Ordinary Differential Equations

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1 Introduction: the notion of ODEs and examples

A general *ordinary differential equation* (shortly, ODE) has a form

$$F\left(x,y,y',...,y^{(n)}\right)=0, \tag{1.1}$$

where $x\in\mathbb{R}$ is an independent variable, $y=y\left(x\right)$ is an unknown function, F is a given function on $n+2$ variables. The number n - the maximal order of the derivative in (1.1), is called the *order* of the ODE. The equation (1.1) is called *differential* because it contains the derivatives of the unknown function. It is called *ordinary* because the derivatives $y^{(k)}$ are ordinary as opposed to partial derivatives. There is a theory of partial differential equations where the unknown function depends on more than one variable and, hence, the partial derivatives are involved, but this is a topic of another lecture course.

The ODEs arise in many areas of Mathematics, as well as in Sciences and Engineering. In most applications, one needs to find explicitly or numerically a solution $y\left(x\right)$ of (1.1) satisfying some additional conditions. There are only a few types of the ODEs where one can explicitly find all the solutions. However, for quite a general class of ODEs one can prove various properties of solutions without evaluating them, including existence of solutions, uniqueness, smoothness, etc.

In Introduction we will be concerned with various examples and specific classes of ODEs of the first and second order, postponing the general theory to the next Chapters.

Consider the differential equation of the first order

$$y'=f\left(x,y\right), \tag{1.2}$$

where $y=y\left(x\right)$ is the unknown real-valued function of a real argument x, and $f\left(x,y\right)$ is a given function of two real variables.

Consider a couple $\left(x,y\right)$ as a point in $\mathbb{R}^2$ and assume that function f is defined on a set $D\subset\mathbb{R}^2$, which is called the *domain* of the function f and of the equation (1.2).

Definition. A real valued function $y\left(x\right)$ defined on an interval $I\subset\mathbb{R}$, is called a (*particular*) solution of (1.2) if

1. $y\left(x\right)$ is differentiable at any $x\in I$,
2. the point $\left(x,y\left(x\right)\right)$ belongs to D for any $x\in I$,
3. and the identity $y'\left(x\right)=f\left(x,y\left(x\right)\right)$ holds for all $x\in I$.

The graph of a particular solution is called an *integral curve* of the equation. Obviously, any integral curve is contained in the domain D. The family of all particular solutions of (1.2) is called the *general* solution.

Usually an ODE cannot be solved explicitly. We will consider some classes of $f\left(x,y\right)$ when one find the general solution to (1.2) in terms of indefinite integration.

Example. Assume that the function f does not depend on y so that (1.2) becomes

$$y'=f\left(x\right).$$

Hence, y must be a *primitive function* of f. Assuming that f is a continuous function on an interval I, we obtain the general solution on I by means of the indefinite integration:

$$y=\int f\left(x\right)dx=F\left(x\right)+C,$$

where $F\left(x\right)$ is a primitive of $f\left(x\right)$ on I and C is an arbitrary constant.

1.1 Separable ODE

Consider a *separable* ODE, that is, an ODE of the form

$$y' = f(x)\, g(y)\,. \tag{1.3}$$

Any separable equation can be solved by means of the *method of separation of variables*. Assume that f and g are continuous and that $g(y)$ does not vanish on some interval J. Then we can find all solutions $y(x)$ of (1.3) taking values in J as follows. Since $g(y) \neq 0$, we can divide (1.3) by $g(y)$, which yields

$$\frac{y'}{g(y)} = f(x)\,.$$

Integrating this equation in x, we obtain

$$\int \frac{y' dx}{g(y)} = \int f(x)\, dx = F(x) + C,$$

where $F(x)$ is a primitive of f. In the left hand side we have $y'dx = dy$ whence

$$\int \frac{y' dx}{g(y)} = \int \frac{dy}{g(y)} = G(y) + C,$$

where G is a primitive of $\frac{1}{g}$. Combining these two identities, we obtain

$$G(y) = F(x) + C. \tag{1.4}$$

Solving this identity in y, we obtain y as a function of x. The identity (1.4) determines all solutions y of (1.3) taking values on J.

Example. Consider the ODE

$$y' = y$$

which has the form (1.3) $f(x) = 1$ and $g(y)$. We have

$$F(x) = \int f(x)\, dx = \int dx = x$$

and, in each of the intervals $(0, +\infty)$ and $(-\infty, 0)$ where g does not vanish,

$$G(y) = \int \frac{dy}{g(y)} = \int \frac{dy}{y} = \ln|y|$$

where we do not write the constant of integration because we need only one primitive function. The equation (1.4) becomes

$$\ln|y| = x + C,$$

whence we obtain $y = C_1 e^x$ where $C_1 = \pm e^C$. Clearly, $y \equiv 0$ is also a solution, so that any function $y = Ce^x$ with any $C \in \mathbb{R}$ solves the equation $y' = y$.

Let us verify that there exists no other solution. Let $y(x)$ be a solution defined on an open interval I. Assume that $y(x)$ takes a positive value somewhere in I, and let

(a,b) be a maximal open interval where $y(x) > 0$. Then either $(a,b) = I$ or one of the points a, b belongs to I, say $a \in I$ and $y(a) = 0$. By the above argument, $y(x) = Ce^x$ in (a,b), where $C > 0$. Since $e^x \neq 0$, this solution does not vanish at a. Hence, the second alternative cannot take place and we conclude that $(a,b) = I$, that is, $y(x) = Ce^x$ in I.

The same argument applies if $y(x) < 0$ for some x. Finally, of $y(x) \equiv 0$ then also $y(x) = Ce^x$ with $C = 0$.

Assume that in the separable equation $y' = f(x)\,g(y)$ the function $g(y)$ vanishes at a sequence of points, say $y_1, y_2, ...$, enumerated in the increasing order. Obviously, the constant function $y(x) = y_k$ is a solution of this ODE. The method of separation of variables allows to obtain all solutions taking values in any of the intervals (y_k, y_{k+1}), where g does not vanish. Then the structure of the general solution requires an additional investigation.

Example. Consider the ODE

$$y' - xy^2 = 2xy,$$

with domain $(x,y) \in \mathbb{R}^2$. Rewriting it in the form

$$y' = x\left(y^2 + 2y\right),$$

we see that it is separable. The function $g(y) = y^2 + 2y$ vanishes at two points $y = 0$ and $y = -2$. Hence, we have two constant solutions $y \equiv 0$ and $y \equiv 2$. The function g does not vanish in any of the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, +\infty)$. In each of these intervals, the method of separation of variables yields

$$\frac{1}{2}\ln\left|\frac{y}{y+2}\right| = \int \frac{dy}{y(y+2)} = \int x\,dx = \frac{x^2}{2} + C$$

whence

$$\frac{y}{y+2} = C_1 e^{x^2}$$

where $C_1 = \pm e^{2C}$. Clearly, C_1 is any non-zero number here. However, since $y \equiv 0$ is a solution, C_1 can be 0 as well. Renaming C_1 to C, we obtain

$$\frac{y}{y+2} = Ce^{x^2}$$

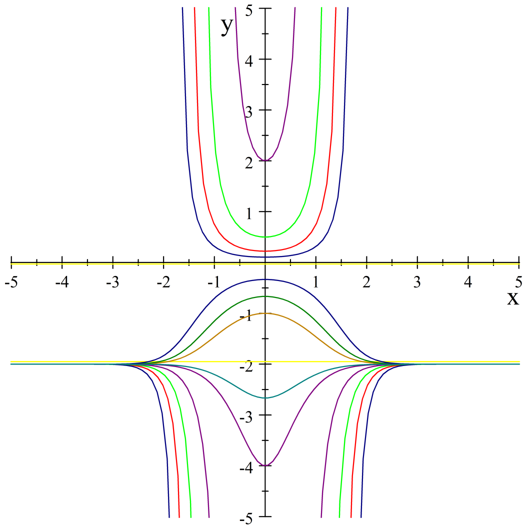
where C is any real number, whence it follows that

$$y = \frac{2Ce^{x^2}}{1 - Ce^{x^2}}. \tag{1.5}$$

We obtain the following solutions:

$$y = \frac{2Ce^{x^2}}{1 - Ce^{x^2}} \quad \text{and} \quad y \equiv -2. \tag{1.6}$$

The integral curves are shown on the diagram:



One can show that this is a general solution.

Example. Consider the equation

$$y' = \sqrt{|y|},$$

which is defined for all $x, y \in \mathbb{R}$. Since the right hand side vanish for $y = 0$, the constant function $y \equiv 0$ is a solution. In the domains $y > 0$ and $y < 0$, the equation can be solved using separation of variables. For example, in the domain $y > 0$, we obtain

$$\int \frac{dy}{\sqrt{y}} = \int dx$$

whence

$$2\sqrt{y} = x + C$$

and

$$y = \frac{1}{4}(x + C)^2, \ \ x > -C$$

(the restriction $x > -C$ comes from the previous line). Similarly, in the domain $y < 0$, we obtain

$$\int \frac{dy}{\sqrt{-y}} = \int dx$$

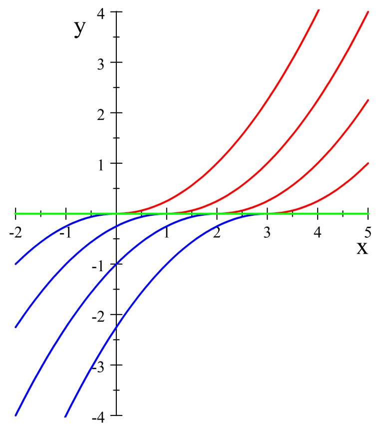
whence

$$-2\sqrt{-y} = x + C$$

and

$$y = -\frac{1}{4}(x + C)^2, \ x < -C.$$

We obtain the following integrals curves:



We see that the integral curves in the domains $y>0$ and $y<0$ touch the line $y=0$. This allows us to construct more solutions as follows: for any couple of reals $a<b$, consider the function

$$y(x)=\begin{cases}-\frac{1}{4}(x-a)^2, & x<a,\\ 0, & a\le x\le b,\\ \frac{1}{4}(x-b)^2, & x>b,\end{cases} \tag{1.7}$$

which is obviously a solution with the domain $\mathbb{R}$. If we allow a to be $-\infty$ and b to be $+\infty$ with the obvious meaning of (1.7) in these cases, then (1.7) represents the general solution to $y'=\sqrt{|y|}$.

1.2 Linear ODE of 1st order

Consider the ODE of the form

$$y'+a(x)\,y=b(x) \tag{1.8}$$

where a and b are given functions of x, defined on a certain interval I. This equation is called *linear* because it depends linearly on y and y'.

A linear ODE can be solved as follows.

Theorem 1.1 (The method of variation of parameter) *Let functions $a(x)$ and $b(x)$ be continuous in an interval I and let $A(x)$ be a primitive of $a(x)$ on I. Then the general solution of the linear ODE (1.8) has the form*

$$y(x)=e^{-A(x)}\int b(x)\,e^{A(x)}dx. \tag{1.9}$$

Note that the function $y(x)$ given by (1.9) is defined on the full interval I.

Proof. Let us make the change of the unknown function $u(x)=y(x)\,e^{A(x)}$, that is,

$$y(x)=u(x)\,e^{-A(x)}. \tag{1.10}$$

Substituting this to the equation (1.8) we obtain

$$\left(ue^{-A}\right)'+aue^{-A}=b,$$

$$u'e^{-A}-ue^{-A}A'+aue^{-A}=b.$$

Since $A' = a$, we see that the two terms in the left hand side cancel out, and we end up with a very simple equation for $u(x)$:

$$u' e^{-A} = b$$

whence $u' = be^{A}$ and

$$u = \int be^{A} dx.$$

Substituting into (1.10), we finish the proof. ■

One may wonder how one can guess to make the change (1.10). Here is the motivation. Consider first the case when $b(x) \equiv 0$. In this case, the equation (1.8) becomes

$$y' + a(x) y = 0$$

and it is called *homogeneous*. Clearly, the homogeneous linear equation is separable. In the domains $y > 0$ and $y < 0$ we have

$$\frac{y'}{y} = -a(x)$$

and

$$\int \frac{dy}{y} = -\int a(x) dx = -A(x) + C.$$

Then $\ln |y| = -A(x) + C$ and

$$y(x) = Ce^{-A(x)} \tag{1.11}$$

where C can be any real (including $C = 0$ that corresponds to the solution $y \equiv 0$).

For a general equation (1.8) replace a constant C in (1.11) by a function $C(x)$ (which was denoted by $u(x)$ in the proof), which will result in the above change. Since we have replaced a constant parameter by a function, this method is called the method of variation of parameter. It applies to the linear equations of higher order as well.

Example. Consider the equation

$$y' + \frac{1}{x} y = e^{x^2} \tag{1.12}$$

in the domain $x > 0$. Then

$$A(x) = \int a(x) dx = \int \frac{dx}{x} = \ln x$$

(we do not add a constant C since $A(x)$ is *one* of the primitives of $a(x)$),

$$y(x) = \frac{1}{x} \int e^{x^2} x dx = \frac{1}{2x} \int e^{x^2} dx^2 = \frac{1}{2x} \left(e^{x^2} + C \right),$$

where C is an arbitrary constant.

Alternatively, one can solve first the homogeneous equation

$$y' + \frac{1}{x} y = 0,$$

using the separable of variables:

$$\begin{aligned}\frac{y'}{y} &= -\frac{1}{x}\\ (\ln y)' &= -(\ln x)'\\ \ln y &= -\ln x + C_1\\ y &= \frac{C}{x}.\end{aligned}$$

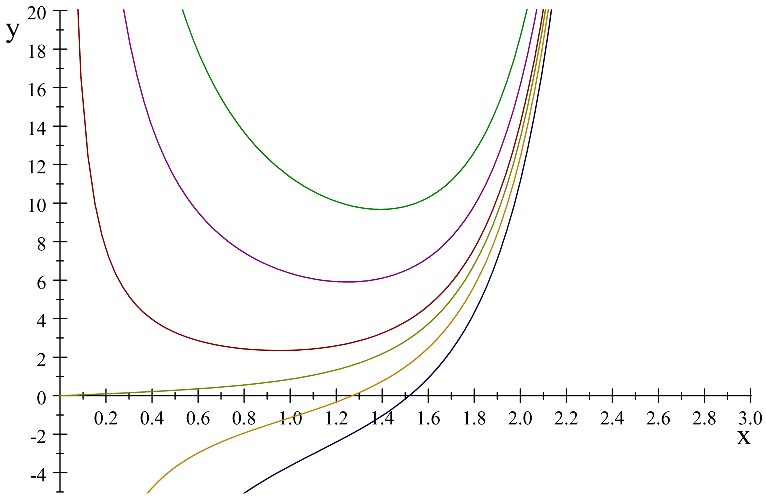
Next, replace the constant C by a function $C(x)$ and substitute into (1.12):

$$\begin{aligned}\left(\frac{C(x)}{x}\right)' + \frac{1}{x}\frac{C}{x} &= e^{x^2},\\ \frac{C'x - C}{x^2} + \frac{C}{x^2} &= e^{x^2}\\ \frac{C'}{x} &= e^{x^2}\\ C' &= e^{x^2}x\\ C(x) &= \int e^{x^2}x\,dx = \frac{1}{2}\left(e^{x^2} + C_0\right).\end{aligned}$$

Hence,

$$y = \frac{C(x)}{x} = \frac{1}{2x}\left(e^{x^2} + C_0\right),$$

where C_0 is an arbitrary constant. The integral curves are shown on the following diagram:



1.3 Quasi-linear ODEs and differential forms

Let Ω be an open subset of $\mathbb{R}^2$.

Definition. Given two functions $a(x,y)$ and $b(x,y)$ in Ω, consider the expression

$$a(x,y)\,dx + b(x,y)\,dy,$$

which is called a *differential form*. The differential form is called *exact* in Ω if there is a differentiable function $F : \Omega \to \mathbb{R}$ such that

$$dF = adx + bdy, \tag{1.13}$$

and *inexact* otherwise. If the form is exact then the function F from (1.13) is called the *integral* of the form.

Observe that not every differential form is exact as one can see from the following statement.

Lemma 1.2 *If functions a, b are continuously differentiable in Ω then the necessary condition for the form $adx + bdy$ to be exact is the identity*

$$a_y = b_x. \tag{1.14}$$

Proof. Indeed, if F is an integral of the differential form $adx + bdy$ then $F_x = a$ and $F_y = b$, whence it follows that the derivatives F_x and F_y are continuously differentiable. By a Schwarz theorem this implies that $F_{xy} = F_{yx}$ whence $a_y = b_x$. ■

Definition. The differential form $adx+bdy$ is called *closed* in Ω if it satisfies the condition $a_y = b_x$ in Ω.

Hence, Lemma 1.2 says that any exact form must be closed. The converse is in general not true, as will be shown later on. However, since it is easier to verify the closedness than the exactness, it is desirable to know under what additional conditions the closedness implies the exactness. Such a result will be stated and proved below.

Example. 1.The form $ydx - xdy$ is not closed because $a_y = 1$ while $b_x = -1$. Hence, it is inexact.

2. The form $ydx + xdy$ is exact because

$$ydx + xdy = d(xy)$$

so that it has an integral $F(x,y) = xy$. Hence, it is also closed, which can be easily verified directly by (1.14).

The form $2xydx + (x^2 + y^2)\,dy$ is exact because

$$2xydx + \left(x^2 + y^2\right)dy = d\left(x^2y + \frac{y^3}{3}\right)$$

so that it has an integral $F(x,y) = x^2y + \frac{y^3}{3}$ (it will be explained later how one can obtain this integral).

If the differential form $adx + bdy$ is exact then this allows to solve easily the following differential equation:

$$a(x,y) + b(x,y)\,y' = 0, \tag{1.15}$$

as it is stated in the next Theorem.

The ODE (1.15) is called *quasi-linear* because it is linear with respect to y' but not necessarily linear with respect to y. Using $y' = \frac{dy}{dx}$, one can write (1.15) in the form

$$a(x,y)\,dx + b(x,y)\,dy = 0,$$

which explains why the equation (1.15) is related to the differential form $adx + bdy$. We say that the equation (1.15) is exact (resp. closed) if the corresponding differential form $adx + bdy$ is exact (reps. closed).

Theorem 1.3 *Let Ω be an open subset of $\mathbb{R}^2$, a, b be continuous functions on Ω, such that the form $adx + bdy$ is exact. Let F be an integral of this form and let $y(x)$ be a differentiable function defined on an interval $I \subset \mathbb{R}$ such that $(x, y(x)) \in \Omega$ for any $x \in I$ (that is, the graph of y is contained in Ω). Then y is a solution of the equation* (1.15) *if and only if*

$$F(x, y(x)) = \text{const} \quad \textit{on } I \tag{1.16}$$

(that is, if function F remains constant on the graph of y).

The identity (1.16) can be considered as (an implicit form of) the general solution of (1.15). The function F is also referred to as the integral of the ODE (1.15).

Proof. The hypothesis that the graph of $y(x)$ is contained in Ω implies that the composite function $F(x, y(x))$ is defined on I. By the chain rule, we have

$$\frac{d}{dx}F(x, y(x)) = F_x + F_y y' = a + by'.$$

Hence, the equation $a + by' = 0$ is equivalent to $\frac{d}{dx}F(x, y(x)) = 0$ on I, and the latter is equivalent to $F(x, y(x)) = \text{const}$. ■

Example. 1. The equation

$$y + xy' = 0$$

is exact because $ydx + xdy = d(xy)$. Hence, the general solution is given by $xy = C$.

2. The equation

$$2xy + \left(x^2 + y^2\right) y' = 0$$

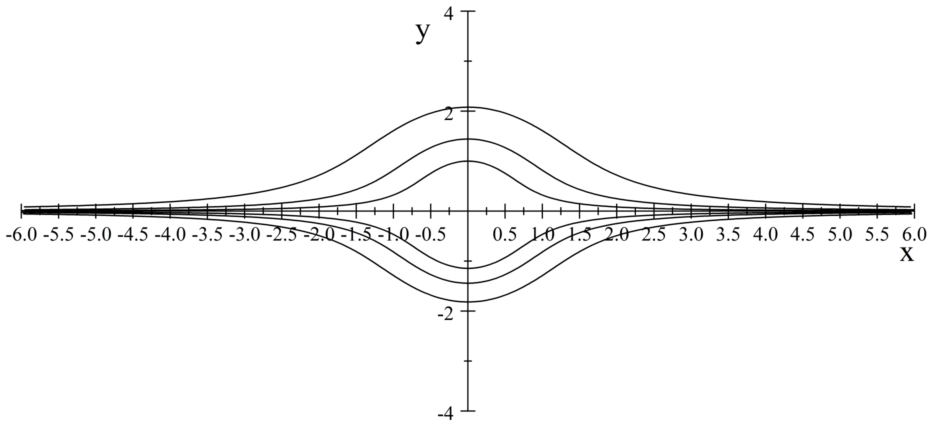
is exact because

$$2xydx + \left(x^2 + y^2\right) dy = d\left(x^2 y + \frac{y^3}{3}\right)$$

as in the previous Example. Hence, the general solution is given by

$$x^2 y + \frac{y^3}{3} = C.$$

Some of the integral curves of this equation are shown on the diagram:

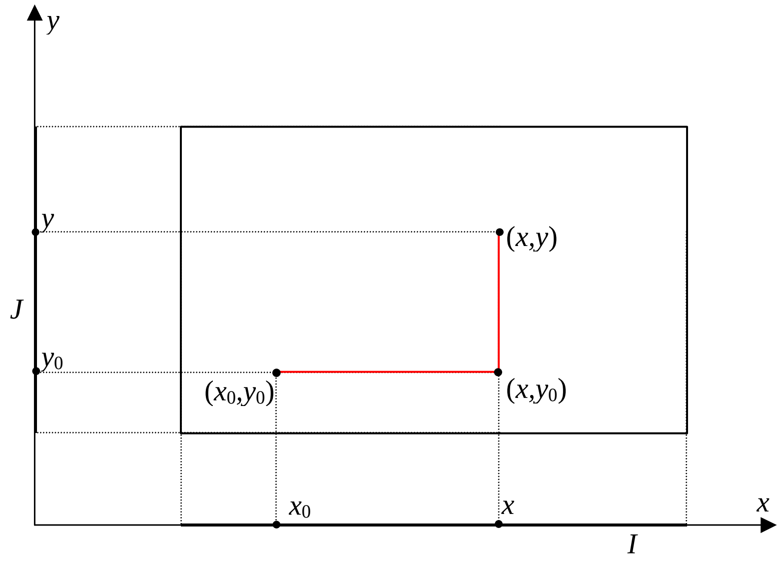


We say that a set $\Omega \subset \mathbb{R}^2$ is a *rectangle* (box) if it has the form $I \times J$ where I and J are intervals in $\mathbb{R}$. A rectangle is open if both I and J are open intervals. The following theorem provides the answer to the question how to decide whether a given differential form is exact in a rectangle.

Theorem 1.4 (The Poincaré lemma) *Let Ω be an open rectangle in $\mathbb{R}^2$. Let a, b be continuously differentiable functions on Ω such that $a_y \equiv b_x$. Then the differential form $adx + bdy$ is exact in Ω.*

It follows from Lemma 1.2 and Theorem 1.4 that if Ω is a rectangle then the form $adx + bdy$ is exact in Ω if and only if it is closed.

Proof. First we try to obtain an explicit formula for the integral F assuming that it exists. Then we use this formula to prove the existence of the integral. Fix some reference point (x_0, y_0) and assume without loss of generality that and $F(x_0, y_0) = 0$ (this can always be achieved by adding a constant to F). For any point $(x, y) \in \Omega$, also the point (x, y_0) belongs Ω; moreover, the intervals $[(x_0, y_0), (x, y_0)]$ and $[(x, y_0), (x, y)]$ are contained in Ω because Ω is a rectangle (see the diagram).



Since $F_x = a$ and $F_y = b$, we obtain by the fundamental theorem of calculus that

$$F(x, y_0) = F(x, y_0) - F(x_0, y_0) = \int_{x_0}^{x} F_x(s, y_0)\, ds = \int_{x_0}^{x} a(s, y_0)\, ds$$

and

$$F(x, y) - F(x, y_0) = \int_{y_0}^{y} F_y(x, t)\, dt = \int_{y_0}^{y} b(x, t)\, dt,$$

whence

$$\boxed{F(x, y) = \int_{x_0}^{x} a(s, y_0)\, ds + \int_{y_0}^{y} b(x, t)\, dt}. \tag{1.17}$$

Now we start the actual proof where we assume that the form $adx + bdy$ is closed and use the formula (1.17) to *define* function $F(x, y)$. We need to show that F is the integral of the differential form $adx + bdy$.

It suffices to verify the identities

$$F_x = a \quad \text{and} \quad F_y = b.$$

It follows from (1.17) that

$$F_y = \frac{\partial}{\partial y} \int_{y_0}^{y} b(x, t)\, dt = b(x, y).$$

Next, we have

$$\begin{aligned} F_x &= \frac{\partial}{\partial x} \int_{x_0}^{x} a(s, y_0)\, ds + \frac{\partial}{\partial x} \int_{y_0}^{y} b(x, t)\, dt \\ &= a(x, y_0) + \int_{y_0}^{y} b_x(x, t)\, dt. \end{aligned} \tag{1.18}$$

Using the hypothesis $b_x = a_y$, we obtain from (1.18)

$$\begin{aligned} F_x &= a(x, y_0) + \int_{y_0}^{y} a_y(x, t)\, dt \\ &= a(x, y_0) + (a(x, y) - a(x, y_0)) \\ &= a(x, y), \end{aligned}$$

which finishes the proof. ■

Consider some examples to Theorem 1.4.

Example. Consider again the differential form

$$2xydx + \left(x^2 + y^2\right) dy$$

in $\Omega = \mathbb{R}^2$. Since

$$a = 2xy, \qquad b = x^2 + y^2$$

$$a_y = (2xy)_y = 2x = \left(x^2 + y^2\right)_x = b_x,$$

we conclude by Theorem 1.4 that the given form is exact. The integral F can be found by (1.17) taking $x_0 = y_0 = 0$:

$$F(x,y) = \int_0^x 2s0ds + \int_0^y \left(x^2+t^2\right) dt = x^2y + \frac{y^3}{3},$$

as it was observed above.

Example. Consider the differential form

$$\frac{-ydx + xdy}{x^2+y^2} \tag{1.19}$$

in $\Omega = \mathbb{R}^2 \setminus \{0\}$. This form satisfies the condition $a_y = b_x$ because

$$a_y = -\left(\frac{y}{x^2+y^2}\right)_y = -\frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

and

$$b_x = \left(\frac{x}{x^2+y^2}\right)_x = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

By Theorem 1.4 we conclude that the given form is exact in any rectangular subdomain of Ω. However, Ω itself is not a rectangle, and let us show that the form is inexact in Ω. To that end, consider the function $\theta(x,y)$ which is the polar angle that is defined in the domain

$$\Omega' = \mathbb{R}^2 \setminus \{(x,0) : x \le 0\}$$

by the conditions

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \theta \in (-\pi, \pi),$$

where $r = \sqrt{x^2+y^2}$. Let us show that in Ω'

$$d\theta = \frac{-ydx + xdy}{x^2+y^2}, \tag{1.20}$$

that is, θ is the integral of (1.19) in Ω'. In the half-plane $\{x > 0\}$ we have $\tan\theta = \frac{y}{x}$ and $\theta \in (-\pi/2, \pi/2)$ whence

$$\theta = \arctan\frac{y}{x}.$$

Then (1.20) follows by differentiation of the arctan:

$$d\theta = \frac{1}{1+(y/x)^2}\frac{xdy - ydx}{x^2} = \frac{-ydx + xdy}{x^2+y^2}.$$

In the half-plane $\{y > 0\}$ we have $\cot\theta = \frac{x}{y}$ and $\theta \in (0, \pi)$ whence

$$\theta = \operatorname{arccot}\frac{x}{y}$$

and (1.20) follows again. Finally, in the half-plane $\{y < 0\}$ we have $\cot\theta = \frac{x}{y}$ and $\theta \in (-\pi, 0)$ whence

$$\theta = -\operatorname{arccot}\left(-\frac{x}{y}\right),$$

and (1.20) follows again. Since Ω' is the union of the three half-planes $\{x > 0\}$, $\{y > 0\}$, $\{y < 0\}$, we conclude that (1.20) holds in Ω' and, hence, the form (1.19) is exact in Ω'.

Now we can prove that the form (1.19) is inexact in Ω. Assume from the contrary that it is exact in Ω and that F is its integral in Ω, that is,

$$dF = \frac{-ydx + xdy}{x^2 + y^2}.$$

Then $dF = d\theta$ in Ω' whence it follows that $d(F - \theta) = 0$ and, hence, $F = \theta + \text{const}$ in Ω'. It follows from this identity that function θ can be extended from Ω' to a continuous function on Ω, which however is not true, because the limits of θ when approaching the point $(-1, 0)$ (or any other point $(x, 0)$ with $x < 0$) from above and below are different: π and $-\pi$ respectively.

The moral of this example is that the statement of Theorem 1.4 is not true for an arbitrary open set Ω. It is possible to show that the statement of Theorem 1.4 is true if and only if the set Ω is *simply connected*, that is, if any closed curve in Ω can be continuously deformed to a point while staying in Ω. Obviously, the rectangles are simply connected (as well as Ω'), while the set $\Omega = \mathbb{R}^2 \setminus \{0\}$ is not simply connected.

1.4 Integrating factor

Consider again the quasilinear equation

$$a(x, y) + b(x, y)\, y' = 0 \tag{1.21}$$

and assume that it is *inexact*.

Write this equation in the form

$$adx + bdy = 0.$$

After multiplying by a non-zero function $M(x, y)$, we obtain an equivalent equation

$$Madx + Mbdy = 0,$$

which may become exact, provided function M is suitably chosen.

Definition. A function $M(x, y)$ is called the *integrating factor* for the differential equation (1.21) in Ω if M is a non-zero function in Ω such that the form $Madx + Mbdy$ is exact in Ω.

If one has found an integrating factor then multiplying (1.21) by M the problem amounts to the case of Theorem 1.3.

Example. Consider the ODE

$$y' = \frac{y}{4x^2y + x},$$

in the domain $\{x > 0, y > 0\}$ and write it in the form

$$ydx - \left(4x^2y + x\right) dy = 0.$$

Clearly, this equation is not closed. However, dividing it by x^2, we obtain the equation

$$\frac{y}{x^2}dx - \left(4y + \frac{1}{x}\right) dy = 0,$$

which is closed

$$\left(\frac{y}{x^2}\right)_y = \frac{1}{x^2} = -\left(4y+\frac{1}{x}\right)_x .$$

Hence, this equation is exact in domain $\{x>0, y>0\}$. Taking in (1.17) $x_0 = y_0 = 1$, we obtain the following integral of the differential form:

$$F(x,y) = \int_1^x \frac{1}{s^2}ds - \int_1^y \left(4t+\frac{1}{x}\right)dt = 3 - 2y^2 - \frac{y}{x}.$$

By Theorem 1.3, the general solution is given by the identity

$$2y^2 + \frac{y}{x} = C.$$

There are no regular methods for finding the integrating factors, though.

1.5 Second order ODE

For higher order ODEs we will use different notation: the independent variable will be denoted by t and the unknown function by $x(t)$. In this notation, a general second order ODE, resolved with respect to x'' has the form

$$x'' = f(t,x,x'),$$

where f is a given function of three variables. We consider here some problems that amount to a second order ODE.

1.5.1 Newtons' second law

Consider movement of a point particle along a straight line and let its coordinate at time t be $x(t)$. The velocity of the particle is $v(t) = x'(t)$ and the acceleration is $a(t) = x''(t)$. The Newton's second law says that at any time

$$mx'' = F, \tag{1.22}$$

where m is the mass of the particle and F is the force acting on the particle. In general, F is a function of t, x, x', that is, $F = F(t,x,x')$ so that (1.22) can be regarded as a second order ODE for $x(t)$.

The force F is called *conservative* if F depends only on the position x. For example, the gravitational, elastic, and electrostatic forces are conservative, while friction and the air resistance are non-conservative as they depend on the velocity. Assuming that $F = F(x)$, let $U(x)$ be a primitive function of $-F(x)$. The function U is called the *potential* of the force F. Multiplying the equation (1.22) by x' and integrating in t, we obtain

$$m\int x''x'dt = \int F(x)\,x'dt,$$

$$\frac{m}{2}\int \frac{d}{dt}(x')^2\,dt = \int F(x)\,dx,$$

$$\frac{m\left(x'\right)^2}{2} = -U\left(x\right) + C$$

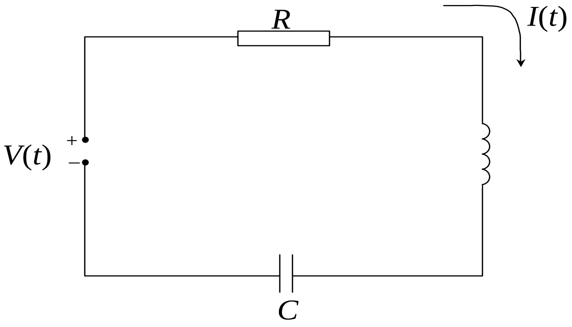
and

$$\frac{mv^2}{2} + U\left(x\right) = C.$$

The sum $\frac{mv^2}{2} + U\left(x\right)$ is called the *mechanical energy* of the particle (which is the sum of the *kinetic* energy and the *potential* energy). Hence, we have obtained the *law of conservation of energy*: the total mechanical energy of the particle in a conservative field remains constant.

1.5.2 Electrical circuit

Consider an RLC-circuit that is, an electrical circuit where a resistor, an inductor and a capacitor are connected in a series:



Denote by R the resistance of the resistor, by L the inductance of the inductor, and by C the capacitance of the capacitor. Let the circuit contain a power source with the voltage $V\left(t\right)$ depending in time t. Denote by $I\left(t\right)$ the current in the circuit at time t. Using the laws of electromagnetism, we obtain that the voltage drop v_R on the resistor R is equal to

$$v_R = RI$$

(Ohm's law), and the voltage drop v_L on the inductor is equal to

$$v_L = L\frac{dI}{dt}$$

(Faraday's law). The voltage drop v_C on the capacitor is equal to

$$v_C = \frac{Q}{C},$$

where Q is the charge of the capacitor; also we have $Q' = I$. By Kirchhoff's law, we obtain

$$v_R + v_L + v_C = V\left(t\right)$$

whence

$$RI + LI' + \frac{Q}{C} = V\left(t\right).$$

Differentiating in t, we obtain

$$LI'' + RI' + \frac{I}{C} = V', \tag{1.23}$$

which is a second order ODE with respect to $I(t)$. We will come back to this equation after having developed the theory of linear ODEs.

1.6 Higher order ODE and normal systems

A general ODE of the order n resolved with respect to the highest derivative can be written in the form

$$y^{(n)} = F\left(t, y, ..., y^{(n-1)}\right), \tag{1.24}$$

where t is an independent variable and $y(t)$ is an unknown function. It is frequently more convenient to replace this equation by a system of ODEs of the 1^{st} order.

Let $x(t)$ be a vector function of a real variable t, which takes values in $\mathbb{R}^n$. Denote by x_k the components of x. Then the derivative $x'(t)$ is defined component-wise by

$$x' = \left(x_1', x_2', ..., x_n'\right).$$

Consider now a *vector ODE of the first order*

$$x' = f(t, x), \tag{1.25}$$

where f is a given function of $n+1$ variables, which takes values in $\mathbb{R}^n$, that is, $f : \Omega \to \mathbb{R}^n$ where Ω is an open subset of $\mathbb{R}^{n+1}$. Here the couple (t, x) is identified with a point in $\mathbb{R}^{n+1}$ as follows:

$$(t, x) = (t, x_1, ..., x_n).$$

Denoting by f_k the components of f, we can rewrite the vector equation (1.25) as a system of n scalar equations

$$\begin{cases} x_1' = f_1(t, x_1, ..., x_n) \\ ... \\ x_k' = f_k(t, x_1, ..., x_n) \\ ... \\ x_n' = f_n(t, x_1, ..., x_n) \end{cases} \tag{1.26}$$

A system of ODEs of the form (1.25) or (1.26) is called the *normal system*. As in the case of the scalar ODEs, define a solution of the normal system (1.25) as a differentiable function $x : I \to \mathbb{R}^n$, where I is an interval in $\mathbb{R}$, such that $(t, x(t)) \in \Omega$ for all $t \in I$ and $x'(t) = f(t, x(t))$ for all $t \in I$.

Let us show how the equation (1.24) can be reduced to the normal system (1.26). Indeed, with any function $y(t)$ let us associate the vector-function

$$x = \left(y, y', ..., y^{(n-1)}\right),$$

which takes values in $\mathbb{R}^n$. That is, we have

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \ ..., \quad x_n = y^{(n-1)}.$$

Obviously,

$$x' = \left(y', y'', ..., y^{(n)}\right),$$

and using (1.24) we obtain a normal system of equations

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \dots \\ x_{n-1}' = x_n \\ x_n' = F\left(t, x_1, ...x_n\right) \end{cases} \tag{1.27}$$

Conversely, the system (1.27) implies

$$x_1^{(n)} = x_n' = F\left(t, x_1, ...x_n\right) = F\left(t, x_1, x_1', .., x_1^{(n-1)}\right)$$

so that we obtain equation (1.24) with respect to $y = x_1$. Hence, the equation (1.24) is equivalent to the normal system (1.27).

Obviously, we can rewrite this system as a vector equation (1.25) with the function f as follows:

$$f\left(t, x\right) = \left(x_2, x_3, ..., x_n, F\left(t, x_1, ..., x_n\right)\right). \tag{1.28}$$

Example. Consider the second order equation

$$y'' = F\left(t, y, y'\right).$$

Setting $x = (y, y')$ we obtain

$$x' = \left(y', y''\right)$$

whence

$$\begin{cases} x_1' = x_2 \\ x_2' = F\left(t, x_1, x_2\right) \end{cases}$$

1.7 Initial value problem

Usually one needs to solve a vector ODE (1.25) with an additional condition that $x\left(t_0\right) = x_0$ where t_0 and x_0 are given. In this case we speak about the *initial value problem*

$$\begin{cases} x' = f\left(t, x\right), \\ x\left(t_0\right) = x_0, \end{cases}$$

shortly IVP. We will see in the next Chapters that, under some mild assumptions about f, this problem has a unique solution.

Let us state the IVP for the scalar equation (1.24) of order n. Using as before the notation $x = \left(y, y', ..., y^{(n-1)}\right)$, we see that the initial condition $x\left(t_0\right) = x_0$ amounts to n scalar scalar conditions

$$\begin{cases} y\left(t_0\right) = y_0 \\ y'\left(t_0\right) = y_1 \\ \dots \\ y^{(n-1)}\left(t_0\right) = y_{n-1} \end{cases}$$

where $(y_0, ..., y_{n-1}) = x_0$ are given values. Hence, the initial value problem IVP for the scalar equation of the order n can be stated as follows:

$$\begin{cases} y^{(n)} = F\left(t, y, y', ..., y^{(n-1)}\right) \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \\ ... \\ y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

2 Linear equations and systems

A normal linear system of ODEs is the following vector equation

$$x' = A(t)\, x + B(t), \tag{2.1}$$

where $A : I \to \mathbb{R}^{n\times n}$, $B : I \to \mathbb{R}^n$, I is an interval in $\mathbb{R}$, and $x = x(t)$ is an unknown function with values in $\mathbb{R}^n$.

In other words, for each $t \in I$, $A(t)$ is a linear operator in $\mathbb{R}^n$, while $A(t)\, x$ and $B(t)$ are the vectors in $\mathbb{R}^n$. In the coordinate form, (2.1) is equivalent to the following system of linear equations

$$x_i' = \sum_{l=1}^{n} A_{ij}(t)\, x_j + B_i(t), \quad i = 1, ..., n,$$

where A_{ij} and B_i are the components of A and B, respectively. We will consider the ODE (2.1) only when $A(t)$ and $B(t)$ are continuous in t, that is, when the mappings $A : I \to \mathbb{R}^{n\times n}$ and $B : I \to \mathbb{R}^n$ are continuous. It is easy to show that the continuity of these mappings is equivalent to the continuity of all the components $A_{ij}(t)$ and $B_i(t)$, respectively

2.1 Existence of solutions for normal systems

Theorem 2.1 *In the above notation, let $A(t)$ and $B(t)$ be continuous in $t \in I$. Then, for any $t_0 \in I$ and $x_0 \in \mathbb{R}^n$, the IVP*

$$\begin{cases} x' = A(t)\, x + B(t) \\ x(t_0) = x_0 \end{cases} \tag{2.2}$$

has a solution $x(t)$ defined on I, and this solution is unique.

Before we start the proof, let us prove the following useful lemma.

Lemma 2.2 (The Gronwall inequality) *Let $z(t)$ be a non-negative continuous function on $[t_0, t_1]$ where $t_0 < t_1$. Assume that there are constants $C, L \geq 0$ such that*

$$z(t) \leq C + L \int_{t_0}^{t} z(s)\, ds \tag{2.3}$$

for all $t \in [t_0, t_1]$. Then

$$z(t) \leq C \exp\left(L(t - t_0)\right) \tag{2.4}$$

for all $t \in [t_0, t]$.

Proof. It suffices to prove the statement the case when C is strictly positive, which implies the validity of the statement also in the case $C = 0$. Indeed, if (2.3) holds with $C = 0$ then it holds with any $C > 0$. Therefore, (2.4) holds with any $C > 0$, whence it follows that it holds with $C = 0$.

Hence, assume in the sequel that $C > 0$. This implies that the right hand side of (2.3) is positive. Set

$$F(t) = C + L \int_{t_0}^{t} z(s)\, ds$$

and observe that F is differentiable and $F' = Lz$. It follows from (2.3) that $z \le F$ whence

$$F' = Lz \le LF.$$

This is a differential inequality for F that can be solved similarly to the separable ODE. Since $F > 0$, dividing by F we obtain

$$\frac{F'}{F} \le L,$$

whence by integration

$$\ln \frac{F(t)}{F(t_0)} = \int_{t_0}^{t} \frac{F'(s)}{F(s)} ds \le \int_{t_0}^{t} L ds = L(t - t_0),$$

for all $t \in [t_0, t_1]$. It follows that

$$F(t) \le F(t_0) \exp(L(t - t_0)) = C \exp(L(t - t_0)).$$

Using again (2.3), that is, $z \le F$, we obtain (2.4). ■

Proof of Theorem 2.1. Let us fix some bounded closed interval $[\alpha, \beta] \subset I$ such that $t_0 \in [\alpha, \beta]$. The proof consists of the following three parts.

1. the uniqueness of a solution on $[\alpha, \beta]$;
2. the existence of a solution on $[\alpha, \beta]$;
3. the existence and uniqueness of a solution on I.

We start with the observation that if $x(t)$ is a solution of (2.2) on I then, for all $t \in I$,

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^{t} x'(s)\, ds \\ &= x_0 + \int_{t_0}^{t} (A(s)\, x(s) + B(s))\, ds. \end{aligned} \tag{2.5}$$

In particular, if $x(t)$ and $y(t)$ are two solutions of IVP (2.2) on the interval $[a, \beta]$, then they both satisfy (2.5) for all $t \in [\alpha, \beta]$ whence

$$x(t) - y(t) = \int_{t_0}^{t} A(s)(x(s) - y(s))\, ds.$$

Setting

$$z(t) = \|x(t) - y(t)\|$$

and using that

$$\|A(y-x)\| \leq \|A\| \|y-x\| = \|A\| z,$$

we obtain

$$z(t) \leq \int_{t_0}^{t} \|A(s)\| z(s) \, ds,$$

for all $t \in [t_0, \beta]$, and a similar inequality for $t \in [\alpha, t_0]$ where the order of integration should be reversed. Set

$$a = \sup_{s \in [\alpha,\beta]} \|A(s)\| . \tag{2.6}$$

Since $s \mapsto A(s)$ is a continuous function, $s \mapsto \|A(s)\|$ is also continuous; hence, it is bounded on the interval $[\alpha, \beta]$ so that $a < \infty$. It follows that, for all $t \in [t_0, \beta]$,

$$z(t) \leq a \int_{t_0}^{t} z(s) \, ds,$$

whence by Lemma 2.2 $z(t) \leq 0$ and, hence, $z(t) = 0$. By a similar argument, the same holds for all $t \in [\alpha, t_0]$ so that $z(t) \equiv 0$ on $[\alpha, \beta]$, which implies the identity of the solutions $x(t)$ and $y(t)$ on $[\alpha, \beta]$.

In the second part, consider a sequence $\{x_k(t)\}_{k=0}^{\infty}$ of functions on $[\alpha, \beta]$ defined inductively by

$$x_0(t) \equiv x_0$$

and

$$x_k(t) = x_0 + \int_{t_0}^{t} (A(s) x_{k-1}(s) + B(s)) \, ds, \quad k \geq 1. \tag{2.7}$$

We will prove that the sequence $\{x_k\}_{k=0}^{\infty}$ converges on $[\alpha, \beta]$ to a solution of (2.2) as $k \to \infty$.

Using the identity (2.7) and

$$x_{k-1}(t) = x_0 + \int_{t_0}^{t} (A(s) x_{k-2}(s) + B(s)) \, ds$$

we obtain, for any $k \geq 2$ and $t \in [t_0, \beta]$,

$$\begin{aligned} \|x_k(t) - x_{k-1}(t)\| &\leq \int_{t_0}^{t} \|A(s)\| \|x_{k-1}(s) - x_{k-2}(s)\| \, ds \\ &\leq a \int_{t_0}^{t} \|x_{k-1}(s) - x_{k-2}(s)\| \, ds \end{aligned}$$

where a is defined by (2.6). Denoting

$$z_k(t) = \|x_k(t) - x_{k-1}(t)\| ,$$

we obtain the recursive inequality

$$z_k(t) \leq a \int_{t_0}^{t} z_{k-1}(s) \, ds.$$

In order to be able to use it, let us first estimate $z_1(t) = \|x_1(t) - x_0(t)\|$. By definition, we have, for all $t \in [t_0, \beta]$,

$$z_1(t) = \left\| \int_{t_0}^{t} (A(s)\,x_0 + B(s))\,ds \right\| \leq b\,(t - t_0),$$

where

$$b = \sup_{s \in [\alpha,\beta]} \|A(s)\,x_0 + B(s)\| < \infty.$$

It follows by induction that

$$\begin{array}{rcl} z_2(t) & \leq & ab \displaystyle\int_{t_0}^{t} (s - t_0)\,ds = ab\frac{(t-t_0)^2}{2}, \\ z_3(t) & \leq & a^2 b \displaystyle\int_{t_0}^{t} \frac{(s-t_0)^2}{2}\,ds = a^2 b\frac{(t-t_0)^3}{3!}, \\ & & \ldots \\ z_k(t) & \leq & a^{k-1} b\dfrac{(t-t_0)^k}{k!}. \end{array}$$

Setting $c = \max(a, b)$ and using the same argument for $t \in [\alpha, t_0]$, rewrite this inequality in the form

$$\|x_k(t) - x_{k-1}(t)\| \leq \frac{(c\,|t - t_0|)^k}{k!},$$

for all $t \in [\alpha, \beta]$. Since the series

$$\sum_k \frac{(c\,|t-t_0|)^k}{k!}$$

is the exponential series and, hence, is convergent for all t, in particular, uniformly in any bounded interval of t, we obtain by the comparison test, that the series

$$\sum_{k=1}^{\infty} \|x_k(t) - x_{k-1}(t)\|$$

converges uniformly in $t \in [\alpha, \beta]$, which implies that also the series

$$\sum_{k=1}^{\infty} (x_k(t) - x_{k-1}(t))$$

converges uniformly in $t \in [\alpha, \beta]$. Since the N-th partial sum of this series is $x_N(t) - x_0$, we conclude that the sequence $\{x_k(t)\}$ converges uniformly in $t \in [\alpha, \beta]$ as $k \to \infty$. Setting

$$x(t) = \lim_{k \to \infty} x_k(t)$$

and passing in the identity

$$x_k(t) = x_0 + \int_{t_0}^{t} (A(s)\,x_{k-1}(s) + B(s))\,ds$$

to the limit as $k \to \infty$, obtain

$$x(t) = x_0 + \int_{t_0}^{t} (A(s)\, x(s) + B(s))\, ds. \tag{2.8}$$

This implies that $x(t)$ solves the given IVP (2.2) on $[\alpha, \beta]$. Indeed, $x(t)$ is continuous on $[\alpha, \beta]$ as a uniform limit of continuous functions; it follows that the right hand side of (2.8) is a differentiable function of t, whence

$$x' = \frac{d}{dt}\left(x_0 + \int_{t_0}^{t} (A(s)\, x(s) + B(s))\, ds.\right) = A(t)\, x(t) + B(t)\,.$$

Finally, it is clear from (2.8) that $x(t_0) = x_0$.

Having constructed the solution of (2.2) on any bounded closed interval $[\alpha, \beta] \subset I$, let us extend it to the whole interval I as follows. For any interval I, there is an increasing sequence of bounded closed intervals $\{[\alpha_l, \beta_l]\}_{l=1}^{\infty}$ such that their union is I; furthermore, we can assume that $t_0 \in [\alpha_l, \beta_l]$ for all l. Denote by $x_l(t)$ be the solution of (2.2) on $[\alpha_l, \beta_l]$. Then $x_{l+1}(t)$ is also the solution of (2.2) on $[\alpha_l, \beta_l]$ whence it follows by the uniqueness statement of the first part that $x_{l+1}(t) = x_l(t)$ on $[\alpha_l, \beta_l]$. Hence, in the sequence $\{x_l(t)\}$ any function is an extension of the previous function to a larger interval, which implies that the function

$$x(t) := x_l(t) \text{ if } t \in [\alpha_l, \beta_l]$$

is well-defined for all $t \in I$ and is a solution of IVP (2.2). Finally, this solution is unique on I because the uniqueness takes place in each of the intervals $[\alpha_l, \beta_l]$. ■