE) + (LE) &\Rightarrow (V) \text{ by [19, Theorem 3.2]}, \\ (UE) + (LE) &\Rightarrow (J) \text{ by Lemma 4.9,} \\ (UE) + (NLE) &\Rightarrow (R_\geq) \text{ by [23, Theorem 6.17]}. \end{aligned}$$

To prove the opposite implication

$$(V) + (J) + (R_\geq) \Rightarrow (UE) + (LE),$$

note that

$$\begin{aligned} (V) + (J_\geq) &\Rightarrow (R_1) \text{ by [23, Proposition 6.2]}, \\ (V_\leq) + (J_\leq) + (R_\geq) + (R_1) &\Rightarrow (R_2) \text{ by [23, Proposition 6.9].}^2 \end{aligned}$$

Therefore, we obtain (LE) by Lemma 5.3, while (UE) holds by [23, Theorem 6.13]. \square

²In the proof of [23, Proposition 6.9], the function ϕ is a cutoff function of the pair $(\frac{1}{2}B, B)$. In fact, since $\text{supp}g \cap \frac{1}{2}B \subset \text{supp}f \cap \frac{1}{2}B$ is precompact by compactness of $\text{supp}f$, ϕ can be replaced by a function in $\text{cutoff}(\text{supp}g \cap \frac{1}{2}B, B)$. Then, the rest of proof still works. In other words, we can drop the assumption that all balls in M are precompact.

6. APPENDIX

Lemma 6.1. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet Form. Given $T \in (0, \infty]$ and an open set $U \subset M$, we fix a function $f \in \mathcal{F} \cap L^\infty$ such that $f|_U \equiv \|f\|_{L^\infty}$ and assume for all $t \in (0, T)$, the function $u(t, \cdot)$ has the Fréchet derivative $\partial_t u$ and satisfies the following equation:*

$$\begin{cases} (\partial_t u, \varphi) + \mathcal{E}(u, \varphi) \leq \mathcal{E}(f, \varphi), & \forall 0 \leq \varphi \in \mathcal{F}(U), \\ u_+(t, \cdot) \in \mathcal{F}(U), & t \in (0, T), \\ u_+(t, \cdot) \xrightarrow{L^2} 0, & \text{as } t \rightarrow 0. \end{cases} \quad (6.1)$$

Then, for any $t \in (0, T)$,

$$\|u_+(t, \cdot)\|_{L^2(U)} \leq 2 \int_0^t \mathcal{E}(f, u_+(s, \cdot)) ds. \quad (6.2)$$

Proof. The proof is motivated by [21, Propostion 5.2]. Let $\Phi : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function such Φ satisfies the following three conditions for some constant $C > 0$:

- (i). $\Phi(r) = 0$ for all $r \leq 0$;
- (ii). $0 \leq \Phi'(r) \leq C$ for all $r > 0$;
- (iii). $|\Phi''(r)| \leq C$ for all $r \geq 0$.

Moreover, let also the function $\Psi(r) = \Phi'(r)r$ satisfy the same conditions (i) – (iii).

To shorten the notation, we write $u(t, \cdot)$ as u and $u_+(t, \cdot)$ as u_+ . Since $u_+ \in \mathcal{F}(U)$, we have $\Phi(u) = \Phi(u_+) \in \mathcal{F}(U)$. Setting $\varphi = \Phi(u)$ in (6.1), we obtain

$$(\partial_t u, \Phi(u)) + \mathcal{E}(u, \Phi(u)) \leq \mathcal{E}(f, \Phi(u)).$$

Since $\mathcal{E}(u, \Phi(u)) \geq 0$ by [17, (4.2), p.113], we obtain

$$(\partial_t u, \Phi(u)) \leq \mathcal{E}(f, \Phi(u)).$$

By the above inequality, a similar inequality for the function Ψ , and chain rule (see also the proof of [21, Propostion 5.2]), we obtain

$$(u, \Phi(u))' = (\partial_t u, \Phi(u)) + (\partial_t u, \Psi(u)) \leq \mathcal{E}(f, \Phi(u)) + \mathcal{E}(f, \Psi(u)). \quad (6.3)$$

By the properties (i) – (ii), $\Phi(u_+) \leq Cu_+$ and so, by the initial condition in (6.1)

$$(u, \Phi(u)) = (u_+, \Phi(u_+)) \leq C\|u_+\|_{L^2(U)} \rightarrow 0 \text{ as } t \rightarrow 0.$$

Hence, by the above formula and (6.3), we obtain for all $t \in (0, T)$,

$$(u, \Phi(u)) \leq \int_0^t \mathcal{E}(f, \Phi(u(s, \cdot))) ds + \int_0^t \mathcal{E}(f, \Psi(u(s, \cdot))) ds. \quad (6.4)$$

Now let us define sequences $\{\Phi_k\}$ and $\{\Psi_k\}$ of functions as follows. Choose a smooth function η on \mathbb{R} satisfying:

$$\eta(r) = \begin{cases} r, & \text{if } r \geq 1, \\ 0, & \text{if } r \leq 0, \end{cases}$$

and observe that the function $\tilde{\eta}(r) = \eta'(r)r$ satisfies the same identity. For any positive integer k , set

$$\Phi_k(r) := \frac{1}{k}\eta(kr) \quad \text{and} \quad \Psi_k(r) := \Phi_k'(r)r = \frac{1}{k}\tilde{\eta}(kr).$$

Clearly, both sequences $\{\Phi_k\}$ and $\{\Psi_k\}$ satisfy the following properties

- (a) $\Phi_k(r) \rightarrow r_+$ uniformly in r as $k \rightarrow \infty$;
- (b) $\Phi_k(r) = 0$, for $r \leq 0$ and $k \geq 1$;
- (c) $\Phi_k' \geq 0$ and $\Phi_k'' \geq 0$, for $k \geq 1$;
- (d) $C := \sup_m \sup_{\mathbb{R}} \Phi_m' < \infty$;
- (e) $\sup_{\mathbb{R}} \Phi_k'' < \infty$, for $k \geq 1$.

By properties (b) and (d), we have for all $s \in (0, T)$,

$$|\Phi_k(u(s, \cdot))| \leq C u_+(s, \cdot). \quad (6.5)$$

Since $u_+(s, \cdot) \in \mathcal{F}$, by property (a) and dominated convergence theorem, we obtain

$$\Phi_k(u(s, \cdot)) \xrightarrow{L^2} u_+(s, \cdot), \quad \text{as } k \rightarrow \infty. \quad (6.6)$$

Let us verify that also

$$\Phi_k(u(s, \cdot)) \rightarrow u_+(s, \cdot) \text{ weakly in } \mathcal{E} \text{ as } k \rightarrow \infty. \quad (6.7)$$

Indeed, since by (b) and (d) the function $C^{-1}\Phi_k(u)$ is a normal contraction of u_+ , we obtain

$$\sup_k \mathcal{E}(\Phi_k(u(s, \cdot)), \Phi_k(u(s, \cdot))) \leq C^2 \mathcal{E}(u_+(s, \cdot), u_+(s, \cdot)) < \infty.$$

Hence, (6.7) follows from the above inequality, (6.6) and [30, Lemma 2.12].

Since for every $k \geq 1$, the functions Φ_k and Ψ_k satisfy properties (i) – (iii), we can apply (6.4) for Φ_k and obtain, using also (6.6), that, for any $t \in (0, T)$,

$$\begin{aligned} \|u_+(t, \cdot)\|_{L^2(U)} &= \lim_{k \rightarrow \infty} (u, \Phi_k(u)) \\ &\leq \lim_{k \rightarrow \infty} \int_0^t \mathcal{E}(f, \Phi_k(u(s, \cdot))) ds + \lim_{k \rightarrow \infty} \int_0^t \mathcal{E}(f, \Psi_k(u(s, \cdot))) ds. \end{aligned} \quad (6.8)$$

Note that since $f|_U \equiv \|f\|_{L^\infty}$, we obtain by Markov property of P_t that $f - P_t f \geq 0$ μ -a.e. in U , which implies that, for any $0 \leq w \in \mathcal{F}(U)$,

$$\mathcal{E}(f, w) = \lim_{t \rightarrow 0} \frac{1}{t} (f - P_t f, w) \geq 0.$$

Using (6.5), we further obtain

$$\begin{aligned} \mathcal{E}(f, \Phi_k(u(s, \cdot))) &= \lim_{t \rightarrow 0} \frac{1}{t} (f - P_t f, \Phi_k(u(s, \cdot))) \\ &\leq C \lim_{t \rightarrow 0} \frac{1}{t} (f - P_t f, u_+(s, \cdot)) = C \mathcal{E}(f, u_+(s, \cdot)). \end{aligned} \quad (6.9)$$

Now we can prove (6.2). It suffices to consider the case that the integral in (6.2) is finite, that is,

$$\int_0^t \mathcal{E}(f, u_+(s, \cdot)) ds < \infty. \quad (6.10)$$

Observing that by (6.7)

$$\mathcal{E}(f, \Phi_k(u(s, \cdot))) \rightarrow \mathcal{E}(f, u_+(s, \cdot))$$

and using the domination conditions (6.9) and (6.10), we conclude by the dominated convergence theorem that

$$\lim_{k \rightarrow \infty} \int_0^t \mathcal{E}(f, \Phi_k(u(s, \cdot))) ds = \int_0^t \mathcal{E}(f, u_+(s, \cdot)) ds.$$

A similar result holds for the second term in (6.8), which yields (6.2). \square

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