# Existence of stable-like heat kernels and stability of the critical index on metric measure spaces

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**Abstract**

Let $(\mathcal{X}, d, \mu)$ be a metric measure space satisfying the volume doubling condition. We introduce a new critical index $\beta^\sharp$ of the metric measure space, which is defined to be the supremum of all possible values of $\beta$ such that there exists a stochastically complete continuous heat kernel on $(\mathcal{X}, d, \mu)$ satisfying *two-sided* stable-like estimate of index $\beta$. This critical index $\beta^\sharp$ is proved to be invariant under *quasi-isometry* of two metric measure spaces. To achieve this, we prove that $\beta^\sharp$ can be equivalently defined by means of a certain *Andres-Barlow condition*. Moreover, for every $\beta < \beta^\sharp$, there exists a stochastically complete continuous heat kernel satisfying the two-sided stable-like estimate of index $\beta$, while for every $\beta > \beta^\sharp$ such heat kernel does not exist. In contrast to that, a heat kernel, satisfying only *upper* stable-like estimate of index $\beta$, exists for any $\beta \in (0, \infty)$. We construct such a heat kernel by using a dyadic decomposition of the space $\mathcal{X}$ and the associated ultra-metric. Further, using adjacent dyadic decompositions, we prove that, for any $\beta \in (0, \infty)$, there exists a finite family of heat kernels on $\mathcal{X}$, such that their sum satisfies the two-sided stable-like estimate of index $\beta$ as well as all other properties of a heat kernel except the semigroup property. In addition, we show that $\beta^\sharp$ coincides with another critical index $\beta^*$ defined by means of *Besov spaces*, as well as with the walk dimension $d_w$ provided there exists a heat kernel satisfying the two-sided *sub-Gaussian estimate*. This indicates that $\beta^\sharp$ could be a good candidate for the walk dimension in future attempts to construct a diffusion process on $\mathcal{X}$.

# Contents



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# 1 Introduction

In this paper we are concerned with construction of heat kernels on a metric measure space, which satisfy stable-like estimates. Recall that in $\mathbb{R}^n$ the fractional Laplace operator $(-\Delta)^{\beta/2}$ with $\beta \in (0, 2)$ has a heat kernel $p_t(x, y)$ that satisfies the following estimate: for all $t > 0$ and $x, y \in \mathcal{X}$,

$$p_t(x,y) \simeq \frac{1}{t^{n/\beta}}\left(1 + \frac{|x-y|}{t^{1/\beta}}\right)^{-(n+\beta)},$$

where $\Delta = \sum_{j=1}^{n} \partial_{x_j}^2$ and the sign $\simeq$ means that the ratio of the both sides is bounded from above and below by positive constants for the specified range of variables. In this case $p_t(x, y)$ coincides with the transition density of the symmetric stable Levy process of the index $\beta$. Note also that $(-\Delta)^{\beta/2}$ is an integral operator of the form

$$(-\Delta)^{\beta/2} f(x) = \left(\frac{\beta 2^{\beta-1}\Gamma(\frac{n+\beta}{2})}{\pi^{n/2}\Gamma(1-\frac{\beta}{2})}\right) \text{P.V.} \int_{\mathbb{R}^n} \frac{f(y)-f(x)}{|x-y|^{n+\beta}} d\mu(y)$$

for a certain class of functions $f$ on $\mathbb{R}^n$.

Let $(\mathcal{X}, d)$ be a separable metric space. We always assume that all metric balls

$$B(x,r) := \{y \in \mathcal{X} : d(x,y) < r\}$$

are precompact. In the case when $\operatorname{diam} \mathcal{X} < \infty$, we always assume that $\mathcal{X}$ is compact. Let $\mu$ be a Radon measure on $\mathcal{X}$ with full support. In particular, the volume function

$$V(x,r) := \mu(B(x,r)) \tag{1.1}$$

is finite and positive for all $x \in \mathcal{X}$ and $r \in (0, +\infty)$. We refer to $(\mathcal{X}, d, \mu)$ as a *metric measure space*.

By a *heat kernel* $\{p_t\}_{t>0}$ on $(\mathcal{X}, d, \mu)$ (or simply, on $\mathcal{X}$) we mean that $p_t(x, y)$ that is defined for all $t > 0$ and $x, y \in \mathcal{X}$ and satisfies the following properties, for all values of the arguments involved:

(P1) for any $t > 0$, $p_t(x, y)$ is a measurable non-negative function of $(x, y)$, and

$$\int_{\mathcal{X}} p_t(x, y)\, d\mu(y) \le 1; \tag{1.2}$$

(P2) symmetry: $p_t(x, y) = p_t(y, x)$;

(P3) the semigroup identity:

$$\int_{\mathcal{X}} p_t(x, z) p_s(z, y)\, d\mu(z) = p_{t+s}(x, y);$$

(P4) approximation of identity: for any $f \in L^2(\mathcal{X}) := L^2(\mathcal{X}, \mu)$,

$$\int_{\mathcal{X}} p_t(\cdot, y) f(y)\, d\mu(y) \to f \ \text{ as } \ t \to 0,$$

where the convergence is in the norm of $L^2(\mathcal{X})$.

The heat kernel is said to be *stochastically complete* if the integral in (1.2) is equal to 1 for all $t > 0$ and $x \in M$.

Assume first that the measure $\mu$ is $\alpha$-regular, that is, for all $x \in \mathcal{X}$ and $r > 0$,

$$V(x, r) \simeq r^\alpha. \tag{1.3}$$

Let us ask the following question: does there exist a heat kernel $\{p_t\}_{t>0}$ on $\mathcal{X}$ that satisfies the following *stable-like* estimate for some $\beta > 0$:

$$p_t(x, y) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \tag{1.4}$$

for all $t > 0$ and $x, y \in \mathcal{X}$? And, what is the range of the index $\beta$ for which (1.4) is possible?

The following approach to construction of Dirichlet forms has been widely used in the literature [21, Example 1.2.4]. Given a non-negative symmetric Borel function $J(x, y)$ on $\mathcal{X}\times\mathcal{X}$ (that is called a *jump kernel*), consider the following bilinear form

$$\mathcal{E}(f, g) := \iint_{\mathcal{X}\times\mathcal{X}} (f(x) - f(y))(g(x) - g(y)) J(x, y)\, d\mu(x)\, d\mu(y)$$

in the domain

$$\mathcal{F} := \left\{ f \in L^2(\mathcal{X}) : \ \mathcal{E}(f, f) < \infty \right\}.$$

Assume in prior that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathcal{X})$. Let $\mathcal{L}$ be its (non-negative definite) generator and $P_t = e^{-t\mathcal{L}}$, $t \geq 0$, be the associated heat semigroup acting in $L^2(\mathcal{X})$. Then one can ask if the operator $P_t$ has for any $t > 0$ the integral kernel $p_t(x,y)$ satisfying the estimate (1.4). Since

$$J(x,y) = \lim_{t\to 0} \frac{p_t(x,y)}{2t},$$

the estimate (1.4) implies that

$$J(x,y) \simeq \frac{1}{d(x,y)^{\alpha+\beta}}, \tag{1.5}$$

which is, hence, a necessary condition for (1.4). Chen and Kumagai stated in [15] that when $0 < \beta < 2$, it is easy to check that the bilinear form $(\mathcal{E}, \mathcal{F})$ with the jump kernel (1.5) is a regular Dirichlet form, and then proved the heat kernel of $(\mathcal{E}, \mathcal{F})$ exists and satisfies (1.4).

However, there are many examples of *fractal* metric measure spaces where the same is true also for a certain range of $\beta \geq 2$. Indeed, it is known that, on many families of fractals, there exists a strongly local regular Dirichlet form with the heat kernel $\{p_t\}_{t>0}$ satisfying a *sub-Gaussian* estimate

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/d_w}} \exp\left(-c\left(\frac{(d(x,y))^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right) \tag{1.6}$$

with the *walk dimension* $d_w \geq 2$ (see, for example, [5, 6, 9, 36, 37] and etc.). The sign $\asymp$ means that both $\leq$ and $\geq$ are valid, but with different values of positive constants $C, c$.

In this case, $p_t(x,y)$ coincides with the transition density of a diffusion process on $\mathcal{X}$. Using the subordination techniques, one easily constructs heat kernels of jump processes satisfying (1.4) with any $\beta < d_w$. Besides, it is known that if $\beta > d_w$ then there exist no heat kernel satisfying (1.4) (see [23, Theorem 5.2]).

In order to describe the results of the present paper, we relax the assumption (1.3) as follows. A metric measure space $(\mathcal{X}, d, \mu)$ is said to satisfy the *volume doubling condition* (VD) if there exists a positive constant $C_D$ such that, for all $x \in \mathcal{X}$ and $r > 0$,

$$V(x, 2r) \leq C_D V(x,r). \tag{1.7}$$

The volume doubling condition (VD) is equivalent to the following: there exist positive constants $C'_D$ and $\alpha_+$ such that, for all $x,\ y \in \mathcal{X}$ and $0 < r \leq R < \infty$,

$$\frac{V(x,R)}{V(y,r)} \leq C'_D \left(\frac{d(x,y)+R}{r}\right)^{\alpha_+}. \tag{1.8}$$

The triple $(\mathcal{X}, d, \mu)$ is said to satisfy the *reverse volume doubling condition* (RVD) if there exist positive constants $C_{RD}$ and $\alpha_-$ such that, for all $x \in \mathcal{X}$ and $0 < r \leq R < \operatorname{diam}(\mathcal{X})$,

$$\frac{V(x,R)}{V(x,r)} \geq C_{RD} \left(\frac{R}{r}\right)^{\alpha_-}. \tag{1.9}$$

If $(\mathcal{X}, d, \mu)$ is connected and satisfies (VD), then we have by [31, Proposition 5.2] that (RVD) holds whenever $\mathcal{X} \setminus B(x,R)$ is non-empty.

Metric measure spaces satisfying (VD) are usually referred to as *doubling metric measure spaces*. Such spaces occur frequently in analysis and include, in particular, the Euclidean spaces $\mathbb{R}^n$ with measures from a certain large class, Riemannian manifolds of non-negative Ricci curvature, nilpotent Lie groups with polynomial growth, many fractals, and etc. (see, for instance, [5, 6, 18, 19, 20, 22, 30, 31, 32, 36, 40, 41]).

Under the condition (VD), we pose the question of the existence of a heat kernel $\{p_t\}_{t>0}$ on $\mathcal{X}$ satisfying the following stable-like two-sided estimate for some $\beta > 0$:

$$p_t(x,y) \simeq \frac{1}{V(x, t^{1/\beta} + d(x,y))}\left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta} \quad \text{for all } \ t \in (0,\infty) \ \text{ and } \ x,y \in \mathcal{X}. \tag{1.10}$$

For any given $\beta \in (0,\infty)$, we refer to (1.10) as $(\mathrm{ULE})_\beta$ (*Upper and Lower Estimate*). We also say that $(\mathrm{UE})_\beta$ (resp. $(\mathrm{LE})_\beta$) is satisfied, if the upper (resp. lower) estimate in (1.10) holds.

Clearly, if $\mu$ is $\alpha$-regular, then (1.10) is equivalent to (1.4). Note that there are heat kernels $\{p_t\}_{t>0}$ satisfying (1.10) even on bounded metric spaces (for example, fractals).

**Definition 1.1.** Define the *critical index* that relates the possible values of $\beta$ in $(\mathrm{ULE})_\beta$ as follows:

$$\begin{aligned}\beta^{\#} := \beta^{\#}(\mathcal{X}, d, \mu) := \sup\{\beta > 0 :\ & \text{there exists a stochastically complete} \\ & \text{continuous heat kernel } \{p_t\}_{t>0} \text{ on } \mathcal{X} \text{ satisfying } (\mathrm{ULE})_\beta\}.\end{aligned}$$

Here continuity means that for any $t > 0$, $p_t(x,y)$ is jointly continuous in $(x,y) \in \mathcal{X}$.

The following theorem gives an equivalent condition to $(\mathrm{ULE})_\beta$ and a new characterization of $\beta^\sharp$ via the *Andres-Barlow* condition $(\mathrm{AB})_\beta$ (see Definition 3.2 below for its precise definition).

**Theorem 1.2.** *Assume that $(\mathcal{X}, d, \mu)$ satisfies (VD) and (RVD). Given any $\beta > 0$, the following two conditions are equivalent:*

(i) *The Andres-Barlow condition $(\mathrm{AB})_\beta$ holds;*

(ii) *There exists a stochastically complete continuous heat kernel $\{p_t\}_{t>0}$ on $\mathcal{X}$ satisfying $(\mathrm{ULE})_\beta$.*

*Consequently, the critical index satisfies*

$$\beta^{\#}(\mathcal{X}, d, \mu) = \sup\{\beta > 0 :\ (\mathrm{AB})_\beta \textit{ is satisfied on } (\mathcal{X}, d, \mu)\,\}. \tag{1.11}$$

The *Andres-Barlow* condition $(\mathrm{AB})_\beta$ ensures the existence of certain cutoff functions in the metric measure space $(\mathcal{X}, d, \mu)$. Note that $(\mathrm{AB})_\beta$ is always satisfied when $\beta < 2$, which, together with (1.11), implies that $\beta^{\#} \geq 2$. If $(\mathcal{X}, d, \mu)$ is an ultra-metric space satisfying (VD) and (RVD), then $\beta^\sharp = \infty$.

Let us remark that Theorem 1.2 improves the result in [12]. Indeed, the result in [12] also shows that (ii) implies (i) in Theorem 1.2 but under the condition that $\{p_t\}_{t>0}$ is the heat kernel of some *regular* Dirichlet form of pure jump type. Now, this condition is not needed in Theorem 1.2. Undoubtedly, the proof of Theorem 1.2 relies on a very deep theory which basically says that the upper estimate $(\mathrm{UE})_\beta$ implies the regularity of the associated Dirichlet form of the heat kernel. The precise statement of this latter result is presented in Theorem 2.1 below, and the whole Section 2 is devoted to the proof of Theorem 2.1, via using both probability and analytic methods, respectively. With help of Theorem 2.1, we then prove Theorem 1.2 in Subsection 3.3.

An important consequence of Theorem 1.2 is that the critical index $\beta^{\#}$ is invariant under *quasi-isometry* of two metric measure spaces. In this paper, we say that $(\mathcal{X}, d, \mu)$ is quasi-isometric to $(\mathcal{X}, d', \mu')$ if and only if $d \simeq d'$ and $\mu \simeq \mu'$.

**Theorem 1.3.** *Let $(\mathcal{X}, d, \mu)$ satisfy (VD) and (RVD). If the two metric measure spaces $(\mathcal{X}, d, \mu)$ and $(\mathcal{X}, d', \mu')$ are quasi-isometric, then*

$$\beta^{\#}(\mathcal{X}, d, \mu) = \beta^{\#}(\mathcal{X}, d', \mu').$$

This stability property of $\beta^{\sharp}$ will be proved in Subsection 3.4, which is indeed a consequence of Theorem 1.2 and Definition 1.1.

Obviously, if $\beta > \beta^{\sharp}$, then there does not exist a stochastically complete continuous heat kernel on $\mathcal{X}$ satisfying $(\mathrm{ULE})_\beta$. The main issue for the following theorem is that the set of $\beta > 0$ for which there exists a stochastically complete continuous heat kernel on $\mathcal{X}$ satisfying $(\mathrm{ULE})_\beta$ is an interval. Indeed, its proof follows from the subordination theory and is given in Subsection 4.1 below.

**Theorem 1.4.** *Assume that $(\mathcal{X}, d, \mu)$ satisfies (VD). Then, for any $\beta \in (0, \beta^{\#})$, there exists a s-tochastically complete continuous heat kernel on $\mathcal{X}$ satisfying $(\mathrm{ULE})_\beta$.*

Let us compare $\beta^{\#}$ with the following *critical exponent* $\beta^*$ of Besov spaces $\{\Lambda^{\beta/2}_{2,\infty}(\mathcal{X})\}_{\beta>0}$, where

$$\beta^* := \sup\left\{\beta > 0 : \ \Lambda^{\beta/2}_{2,\infty}(\mathcal{X}) \text{ is dense in } L^2(\mathcal{X})\right\}. \tag{1.12}$$

The precise definition of the Besov space $\Lambda^{\beta/2}_{2,\infty}(\mathcal{X})$ is given in Definition 3.1 below. Assuming for simplicity that the metric measure space $(\mathcal{X}, d, \mu)$ is $\alpha$-regular. Then the critical exponent $\beta^*$ in (1.12) is exactly the walk dimension $d_w$ that appears in the sub-Gaussian heat kernel estimate (1.6) (see [23, Section 5.1]). The identity (1.12) can therefore always be used as the definition of the walk dimension, even if there is no sub-Gaussian heat kernel on $\mathcal{X}$.

Applying Theorem 1.4 and the subordination theory of heat kernels, we can get the following relationship between $\beta^{\sharp}$ and $\beta^*$.

**Theorem 1.5.** *Assume that $(\mathcal{X}, d, \mu)$ satisfies (VD). If there is a stochastically complete continuous heat kernel $\{q_t\}_{t>0}$ on $\mathcal{X}$ satisfying the following sub-Gaussian estimate $(\mathrm{SG})_{d_w}$ with $d_w > 1$:*

$$q_t(x, y) \asymp \frac{C}{V(x, t^{1/d_w} + d(x, y))} \exp\left(-c\left(\frac{d(x, y)}{t^{1/d_w}}\right)^{\frac{d_w}{d_w-1}}\right) \quad \textit{for all } \ t \in (0, \infty) \ \textit{ and } \ x, y \in \mathcal{X}, \tag{1.13}$$

*then*

$$\beta^{\sharp} = d_w = \beta^*. \tag{1.14}$$

We will show Theorem 1.5 in Subsection 4.2. Its proof is divided into two parts: the first part is to apply Theorem 1.4 to prove $\beta^{\#} = d_w$; the second part is to apply the idea in [23, Section 5.1] to prove $d_w = \beta^*$.

According to Theorems 1.3 and 1.5, we remark that the critical index $\beta^{\#}$ could be a good candidate for the walk dimension in future attempts to construct a diffusion process on $\mathcal{X}$. Or, one can ask whether or not there exists a Sub-Gaussian heat kernel $\{p_t\}_{t>0}$ satisfying $(\mathrm{SG})_{d_w}$ when $d_w = \beta^{\#}$. On the other hand, it would also be interesting to construct an example of a doubing space with $\beta^{\#} < \beta^{*}$. By Theorem 1.5, such spaces can not have a sub-Gaussian heat kernel. Spaces without sub-Gaussian heat kernels are known in [13].

In contrast to Theorem 1.4 and the two-sided condition $(\mathrm{ULE})_\beta$ that is used in the definition of $\beta^{\sharp}$, we show in the next theorem that for any $\beta \in (0,\infty)$ there always exists a non-trivial heat kernel satisfying only the *upper* estimate $(\mathrm{UE})_\beta$.

**Theorem 1.6.** *Assume that $(\mathcal{X}, d, \mu)$ satisfies (VD). Then, for any $\beta > 0$, there exists a stochastically complete heat kernel $\{p_t\}_{t>0}$ on $\mathcal{X}$ satisfying the upper estimate*

$$p_t(x,y) \leq \frac{C}{V(x, t^{1/\beta} + d(x,y))}\left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta}, \tag{1.15}$$

*as well as the on-diagonal estimate*

$$p_t(x,x) \simeq \frac{1}{V(x, t^{1/\beta})}, \tag{1.16}$$

*for all $t > 0$ and $x, y \in \mathcal{X}$.*

Let us emphasize that Theorem 1.6 does not require any upper bound of $\beta$. Hence, the necessity of the restriction $\beta < \beta^{\#}$ for the existence of a stable-like heat kernel is dictated by the *off-diagonal lower* bound of the heat kernel and, possibly, by continuity. Note also that if a stochastically complete heat kernel comes from a *regular Dirichlet form* and satisfies $(\mathrm{ULE})_\beta$ then it is continuous; see [16, Theorem 1.13 and Lemma 5.6] and [24, Theorem 1.12].

The proof of Theorem 1.6 is given in Subsection 5.3 (see Theorem 5.6). It uses the dyadic decomposition of metric spaces from [4, 33, 35] and the construction of heat kernels from [10] on ultra-metric spaces.

In addition to Theorem 1.6, we prove the following curious result about *families* of heat kernels that is based on *adjacent* systems of dyadic decompositions from [33] (see also Subsection 5.4).

**Theorem 1.7.** *Assume that $(\mathcal{X}, d, \mu)$ satisfies (VD). Then, for any $\beta > 0$, there exists a finite family $\{\{p_t^{(k)}\}_{t>0} : k = 1, \ldots, K\}$ of stochastically complete heat kernels on $\mathcal{X}$ satisfying the two-sided estimate*

$$\sum_{k=1}^{K} p_t^{(k)}(x,y) \simeq \frac{C}{V(x, t^{1/\beta} + d(x,y))}\left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta},$$

*where $K$ is a natural number that depends on the doubling constant $C_D$ in (VD).*

Theorem 1.7 will be proved in Subsection 5.5 (see Theorem 5.9). In fact, all heat kernels $\{\{p_t^{(k)}\}_{t>0} : k = 1, \ldots, K\}$ here satisfy both (1.15) and (1.16). Clearly, if $\beta > \beta^{\#}$ then each individual heat kernel $\{p_t^{(k)}\}_{t>0}$ does not satisfy the stable-like *off-diagonal lower* bound, but their sum does. In addition, observe that the function

$$q_t(x,y) := \frac{1}{K}\sum_{k=1}^{K} p_t^{(k)}(x,y) \quad \text{for all } \ t \in (0,\infty) \ \text{ and } \ x, y \in \mathcal{X}$$

satisfies $(\mathrm{ULE})_\beta$ and all properties of a stochastically complete heat kernel, except the semigroup property.

**Notation.** We use the following notation throughout the paper.

- For a set $E \subseteq \mathcal{X}$, denote by $E^{\complement} := \mathcal{X} \setminus E$.
- For any $p \in (0,\infty]$, let $L^p_c(\mathcal{X}) := \{f \in L^p(\mathcal{X}) : \ \operatorname{supp} f$ is compact$\}$, where $\operatorname{supp} f$ is the complement of the largest open set where $f = 0$ $\mu$-a.e.
- $C(\mathcal{X})$ is the space of all continuous functions on $\mathcal{X}$, and $C_c(\mathcal{X})$ is the subspace of $C(\mathcal{X})$ consisting of functions with compact supports. Denote by $C_\infty(\mathcal{X})$ the subspace of $C(\mathcal{X})$ consisting continuous functions vanishing at infinity, that is,

$$C_\infty(\mathcal{X}) := \{u \in C(\mathcal{X}) : \text{ for any } \varepsilon > 0, \exists \text{ compact } K \text{ s.t. } |u(x)| < \varepsilon \text{ for any } x \notin K\}.$$

  In the case when $\operatorname{diam}\mathcal{X} < \infty$, since we have assumed that $\mathcal{X}$ is compact, it follows that

$$C_\infty(\mathcal{X}) = C_c(\mathcal{X}) = C(\mathcal{X}). \tag{1.17}$$

- For a function or a number $u$, set $u_+ := u \vee 0 := \max\{u, 0\}$ and $u_- := u \wedge 0 := -\max\{-u, 0\}$.
- The letters $C$ and $c$ are used to denote positive constants that are independent of the variables in question, but may vary at each occurrence. The relation $u \lesssim v$ (resp., $u \gtrsim v$) between functions $u$ and $v$ means that $u \le Cv$ (resp., $u \ge Cv$) for a positive constant $C$ and for a specified range of the variables. We write $u \simeq v$ if $u \lesssim v \lesssim u$.

## 2 Regularity of Dirichlet forms under heat kernel upper estimates

A symmetric bilinear form $\mathcal{E}$ with domain $\mathrm{Dom}(\mathcal{E})$ is called a *Dirichlet form* on $L^2(\mathcal{X})$ if $\mathrm{Dom}(\mathcal{E})$ is a dense subspace of $L^2(\mathcal{X})$ and $\mathcal{E}$ is not only *closed* but also *Markovian*. Endow $\mathrm{Dom}(\mathcal{E})$ with the norm

$$\|u\|^2_{\mathcal{E}_1} := \|u\|^2_{L^2} + \mathcal{E}(u,u) =: \mathcal{E}_1(u,u). \tag{2.1}$$

We say that a subset $\mathcal{D} \subseteq \mathrm{Dom}(\mathcal{E}) \cap C_c(\mathcal{X})$ is a *core* of $\mathcal{E}$ if $\mathcal{D}$ is dense in $\mathrm{Dom}(\mathcal{E})$ with respect to the norm $\|\cdot\|_{\mathcal{E}_1}$ and also dense in $C_c(\mathcal{X})$ with respect to sup-norm (or, uniform norm). The Dirichlet form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is called *regular* if it possesses a core. In particular, for a regular Dirichlet form, $\mathrm{Dom}(\mathcal{E}) \cap C_c(\mathcal{X})$ is a core. We refer the reader to [21] for more information on Dirichlet forms.

Let $\{p_t\}_{t>0}$ be a heat kernel on $\mathcal{X}$, and $\{P_t\}_{t>0}$ be the associated heat semigroup that is defined by

$$P_t f(x) = \int_{\mathcal{X}} p_t(x,y) f(y)\, d\mu(y). \tag{2.2}$$

It is known that $\{P_t\}_{t>0}$ is a strongly continuous Markovian semigroup acting in $L^2(\mathcal{X})$. According to [21, Lemma 1.3.2(ii) and Theorem 1.3.1], the heat semigroup $\{P_t\}_{t>0}$ determines uniquely a Dirichlet form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ (see [21, Theorem 1.4.1]). In fact, by [22, Section 4 and Theorem 5.2], we have

$$\begin{cases} \mathcal{E}(u,v) = \lim_{t\to 0} \frac{1}{t}(u - P_t u, v) \quad \text{for all } \ u, v \in \mathrm{Dom}(\mathcal{E}); \\ \mathrm{Dom}(\mathcal{E}) = \{u \in L^2(\mathcal{X}) : \ \mathcal{E}(u,u) < \infty\}. \end{cases} \tag{2.3}$$

The main goal of this section is to drive the regularity of such a Dirichlet form under some very mild assumptions of heat kernel upper estimates.

**Theorem 2.1.** *Assume that* (VD) *is satisfied and* $\beta \in (0,\infty)$. *Let* $\{P_t\}_{t>0}$ *be the semigroup defined in* (2.2)*, with* $\{p_t\}_{t>0}$ *being a stochastically complete heat kernel on* $\mathcal{X}$ *satisfying*

$$0 \le p_t(x,y) \le \frac{C}{V(x, t^{1/\beta} + d(x,y))} \quad \text{for all } \ t \in (0,\infty) \ \text{ and } \ x,y \in \mathcal{X}, \tag{2.4}$$

*and*

$$\int_{B(x,r)^{\complement}} p_t(x,y)\,d\mu(y) \le C\left(\frac{t^{1/\beta}}{r}\right)^{\gamma} \quad \text{for all } \ t,r \in (0,\infty) \ \text{ and } \ x \in \mathcal{X}, \tag{2.5}$$

*for some constants* $C > 0$ *and* $\gamma > 0$. *If, for any* $t > 0$ *and* $x \in \mathcal{X}$, $p_t(x,\cdot)$ *is continuous on* $\mathcal{X}$, *then the Dirichlet form* $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ *given by* (2.3) *is regular.*

**Remark 2.2.** Given a heat kernel $\{p_t\}_{t>0}$ on $\mathcal{X}$ that satisfies $(\mathrm{UE})_\beta$, it satisfies both (2.4) and (2.5); see the proof of Proposition 3.9 below.

**Remark 2.3.** Let $\{p_t\}_{t>0}$ be a heat kernel on $\mathcal{X}$ satisfying $(\mathrm{UE})_\beta$. Assume that (VD) is satisfied and for any $t > 0$ and $x \in \mathcal{X}$, $p_t(x,\cdot)$ is continuous. Then $p_t(x,y)$ is jointly continuous in $(x,y) \in \mathcal{X}\times\mathcal{X}$.

Indeed, fix $x,y \in \mathcal{X}$ and $t > 0$. Consider $x' \in B(x,t^{1/\beta})$ and $y' \in B(y,t^{1/\beta})$. Note that

$$|p_{2t}(x,y) - p_{2t}(x',y')| \le |p_{2t}(x,y) - p_{2t}(x',y)| + |p_{2t}(x',y) - p_{2t}(x',y')| =: \mathrm{I}_1 + \mathrm{I}_2.$$

By the assumption, we have

$$\lim_{x'\to x} \mathrm{I}_1 = 0.$$

For $\mathrm{I}_2$, by the semigroup property, we write

$$\mathrm{I}_2 \le \int_{\mathcal{X}} p_t(x',z)|p_t(z,y) - p_t(z,y')|\,d\mu(z). \tag{2.6}$$

Since $d(x,x') < t^{1/\beta}$, we have $t^{1/\beta} + d(x',z) \le 2t^{1/\beta} + d(x,z) \le 3t^{1/\beta} + d(x',z)$ and then, by (1.8),

$$\frac{V(x, t^{1/\beta} + d(x,z))}{V(x', t^{1/\beta} + d(x',z))} \le C'_D\left(\frac{d(x,x') + 2t^{1/\beta} + d(x,z)}{t^{1/\beta} + d(x',z)}\right)^{\alpha_+} \lesssim 1.$$

Hence, by $(\mathrm{UE})_\beta$, we have

$$p_t(x',z) \lesssim \frac{1}{V(x', t^{1/\beta} + d(x',z))}\left(\frac{t^{1/\beta} + d(x',z)}{t^{1/\beta}}\right)^{\beta} \lesssim \frac{1}{V(x, t^{1/\beta} + d(x,z))}\left(\frac{t^{1/\beta} + d(x,z)}{t^{1/\beta}}\right)^{\beta}.$$

In a similarly way, by the fact that $d(y,y') < t^{1/\beta}$, we also have

$$|p_t(z,y) - p_t(z,y')| \lesssim \frac{1}{V(y, t^{1/\beta} + d(y,z))} \lesssim \frac{1}{V(y, t^{1/\beta})}.$$

Note that by (VD) and direct computation, the function

$$z \mapsto \frac{1}{V(x, t^{1/\beta} + d(x,z))}\left(\frac{t^{1/\beta} + d(x,z)}{t^{1/\beta}}\right)^{\beta} \frac{1}{V(y, t^{1/\beta})}$$

is integrable. Then, by (2.6) and dominated convergence theorem, we obtain that

$$\lim_{(x',y')\to(x,y)} \mathrm{I}_2 = 0.$$

Finally, combining the estimates of $I_1$ and $I_2$ yields

$$\lim_{(x',y')\to(x,y)} |p_{2t}(x,y) - p_{2t}(x',y')| = 0.$$

Thus, each $p_t(x,y)$ is jointly continuous in $(x,y) \in \mathcal{X} \times \mathcal{X}$.

We will prove Theorem 2.1 by using three different methods, which are presented in Subsections 2.2-2.4-2.5, respectively. For the third method in Subsection 2.5, we need an additional *non-collapsing condition* (NC), that is,

$$\inf_{x\in\mathcal{X}} V(x,1) > 0. \tag{2.7}$$

**Remark 2.4.** Let $T > 0$. By the arguments in the proofs (see Subsections 2.2-2.4-2.5) of Theorem 2.1, if both (2.4) and (2.5) hold only for all $t \in (0,T)$ and if for any $t \in (0,T)$ and $x \in \mathcal{X}$, $p_t(x,\cdot)$ is continuous, then the conclusion of Theorem 2.1 still holds.

## 2.1 Preparations

Let us begin with the notion of cutoff functions.

**Definition 2.5.** Let $U$ be an open set of $\mathcal{X}$ and $A$ be any Borel set of $U$. A function $\phi \in C_c(\mathcal{X})$ is called a *cutoff function* of the pair $(A,U)$ if it satisfies the following properties (see Figure 1):

$$\text{(i)}\ \ 0 \le \phi \le 1 \text{ on } \mathcal{X}; \quad \text{(ii)}\ \ \phi = 1 \text{ on } A; \quad \text{(iii)}\ \ \phi \equiv 0 \text{ on } U^{\complement}.$$

Denote by cutoff$(A,U)$ the collection of all cutoff functions of the pair $(A,U)$.

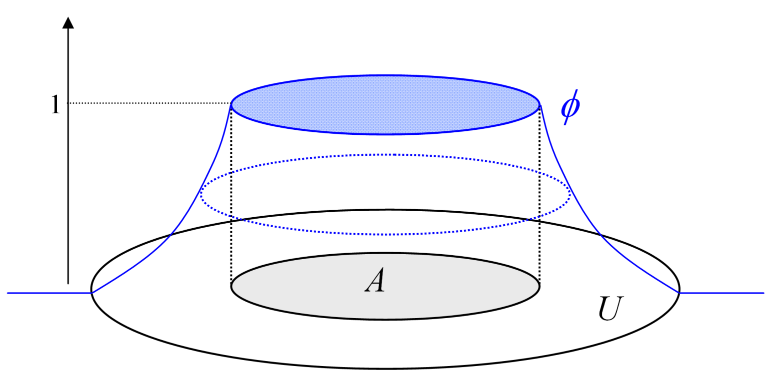


Figure 1: A cutoff function $\phi$ of a pair $(A,U)$.

Now, we recall the following regularity result from [21, p.29, Lemma 1.4.2(i)].

**Lemma 2.6** ([21]). *The Dirichlet form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ on $L^2(\mathcal{X})$ is regular provided that the following hold:*

(i) $\mathrm{Dom}(\mathcal{E}) \cap C_\infty(\mathcal{X})$ *is dense in* $C_\infty(\mathcal{X})$ *with respect to the uniform norm;*

(ii) $\mathrm{Dom}(\mathcal{E}) \cap C_\infty(\mathcal{X})$ *is dense in* $\mathrm{Dom}(\mathcal{E})$ *with respect to the norm* $\|\cdot\|_{\mathcal{E}_1}$ *in* (2.1).

**Remark 2.7.** Regarding Lemma 2.6(i), we observe that the following three statements are equivalent:

(i) $\mathrm{Dom}(\mathcal{E}) \cap C_c(\mathcal{X})$ is dense in $C_c(\mathcal{X})$ with respect to the uniform norm;

(ii) $\mathrm{Dom}(\mathcal{E}) \cap C_\infty(\mathcal{X})$ is dense in $C_\infty(\mathcal{X})$ with respect to the uniform norm;

(iii) For any compact set $K$ and open set $U$ with $K \subseteq U$, $\mathrm{Dom}(\mathcal{E}) \cap \mathrm{cutoff}(K, U) \neq \emptyset$.

Indeed, it is obvious that "(i) $\Rightarrow$ (ii)" because $C_c(\mathcal{X})$ is a dense subspace of $C_\infty(\mathcal{X})$ (by multiplying a cutoff function) with respect to the uniform norm. The proofs of "(ii) $\Rightarrow$ (iii)" and "(iii) $\Rightarrow$ (i)" are contained in [12, Proposition 4.1] and [12, Proposition 3.8], respectively.

Next, we show the following list of properties of the heat semigroup associated with a heat kernel satisfying (2.4) and (2.5), which are the key ingredients for the proof of Theorem 2.1.

**Proposition 2.8.** *Under the hypothesis of Theorem 2.1, the following assertions are true:*

(i) *If $f \in L^p(\mathcal{X})$ for some $p \in [1, \infty)$, then $P_t f \in C(\mathcal{X})$ for all $t > 0$.*

(ii) *If $f \in L^1(\mathcal{X})$ has bounded support, then $P_t f \in C_\infty(\mathcal{X})$ for all $t > 0$.*

(iii) *If $f \in C_\infty(\mathcal{X})$, then $P_t f \in C_\infty(\mathcal{X})$ for all $t > 0$ and $\lim_{t\to 0} \|P_t f - f\|_{L^\infty(\mathcal{X})} = 0$.*

*Proof.* To show (i), we fix $x \in \mathcal{X}$, $t > 0$ and $f \in L^p(\mathcal{X})$. By the Hölder inequality (with respect to the measure $|p_t(x,z) - p_t(y,z)|\, d\mu(z)$) and stochastic completeness of the heat kernel, we obtain that for any $y \in \mathcal{X}$,

$$\begin{aligned}
|P_t f(x) - P_t f(y)| &= \left| \int_{\mathcal{X}} (p_t(x,z) - p_t(y,z)) f(z)\, d\mu(z) \right| \\
&\le \left( \int_{\mathcal{X}} |p_t(x,z) - p_t(y,z)|\, d\mu(z) \right)^{\frac{p-1}{p}} \left( \int_{\mathcal{X}} |p_t(x,z) - p_t(y,z)| |f(z)|^p\, d\mu(z) \right)^{\frac{1}{p}} \\
&\le 2^{\frac{p-1}{p}} \left( \int_{\mathcal{X}} |p_t(x,z) - p_t(y,z)| |f(z)|^p\, d\mu(z) \right)^{\frac{1}{p}}. 
\end{aligned} \tag{2.8}$$

For any $y \in B(x, t^{1/\beta})$, by (VD) (see also (1.8)), we see that

$$V(x, t^{1/\beta}) \simeq V(y, t^{1/\beta}),$$

which, along with (2.4), implies

$$|p_t(x,z) - p_t(y,z)| \lesssim \frac{1}{V(x,t^{1/\beta})} + \frac{1}{V(y,t^{1/\beta})} \simeq \frac{1}{V(x,t^{1/\beta})}.$$

Hence, by $f \in L^p(\mathcal{X})$ and the fact that $p_t(\cdot, z)$ is continuous on $\mathcal{X}$, we apply the dominated convergence theorem for (2.8) to prove that

$$\lim_{y\to x} |P_t f(x) - P_t f(y)| = 0,$$

that is, $P_t f \in C(\mathcal{X})$ since $x \in \mathcal{X}$ can be arbitrary.

Now we show (ii). Fix $t > 0$ and $f \in L^1(\mathcal{X})$ such that $\operatorname{supp} f \subseteq B(x_o, R)$ for some point $x_o \in \mathcal{X}$ and $R \in (0, \infty)$. According to (i), we know that $P_t f \in C(\mathcal{X})$. Under the case $\operatorname{diam} \mathcal{X} < \infty$, we have by (1.17) that $P_t f \in C(\mathcal{X}) = C_\infty(\mathcal{X})$. So, it remains to validate that $P_t f$ vanishes at infinity under the assumption of $\operatorname{diam} \mathcal{X} = \infty$. To this end, for all $y \in B(x_o, R)$ and $x \in B(x_o, 2R)^{\complement}$, we have

$$d(x, y) \geq d(x_o, x) - d(x, y) \geq \frac{1}{2} d(x_o, x) \geq R,$$

which implies $B(x_o, R) \subseteq B(x, R)^{\complement}$ and, by (VD) and (2.4),

$$p_t(x, y) \lesssim \frac{1}{V(x, d(x, y))} \simeq \frac{1}{V(y, d(x, y))} \lesssim \frac{1}{V(y, R)} \simeq \frac{1}{V(x_o, R)}.$$

Now, for any $n \in \mathbb{N}$, set

$$f_n := f \mathbf{1}_{\{|f| \leq n\}},$$

which is supported in $B(x_o, R)$. Clearly, the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to $f$ both in $L^1(\mathcal{X})$ and pointwise. By these and (2.5), for any $x \in B(x_o, 2R)^{\complement}$, we obtain

$$\begin{aligned}
|P_t f(x)| &\leq \int_{B(x_o, R)} p_t(x, y)|f(y) - f_n(y)| \, d\mu(y) + \int_{B(x_o, R)} p_t(x, y)|f_n(y)| \, d\mu(y) \\
&\lesssim \frac{1}{V(x_o, R)} \|f_n - f\|_{L^1(\mathcal{X})} + \|f_n\|_{L^\infty(\mathcal{X})} \int_{B(x, R)^{\complement}} p_t(x, y) \, d\mu(y) \\
&\lesssim \frac{1}{V(x_o, R)} \|f_n - f\|_{L^1(\mathcal{X})} + n \left( \frac{t^{1/\beta}}{R} \right)^\gamma . \qquad (2.9)
\end{aligned}$$

Note that the implicit constants in the estimate (2.9) are independent of $x \in B(x_o, 2R)^{\complement}$, $n, t, R$ and $x_o$. So, in both sides of (2.9), first letting $R \to \infty$ and then letting $n \to \infty$, we conclude that

$$\lim_{R\to\infty} \sup_{x \in B(x_o, 2R)^{\complement}} |P_t f(x)| = 0,$$

that is, $P_t f \in C_\infty(\mathcal{X})$.

It remains to verify (iii). Fix $f \in C_\infty(\mathcal{X})$ and $t > 0$. For any $\varepsilon > 0$, there exists a compact set $K \subseteq \mathcal{X}$ such that $|f(x)| < \varepsilon$ for all $x \in K^{\complement}$. By this property and the stochastic completeness of the heat kernel, we have for all $x \in \mathcal{X}$ that

$$|P_t(f \mathbf{1}_{K^{\complement}})(x)| \leq \int_{K^{\complement}} p_t(x, y)|f(y)| \, d\mu(y) \leq \varepsilon \int_{K^{\complement}} p_t(x, y) \, d\mu(y) \leq \varepsilon \qquad (2.10)$$

and, hence, for any $x, y \in \mathcal{X}$,

$$|P_t f(x) - P_t f(y)| \leq |P_t(f \mathbf{1}_K)(x) - P_t(f \mathbf{1}_K)(y)| + |P_t(f 1_{K^{\complement}})(x)| + |P_t(f 1_{K^{\complement}})(y)|$$

$$\leq |P_t(f\mathbf{1}_K)(x) - P_t(f\mathbf{1}_K)(y)| + 2\varepsilon.$$

Since $f\mathbf{1}_K \in L^1(\mathcal{X})$ has compact support, by (ii), we obtain that $P_t(f\mathbf{1}_K) \in C_\infty(\mathcal{X})$. Combining this and the above inequality yields

$$\limsup_{y\to x} |P_t f(x) - P_t f(y)| \leq 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have proved that $P_t f \in C(\mathcal{X})$.

Again, using the fact $P_t(f\mathbf{1}_K) \in C_\infty(\mathcal{X})$, we find that for any $\varepsilon > 0$, there exists a compact set $K_1 \subseteq \mathcal{X}$ such that $|P_t(f\mathbf{1}_K)(x)| < \varepsilon$ for all $x \in K_1^{\complement}$. By this and (2.10), we see that for all $x \in K_1^{\complement}$,

$$|P_t f(x)| \leq |P_t(f\mathbf{1}_K)(x)| + |P_t(f 1_{K^{\complement}})(x)| \leq |P_t(f\mathbf{1}_K)(x)| + \varepsilon < 2\varepsilon,$$

which alternatively says that $P_t f \in C_\infty(\mathcal{X})$.

Note that $f \in C_\infty(\mathcal{X})$ implies that $f$ is bounded and uniformly continuous on $\mathcal{X}$. With the latter fact, we have that for any $\varepsilon > 0$, there is $\delta > 0$ such that for all $x, y \in \mathcal{X}$ with $d(x, y) < \delta$,

$$|f(x) - f(y)| \leq \varepsilon.$$

Then, it follows from the stochastic completeness of $\{p_t\}_{t>0}$ and (2.5) that

$$\begin{aligned}
|P_t f(x) - f(x)| &= \left| \int_{\mathcal{X}} p_t(x, y)(f(y) - f(x))\, d\mu(y) \right| \\
&\leq \int_{\mathcal{X}} p_t(x, y)|f(y) - f(x)|\, d\mu(y) \\
&\leq \left( \int_{B(x,\delta)} + \int_{B(x,\delta)^{\complement}} \right) p_t(x, y)|f(y) - f(x)|\, d\mu(y) \\
&\leq \varepsilon + 2C \left( \frac{t^{1/\beta}}{\delta} \right)^{\gamma} \|f\|_{L^\infty(\mathcal{X})},
\end{aligned}$$

which implies that $\lim_{t\to 0} \|P_t f - f\|_{L^\infty(\mathcal{X})} = 0$ since $\varepsilon > 0$ is arbitrary. □

## 2.2 Proof of Theorem 2.1: a probability method

**Theorem 2.9.** *Let $\{T_t\}_{t\geq 0}$ be a strongly continuous Markov semigroup in $L^2(\mathcal{X})$. Assume that*

(a) *for any $f \in C_\infty(\mathcal{X})$, $T_t f \in C_\infty(\mathcal{X})$ for any $t > 0$ and $\lim_{t\to 0} \|T_t f - f\|_{L^\infty(\mathcal{X})} = 0$;*

(b) *$T_t(L^2_c(\mathcal{X})) \subseteq C_\infty(\mathcal{X})$ for any $t > 0$.*

*Then, the Dirichlet form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ determined by*

$$\begin{cases} \mathcal{E}(f, g) := \lim_{t\to 0} \frac{1}{t}(f - T_t f, g) \ \text{for all } f, g \in \mathrm{Dom}(\mathcal{E}); \\ \mathrm{Dom}(\mathcal{E}) := \{f \in L^2(\mathcal{X}) :\ \mathcal{E}(f, f) < \infty\}, \end{cases} \tag{2.11}$$

*is regular.*

*Proof.* To obtain the regularity of $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$, it suffices to validate the two items stated in Lemma 2.6.

For any $f \in C_\infty(\mathcal{X})$ and $t \in (0, \infty)$, it follows from [21, Lemma 1.3.3(i)] that $T_t f \in \mathrm{Dom}(\mathcal{E})$. This, together with (a), shows that any $f \in C_\infty(\mathcal{X})$ can be approximated by functions in $\mathrm{Dom}(\mathcal{E}) \cap C_\infty(\mathcal{X})$ with respect to $\|\cdot\|_{L^\infty(\mathcal{X})}$. Thus, Lemma 2.6(i) holds.

Now, we will employ a probability argument to verify that Lemma 2.6(ii) holds. It is clear that $\{T_t\}_{t\geq 0}$ determines a sub-Markov transition function $T_t(x, A)$ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ with $T_0(x, \cdot) = \delta_x(\cdot)$, where $\mathcal{B}(\mathcal{X})$ denotes the Borel sets on $\mathcal{X}$ and $\delta_x$ is the Dirac measure at the point $x$. Hence, by condition (a), it follows from [11, Theorem 9.4, p. 46] that there exists a Hunt process $Y_t$ with state space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and the transition function $T_t(x, A)$.

Moreover, it follows from [2, Remark (ii), p. 247] that any Hunt process is a *right process*. See also the following flowchart for relations among various processes.

$\boxed{\text{Hunt processes [14, Definition A.1.23, p. 405]}}$

$\downarrow$

$\boxed{\text{Borel standard right processes [14, Definition A.1.23, p. 405]}}$

$\downarrow$

$\boxed{\text{Borel right processes [14, Definition A.1.17, p. 401]}}$

$\downarrow$ by [14, Theorem A.1.37, p. 410]

$\boxed{\text{right processes [14, Definition A.1.36, p. 410]}}$

Then, it follows from the second part of [14, Theorem 1.5.3, p. 35] or [38, Theorem 6.7, p. 142] that the Dirichlet form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ in (2.11) of $Y_t$ is *quasi-regular* (see the definition of quasi-regular Dirichlet forms in [14, Definition 1.3.8, p. 26]). It remains to prove that $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is actually regular (not only quasi-regular).

Indeed, by the definition of quasi-regular Dirichlet forms, for any function $f \in \mathrm{Dom}(\mathcal{E})$, there is a sequence of functions $\{f_n\}_{n\geq 1} \subseteq \mathrm{Dom}(\mathcal{E})$ with compact supports such that

$$\lim_{n\to\infty} \|f_n - f\|_{\mathcal{E}_1} = 0.$$

Note also that each $f_n \in L^2_c(\mathcal{X})$. Hence, by condition (b) and the fact $T_t(L^2_c(\mathcal{X})) \subseteq \mathrm{Dom}(\mathcal{E})$ (see [21, p. 22, Lemma 1.3.3(i)]), we have that $T_t f_n \in C_\infty(\mathcal{X}) \cap \mathrm{Dom}(\mathcal{E})$ for any $t > 0$ and $n \geq 1$. Moreover, by [21, p. 22, Lemma 1.3.3(iii)], we have

$$\lim_{t\to 0} \|T_t f_n - f_n\|_{\mathcal{E}_1} = 0.$$

Combining the above two formulae, we obtain that $f \in \mathrm{Dom}(\mathcal{E})$ can be approximated by functions $T_t f_n$ with respect to the $\mathcal{E}_1$-norm. Thus, we obtain that Lemma 2.6(ii) holds. □

*Proof of Theorem 2.1.* To obtain the regularity of $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ in (2.3), we are about to apply Theorem 2.9 with $T_t = P_t$.

Indeed, it follows directly from Proposition 2.8(iii) that condition (a) of Theorem 2.9 holds. Moreover, by Proposition 2.8(ii) and the fact that $L^2_c(\mathcal{X}) \subseteq L^1_c(\mathcal{X})$, we see that condition (b) of Theorem 2.9 also holds. This finishes the proof of Theorem 2.1 by Theorem 2.9. □

### 2.3 Weakly regular Dirichlet forms

In this subsection, we introduce the notion of *weakly regular* Dirichlet forms and build a general theory that ensures weakly regular Dirichlet forms to be regular (see Proposition 2.19 below), which will not only yield an alternative proof of Theorem 2.1 but also has its own interest.

**Definition 2.10.** Let $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ be a Dirichlet form on $L^2(\mathcal{X})$. We say that $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is *weakly regular* if it satisfies:

$$\text{(i) } \mathrm{Dom}(\mathcal{E}) \cap C_c(\mathcal{X}) \text{ is dense in } C_c(\mathcal{X}) \text{ with respect to the uniform norm;} \tag{2.12}$$

$$\text{(ii) } \mathrm{Dom}(\mathcal{E}) \cap C(\mathcal{X}) \text{ is dense in } \mathrm{Dom}(\mathcal{E}) \text{ with respect to the norm } \|\cdot\|_{\mathcal{E}_1} \text{ in (2.1).} \tag{2.13}$$

**Remark 2.11.** Based on Lemma 2.6 and Remark 2.7, if $C(\mathcal{X})$ in (2.13) is replaced by $C_c(\mathcal{X})$ or $C_\infty(\mathcal{X})$, then weakly regular Dirichlet forms will become regular. In particular, any regular Dirichlet form is weakly regular. Under the case $\operatorname{diam} \mathcal{X} < \infty$, it follows from (1.17) that a Dirichlet form is weakly regular if and only if it is regular.

Suppose that $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is a Dirichlet form and $U \subseteq \mathcal{X}$ is a non-empty open set. Let

$$\mathcal{F}(U) := \overline{\mathrm{Dom}(\mathcal{E}) \cap C_c(U)}^{\|\cdot\|_{\mathcal{E}_1}}, \tag{2.14}$$

where we recall that $\mathcal{E}_1(u,v) = \mathcal{E}(u,v) + (u,v)$ for all $u, v \in \mathrm{Dom}(\mathcal{E})$. Clearly, $\mathcal{F}(U) \subseteq \mathrm{Dom}(\mathcal{E})$.

**Proposition 2.12.** *Under the condition (2.12), for any non-empty open set $U \subseteq \mathcal{X}$, $(\mathcal{E}, \mathcal{F}(U))$ is a regular Dirichlet form on $L^2(U)$.*

*Proof.* We shall prove that $\mathrm{Dom}(\mathcal{E}) \cap C_c(U)$ is dense both in $C_c(U)$ and $\mathcal{F}(U)$. This will automatically imply that $\mathcal{F}(U)$ is dense in $L^2(U)$ since so is $C_c(U)$.

Indeed, it follows from the definition (2.14) that $\mathrm{Dom}(\mathcal{E}) \cap C_c(U) \subseteq \mathcal{F}(U) \cap C_c(U)$ is dense in $\mathcal{F}(U)$ with respect to $\mathcal{E}_1$.

Now, let us show that $\mathrm{Dom}(\mathcal{E}) \cap C_c(U)$ is dense in $C_c(U)$. Fix $u \in C_c(U)$ and $\varepsilon > 0$. Suppose that $K = \operatorname{supp} u$. It follows from (2.12) that there exists $v \in \mathrm{Dom}(\mathcal{E}) \cap C_c(\mathcal{X})$ such that

$$\|u - v\|_{L^\infty(\mathcal{X})} < \varepsilon.$$

In particular, $|v(x)| < \varepsilon$ for all $x \notin K$. Moreover, it follows from [21, Theorem 1.4.2(iv), p. 28] that

$$v^{(\varepsilon)} := v - ((-\varepsilon) \vee v) \wedge \varepsilon \in \mathrm{Dom}(\mathcal{E}) \cap C(\mathcal{X}).$$

Note that $v^{(\varepsilon)}(x) = 0$ for all $x \notin K$ and

$$\|v - v^{(\varepsilon)}\|_{L^\infty(\mathcal{X})} < \varepsilon.$$

Hence, we obtain that $v^{(\varepsilon)} \in C_c(U)$ and

$$\|u - v^{(\varepsilon)}\|_{L^\infty(\mathcal{X})} \le \|u - v\|_{L^\infty(\mathcal{X})} + \|v - v^{(\varepsilon)}\|_{L^\infty(\mathcal{X})} < 2\varepsilon.$$

Since $\varepsilon$ can be arbitrary, we have proved that $\mathrm{Dom}(\mathcal{E}) \cap C_c(U)$ is dense in $C_c(U)$. □

Let $U \subseteq \mathcal{X}$ be a non-empty open set. Under (2.12), we denote by $\{P_t^U\}_{t\geq 0}$ the associated heat semigroup of the regular Dirichlet form $(\mathcal{E}, \mathcal{F}(U))$. According to [21, Lemma 1.3.4], the domain $\mathcal{F}(U)$ of the Dirichlet form $(\mathcal{E}, \mathcal{F}(U))$ satisfies

$$\mathcal{F}(U) = \left\{u \in L^2(U) : \lim_{t\to 0} t^{-1}(u - P_t^U u, u) < \infty\right\}. \tag{2.15}$$

However, even under the case $U = \mathcal{X}$, we remark that it might happen that

$$P_t^{\mathcal{X}} \neq P_t.$$

The following proposition strengthens [29, Lemma 4.4], in which instead of regularity of Dirichlet forms we now use only the weakly regular.

**Proposition 2.13.** *Let $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ be a weakly regular Dirichlet form. For any non-empty open set $U \subseteq \mathcal{X}$ and for any $u \in \mathrm{Dom}(\mathcal{E})$, the followings are equivalent:*

(i) $u_+ \in \mathcal{F}(U)$;

(ii) $u \leq v$ *for some* $v \in \mathcal{F}(U)$.

*Proof.* To obtain (i) $\Rightarrow$ (ii), one can take $v = |u|$, which is in $\mathcal{F}(U)$ whenever $u \in \mathcal{F}(U)$. To prove (ii) $\Rightarrow$ (i), via taking some $v \in \mathcal{F}(U)$ such that $u \leq v$, we divide the proof of $u_+ \in \mathcal{F}(U)$ into three cases.

**Case 1:** $u \in \mathrm{Dom}(\mathcal{E}) \cap C(\mathcal{X})$ and $v \in \mathcal{F}(U) \cap C_c(U)$. In this case, we have by $u \leq v$ that

$$u_+ \leq v_+ \quad \text{and} \quad u_+ \in \mathrm{Dom}(\mathcal{E}) \cap C(\mathcal{X}).$$

Moreover, $u_+(x) = 0$ for all $x \notin \operatorname{supp} v$, which shows that $u_+ \in C_c(U)$, that is,

$$u_+ \in \mathrm{Dom}(\mathcal{E}) \cap C_c(U) \subseteq \mathcal{F}(U).$$

**Case 2:** $u \in \mathrm{Dom}(\mathcal{E})$ and $v \in \mathcal{F}(U) \cap C_c(U)$. In this case, it follows from (2.13) that there exists a sequence $\{u_n\} \subseteq \mathrm{Dom}(\mathcal{E}) \cap C(\mathcal{X})$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{\mathcal{E}_1} = 0.$$

The result in **Case 1** shows that each $(u_n \wedge v)_+ \in \mathcal{F}(U)$ since $u_n \wedge v \in \mathrm{Dom}(\mathcal{E}) \cap C(\mathcal{X})$ and $u_n \wedge v \leq v$. Moreover, for any $n \in \mathbb{N}$, by the normal contraction and bilinear properties of the Dirichlet form (see [21, pp. 3-5]) property, we have

$$\begin{aligned}
\mathcal{E}((u_n \wedge v)_+, (u_n \wedge v)_+) &\leq \mathcal{E}(u_n \wedge v,\, u_n \wedge v) \\
&\leq \mathcal{E}(u_n \wedge v,\, u_n \wedge v) + \mathcal{E}(u_n \vee v,\, u_n \vee v) \\
&= \frac{1}{2}\left(\mathcal{E}(u_n + v,\, u_n + v) + \mathcal{E}(|u_n - v|,\, |u_n - v|)\right) \\
&\leq \mathcal{E}(u_n, u_n) + \mathcal{E}(v, v)
\end{aligned}$$

and, hence,

$$\sup_{n\in\mathbb{N}} \mathcal{E}((u_n \wedge v)_+,\ (u_n \wedge v)_+) \le \sup_{n\in\mathbb{N}} \mathcal{E}(u_n, u_n) + \mathcal{E}(v, v) < \infty.$$

Further, from the fact that $(u_n \wedge v)_+ \to (u \wedge v)_+ = u_+$ in $L^2(\mathcal{X})$ as $n \to \infty$ and [38, Lemma 2.12, p. 21], it follows that

$$(u_n \wedge v)_+ \to u_+ \quad \text{weakly with respect to } \mathcal{E}_1 \text{ as } n \to \infty.$$

Since $\mathcal{F}(U)$ is also weakly closed, we obtain that $u_+ \in \mathcal{F}(U)$.

**Case 3:** $u \in \mathrm{Dom}(\mathcal{E})$ and $v \in \mathcal{F}(U)$. In this case, by (2.14), there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subseteq \mathrm{Dom}(\mathcal{E}) \cap C_c(U)$ such that

$$\lim_{n\to\infty} \|v_n - v\|_{\mathcal{E}_1} = 0.$$

The result in **Case 2** shows that each $(u \wedge v_n)_+ \in \mathcal{F}(U)$ since $u \wedge v_n \in \mathrm{Dom}(\mathcal{E})$ and $u \wedge v_n \le v_n$ with $v_n \in \mathcal{F}(U) \cap C_c(U)$.

Similar to the arguments in **Case 2**, we also obtain that $(u \wedge v_n)_+ \to (u \wedge v)_+ = u_+$ weakly with respect to $\mathcal{E}_1$ as $n \to \infty$. Again, since $\mathcal{F}(U)$ is weakly closed, we obtain that $u_+ \in \mathcal{F}(U)$. □

**Proposition 2.14.** *Let $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ be a weakly regular Dirichlet form. For any non-empty open set $U \subseteq \mathcal{X}$, if both $u \in \mathrm{Dom}(\mathcal{E})$ and $\phi \in \mathcal{F}(U)$ are bounded functions, then $\phi u \in \mathcal{F}(U)$.*

*Proof.* Since $u \in \mathrm{Dom}(\mathcal{E})$ and $\phi \in \mathcal{F}(U) \subseteq \mathrm{Dom}(\mathcal{E})$ are bounded functions, it follows from [21, Theorem 1.4.2(ii), p. 28] that $\phi u \in \mathrm{Dom}(\mathcal{E})$. Observing that

$$\phi u \le |\phi u| \le \|u\|_{L^\infty(\mathcal{X})} |\phi| \in \mathcal{F}(U),$$

we then derive from Proposition 2.13 that $(\phi u)_+ \in \mathcal{F}(U)$.

Similar arguments also show that $(\phi u)_- \in \mathcal{F}(U)$. Finally, we have $\phi u \in \mathcal{F}(U)$. □

Let us now recall the notions of *subcaloric* and *caloric* functions. Let $I$ be an interval in $\mathbb{R}$. A function $u : I \to L^2(\mathcal{X})$ is said to be *weakly differentiable* at $t \in I$, if for any $\varphi \in L^2(\mathcal{X})$, the function $(u(\cdot), \varphi)$ is differentiable at $t$, that is, the limit

$$\lim_{\varepsilon\to 0} \left( \frac{u(t+\varepsilon) - u(t)}{\varepsilon},\ \varphi \right)$$

exists. In this case, by the principle of uniform boundedness, there exists some $w \in L^2(\mathcal{X})$ such that for any $\varphi \in L^2(\mathcal{X})$,

$$\lim_{\varepsilon\to 0} \left( \frac{u(t+\varepsilon) - u(t)}{\varepsilon},\ \varphi \right) = (w, \varphi). \tag{2.16}$$

The function $w$ is called the *weak derivative* of $u$ at $t$, and we write $\partial_t u = w$ or $u'(t) = w$. In this case, we have

$$\sup_{\varepsilon\in(0,1]} \varepsilon^{-1} \|u(t+\varepsilon) - u(t)\|_{L^2(\mathcal{X})} < \infty,$$

which also implies

$$\lim_{\varepsilon\to 0} u(t+\varepsilon) = u(t) \quad \text{in } L^2(\mathcal{X}). \tag{2.17}$$

**Definition 2.15.** For an open subset $\Omega \subseteq \mathcal{X}$, a function $u : I \to \mathrm{Dom}(\mathcal{E})$ is called *subcaloric* in $I \times \Omega$ if $u$ is weakly differentiable in $L^2(\mathcal{X})$ at any $t \in I$ and if, for any $t \in I$ and any non-negative $\varphi \in \mathcal{F}(\Omega)$,

$$(\partial_t u,\ \varphi) + \mathcal{E}(u(t,\cdot),\ \varphi) \le 0.$$

A function $u$ is said to be *caloric* in $I \times \Omega$ if the above inequality is replaced by equality, that is,

$$(\partial_t u, \varphi) + \mathcal{E}(u(t,\cdot), \varphi) = 0.$$

In a similar manner, $u$ is said to be *supercaloric* in $I \times \Omega$ if $-u$ is subcaloric.

**Remark 2.16.** If $u$ is a subcaloric or supercaloric function in $I \times \Omega$, then it follows from (2.17) that there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subseteq (0, 1)$ such that

$$\lim_{n\to 0} u(t + \varepsilon_n, x) = u(t, x) \quad \mu\text{-a.a. } x \in \Omega. \tag{2.18}$$

Note that for any $f \in L^2(\Omega)$, the function $u(t,\cdot) = P_t^\Omega f(\cdot)$ is caloric in $(0,\infty)\times\Omega$. In this case, there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subseteq (0, 1)$ such that (2.18) is true for $u(t,\cdot) = P_t^\Omega f(\cdot)$.

The following parabolic maximal principle comes from [29, Proposition 4.11, p. 117], whose proof does **NOT** use regularity of the corresponding Dirichlet form. Hence, it is true for all (not necessarily regular) Dirichlet forms.

**Proposition 2.17** ([29]). *Fix $T \in (0,\infty]$ and an open set $\Omega \subseteq \mathcal{X}$. Assume that $u : (0, T) \to \mathrm{Dom}(\mathcal{E})$ is a subcaloric function in $(0, T)\times\Omega$ such that the following hold:*

(i) *$u_+(t,\cdot) \in \mathcal{F}(\Omega)$ for all $t \in (0, T)$, where $u_+(t,\cdot) := \max\{u(t,\cdot),\ 0\}$;*

(ii) *$u_+(t,\cdot) \to 0$ in $L^2(\Omega)$ as $t \to 0$.*

*Then $u(t, x) \le 0$ for all $t \in (0, T)$ and $\mu$-a.a. $x \in \Omega$.*

Applying Propositions 2.13-2.14-2.17, we are about to show the following comparison estimates for subcaloric and supercaloric functions, which extends [29, Lemmas 4.16 and 4.18] since now we relax the regularity assumption of the Dirichlet form.

**Proposition 2.18.** *Let $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ be a weakly regular Dirichlet form. Let $U \subseteq \mathcal{X}$ be an open set, $K \subseteq U$ be a compact set and $0 \le f \in L^2(U)$.*

(i) *Assume that $w : (0,\infty) \to \mathrm{Dom}(\mathcal{E})$ is a non-negative supercaloric function in $(0,\infty)\times U$ satisfying:*

$$w(t,\cdot) \xrightarrow{L^2(U)} f(\cdot) \quad \text{as } t \to 0. \tag{2.19}$$

*Then, for any $t \in (0,\infty)$ and $\mu$-a.a $x \in \mathcal{X}$,*

$$w(t, x) \ge P_t^U f(x). \tag{2.20}$$

(ii) *Assume that $u : (0,\infty) \to \mathrm{Dom}(\mathcal{E})$ is a bounded subcaloric function in $(0,\infty)\times U$ satisfying (2.19). Then, for any $t \in (0,\infty)$ and $\mu$-a.a $x \in \mathcal{X}$,*

$$u(t, x) \le P_t^U f(x) + \sup_{s\in(0,t]} \operatorname*{esup}_{z\in K^{\complement}} u(s, z). \tag{2.21}$$

*Proof.* We first show (i). For any $t \in (0, \infty)$ and $x \in \mathcal{X}$, let

$$v(t, x) := P_t^U f(x) - w(t, x).$$

Since $w$ is a supercaloric function in $(0, \infty) \times U$, it follows that $v$ is subcaloric in $(0, \infty) \times U$. According to [21, p. 22, Lemma 1.3.3(i)], it holds that $P_t^U f \in \mathcal{F}(U)$. By this and $w \geq 0$, we have

$$v(t, \cdot) \leq P_t^U f \in \mathcal{F}(U),$$

which implies that $v_+(t, \cdot) \in \mathcal{F}(U)$ by Proposition 2.13. Moreover, noting that

$$v_+(t, \cdot) \leq |P_t^U f(\cdot) - f(\cdot)| + |f(\cdot) - w(t, \cdot)|,$$

we deduce from (2.19) and [21, p. 22, Lemma 1.3.3(iii)] that $v_+(t, \cdot) \to 0$ in $L^2(U)$ as $t \to 0$. Finally, by the parabolic maximal principle in Proposition 2.17, we conclude that (2.20) holds.

Now, we are about to prove (ii). Let $\Omega$ be a precompact open set such that

$$K \subseteq \Omega \subseteq \overline{\Omega} \subseteq U.$$

By (2.12) and Remark 2.7(iii), we can find $\phi \in \mathrm{Dom}(\mathcal{E}) \cap \mathrm{cutoff}(\Omega, U)$. Fix $T \in (0, \infty)$. For any $t \in (0, T)$ and $x \in \mathcal{X}$, let

$$m := \sup_{s \in (0,T]} \operatorname*{esup}_{z \in K^\complement} u(s, z) \quad \text{and} \quad v(t, x) := \phi(x) u(t, x) - m\phi(x) - P_t^U f(x).$$

Note that $P_t^U f \in \mathcal{F}(U) \subseteq \mathrm{Dom}(\mathcal{E})$ (see [21, p. 22, Lemma 1.3.3(i)]). Moreover, since $u(t, \cdot) \in \mathrm{Dom}(\mathcal{E})$ is bounded, it follows from Proposition 2.14 that $\phi(\cdot) u(t, \cdot) \in \mathrm{Dom}(\mathcal{E})$, so does $v(t, \cdot)$.

- Let $0 \leq \psi \in \mathcal{F}(\Omega)$. Since $P_t^U f$ is caloric and $u$ is subcaloric in $(0, T) \times \Omega$, it follows from the definition of $v$ and the fact that $\phi = 1$ on $\Omega$ that for any $t \in (0, T)$,

$$\begin{aligned}
\Big(\partial_t v(t, \cdot), \psi\Big) + \mathcal{E}(v(t, \cdot), \psi) &= \Big(\phi \partial_t u(t, \cdot), \psi\Big) - \Big(\partial_t P_t^U f, \psi\Big) - \mathcal{E}(P_t^U f, \psi) + \mathcal{E}\Big(\phi u(t, \cdot) - m\phi, \psi\Big) \\
&= \Big(\partial_t u(t, \cdot), \psi\Big) + \mathcal{E}\Big(\phi u(t, \cdot) - m\phi, \psi\Big) \\
&= \Big(\partial_t u(t, \cdot), \psi\Big) + \mathcal{E}\Big(u(t, \cdot), \psi\Big) + \mathcal{E}\Big((\phi - 1) u(t, \cdot) - m\phi, \psi\Big) \\
&\leq \mathcal{E}\Big((\phi - 1) u(t, \cdot) - m\phi, \psi\Big). \qquad (2.22)
\end{aligned}$$

Due to the continuity of $\phi$ and the fact that $\operatorname{supp} \psi \subseteq \overline{\Omega}$, we have $\phi = 1$ on $\overline{\Omega}$ and, hence, $\phi = 1$ on $\operatorname{supp} \psi$. Thus, for any $t > 0$, we have

$$\Big((\phi - 1) u(t, \cdot) - m\phi, \psi\Big) = -(m, \psi),$$

which, along with (1.2) and $\psi \geq 0$, implies that for any $s > 0$,

$$\begin{aligned}
\Big((\phi - 1) u(t, \cdot) - m\phi - P_s((\phi - 1) u(t, \cdot) - m\phi), \psi\Big) &= \Big(-m - P_s((\phi - 1) u(t, \cdot) - m\phi), \psi\Big) \\
&\leq \Big(-m P_s \mathbf{1} - P_s((\phi - 1) u(t, \cdot) - m\phi), \psi\Big) \\
&= -\Big(P_s((\phi - 1)(u(t, \cdot) - m)), \psi\Big)
\end{aligned}$$

$$\leq 0.$$

This last inequality follows from the facts that $\psi \geq 0$ and

$$\begin{cases} (\phi - 1)(u(t,\cdot) - m) \geq 0 & \text{on } K^{\complement}; \\ (\phi - 1)(u(t,\cdot) - m) = 0 & \text{on } K. \end{cases} \tag{2.23}$$

Hence, by (2.3), we obtain

$$\mathcal{E}\Big((\phi - 1)u(t,\cdot) - m\phi,\, \psi\Big) = \lim_{s\to 0} \frac{1}{s}\Big((\phi - 1)u(t,\cdot) - m\phi - P_s((\phi - 1)u(t,\cdot) - m\phi),\, \psi\Big) \leq 0,$$

and then, by (2.22), $v$ is subcaloric in $(0,T)\times\Omega$.

- Take $\widetilde{\phi} \in \mathrm{Dom}(\mathcal{E}) \cap \mathrm{cutoff}(K,\Omega) \subseteq \mathcal{F}(\Omega)$. Similarly to (2.23), by the definition of $m$ and the fact that $\phi \in \mathrm{cutoff}(\Omega, U)$, we have

$$\begin{cases} (\phi - \widetilde{\phi})(u(t,\cdot) - m) \leq 0 & \text{on } K^{\complement}; \\ (\phi - \widetilde{\phi})(u(t,\cdot) - m) = 0 & \text{on } K. \end{cases}$$

This, combined with $P_t^U f \geq 0$ and Proposition 2.14, yields that for all $t \in (0,T)$,

$$v(t,\cdot) = \phi(u(t,\cdot) - m) - P_t^U f \leq \phi(u(t,\cdot) - m) \leq \widetilde{\phi}(u(t,\cdot) - m) \in \mathcal{F}(\Omega).$$

Hence, it follows from Proposition 2.13 that $v_+(t,\cdot) \in \mathcal{F}(\Omega)$ for all $t \in (0,T)$.

- By the definition of $v$ and the fact that $\phi = 1$ on $\Omega$, we write for any $t \in (0,T)$ and $x \in \Omega$,

$$v(t,x) = \phi(x)(u(t,x) - f(x)) - (P_t^U f(x) - f(x)) - m\phi(x),$$

which implies

$$v_+(t,x) \leq \phi(x)|u(t,x) - f(x)| + |P_t^U f(x) - f(x)|.$$

From [21, p. 22, Lemma 1.3.3(iii)], it follows that $P_t^U f \to f$ in $L^2(U)$ as $t \to 0$. By this, $\Omega \subseteq U$ and (2.19), we then derive that $v_+(t,\cdot) \to 0$ in $L^2(\Omega)$ as $t \to 0$.

Therefore, by using Proposition 2.17, we obtain that $v(t,x) \leq 0$ for all $t \in (0,T)$ and $\mu$-a.a. $x \in \Omega$. Because $\phi = 1$ on $\Omega$, this amounts to saying that for all $t \in (0,T)$ and $\mu$-a.a. $x \in \Omega$,

$$u(t,x) \leq P_t^U f(x) + \sup_{s\in(0,T]} \operatorname*{esup}_{z\in K^{\complement}} u(s,z). \tag{2.24}$$

Due to (2.18), via letting $t \to T$ in (2.24), we obtain that (2.24) is valid for $\mu$-a.a. $x \in \Omega$ at the endpoint $t = T$. In addition, note that (2.21) is trivial for $x \in \Omega^{\complement}$ and $T$ is arbitrary. Thus, we finish the proof of (2.21) . □

As a consequence of Propositions 2.12 and 2.18, we give a sufficient condition that ensures a weakly regular Dirichlet form to be regular.

**Proposition 2.19.** *Let $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ be a weakly regular Dirichlet form, and $\{P_t\}_{t>0}$ be the associated heat semigroup. Fix $x_o \in \mathcal{X}$. Suppose that $\mathcal{H}$ is a dense subset of $L^2_+(\mathcal{X}) := \{f \in L^2(\mathcal{X}) : f \geq 0\}$ such that for all $f \in \mathcal{H}$ and $t \in (0, \infty)$,*

$$\lim_{R\to\infty} \sup_{s\in(0,t]} \operatorname*{esup}_{z\in B(x_o,R)^\complement} P_s f(z) = 0. \tag{2.25}$$

*Then, $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is regular.*

*Proof.* Since $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is weakly regular, by taking $U = \mathcal{X}$ in Proposition 2.12, we deduce that $(\mathcal{E}, \mathcal{F}(\mathcal{X}))$ is a regular Dirichlet form on $L^2(\mathcal{X})$ and, moreover, $\mathcal{F}(\mathcal{X})$ can be characterized by (2.15). We will obtain the regularity of $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ by proving $\mathrm{Dom}(\mathcal{E}) = \mathcal{F}(\mathcal{X})$. And, by a comparison of (2.15) and (2.3), this can be achieved by proving that $P_t = P_t^{\mathcal{X}}$ for all $t \in (0, \infty)$, where $\{P_t^{\mathcal{X}}\}_{t\geq 0}$ is the associated heat semigroup of $(\mathcal{E}, \mathcal{F}(\mathcal{X}))$.

Let $0 \leq f \in L^2(\mathcal{X})$. By Remark 2.16, the function $w(t, x) := P_t f(x)$ is a non-negative caloric function in $(0, \infty) \times \mathcal{X}$. Moreover, it follows from [21, p. 22, Lemma 1.3.3(iii)] that $P_t f \to f$ in $L^2(\mathcal{X})$ as $t \to 0$, which implies that (2.19) holds with $U = \mathcal{X}$ therein. So, by (2.20) we obtain that for all $t > 0$ and $\mu$-a.a. $x \in \mathcal{X}$,

$$P_t^{\mathcal{X}} f(x) \leq w(t, x) = P_t f(x). \tag{2.26}$$

Assume in addition that $0 \leq f \in L^2(\mathcal{X})$ is bounded, which implies that $P_t f$ is bounded. In a similar manner, but now we apply (2.21) to the caloric function $u(t, x) := P_t f(x)$, thereby leading to that for any compact set $K \subseteq \mathcal{X}$, for all $t > 0$ and $\mu$-a.a. $x \in \mathcal{X}$,

$$P_t f(x) \leq P_t^{\mathcal{X}} f(x) + \sup_{s\in(0,t]} \operatorname*{esup}_{z\in K^\complement} P_s f(z). \tag{2.27}$$

Moreover, by the standard approximation arguments, we can prove that the above inequality holds for all $0 \leq f \in L^2(\mathcal{X})$ (not necessary for bounded $L^2(\mathcal{X})$-functions).

In (2.26) and (2.27), we take $f \in \mathcal{H}$. Also, take the compact set $K := \overline{B(x_o, R)}$ in (2.27) for $R \in (0, \infty)$. Then, combining (2.26) and (2.27) yields that for all $t \in (0, \infty)$ and $\mu$-a.a. $x \in \mathcal{X}$,

$$P_t^{\mathcal{X}} f(x) \leq P_t f(x) \leq P_t^{\mathcal{X}} f(x) + \sup_{s\in(0,t]} \operatorname*{esup}_{z\in B(x_o,R)^\complement} P_s f(z).$$

Applying (2.25) to the above inequality, we obtain that for any $f \in \mathcal{H}$ and $t > 0$,

$$P_t f = P_t^{\mathcal{X}} f \quad \mu\text{-a.e. on } \mathcal{X}.$$

Since $\mathcal{H}$ is dense in $L^2_+(\mathcal{X})$, it follows that the above identity holds for all $f \in L^2_+(\mathcal{X})$, and hence, for all $f \in L^2(\mathcal{X})$. This proves the desired identity of $P_t = P_t^{\mathcal{X}}$ for all $t \in (0, \infty)$. Thus, we complete the proof of Proposition 2.19. □

## 2.4 Proof of Theorem 2.1: an analytic method

As a consequence of Proposition 2.8, we show that the Dirichlet form in Theorem 2.1 is weakly regular.

**Proposition 2.20.** *Under the hypothesis of Theorem 2.1, the Dirichlet form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ given by (2.3) is weakly regular.*

*Proof.* Let $f \in C_\infty(\mathcal{X})$ and $t \in (0, \infty)$. By [21, Lemma 1.3.3(i)], each $P_t f \in \mathrm{Dom}(\mathcal{E})$. Meanwhile, by Proposition 2.8(iii), every $P_t f \in C_\infty(\mathcal{X})$ and $\lim_{t\to 0} \|P_t f - f\|_{L^\infty(\mathcal{X})} = 0$. Thus, $\mathrm{Dom}(\mathcal{E}) \cap C_\infty(\mathcal{X})$ is dense in $C_\infty(\mathcal{X})$ with respect to $\|\cdot\|_{L^\infty(\mathcal{X})}$. This, combined with Remark 2.7, gives (2.12).

Let $f \in \mathrm{Dom}(\mathcal{E})$. Then Proposition 2.8(i) says that every $P_t f \in C(\mathcal{X})$. According to [21, p. 22, Lemma 1.3.3(i) and (iii)], every $P_t f \in \mathrm{Dom}(\mathcal{E})$ and

$$\lim_{t\to 0} \|f - P_t f\|_{\mathcal{E}_1} = 0.$$

This proves that $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ satisfies (2.13). So, we obtain that $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is weakly regular. □

Now, applying the theory of weakly regular Dirichlet forms in Subsection 2.3, we give the second analytic proof of Theorem 2.1 without referring to the deep connection between right processes and quasi-regular Dirichlet froms.

*Proof of Theorem 2.1.* As was proved in Proposition 2.20, the Dirichelt form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ given by (2.3) is weakly regular.

In order to obtain that $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is regular, by Proposition 2.19, it suffices to prove that the semigroup $\{P_t\}_{t>0}$ satisfies (2.25). Indeed, we fix $x_o \in \mathcal{X}$ and choose

$$\mathcal{H} = \{f \in C_c(\mathcal{X}) : \ f \geq 0\}.$$

Let $f \in \mathcal{H}$ such that $\mathrm{supp}\, f \subseteq B(x_o, R)$ for some large $R > 0$. For any $z \in B(x_o, 2R)^\complement$ and $y \in B(x_o, R)$, observing that

$$B(x_o, R) \subseteq B(z, R)^\complement,$$

we then derive from (2.5) that for any $0 < s < t$,

$$\begin{aligned}
\operatorname*{esup}_{z\in B(x_o, 2R)^\complement} |P_s f(z)| &= \operatorname*{esup}_{z\in B(x_o, 2R)^\complement} \left| \int_{B(x_o, R)} p_s(z, y) f(y)\, d\mu(y) \right| \\
&\leq \|f\|_{L^\infty(\mathcal{X})} \operatorname*{esup}_{z\in B(x_o, 2R)^\complement} \int_{B(z, R)^\complement} p_s(z, y)\, d\mu(y) \\
&\lesssim \left(\frac{t^{1/\beta}}{R}\right)^\gamma \|f\|_{L^\infty(\mathcal{X})}.
\end{aligned}$$

This induces (2.25) by letting $R \to \infty$. Further, an application of Proposition 2.19 yields that $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is regular. □

## 2.5 Proof of Theorem 2.1: a simpler analytic method under (NC)

In this subsection, under the additional condition (NC), we give a third proof of the Theorem 2.1. This proof is simpler than the second one, and unlike the first one as it does NOT use a deep theory related to stochastic processes.

**Theorem 2.21.** *Under the hypothesis of Theorem 2.1, assume further that (NC) holds. Then the following statements hold:*

(i) *$P_t f \in C_\infty(\mathcal{X})$ for all $f \in L^1(\mathcal{X})$ and $t \in (0, \infty)$;*

(ii) $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ *given by* (2.3) *is a regular Dirichlet form.*

*Proof.* Let us show (i) and (ii) in turn.

**Proof of (i).** Fix $f \in L^1(\mathcal{X})$ and $t \in (0,\infty)$. By Proposition 2.8(i), we have that $P_t f \in C(\mathcal{X})$. Thus, it suffices to prove that $P_t f$ vanishes at infinity.

Fix $x_o \in \mathcal{X}$ and $R \in (0,\infty)$. If $t \in (0,1)$, then we have by (VD) (see (1.8)) and (NC) that

$$C_D' V(x, t^{1/\beta}) \geq t^{\alpha_+/\beta} V(x,1) \geq c t^{\alpha_+/\beta},$$

where $C_D'$ and $\alpha_+$ are as in (1.8), and $c$ is the constant given in (2.7). If $t \geq 1$, then by (NC),

$$V(x, t^{1/\beta}) \geq V(x,1) \geq c.$$

Combining the above two formulae, we then derive from (2.4) that

$$p_t(x,y) \lesssim \frac{1}{V(x,t^{1/\beta})} \lesssim \frac{1}{\min\{1, t^{\alpha_+/\beta}\}} \quad \text{for all} \;\; t \in (0,\infty) \;\; \text{and} \;\; x,y \in \mathcal{X},$$

thereby leading to that

$$\int_{B(x_o,R)^\complement} p_t(x,y)\,|f(y)|\;d\mu(y) \lesssim \frac{1}{\min\{1, t^{\alpha_+/\beta}\}} \int_{B(x_o,R)^\complement} |f(y)|\,d\mu(y). \tag{2.28}$$

Next, for any $n \in \mathbb{N}$, we set $f_n := f\mathbf{1}_{\{|f|\leq n\}}$, which converges to $f$ both in $L^1(\mathcal{X})$ and pointwise as $n \to \infty$. As it has been proved in (2.9), for all $t \in (0,\infty)$ and $x \notin B(x_o, 2R)$, we still have

$$\begin{aligned}\int_{B(x_o,R)} p_t(x,y)|f(y)|\,d\mu(y) &\leq \int_{B(x_o,R)} p_t(x,y)|f(y) - f_n(y)|\,d\mu(y) + \int_{B(x_o,R)} p_t(x,y)|f_n(y)|\,d\mu(y)\\ &\lesssim \frac{1}{V(x_o,R)}\|f_n - f\|_{L^1(\mathcal{X})} + n\left(\frac{t^{1/\beta}}{R}\right)^{\gamma}.\end{aligned} \tag{2.29}$$

Combining (2.28) and (2.29), we deduce that for all $x \notin B(x_o,2R)$ and $t \in (0,\infty)$,

$$\begin{aligned}|P_t f(x)| &\leq \left(\int_{B(x_o,R)} + \int_{B(x_o,R)^\complement}\right) p_t(x,y)|f(y)|d\mu(y)\\ &\lesssim \frac{1}{V(x_o,R)}\|f_n - f\|_{L^1(\mathcal{X})} + n\left(\frac{t^{1/\beta}}{R}\right)^{\gamma} + \frac{1}{\min\{1, t^{\alpha_+/\beta}\}} \int_{B(x_o,R)^\complement} |f(y)|\,d\mu(y),\end{aligned}$$

which implies

$$\lim_{R\to\infty} \sup_{x \in B(x_o,2R)^\complement} |P_t f(x)| = 0$$

by first letting $R \to \infty$ and the letting $n \to \infty$. This proves that $P_t f$ vanishes at infinity and, hence, $P_t f \in C_\infty(\mathcal{X})$.

**Proof of (ii).** As in the first part of the proof of Proposition 2.20, we can derive from Proposition 2.8(iii), [21, Lemma 1.3.3(i)] and Remark 2.7 that $\mathrm{Dom}(\mathcal{E}) \cap C_c(\mathcal{X})$ is dense in $C_c(\mathcal{X})$ with respect to $\|\cdot\|_{L^\infty(\mathcal{X})}$. Based on this and Lemma 2.6, we only need to show that any function in $\mathrm{Dom}(\mathcal{E})$ can be approximated by functions in $C_\infty(\mathcal{X}) \cap \mathrm{Dom}(\mathcal{E})$ with respect to $\|\cdot\|_{\mathcal{E}_1}$.

Let $u \in \mathrm{Dom}(\mathcal{E})$. By [21, p. 22, Lemma 1.3.3(iii)], we know that $P_t u \in \mathrm{Dom}(\mathcal{E})$ and $P_t u \to u$ with respect to $\|\cdot\|_{\mathcal{E}_1}$ as $t \to 0$. Since $u \in L^2(\mathcal{X})$, it follows from Proposition 2.8(i) that $P_t u \in C(\mathcal{X})$ for all $t > 0$. Moreover, for any $t > 0$ and $x \in \mathcal{X}$, by the Hölder inequality with respect to the measure $p_t(x,z)\,d\mu(z)$ and the stochastic completeness of the heat kernel, we obtain

$$|P_t u(x)| \le \int_{\mathcal{X}} p_t(x,z)|u(z)|\,d\mu(z) \le \left(\int_{\mathcal{X}} p_t(x,z)\,d\mu(z)\right)^{1/2} \left(\int_{\mathcal{X}} p_t(x,z)|u(z)|^2\,d\mu(z)\right)^{1/2} = \sqrt{P_t(u^2)(x)}.$$

Using (NC) and the fact that $u^2 \in L^1(\mathcal{X})$, we then derive from (i) that $P_t(u^2) \in C_\infty(\mathcal{X})$. Hence, the last inequality implies that $P_t u$ vanishes at infinity, that is, $P_t u \in C_\infty(\mathcal{X})$.

Altogether, given any $u \in \mathrm{Dom}(\mathcal{E})$, we obtain that $P_t u \to u$ with respect to $\|\cdot\|_{\mathcal{E}_1}$ as $t \to 0$, with every $P_t u \in C_\infty(\mathcal{X}) \cap \mathrm{Dom}(\mathcal{E})$. This ends the proof of (ii). □

# 3 Stability of the critical index

Our aim in this section is to prove Theorems 1.2 and 1.3. In Subsection 3.1, we present some preliminary materials including the definition of the condition $(\mathrm{AB})_\beta$. Further, in Subsection 3.2, we show that if a heat kernel satisfies $(\mathrm{ULE})_\beta$ then the associated Dirichlet form is regular by Theorem 2.1 and is also of pure jump type. With these, the proof of Theorem 1.2 is presented in Subsection 3.3. As a consequence of Theorem 1.2, we then show Theorem 1.3 in Subsection 3.4.

## 3.1 Condition $(\mathrm{AB})_\beta$ and the jump kernel

Let $(\mathcal{X}, d, \mu)$ be a metric measure space satisfying (VD). We introduce some necessary notions. Set

$$V(x,y) := V(x, d(x,y)) + V(y, d(x,y))$$

and, for any $\beta \in (0, +\infty)$, consider the *standard jump kernel*

$$J_\beta(x,y) := \frac{1}{V(x,y)\,d(x,y)^\beta}. \tag{3.1}$$

By (VD), we have $V(x,y) \simeq V(x, d(x,y)) \simeq V(y, d(x,y))$, whence

$$J_\beta(x,y) \simeq \frac{1}{V(x,d(x,y))d(x,y)^\beta}.$$

Note that if $\mu$ is $\alpha$-regular then $V(x,y) \simeq d(x,y)^\alpha$ and, hence,

$$J_\beta(x,y) \simeq \frac{1}{d(x,y)^{\alpha+\beta}}.$$

Based on the standard jump kernel in (3.1), we give the definitions of Besov spaces.

**Definition 3.1.** For $s \in (0,\infty)$, the *homogeneous Besov spaces* $\dot{\Lambda}^s_{2,2}(\mathcal{X})$ and $\dot{\Lambda}^s_{2,\infty}(\mathcal{X})$ are respectively defined to be the collection of all locally integrable functions $f$ on $\mathcal{X}$ such that

$$\|f\|_{\dot{\Lambda}^s_{2,2}(\mathcal{X})} := \left(\iint_{\mathcal{X}\times\mathcal{X}} |f(x)-f(y)|^2 J_{2s}(x,y)\,d\mu(x)\,d\mu(y)\right)^{1/2} < \infty$$

and

$$\|f\|_{\dot{\Lambda}^s_{2,\infty}(X)} := \left(\sup_{r\in(0,\infty)} \int_X \left(\frac{1}{V(x,r)} \int_{B(x,r)} \frac{|f(x)-f(y)|^2}{r^{2s}} \, d\mu(y)\right) d\mu(x)\right)^{1/2} < \infty.$$

Moreover, define the corresponding *inhomogeneous Besov spaces*

$$\Lambda^s_{2,2}(X) = \left\{ f \in L^2(X) : \|f\|_{\Lambda^s_{2,2}(X)} := \|f\|_{L^2(X)} + \|f\|_{\dot{\Lambda}^s_{2,2}(X)} < \infty \right\}$$

and

$$\Lambda^s_{2,\infty}(X) = \left\{ f \in L^2(X) : \|f\|_{\Lambda^s_{2,\infty}(X)} := \|f\|_{L^2(X)} + \|f\|_{\dot{\Lambda}^s_{2,\infty}(X)} < \infty \right\}.$$

It is obvious that $\dot{\Lambda}^s_{2,2}(X) \subseteq \dot{\Lambda}^s_{2,\infty}(X)$ and $\Lambda^s_{2,2}(X) \subseteq \Lambda^s_{2,\infty}(X)$.

Let us state the condition $(AB)_\beta$ that was introduced in [24]. This condition is named after Andres and Barlow, because they first introduced in [3] a similar condition for *local* Dirichlet forms. That condition, denoted by *cutoff Sobolev inequality in annuli*, was a simplified version of the cutoff Sobolev inequality introduced by Barlow and Bass (see [7, 8]).

**Definition 3.2.** For any given $\beta \in (0,\infty)$, we say that the condition $(AB)_\beta$ holds, if there exist $\zeta \geq 0$ and $C > 0$ such that, for any function

$$u \in \left(\Lambda^{\beta/2}_{2,2}(X) + \{\text{const}\}\right) \cap L^\infty(X),$$

and for any three concentric balls

$$B_0 = B(x_0, R), \quad B = B(x_0, R+r), \quad \Omega = B(x_0, R'), \tag{3.2}$$

with $x_0 \in X$ and $0 < R < R + r < R' < \operatorname{diam}(X)$, there exists a function $\phi \in \operatorname{cutoff}(B_0, B)$ such that

$$\begin{aligned} &\iint_{\Omega\times\Omega} |u(x)|^2 |\phi(x)-\phi(y)|^2 J_\beta(x,y) \, d\mu(x)\, d\mu(y) \\ &\leq \zeta \iint_{B\times B} |\phi(x)|^2 |u(x)-u(y)|^2 J_\beta(x,y)\, d\mu(x)\, d\mu(y) + \frac{C}{r^\beta}\int_\Omega |u(x)|^2 \, d\mu(x). \end{aligned} \tag{3.3}$$

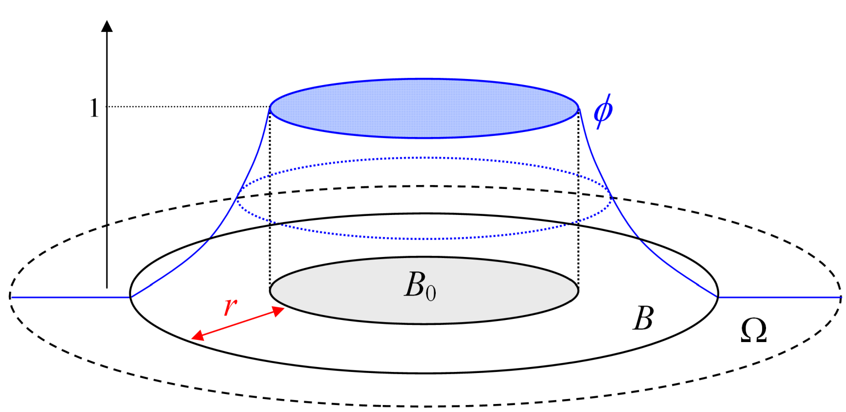


Figure 2: Balls $B_0, B, \Omega$ and a cutoff function $\phi$.

It was proved in [12, Lemma 2.2 and Remark 2.3] that $(\mathrm{AB})_\beta$ always holds when $0 < \beta < 2$ (by using a standard bump function $\phi$ of two concentric balls); see also [16, Remark 1.7], [24, the proof of Corollary 2.12] and [26, Example 4.2]. However, so far there are no regular methods for a direct proof of $(\mathrm{AB})_\beta$ when $\beta \geq 2$, except for the following example (see also [26, Example 4.1]).

**Example 3.3.** Let $(\mathcal{X}, d)$ be an *ultra-metric* space, that is, $d$ satisfies the ultra-metric inequality

$$d(x, y) \leq \max(d(x, z), d(y, z)) \quad \text{for all } x, y, z \in \mathcal{X}. \tag{3.4}$$

A typical example of an ultra-metric space is the $p$-adic field (see, for example, [10, 42, 43]).

Suppose that $\mu$ is a measure on the ultra-metric space $(\mathcal{X}, d)$ satisfying (VD). Consider the three concentric balls $B_0, B$ and $\Omega$ as in (3.2). Let $\phi := \mathbf{1}_B$ be the characteristic function of the ball $B$. By the properties of ultra-metric, this function is continuous and, hence, is a cutoff function of the pair $(B_0, B)$. Let us verify (3.3). In fact, it suffices to prove the following inequality:

$$\sup_{x \in \mathcal{X}} \int_{\mathcal{X}} |\phi(x) - \phi(y)|^2 J_\beta(x, y)\, d\mu(y) \lesssim r^{-\beta}. \tag{3.5}$$

Indeed, if $x \in B$, then, by (VD) and a direct computation (see [27, Proposition 3.1] or [12, Lemma 2.1]), we have

$$\sup_{x \in B} \int_{\mathcal{X}} |\phi(x) - \phi(y)|^2 J_\beta(x, y)\, d\mu(y) = \sup_{x \in B} \int_{B(x, R+r)^\complement} |\phi(x) - \phi(y)|^2 J_\beta(x, y)\, d\mu(y) \lesssim r^{-\beta}.$$

If $x \notin B$ and $y \in B$ then

$$J_\beta(x, y) \leq \frac{1}{(R + r)^\beta V(y, d(x, y))} \leq \frac{1}{r^\beta \mu(B)},$$

and hence, we have

$$\sup_{x \in B^\complement} \int_{\mathcal{X}} |\phi(x) - \phi(y)|^2 J_\beta(x, y)\, d\mu(y) = \sup_{x \in B^\complement} \int_B J_\beta(x, y)\, d\mu(y) \leq \int_B \frac{1}{r^\beta \mu(B)}\, d\mu = r^{-\beta}.$$

Combining the above two cases, we obtain (3.5) and then

$$\begin{aligned} \iint_{\Omega \times \Omega} |u(x)|^2 |\phi(x) - \phi(y)|^2 J_\beta(x, y)\, d\mu(x)\, d\mu(y) &\leq \sup_{z \in \mathcal{X}} \int_{\mathcal{X}} \phi(z) - \phi(y)|^2 J_\beta(x, y)\, d\mu(y) \int_\Omega |u(x)|^2\, d\mu(x) \\ &\lesssim \frac{1}{r^\beta} \int_\Omega |u|^2\, d\mu, \end{aligned}$$

which is (3.3) (with $\zeta = 0$ therein). Hence, on a doubling ultra-metric space, the condition $(\mathrm{AB})_\beta$ is satisfied for all $\beta \in (0, +\infty)$.

**Remark 3.4.** In view of Example 3.3 and Theorem 1.2, we see that if $(\mathcal{X}, d, \mu)$ is an ultra-metric measure space satisfying (VD) and (RVD), then $\beta^\sharp = \infty$.

### 3.2 Jump type Dirichlet forms

We start with the following definition.

**Definition 3.5.** For any $\beta > 0$, we say that the jump kernel $J(x,y)$ satisfies the condition $(\mathrm{J})_\beta$ if

$$J(x,y) \simeq J_\beta(x,y), \tag{3.6}$$

where $J_\beta$ is the standard jump kernel defined in (3.1). We also say that $J(x,y)$ satisfies $(\mathrm{J}_\leq)_\beta$ (resp. $(\mathrm{J}_\geq)_\beta$) if the upper (resp. lower) bound in (3.6) is satisfied.

**Lemma 3.6.** *Assume that* $(\mathcal{X}, d, \mu)$ *is a metric measure space satisfying* (VD). *Let* $(\mathcal{E}, \mathcal{F})$ *be a regular Dirichlet form where the jump part is given by the jumping measure* $dj(x,y)$*. If the heat kernel* $p_t(x,y)$ *of* $(\mathcal{E}, \mathcal{F})$ *exists and satisfies* $(\mathrm{UE})_\beta$ (that is, (1.15))*, then the jump kernel*

$$J(x,y) := \frac{dj(x,y)}{d\mu(x)\,d\mu(y)} \tag{3.7}$$

*exists and satisfies* $(\mathrm{J}_\leq)_\beta$*. If the heat kernel satisfies the two-sided estimate* $(\mathrm{ULE})_\beta$ *then the jump kernel* $J(x,y)$ *in* (3.7) *satisfies* $(\mathrm{J})_\beta$*. If in addition* $(\mathcal{E}, \mathcal{F})$ *is stochastically complete then it is of pure jump type.*

*Proof.* By the Beurling-Deny decomposition (see [21, Theorem 3.2.1, Lemma 4.5.2]), we have

$$\mathcal{E} = \mathcal{E}^L + \mathcal{E}^J + \mathcal{E}^K,$$

where $\mathcal{E}^L$ is the local part, $\mathcal{E}^K$ is the killing part, and $\mathcal{E}^J$ is the jump part:

$$\mathcal{E}^J(f,g) = \int_{\mathcal{X}}\int_{\mathcal{X}} (f(x)-f(y))(g(x)-g(y))\,dj(x,y). \tag{3.8}$$

For any functions $f, g \in \mathcal{F}$ we have by (2.3) (see also [21, Eq. (1.3.17), p. 23]) that

$$\mathcal{E}(f,g) = \lim_{t\to 0} \frac{1}{t}(f - P_t f, g). \tag{3.9}$$

Given any two disjoint precompact open sets $A, B$ in $\mathcal{X}$, let $f, g \in \mathcal{F} \cap C_c(\mathcal{X})$ supported in $A$ and $B$, respectively. Then, by (3.9), we have

$$\mathcal{E}(f,g) = -\lim_{t\to 0} \frac{1}{t} \int_A \int_B f(x)g(y)p_t(x,y)\,d\mu(x)\,d\mu(y),$$

and, by (3.8),

$$\mathcal{E}(f,g) = \mathcal{E}^J(f,g) = -2\int_A\int_B f(x)g(y)\,dj(x,y).$$

In other words, if $t \to 0$, then

$$\int_A\int_B f(x)g(y)\frac{p_t(x,y)}{2t}\,d\mu(x)\,d\mu(y) \to \int_A\int_B f(x)g(y)\,dj(x,y). \tag{3.10}$$

By $(\mathrm{UE})_\beta$, we have for all $t \in (0, \infty)$ and $x, y \in \mathcal{X}$ that

$$\frac{p_t(x,y)}{2t} \leq \frac{C}{V(x, t^{1/\beta} + d(x,y))} \left(\frac{1}{t^{1/\beta} + d(x,y)}\right)^\beta \leq \frac{C}{V(x,y)d(x,y)^\beta}, \tag{3.11}$$

which implies from (3.10) that

$$dj(x,y) \leq \frac{C}{V(x,y)d(x,y)^\beta}\, d\mu(x)\, d\mu(y)$$

and, hence, the Radon-Nikodym derivative $J(x,y)$ in (3.7) exists and satisfies $(\mathrm{J}_\leq)_\beta$. See also [28, Lemma 10.3(2) for $q = \infty$] for that $(\mathrm{UE})_\beta$ implies the existence of jump kernel $J(x,y)$ and $(\mathrm{J}_\leq)_\beta$.

If in addition the lower bound in $(\mathrm{LE})_\beta$ is satisfied, then, for all $x, y \in \mathcal{X}$,

$$\begin{aligned}\liminf_{t\to 0} \frac{p_t(x,y)}{2t} &\geq \liminf_{t\to 0} \frac{c}{V(x, t^{1/\beta} + d(x,y))} \left(\frac{1}{t^{1/\beta} + d(x,y)}\right)^\beta \\ &\geq \frac{c}{V(x,y)d(x,y)^\beta},\end{aligned}$$

which, together with (3.10) and the Fatou lemma, implies that for non-negative functions $f, g \in \mathcal{F} \cap C_c(\mathcal{X})$,

$$\begin{aligned}\int_A \int_B f(x)g(y)\, dj(x,y) &\geq \int_A \int_B f(x)g(y) \liminf_{t\to 0} \frac{p_t(x,y)}{2t}\, d\mu(x)\, d\mu(y) \\ &\geq \int_A \int_B f(x)g(y) \frac{c}{V(x,y)d(x,y)^\beta}\, d\mu(x)\, d\mu(y).\end{aligned}$$

This proves $(\mathrm{J}_\geq)_\beta$. See also [25, Lemma 8.8, p. 763] for that $(\mathrm{LE})_\beta$ implies $(\mathrm{J}_\geq)_\beta$.

In conclusion, the above arguments not only show that $(\mathrm{ULE})_\beta$ implies $(\mathrm{J})_\beta$, but also implies from (3.11) and $(\mathrm{J}_\geq)_\beta$ that

$$\frac{p_t(x,y)}{2t} \leq CJ(x,y) \quad \text{for all } \ t \in (0,\infty) \ \text{ and } \ x, y \in \mathcal{X},$$

for a universal constant $C > 0$. If, in addition, $(\mathcal{E}, \mathcal{F})$ is stochastically complete, then $\mathcal{E}^K \equiv 0$ and the formula in (3.9) implies that for any $f \in \mathcal{F}$,

$$\mathcal{E}(f,f) = \lim_{t\to 0} \int_{\mathcal{X}} \int_{\mathcal{X}} |f(x) - f(y)|^2 \frac{p_t(x,y)}{2t}\, d\mu(x)\, d\mu(y).$$

Combining the above two formulae yields that for any $f \in \mathcal{F}$,

$$\mathcal{E}(f,f) \leq C\mathcal{E}^J(f,f),$$

thereby leading to

$$\mathcal{E}^L(f,f) = \mathcal{E}(f,f) - \mathcal{E}^J(f,f) - \mathcal{E}^K(f,f) \leq C\mathcal{E}^J(f,f). \tag{3.12}$$

Let us use (3.12) to prove $\mathcal{E}^L \equiv 0$. Indeed, denote by $\Gamma(u,u)$ the *energy measure* of $\mathcal{E}^L$ for a bounded function $u \in \mathcal{F}$. By the arguments in [39, p. 389], we have for all $\phi, u \in \mathcal{F} \cap L^\infty(\mathcal{X})$ and $\lambda \geq 0$,

$$\begin{aligned}
&\lambda^2 \int_{\mathcal{X}} \phi^2 d\Gamma(u,u) + \int_{\mathcal{X}} d\Gamma(\phi,\phi) \\
&\quad = \mathcal{E}^L(\phi\cos(\lambda u), \phi\cos(\lambda u)) + \mathcal{E}^L(\phi\sin(\lambda u), \phi\sin(\lambda u)) \\
&\quad \lesssim \mathcal{E}^J(\phi\cos(\lambda u), \phi\cos(\lambda u)) + \mathcal{E}^J(\phi\sin(\lambda u), \phi\sin(\lambda u)) \quad \text{(by (3.12))} \\
&\quad = \int_{\mathcal{X}}\int_{\mathcal{X}} \Big(\phi^2(x) + \phi^2(y) - 2\phi(x)\phi(y)\cos(\lambda(u(x)-u(y)))\Big) J(x,y)\, d\mu(x)\, d\mu(y) \\
&\quad = \int_{\mathcal{X}}\int_{\mathcal{X}} \Big(|\phi(x)-\phi(y)|^2 + 2\phi(x)\phi(y)(1-\cos(\lambda(u(x)-u(y))))\Big) J(x,y)\, d\mu(x)\, d\mu(y) \\
&\quad \lesssim \int_{\mathcal{X}}\int_{\mathcal{X}} \Big(|\phi(x)-\phi(y)|^2 + \|\phi\|^2_{L^\infty}(\lambda^2|u(x)-u(y)|^2 \wedge 2)\Big) J(x,y)\, d\mu(x)\, d\mu(y).
\end{aligned}$$

Dividing both sides of the above inequality by $\lambda^2$ and passing to the limit as $\lambda \to \infty$, we obtain that $\int_{\mathcal{X}} \phi^2\, d\Gamma(u,u) = 0$. Since $\phi, u$ are arbitrary, it follows that $\mathcal{E}^L \equiv 0$, that is, $\mathcal{E}$ is of pure jump type. □

### 3.3 Proof of Theorem 1.2

Let us return to a general setting. Given a symmetric non-negative jump kernel $J(x,y)$ on $\mathcal{X}$, define the bilinear form

$$\mathcal{E}(u,v) := \int_{\mathcal{X}}\int_{\mathcal{X}} (u(x)-u(y))(v(x)-v(y))J(x,y)\, d\mu(x)\, d\mu(y) \tag{3.13}$$

with domain

$$\mathcal{F} := \Big\{u \in L^2(\mathcal{X}) :\ \mathcal{E}(u,u) < \infty\Big\}. \tag{3.14}$$

It follows from the Fatou Lemma that $\mathcal{F}$ is complete with respect to the norm $\|\cdot\|_{\mathcal{E}_1}$ given in (2.1); see also [21, Example 1.2.4, p. 14]. Note that $(\mathcal{E},\mathcal{F})$ becomes a Dirichlet form only if the domain $\mathcal{F}$ is dense in $L^2(\mathcal{X},\mu)$.

The following theorem plays an essential role in the proof of Theorem 1.2.

**Theorem 3.7** ([12]). *Assume that* (VD) *and* (RVD) *are satisfied. Then, for any $\beta \in (0,\infty)$, the following two conditions are equivalent:*

(i) $(\mathrm{AB})_\beta$

(ii) *For any/some jump kernel $J$ satisfying $(\mathrm{J})_\beta$, the bilinear form $(\mathcal{E},\mathcal{F})$ given by (3.13)-(3.14) is a regular Dirichlet form whose heat kernel exists and satisfies $(\mathrm{ULE})_\beta$.*

*Moreover, if* (i) or (ii) *holds, then the heat kernel $\{p_t\}_{t>0}$ of $(\mathcal{E},\mathcal{F})$ is stochastically complete and jointly continuous on $\mathcal{X}\times\mathcal{X}$ for any $t > 0$.*

**Remark 3.8.** Let $\beta > 0$ and $J_\beta$ be the standard jump kernel in (3.1). Consider the bilinear form

$$\mathcal{E}_\beta(u,v) := \int_{\mathcal{X}} \int_{\mathcal{X}} (u(x)-u(y))(v(x)-v(y)) J_\beta(x,y)\, d\mu(y)\, d\mu(x) \tag{3.15}$$

with the domain

$$\mathcal{F}_\beta := \left\{ u \in L^2(\mathcal{X}) : \ \mathcal{E}_\beta(u,u) < \infty \right\} = \Lambda^{\beta/2}_{2,2}(\mathcal{X}). \tag{3.16}$$

By Theorem 3.7, the hypothesis $(\mathrm{AB})_\beta$ is equivalent to the fact that the bilinear form $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ is a regular Dirichlet form whose heat kernel exists and satisfies $(\mathrm{ULE})_\beta$.

We give the following regularity result, which is a consequence of Theorem 2.1.

**Proposition 3.9.** *Assume that* (VD) *is satisfied and $\beta, \gamma \in (0,\infty)$. Let $\{P_t\}_{t>0}$ be the semigroup defined in* (2.2)*, with $\{p_t\}_{t>0}$ being a stochastically complete continuous heat kernel on $\mathcal{X}$ satisfying*

$$0 \le p_t(x,y) \le \frac{C}{V(x, t^{1/\beta} + d(x,y))} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\gamma} \quad \text{for all } \ t \in (0,\infty) \ \text{ and } \ x,y \in \mathcal{X}, \tag{3.17}$$

*for some constant $C > 0$. Then, the Dirichlet form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ given by* (2.3) *is regular. In particular, this last statement is true provided that $\{p_t\}_{t>0}$ satisfies $(\mathrm{UE})_\beta$.*

*Proof.* Note that $(\mathrm{UE})_\beta$ implies (3.17). So, it suffices to show the regularity of $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ under (3.17). To this end, we need to apply Theorem 2.1 via verifying that (3.17) implies (2.4) and (2.5).

It is obvious that (3.17) implies (2.4). By a direct computation, it follows from (VD) and (3.17) that for any $x \in \mathcal{X}$ and $t, r > 0$,

$$\begin{aligned}
\int_{B(x,r)^\complement} p_t(x,y)\, d\mu(y) &\lesssim \int_{B(x,r)^\complement} \frac{1}{V(x, t^{1/\beta} + d(x,y))} \left(\frac{t^{1/\beta}}{t^{1/\beta} + d(x,y)}\right)^{\gamma} d\mu(y) \\
&\lesssim \left(\frac{t^{1/\beta}}{r}\right)^{\frac{\gamma}{2}} \int_{\mathcal{X}} \frac{1}{V(x, t^{1/\beta} + d(x,y))} \left(\frac{t^{1/\beta}}{t^{1/\beta} + d(x,y)}\right)^{\frac{\gamma}{2}} d\mu(y) \\
&\lesssim \left(\frac{t^{1/\beta}}{r}\right)^{\frac{\gamma}{2}},
\end{aligned}$$

which gives the desired condition (2.5). □

*Proof of Theorem 1.2.* By Theorem 3.7, it suffices to prove that (ii) implies (i).

Suppose that $\beta > 0$ and there exists a stochastically complete continuous heat kernel $\{p_t\}_{t>0}$ on $\mathcal{X}$ satisfying $(\mathrm{ULE})_\beta$. Let $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ be the Dirichlet form determined by the heat kernel $\{p_t\}_{t>0}$ (given in (2.3)). It follows from Proposition 3.9 (see also Theorem 2.1) that $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is regular. Moreover, it follows from Lemma 3.6 that $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is of pure jump type, and the jump kernel $J(x,y)$ exists and satisfies $(\mathrm{J})_\beta$. In particular, $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ coincides with that in (3.13)-(3.14). Therefore, Theorem 3.7(ii) is satisfied, which implies $(\mathrm{AB})_\beta$. This finishes the proof of Theorem 1.2. □

By Theorems 3.7 and 1.2, we have the following characterizations of the critical index.

**Corollary 3.10.** *Assume that* (VD) *and* (RVD) *are satisfied on* $(\mathcal{X}, d, \mu)$. *Then* $\beta^\sharp := \beta^\sharp(\mathcal{X}, d, \mu)$ *has the following equivalent expressions:*

$$\begin{aligned}\beta^\sharp &= \sup\{\beta > 0 : (\mathcal{E}_\beta, \mathcal{F}_\beta) \text{ in (3.15)-(3.16) is a regular Dirichlet form, whose heat} \\ &\qquad \text{kernel is stochastically complete, continuous and satisfies } (\mathrm{ULE})_\beta\} \\ &= \sup\{\beta > 0 : \text{there exists a regular pure jump type Dirichlet form, whose jump} \\ &\qquad \text{kernel exists and satisfies } (\mathrm{J})_\beta \text{ and whose heat kernel is stochastically} \\ &\qquad \text{complete, continuous and satisfies } (\mathrm{ULE})_\beta\}.\end{aligned}$$

### 3.4 Proof of Theorem 1.3

In this subsection, we will utilize Theorem 1.2 to prove Theorem 1.3, that is, to prove the quasi-isometric invariance of the critical index $\beta^\sharp$.

*Proof of Theorem 1.3.* Let $(\mathcal{X}, d', \mu')$ be a metric measure space that is quasi-isometric to $(\mathcal{X}, d, \mu)$, that is, there exists $c > 1$ such that

$$c^{-1}d' \le d \le cd' \quad \text{and} \quad c^{-1} \le \frac{d\mu}{d\mu'} \le c. \tag{3.18}$$

For any $x \in \mathcal{X}$ and $r > 0$, denote by $B_d(x, r)$ and $B_{d'}(x, r)$ the metric balls in $(\mathcal{X}, d)$ and $(\mathcal{X}, d')$, respectively.

Let us first show that, if $\mu \simeq \mu'$ but the metric is unchanged, then $\beta^\sharp$ is invariant. By the second formula in (3.18), we have

$$(\mathcal{X}, d, \mu) \text{ satisfies (VD) and (RVD)} \quad \Leftrightarrow \quad (\mathcal{X}, d, \mu') \text{ satisfies (VD) and (RVD).}$$

In this case, note that $(\mathrm{AB})_\beta$ holds on $(\mathcal{X}, d, \mu)$ if and only if it holds on $(\mathcal{X}, d, \mu')$. This, along with (1.11) in Theorem 1.2, yields

$$\beta^\#(\mathcal{X}, d, \mu) = \beta^\#(\mathcal{X}, d, \mu'). \tag{3.19}$$

Next, we show that if $d \simeq d'$ but the measure is unchanged, then $\beta^\sharp$ is also invariant. We use the notation

$$\mathrm{diam}(\mathcal{X}, d) := \sup\{d(x, y) : x, y \in \mathcal{X}\}.$$

In a similar manner, we define $\mathrm{diam}(\mathcal{X}, d')$. Since $d \simeq d'$, it follows that

$$R_d := \mathrm{diam}(\mathcal{X}, d) < \infty \quad \Leftrightarrow \quad R_{d'} := \mathrm{diam}(\mathcal{X}, d') < \infty$$

and, by the first formula in (3.18),

$$c^{-1}R_d \le R_{d'} \le cR_d.$$

Since $(\mathcal{X}, d, \mu)$ satisfies (VD) (see also (1.8)), for any $x \in \mathcal{X}$ and $r > 0$, we have

$$\frac{\mu(B_{d'}(x, 2r))}{\mu(B_{d'}(x, r))} \le \frac{\mu(B_d(x, 2cr))}{\mu(B_d(x, c^{-1}r))} \lesssim 1. \tag{3.20}$$

Let $x \in \mathcal{X}$ and $0 < r \le R < R_{d'}$. Since $(\mathcal{X}, d, \mu)$ satisfies (RVD), if $cr \le c^{-1}R$, then $c^{-1}R < c^{-1}R_{d'} \le R_d$ and

$$\frac{\mu(B_{d'}(x,R))}{\mu(B_{d'}(x,r))} \ge \frac{\mu(B_d(x,c^{-1}R))}{\mu(B_d(x,cr))} \gtrsim \left(\frac{R}{r}\right)^{\alpha_-}, \tag{3.21}$$

where $\alpha_-$ is the same constant as in (RVD). If $cr \ge c^{-1}R$, then $R \simeq r$, so that (3.21) still holds since

$$\frac{\mu(B_{d'}(x,R))}{\mu(B_{d'}(x,r))} \ge 1 \simeq \left(\frac{R}{r}\right)^{\alpha_-}. \tag{3.22}$$

Due to the symmetry, the above formulae (3.20)-(3.21)-(3.22) imply that

$$(\mathcal{X}, d, \mu) \text{ satisfies (VD) and (RVD)} \quad \Leftrightarrow \quad (\mathcal{X}, d', \mu) \text{ satisfies (VD) and (RVD)}.$$

Let $\beta < \beta^{\#}(\mathcal{X}, d, \mu)$ such that there exists a stochastically complete continuous heat kernel $\{p_t\}_{t>0}$ on $(\mathcal{X}, d, \mu)$ satisfying $(\mathrm{ULE})_\beta$. It is obvious that $\{p_t\}_{t>0}$ is also stochastically complete and continuous on $(\mathcal{X}, d', \mu)$. Moreover, using $d \simeq d'$ and volume doubling property of $(\mathcal{X}, d', \mu)$, we obtain that $\{p_t\}_{t>0}$ also satisfies $(\mathrm{ULE})_\beta$ on $(\mathcal{X}, d', \mu)$ (that is, replacing the metric $d$ in $(\mathrm{ULE})_\beta$ with $d'$). From this and the definition of $\beta^{\#}(\mathcal{X}, d', \mu)$, it follows that $\beta \le \beta^{\#}(\mathcal{X}, d', \mu)$, and hence

$$\beta^{\#}(\mathcal{X}, d, \mu) \le \beta^{\#}(\mathcal{X}, d', \mu).$$

The opposite inequality can be proved via exchanging the roles of $d$ and $d'$. Therefore, we obtain

$$\beta^{\#}(\mathcal{X}, d, \mu) = \beta^{\#}(\mathcal{X}, d', \mu). \tag{3.23}$$

Finally, first using (3.19), and then using (3.23) (with $\mu$ therein taken to be $\mu'$), we arrive at the identity

$$\beta^{\#}(\mathcal{X}, d, \mu) = \beta^{\#}(\mathcal{X}, d, \mu') = \beta^{\#}(\mathcal{X}, d', \mu').$$

This finishes the proof of Theorem 1.3. □

# 4 Relations of the critical index to the walk dimension

The aim of this section is to show Theorems 1.4 and 1.5, whose proofs will rely on the subordination theory of heat kernels that are basically from [22, 23].

## 4.1 Proof of Theorem 1.4

This subsection is mainly motivated by [22, Section 5.4] (see also [23, Section 4.3]), which deals with the $\alpha$-regular metric measure space but now we assume the general doubling condition (VD).

Suppose that $\{p_t\}_{t>0}$ is a heat kernel. For any $\delta \in (0,1)$, let $\{\eta_t^{(\delta)}\}_{t>0}$ be a $\delta$-*stable subordinator*, that is, for any $t > 0$, $\eta_t^{(\delta)}$ is a positive function defined on $(0, \infty)$ such that

$$e^{-t\lambda^\delta} = \int_0^\infty \eta_t^{(\delta)}(s) e^{-s\lambda}\, ds \quad \text{for all } \lambda \in [0, \infty).$$

Then, by the functional calculus, the function

$$p_t^{(\delta)}(x,y) = \int_0^\infty \eta_t^{(\delta)}(s) p_s(x,y)\, ds \quad \text{for all } \ t \in (0,\infty) \ \text{ and } \ x,y \in \mathcal{X}, \tag{4.1}$$

defines a new heat kernel on $\mathcal{X}$ (see, for example, [22, Section 5.4]). We call $\{p_t^{(\delta)}\}_{t>0}$ the *subordinated heat kernel to* $\{p_t\}_{t>0}$. It follows from (2.2) and (2.3) that the heat kernel $\{p_t^{(\delta)}\}_{t>0}$ determines uniquely a Dirichlet form $(\mathcal{E}^{(\delta)}, \mathrm{Dom}(\mathcal{E}^{(\delta)}))$, which is called the *subordinated Dirichlet form*.

**Proposition 4.1.** *Let $(\mathcal{X}, d, \mu)$ be a metric measure space satisfying* (VD). *Assume that $\{p_t\}_{t>0}$ is a stochastically complete heat kernel satisfying* (ULE)$_\beta$ *for some $\beta \in (0,\infty)$. Then, for any $\delta \in (0,1)$ the subordinated heat kernel $\{p_t^{(\delta)}\}_{t>0}$ determined by* (4.1) *is also a stochastically complete heat kernel satisfying* (ULE)$_{\beta'}$ *with $\beta' := \delta\beta$, that is, for all $t \in (0,\infty)$ and $x,y \in \mathcal{X}$,*

$$p_t^{(\delta)}(x,y) \simeq \frac{1}{V(x, t^{1/\beta'} + d(x,y))} \left(1 + \frac{d(x,y)}{t^{1/\beta'}}\right)^{-\beta'}. \tag{4.2}$$

*If, in addition, the heat kernel $\{p_t\}_{t>0}$ is continuous, then $p_t^{(\delta)}$ is also continuous.*

*Proof.* If $(\mathcal{X}, d, \mu)$ is $\alpha$-regular, then (4.2) has been proved in [22, Lemma 5.4]. Now, we assume only (VD).

It is known that (see [44] or [22, Section 5.4]) the subordinator $\{\eta_t^{(\delta)}\}_{t>0}$ satisfies the scaling property

$$\eta_t^{(\delta)}(s) = \frac{1}{t^{1/\delta}} \eta_1^{(\delta)}\left(\frac{s}{t^{1/\delta}}\right) \quad \text{for all } s \in (0,\infty), \tag{4.3}$$

and the fast decay property at infinity

$$\int_0^\infty s^{-\gamma} \eta_1^{(\delta)}(s)\, ds < \infty \quad \text{for all } \gamma \in (0,\infty), \tag{4.4}$$

as well as the estimates

$$\eta_t^{(\delta)}(s) \lesssim \frac{t}{s^{1+\delta}} \quad \text{for all } \ s,t \in (0,\infty), \tag{4.5}$$

and

$$\eta_t^{(\delta)}(s) \simeq \frac{t}{s^{1+\delta}} \quad \text{when } \ s \geq t^{1/\delta}. \tag{4.6}$$

Also, $\{\eta_t^{(\delta)}\}_{t>0}$ satisfies that for all $t \in (0,\infty)$,

$$\int_0^\infty \eta_t^{(\delta)}(s)\, ds = 1. \tag{4.7}$$

From (4.1) and (4.7), it follows that if $\{p_t\}_{t>0}$ is stochastically complete, then so is $\{p_t^{(\delta)}\}_{t>0}$.

The remaining proof is divided into three steps.

**Step 1: proof of the lower estimate of (4.2).** For any $t \in (0, \infty)$ and $x, y \in \mathcal{X}$, we set

$$r := t^{1/\delta} + d(x, y)^\beta.$$

By this, the lower estimate (LE)$_\beta$ and (VD), we derive that for any $s \in [r, 2r]$ and $x, y \in \mathcal{X}$,

$$\begin{aligned} p_s(x, y) &\gtrsim \frac{1}{V(x, s^{1/\beta} + d(x, y))} \left(1 + \frac{d(x, y)}{s^{1/\beta}}\right)^{-\beta} \\ &\simeq \frac{1}{V(x, r^{1/\beta} + d(x, y))} \left(1 + \frac{d(x, y)}{r^{1/\beta}}\right)^{-\beta} \\ &\simeq \frac{1}{V(x, r^{1/\beta})}. \end{aligned} \tag{4.8}$$

Applying (4.8), (4.1), (4.6) and (VD), we derive that for any $t \in (0, \infty)$ and $x, y \in \mathcal{X}$,

$$\begin{aligned} p_t^{(\delta)}(x, y) &\geq \int_r^{2r} \eta_t^{(\delta)}(s) p_s(x, y)\, ds \quad \text{(by (4.1))} \\ &\gtrsim \int_r^{2r} \frac{t}{s^{1+\delta}} \frac{1}{V(x, r^{1/\beta})}\, ds \quad \text{(by (4.6))} \\ &\simeq \frac{t}{r^\delta} \frac{1}{V(x, r^{1/\beta})} \\ &\simeq \frac{t}{t + d(x, y)^{\beta'}} \frac{1}{V(x, t^{1/\beta'} + d(x, y))} \quad \text{(by (VD))} \\ &\simeq \frac{1}{V(x, t^{1/\beta'} + d(x, y))} \left(1 + \frac{d(x, y)}{t^{1/\beta'}}\right)^{-\beta'}. \end{aligned} \tag{4.9}$$

**Step 2: proof of the upper estimate of (4.2).** By (UE)$_\beta$, for all $s \in (0, \infty)$ and $x, y \in \mathcal{X}$, we have

$$p_s(x, y) \lesssim \frac{1}{V(x, s^{1/\beta})}. \tag{4.10}$$

If $s > t^{1/\delta}$, then it follows from $t^{1/(\delta\beta)} = t^{1/\beta'}$ that

$$V(x, s^{1/\beta}) \geq V(x, t^{1/(\delta\beta)}).$$

If $s \leq t^{1/\delta}$, then we have $s^{1/\beta} \leq t^{1/(\delta\beta)} = t^{1/\beta'}$, which, together with (1.8), implies

$$V(x, s^{1/\beta}) \geq (C_D')^{-1} \left(\frac{s}{t^{1/\delta}}\right)^{\alpha_+/\beta} V(x, t^{1/(\delta\beta)}).$$

Substituting the last two estimates into (4.10) gives that for all $s \in (0, \infty)$ and $x, y \in \mathcal{X}$,

$$p_s(x, y) \lesssim \frac{1}{V(x, t^{1/(\delta\beta)})} \left(\frac{t^{1/\delta}}{s} \vee 1\right)^{\alpha_+/\beta}. \tag{4.11}$$

When $t > d(x,y)^{\beta'}$, combining (4.11), (4.1), (4.7), (4.3) and (4.4), we derive

$$\begin{aligned}
p_t^{(\delta)}(x,y) &= \left(\int_0^{t^{1/\delta}} + \int_{t^{1/\delta}}^{\infty}\right)\eta_t^{(\delta)}(s)p_s(x,y)\,ds \\
&\lesssim \int_0^{t^{1/\delta}} \frac{1}{t^{1/\delta}}\eta_1^{(\delta)}\left(\frac{s}{t^{1/\delta}}\right)\left(\frac{t^{1/\beta}}{s}\right)^{\alpha_+}\frac{1}{V(x,t^{1/(\delta\beta)})}\,ds + \frac{1}{V(x,t^{1/(\delta\beta)})}\int_{t^{1/\delta}}^{\infty}\eta_t^{(\delta)}(s)\,ds \quad \text{(by (4.3))} \\
&\lesssim \frac{1}{V(x,t^{1/\beta'})}\left(\int_0^1 \tau^{-\alpha_+}\eta_1^{(\delta)}(\tau)\,ds + 1\right) \quad \text{(by (4.7))} \\
&\lesssim \frac{1}{V(x,t^{1/\beta'})} \quad \text{(by (4.4))}. 
\end{aligned} \tag{4.12}$$

In the opposite case $t \le d(x,y)^{\beta'}$, we obtain by (4.5), (VD), $(\mathrm{UE})_\beta$ and a change of variables $\tau = d(x,y)/s^{1/\beta}$, that

$$\begin{aligned}
p_t^{(\delta)}(x,y) &\lesssim \int_0^{\infty} \frac{t}{s^{1+\delta}}\frac{1}{V(x,s^{1/\beta}+d(x,y))}\left(1+\frac{d(x,y)}{s^{1/\beta}}\right)^{-\beta} ds \quad \text{(by (4.5) and } (\mathrm{UE})_\beta) \\
&\lesssim \frac{t}{V(x,d(x,y))}\int_0^{\infty}\frac{1}{s^{1+\delta}}\left(1+\frac{d(x,y)}{s^{1/\beta}}\right)^{-\beta} ds \\
&\simeq \frac{1}{V(x,d(x,y))}\frac{t}{d(x,y)^{\delta\beta}}\int_0^{\infty}\tau^{\delta\beta-1}(1+\tau)^{-\beta}\,d\tau \\
&\simeq \frac{1}{V(x,d(x,y))}\frac{t}{d(x,y)^{\beta'}} \\
&\simeq \frac{1}{V(x,t^{1/\beta'}+d(x,y)))}\left(1+\frac{d(x,y)}{t^{1/\beta'}}\right)^{-\beta'} \quad \text{(by (VD))}.
\end{aligned}$$

Hence, in all cases we obtain the desired upper estimate of (4.2).

**Step 3: proof of the continuity of $p_t^{(\delta)}$.** Fix $t \in (0,\infty)$ and $x, x', y, y' \in \mathcal{X}$ with $d(x,x') < t^{1/(\delta\beta)}$. The inequality (4.11) yields that for all $s \in (0,\infty)$,

$$\begin{aligned}
|p_s(x,y) - p_s(x',y')| &\lesssim \left(\frac{1}{V(x,t^{1/(\delta\beta)})} + \frac{1}{V(x',t^{1/(\delta\beta)})}\right)\left(\frac{t^{1/\delta}}{s}\vee 1\right)^{\alpha_+/\beta} \\
&\lesssim \frac{1}{V(x,t^{1/(\delta\beta)})}\left(\frac{t^{1/\delta}}{s}\vee 1\right)^{\alpha_+/\beta}. \quad \text{(by (VD))}
\end{aligned}$$

Moreover, by the computation in (4.12), we see that the function

$$s \mapsto \eta_t^{(\delta)}(s)\frac{1}{V(x,t^{1/(\delta\beta)})}\left(\frac{t^{1/\delta}}{s}\vee 1\right)^{\alpha_+/\beta}$$

is integrable over $(0,\infty)$ with respect to $ds$ uniformly in $y, y'$. Thus, by using the dominated convergence theorem and continuity of $p_t$, we have

$$\lim_{(x',y')\to(x,y)} |p_t^{(\delta)}(x,y) - p_t^{(\delta)}(x',y')| \le \lim_{(x',y')\to(x,y)} \int_0^{\infty} \eta_t^{(\delta)}(s)|p_s(x,y) - p_s(x',y')|\,ds = 0.$$

This implies the continuity of $p_t^{(\delta)}$. Thus, we complete the proof of Proposition 4.1. □

With the subordination of heat kernels in Proposition 4.1, we now prove Theorem 1.4.

*Proof of Theorem 1.4.* Fix any $\beta' \in (0, \beta^{\#})$. By the definition of $\beta^{\#}$, we can choose $\beta \in (\beta', \beta^{\#})$ such that there exists a stochastically complete continuous heat kernel $\{p_t\}_{t>0}$ on $\mathcal{X}$ satisfying $(\mathrm{ULE})_\beta$. Hence, it follows from Proposition 4.1 with $\delta := \beta'/\beta \in (0,1)$ that the $\delta$-subordinated heat kernel $\{p_t^{(\delta)}\}_{t>0}$ is a stochastically complete continuous heat kernel on $\mathcal{X}$ satisfying $(\mathrm{ULE})_{\beta'}$. The proof is completed. □

## 4.2 Proof of Theorem 1.5

**Proposition 4.2.** *Assume that* (VD) *is satisfied. Let $\{q_t\}_{t>0}$ be a stochastically complete continuous heat kernel satisfying the two-sided sub-Gaussian estimate*

$$q_t(x,y) \asymp \frac{C}{V(x, t^{1/\beta} + d(x,y))} \exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right) \quad \text{for all } t \in (0,\infty) \text{ and } x,y \in \mathcal{X}. \tag{4.13}$$

*Then, for any $\delta \in (0,1)$, the subordinated heat kernel $\{q_t^{(\delta)}\}_{t>0}$ determined by $\{q_t\}_{t>0}$ in* (4.1) *satisfies all conclusions of Proposition 4.1.*

*Proof.* Since (4.13) holds, it follows that $\{q_t\}_{t>0}$ satisfies $(\mathrm{UE})_\beta$. Thus, the arguments in **Step 2** and **Step 3** of the proof of Proposition 4.1 remain valid. In other words, the subordinated heat kernel $\{q_t^{(\delta)}\}_{t>0}$ satisfies the upper estimate in (4.2) and, moreover, is jointly continuous.

Concerning the arguments in **Step 1** of the proof of Proposition 4.1, for any $t \in (0,\infty)$ and $x,y \in \mathcal{X}$, we again set

$$r := (t^{1/\delta} + d(x,y)^\beta)/8.$$

Instead of (4.8), we now use the lower estimate of (4.13) to derive that for any $s \in [r, 2r]$ and $x,y \in \mathcal{X}$,

$$\begin{aligned} q_s(x,y) &\gtrsim \frac{1}{V(x, s^{1/\beta} + d(x,y))} \exp\left(-c\left(\frac{d(x,y)}{s^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right) \\ &\simeq \frac{1}{V(x, r^{1/\beta} + d(x,y))} \exp\left(-c\left(\frac{d(x,y)}{r^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right) \\ &\simeq \frac{1}{V(x, r^{1/\beta})}, \end{aligned}$$

so that the calculation in (4.9) can be proceeded as before, thereby leading to

$$q_t^{(\delta)}(x,y) \gtrsim \frac{1}{V(x, t^{1/\beta'} + d(x,y))} \left(1 + \frac{d(x,y)}{t^{1/\beta'}}\right)^{-\beta'}.$$

Thus, $\{q_t^{(\delta)}\}_{t>0}$ satisfies the lower estimate in (4.2). This concludes the proof. □

Now, by Theorem 1.4 and Proposition 4.2, we are about to prove the first identity in (1.14) of Theorem 1.5.

*Proof of $\beta^\# = d_w$ in Theorem 1.5.* Let $\{q_t\}_{t>0}$ be a stochastically complete continuous heat kernel on $\mathcal{X}$ satisfying the sub-Gaussian estimate $(\mathrm{SG})_{d_w}$ (see (1.13)). For any $\beta \in (0, d_w)$, upon setting $\delta := \beta/d_w \in (0,1)$, we derive from Proposition 4.2 that the subordinated heat kernel $\{q_t^{(\delta)}\}_{t>0}$ determined by $\{q_t\}_{t>0}$ (in the way of (4.1)) is a stochastically complete continuous heat kernel on $\mathcal{X}$ satisfying $(\mathrm{ULE})_\beta$, which implies by definition of $\beta^\#$ that $\beta \le \beta^\#$ and, hence,

$$d_w \le \beta^\#. \tag{4.14}$$

It remains to prove that $d_w$ is not strictly smaller than $\beta^\#$. Suppose on the contrary that

$$d_w < \beta^\#. \tag{4.15}$$

Then we will deduce a contradiction in the following three steps.

**Step 1: define two regular Dirichlet forms $(\mathcal{E}^{(i)}, \mathrm{Dom}(\mathcal{E}^{(i)}))$ with $i = 1, 2$.** As in (2.3), let $(\mathcal{E}^{(1)}, \mathrm{Dom}(\mathcal{E}^{(1)}))$ be the Dirichlet form determined by the heat kernel $\{q_t\}_{t>0}$. Invoking the stochastically completeness of $\{q_t\}_{t>0}$, we then derive from (2.3) that

$$\mathcal{E}^{(1)}(u,v) = \lim_{t\to 0} \frac{1}{2t} \iint_{\mathcal{X}\times\mathcal{X}} (u(x)-u(y))(v(x)-v(y)) q_t(x,y)\, d\mu(x)\, d\mu(y) \tag{4.16}$$

for all $u, v \in \mathrm{Dom}(\mathcal{E}^{(1)})$, with

$$\mathrm{Dom}(\mathcal{E}^{(1)}) = \{u \in L^2(\mathcal{X}) : \mathcal{E}^{(1)}(u,u) < \infty\}. \tag{4.17}$$

It follows from $(\mathrm{SG})_{d_w}$ that

$$q_t(x,y) \lesssim \frac{1}{V(x, t^{1/d_w} + d(x,y))} \left(1 + \frac{d(x,y)}{t^{1/d_w}}\right)^{-d_w} \tag{4.18}$$

holds uniformly in $t > 0$ and $x, y \in \mathcal{X}$. Then, since $\{q_t\}_{t>0}$ is stochastically complete and continuous, we have by Proposition 3.9 that the Dirichlet form $(\mathcal{E}^{(1)}, \mathrm{Dom}(\mathcal{E}^{(1)}))$ is regular.

Since we have assumed $d_w < \beta^\#$ in (4.15), it follows form Theorem 1.4 that there exists a stochastically complete continuous heat kernel $\{p_t\}_{t>0}$ on $\mathcal{X}$ satisfying $(\mathrm{ULE})_{d_w}$. Again, let $(\mathcal{E}^{(2)}, \mathrm{Dom}(\mathcal{E}^{(2)}))$ be the Dirichlet form determined by the heat kernel $\{p_t\}$ in the way of (2.3), which yields that

$$\mathcal{E}^{(2)}(u,v) = \lim_{t\to 0} \frac{1}{2t} \iint_{\mathcal{X}\times\mathcal{X}} (u(x)-u(y))(v(x)-v(y)) p_t(x,y)\, d\mu(x)\, d\mu(y) \tag{4.19}$$

whenever $u, v \in \mathrm{Dom}(\mathcal{E}^{(2)})$, with

$$\mathrm{Dom}(\mathcal{E}^{(2)}) = \{u \in L^2(\mathcal{X}) : \mathcal{E}^{(2)}(u,u) < \infty\}. \tag{4.20}$$

By Proposition 3.9, $(\mathcal{E}^{(2)}, \mathrm{Dom}(\mathcal{E}^{(2)}))$ is regular.

**Step 2: prove that $(\mathcal{E}^{(1)}, \mathrm{Dom}(\mathcal{E}^{(1)}))$ is strongly local.** By the general theory of Dirichlet forms, $(\mathcal{E}^{(1)}, \mathrm{Dom}(\mathcal{E}^{(1)}))$ can be decomposed into two parts: the strongly local part and the jump part associated with a Radon measure $j^{(1)}$ on $\mathcal{X} \times \mathcal{X} \setminus \text{diag}$. Moreover, following the arguments that lead to (3.10), we obtain that for any two disjoint precompact open sets $A, B \subseteq \mathcal{X}$, and for any non-negative functions $f, g \in \mathrm{Dom}(\mathcal{E}^{(1)}) \cap C_c(\mathcal{X})$ supported in $A$ and $B$, respectively, it holds that

$$\int_A \int_B f(x)g(y) \frac{q_t(x,y)}{2t}\, d\mu(x)\, d\mu(y) \to \int_A \int_B f(x)g(y)\, dj(x,y) \quad \text{as } t \to 0^+.$$

Again, by the upper bound in $(\mathrm{SG})_{d_w}$, we obtain that for any $x \in A$ and $y \in B$ (note that $d(x,y) > 0$ since $A \cap B = \emptyset$),

$$0 \le \frac{q_t(x,y)}{2t} \le \frac{Ct^{-1}}{V(x, t^{1/d_w} + d(x,y))} \exp\left(-c\left(\frac{d(x,y)}{t^{1/d_w}}\right)^{\frac{d_w}{d_w-1}}\right) \to 0 \quad \text{as } t \to 0^+.$$

Combining the above two formulae and using the Fatou lemma, we obtain

$$\int_A \int_B f(x)g(y)\, dj(x,y) \le \int_A \int_B \lim_{t\to 0} f(x)g(y) \frac{q_t(x,y)}{2t}\, d\mu(x)\, d\mu(y) = 0.$$

Since $A, B, f, g$ are arbitrary, we see that $dj \equiv 0$. In other words, $(\mathcal{E}^{(1)}, \mathrm{Dom}(\mathcal{E}^{(1)}))$ is a strongly local, regular Dirichlet form on $L^2(\mathcal{X})$.

**Step 3: obtain a contradiction.** Since $\{p_t\}_{t>0}$ satisfies $(\mathrm{ULE})_{d_w}$, we have by (4.18) that

$$q_t(x,y) \lesssim p_t(x,y) \quad \text{for all } t > 0 \text{ and } x, y \in \mathcal{X}.$$

Consequently, by (4.16)-(4.17) and (4.19)-(4.20), we obtain that

$$\mathcal{E}^{(1)}(u,u) \le C\mathcal{E}^{(2)}(u,u) \quad \text{for all } u \in L^2(\mathcal{X}), \tag{4.21}$$

for some constant $C > 0$, which implies

$$\mathrm{Dom}(\mathcal{E}^{(2)}) \subseteq \mathrm{Dom}(\mathcal{E}^{(1)}).$$

Moreover, by (4.21) and the arguments following (3.12) in the proof of Lemma 3.6, we obtain that

$$\mathcal{E}^{(1)}(u,u) = 0 \quad \text{for all } u \in \mathrm{Dom}(\mathcal{E}^{(2)}) \cap L^\infty(\mathcal{X}).$$

Fix $u \in \mathrm{Dom}(\mathcal{E}^{(2)}) \cap L^\infty(\mathcal{X})$. Since

$$\mathcal{E}^{(1)}(u,u) \ge \frac{1}{2t} \iint_{\mathcal{X}\times\mathcal{X}} |u(x) - u(y)| q_t(x,y)\, d\mu(x)\, d\mu(y)$$

for all $t > 0$, it follows from the lower estimate of $(\mathrm{SG})_{d_w}$ that $|u(x) - u(y)| = 0$ for $\mu \times \mu$-a.a. $(x,y) \in \mathcal{X} \times \mathcal{X}$, which implies that $u$ is an almost everywhere constant function on $\mathcal{X}$. Thus,

$$\mathrm{Dom}(\mathcal{E}^{(2)}) \cap L^\infty(\mathcal{X}) = \{\text{constant functions}\},$$

which is not possible since $\mathrm{Dom}(\mathcal{E}^{(2)}) \cap L^\infty(\mathcal{X})$ is dense in $L^2(\mathcal{X})$ by the fact that $\mathrm{Dom}(\mathcal{E}^{(2)})$ is the domain of the regular Dirichlet form $(\mathcal{E}^{(2)}, \mathrm{Dom}(\mathcal{E}^{(2)}))$.

Therefore, it is not possible to have $d_w < \beta^\#$. Combining this and (4.14) yields $d_w = \beta^\#$, as desired. $\square$

Next, we show the second identity in (1.14) of Theorem 1.5, by using the idea in [23, Section 5.1].

*Proof of $d_w = \beta^*$ in Theorem 1.5.* For simplicity, we rewrite the regular Dirichlet form $(\mathcal{E}^{(1)}, \mathrm{Dom}(\mathcal{E}^{(1)}))$ defined in (4.16)-(4.17) as $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$. For any $u \in L^2(\mathcal{X})$ and $t > 0$, set

$$\mathcal{E}_{(t)}(u,u) := \frac{1}{2t} \int_{\mathcal{X}} \int_{\mathcal{X}} q_t(x,y) |u(x) - u(y)|^2 \, d\mu(x) \, d\mu(y). \tag{4.22}$$

We first prove that $d_w \le \beta^*$. For $r > 0$, let $t := r^{d_w}$. For any $u \in \mathrm{Dom}(\mathcal{E})$, by the lower bound of $q_t$ in $(\mathrm{SG})_{d_w}$, (VD) and (4.22), we obtain

$$\begin{aligned}
\int_{\mathcal{X}} \left( \frac{1}{V(x,r)} \int_{B(x,r)} \frac{|u(x) - u(y)|^2}{r^{d_w}} \, d\mu(y) \right) d\mu(x) &= \frac{1}{t} \int_{\mathcal{X}} \int_{B(x,t^{1/d_w})} \frac{|u(x) - u(y)|^2}{V(x, t^{1/d_w})} \, d\mu(y) \, d\mu(x) \\
&\lesssim \frac{1}{2t} \int_{\mathcal{X}} \int_{B(x,t^{1/d_w})} q_t(x,y) |u(x) - u(y)|^2 \, d\mu(x) \, d\mu(y) \\
&\lesssim \mathcal{E}_{(t)}(u,u).
\end{aligned}$$

Since $\mathcal{E}_{(t)}(u,u) \uparrow \mathcal{E}(u,u)$ as $t \downarrow 0$ (see [23]), this last formula shows that for all $u \in \mathrm{Dom}(\mathcal{E})$,

$$\|u\|^2_{\Lambda^{d_w/2}_{2,\infty}(\mathcal{X})} \lesssim \mathcal{E}(u,u).$$

Hence, we have $\mathrm{Dom}(\mathcal{E}) \subseteq \Lambda^{d_w/2}_{2,\infty}(\mathcal{X})$. Because $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is a regular Dirichlet form, we see that $\mathrm{Dom}(\mathcal{E})$ is dense in $L^2(\mathcal{X})$, so does the Besov space $\Lambda^{d_w/2}_{2,\infty}(\mathcal{X})$. This leads to $d_w \le \beta^*$.

To obtain the opposite inequality $d_w \ge \beta^*$, it suffices to prove that when $\beta > d_w$ the Besov space $\Lambda^{\beta/2}_{2,\infty}(\mathcal{X})$ is not dense in $L^2(\mathcal{X})$. We will prove this by verifying that if $u \in \Lambda^{\beta/2}_{2,\infty}(\mathcal{X})$ then $\mathcal{E}(u,u) = 0$. To this end, for any $t > 0$ and $r > 0$, we decompose $\mathcal{E}_{(t)}(u,u)$ as follows:

$$\begin{aligned}
\mathcal{E}_{(t)}(u,u) &= \frac{1}{2t} \int_{\mathcal{X}} \int_{\mathcal{X}} q_t(x,y) |u(x) - u(y)|^2 \, d\mu(x) \, d\mu(y) \\
&= \frac{1}{2t} \int_{\mathcal{X}} \int_{B(x,r)^\complement} q_t(x,y) |u(x) - u(y)|^2 \, d\mu(x) \, d\mu(y) \\
&\quad + \frac{1}{2t} \int_{\mathcal{X}} \int_{B(x,r)} q_t(x,y) |u(x) - u(y)|^2 \, d\mu(x) \, d\mu(y) \\
&=: A(t) + B(t).
\end{aligned} \tag{4.23}$$

By the upper bound of $q_t(x,y)$ in $(\mathrm{SG})_{d_w}$ and (VD), we have that for any $x \in \mathcal{X}$, $t > 0$ and $r > 0$,

$$\begin{aligned}
\int_{B(x,r)^\complement} q_t(x,y) \, d\mu(y) &\le \sum_{n=0}^{\infty} \int_{B(x,2^{n+1}r) \setminus B(x,2^n r)} \frac{C}{V(x, t^{1/d_w} + d(x,y))} \exp\left( -c \left( \frac{d(x,y)}{t^{1/d_w}} \right)^{\frac{d_w}{d_w - 1}} \right) d\mu(y) \\
&\le \sum_{n=0}^{\infty} \int_{B(x,2^{n+1}r) \setminus B(x,2^n r)} \frac{C}{V(x, 2^n r)} \exp\left( -c \left( \frac{2^n r}{t^{1/d_w}} \right)^{\frac{d_w}{d_w - 1}} \right) d\mu(y) \\
&\le C \sum_{n=0}^{\infty} \frac{V(x, 2^{n+1} r)}{V(x, 2^n r)} \exp\left( -c \left( \frac{2^n r}{t^{1/d_w}} \right)^{\frac{d_w}{d_w - 1}} \right)
\end{aligned}$$

$$\lesssim \sum_{n=0}^{\infty} \frac{t^2}{2^{2nd_w} r^{2d_w}},$$

which implies

$$\begin{aligned}
A(t) &\le \frac{1}{t} \int_{\mathcal{X}} \int_{B(x,\,r)^{\complement}} (u(x)^2 + u(y)^2) q_t(x,y)\, d\mu(y)\, d\mu(x) \\
&\le \frac{2}{t} \int_{\mathcal{X}} u(x)^2 \left( \sup_{z\in\mathcal{X}} \int_{B(z,\,r)^{\complement}} q_t(z,y)\, d\mu(y) \right) d\mu(x) \\
&\lesssim \frac{t}{r^{2d_w}} \|u\|^2_{L^2(\mathcal{X})}.
\end{aligned}$$

Similarly, by the upper bound of $q_t(x,y)$ in $(\mathrm{SG})_{d_w}$ and (VD), we have that for any $t>0$ and $r>0$,

$$\begin{aligned}
B(t) &= \frac{1}{2t} \sum_{n=0}^{\infty} \int_{\mathcal{X}} \int_{B(x,\,2^{-n}r)\setminus B(x,\,2^{-n-1}r)} q_t(x,y) |u(x)-u(y)|^2\, d\mu(x)\, d\mu(y) \\
&\le \frac{C}{2t} \sum_{n=0}^{\infty} \exp\left( -c \left( \frac{2^{-n-1}r}{t^{1/d_w}} \right)^{\frac{d_w}{d_w-1}} \right) \int_{\mathcal{X}} \frac{1}{V(x,2^{-n-1}r)} \int_{B(x,\,2^{-n}r)} |u(x)-u(y)|^2\, d\mu(x)\, d\mu(y) \\
&\lesssim \sum_{n=0}^{\infty} \frac{(2^{-n}r)^{\beta}}{t} \exp\left( -c \left( \frac{2^{-n-1}r}{t^{1/d_w}} \right)^{\frac{d_w}{d_w-1}} \right) \int_{\mathcal{X}} \frac{1}{V(x,2^{-n}r)} \int_{B(x,\,2^{-n}r)} \frac{|u(x)-u(y)|^2}{(2^{-n}r)^{\beta}}\, d\mu(x)\, d\mu(y) \\
&\lesssim \sum_{n=0}^{\infty} \frac{(2^{-n}r)^{\beta}}{t} \exp\left( -c \left( \frac{2^{-n-1}r}{t^{1/d_w}} \right)^{\frac{d_w}{d_w-1}} \right) \|u\|^2_{\dot{\Lambda}^{\beta/2}_{2,\infty}(\mathcal{X})} \\
&\lesssim \sum_{n=0}^{\infty} \frac{(2^{-n}r)^{\beta}}{t} \left( \frac{t^{1/d_w}}{2^{-n-1}r} \right)^{(\beta+d_w)/2} \|u\|^2_{\dot{\Lambda}^{\beta/2}_{2,\infty}(\mathcal{X})} \\
&\lesssim \sum_{n=0}^{\infty} 2^{-\frac{n(\beta-d_w)}{2}} r^{\frac{\beta-d_w}{2}} t^{\frac{\beta-d_w}{2d_w}} \|u\|^2_{\dot{\Lambda}^{\beta/2}_{2,\infty}(\mathcal{X})} \\
&\lesssim r^{\frac{\beta-d_w}{2}} t^{\frac{\beta-d_w}{2d_w}} \|u\|^2_{\dot{\Lambda}^{\beta/2}_{2,\infty}(\mathcal{X})},
\end{aligned}$$

where in the last step we used the assumption $\beta > d_w$. Combining (4.23) and the estimates of $A(t)$ and $B(t)$, we obtain that for any $t>0$ and $r>0$,

$$\mathcal{E}_{(t)}(u,u) \lesssim \frac{t}{r^{2d_w}} \|u\|^2_{L^2(\mathcal{X})} + r^{\frac{\beta-d_w}{2}} t^{\frac{\beta-d_w}{2d_w}} \|u\|^2_{\dot{\Lambda}^{\beta/2}_{2,\infty}(\mathcal{X})}.$$

From this and the fact that $\mathcal{E}_{(t)}(u,u) \uparrow \mathcal{E}(u,u)$ as $t \downarrow 0$ (see [23]), it follows that if $u \in \Lambda^{\beta/2}_{2,\infty}(\mathcal{X})$ then

$$\mathcal{E}(u,u) = \lim_{t\to 0} \mathcal{E}_{(t)}(u,u) = 0,$$

thereby leading to that, for any $t>0$,

$$0 = \mathcal{E}(u,u) \ge \mathcal{E}_{(t)}(u,u) = \frac{1}{2t} \int_{\mathcal{X}} \int_{\mathcal{X}} q_t(x,y) |u(x)-u(y)|^2\, d\mu(x)\, d\mu(y) \ge 0.$$

Since $q_t(x, y) > 0$ by the lower bound of $q_t$ in $(\mathrm{SG})_{d_w}$, we deduce that $|u(x) - u(y)| = 0$ for $\mu\times\mu$-a.a. $(x, y) \in \mathcal{X} \times \mathcal{X}$. In other words, $u$ equals to a constant almost everywhere on $\mathcal{X}$. This gives

$$\Lambda^{\beta/2}_{2,\infty}(\mathcal{X}) = \{\text{constant functions}\}.$$

Thus, $\Lambda^{\beta/2}_{2,\infty}(\mathcal{X})$ can not be dense in $L^2(\mathcal{X})$. Hence, we have $\beta \geq \beta^*$ for all $\beta > d_w$, which implies

$$d_w \geq \beta^*.$$

The proof of $d_w = \beta^*$ is completed. □

# 5 Heat kernel construction by means of an ultra-metric

The main aim of this subsection is to show Theorems 1.6 and 1.7. To this end, we use a system $\mathcal{D}$ of dyadic cubes on $\mathcal{X}$ (see Subsection 5.1) and the family of adjacent dyadic cubes (see Subsection 5.4). The key point is that $\mathcal{D}$ induces an ultra-metric $d_{\mathcal{D}}$ on $\mathcal{X}$ (see Definition 5.3 in Subsection 5.2 below). Next, we apply the method of [10] of heat kernel construction on general ultra-metric spaces to prove Theorem 1.6 in Subsection 5.3. Applying Theorem 1.6, we show Theorem 1.7 in Subsection 5.5.

## 5.1 Dyadic cubes and quadrants

In the Euclidean space $\mathbb{R}^n$ there is a standard system of dyadic cubes

$$\mathcal{D} := \left\{2^{-k}\left([0, 1)^n + \alpha\right) : \ k \in \mathbb{Z}, \ \alpha \in \mathbb{Z}^n\right\}$$

that possesses the following properties: (i) all the cubes with the same side-length form a partition of $\mathbb{R}^n$; (ii) any two different cubes are either disjoint or one of them is contained in another. Construction of an analogous system of dyadic cubes on metric measure spaces satisfying (VD) was done in a seminal work of Christ [17]. Christ's construction was improved in [4, 33, 34, 35], in a very general setting of the so-called *geometric doubling* metric spaces.

In the next theorem we collect necessary for us properties of the dyadic cubes on metric measure spaces (see e.g. [33, Theorem 2.2]).

**Theorem 5.1** ([33])**.** *Let $(\mathcal{X}, d, \mu)$ be a metric measure space satisfying (VD). Fix some positive constants $c_0 \leq C_0$ and $\delta < 1$ such that $12C_0\delta \leq c_0$. Then there exists a family*

$$\{Q^k_\alpha : \ k \in \mathbb{Z}, \ \alpha \in \mathcal{A}_k\}$$

*of Borel subsets of $\mathcal{X}$ that are called dyadic cubes (where $\mathcal{A}_k$ is at most countable index set), which satisfies the following properties:*

(i) *for any $k \in \mathbb{Z}$, the dyadic cubes $\{Q^k_\alpha\}_{\alpha\in\mathcal{A}_k}$ are disjoint, and $\mathcal{X} = \bigcup_{\alpha\in\mathcal{A}_k} Q^k_\alpha$;*

(ii) *any two dyadic cubes $Q^k_\alpha$ and $Q^j_\beta$ with $j \leq k$ are either disjoint or $Q^k_\alpha \subseteq Q^j_\beta$;*

(iii) *for any $k \in \mathbb{Z}$ and $\alpha \in \mathcal{A}_k$, there exists $z_\alpha^k \in Q_\alpha^k$ such that*

$$B(z_\alpha^k, c_1\delta^k) \subseteq Q_\alpha^k \subseteq B(z_\alpha^k, C_1\delta^k),$$

*where $c_1 = \frac{1}{3}c_0$ and $C_1 = 2C_0$.*

If $Q_\beta^{k+1} \subseteq Q_\alpha^k$, then $Q_\beta^{k+1}$ is called a *child* of $Q_\alpha^k$, while $Q_\alpha^k$ is called the *parent* of $Q_\beta^{k+1}$ (see Figure 3). Each dyadic cube has finitely many children (which follows from (iii) and (VD)) and at most one parent (which follows from (i) and (ii)). It also follows from (i) and (ii) that, for any cube $Q_\alpha^k$ and any $j < k$, there exists a unique $\beta \in \mathcal{A}_j$ such that $Q_\alpha^k \subseteq Q_\beta^j$.

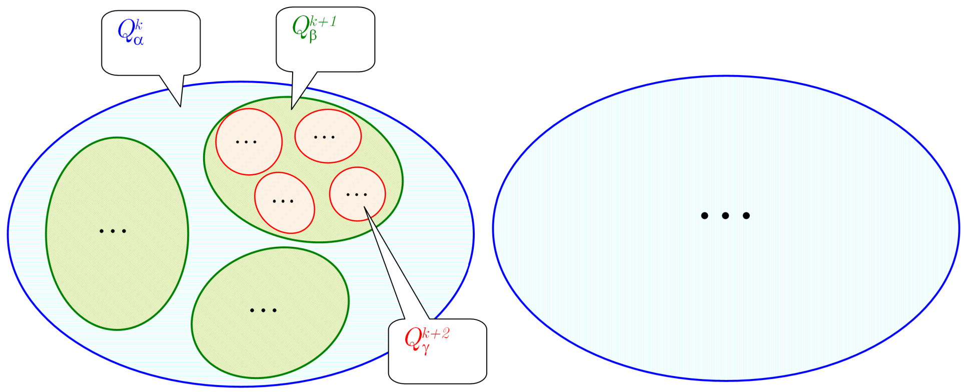


Figure 3: Dyadic cubes

Denote by

$$\mathcal{D} := \{Q_\alpha^k : \ k \in \mathbb{Z}, \ \alpha \in \mathcal{A}_k\}$$

the system of all dyadic cubes on $(\mathcal{X}, d, \mu)$. For any $k \in \mathbb{Z}$, let

$$\mathcal{D}_k := \{Q_\alpha^k : \ \alpha \in \mathcal{A}_k\},$$

that is, $\mathcal{D}_k$ is the set of all dyadic cubes of $k$-th generation. The set $\mathcal{D}$ has a natural structure of a directed tree, where the vertices are the dyadic cubes, and arrows go from parents to children. The sets $\mathcal{D}_k$ are naturally identified as levels of this tree (see Figure 4).

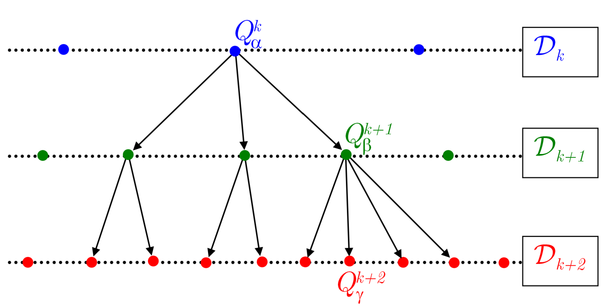


Figure 4: The tree structure on $\mathcal{D}$

The system of dyadic cubes $\mathcal{D}$ gives rise to the notion of a *quadrant* on $\mathcal{X}$: for any $Q \in \mathcal{D}$, the set

$$C(Q) := \bigcup_{\substack{Q' \in \mathcal{D} \\ Q' \supseteq Q}} Q' \tag{5.1}$$

is called a *quadrant* of $\mathcal{X}$ containing $Q$. In other words, $C(Q)$ is the union of all ancestors of $Q$. According to [1, Lemma 2.2], the quadrants have the following properties.

**Lemma 5.2** ([1]). *Let $(\mathcal{X}, d, \mu)$ be a metric measure space satisfying (VD). Suppose that $\mathcal{D}$ is a dyadic system as in Theorem 5.1. Then the family of quadrants defined in (5.1) satisfies the following properties:*

(i) *for each quadrant $C$, the triple $(C, d, \mu)$ satisfies (VD);*

(ii) *any two intersecting quadrants coincide;*

(iii) *$\mathcal{X}$ is a disjoint union of finitely many quadrants*;

(iv) *if $\mu(\mathcal{X}) < \infty$ then $\mathcal{X}$ coincides with one quadrant $C$, where $C$ coincides with some dyadic cube $Q \in \mathcal{D}$;*

(v) *if $\mu(\mathcal{X}) = \infty$ then for every quadrant $C$ we have $\mu(C) = \infty$.*

If two points $x, y$ belong to the same dyadic cube then, clearly, they belong to one quadrant. The converse is also true: if $x, y$ belong to the same quadrant $C$ then there is a dyadic cube containing both $x, y$. Indeed, if $x, y \in C$ then $x \in Q'$ and $y \in Q''$ for some dyadic cubes $Q', Q''$ from $C$ (cf. (5.1)). Since by definition of a quadrant $Q'$ and $Q''$ have a non-empty intersection then one of them contains the other, whence the claim follows.

## 5.2 An ultra-metric induced by the dyadic structure

Recall that a metric $\rho$ on $\mathcal{X}$ is called an ultra-metric (see also (3.4)) if it satisfies the following stronger version of the triangle inequality: for any $x, y, z \in \mathcal{X}$,

$$\rho(x, y) \leq \max\{\rho(x, z), \rho(z, y)\}. \tag{5.2}$$

Of course, any ultra-metric is a metric. Usually a metric must take non-negative real values, but we will allow an ultra-metric to take also the value $+\infty$.

The dyadic system $\mathcal{D}$ on $(\mathcal{X}, d, \mu)$ that is stated in Theorem 5.1 determines naturally an ultra-metric $d_{\mathcal{D}}$ on $\mathcal{X}$ as follows. For any two distinct points $x, y \in \mathcal{X}$ that belong to the same quadrant, denote by $Q_{x,y}$ the smallest dyadic cube from $\mathcal{D}$ containing both $x$ and $y$; then denote by $k_{x,y}$ the unique integer $k$ such that $Q_{x,y} \in \mathcal{D}_k$.

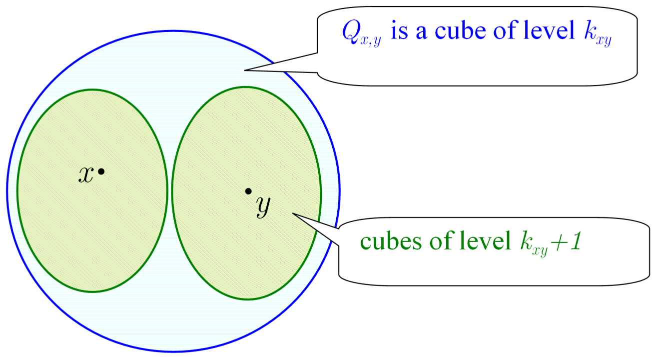


Figure 5: Cube $Q_{x,y}$

If $x, y$ do not belong to the same quadrant then we set $k_{x,y} = -\infty$. Finally, if $x = y$ then we set $k_{x,y} = +\infty$.

**Definition 5.3.** For any $x, y \in \mathcal{X}$, set

$$d_{\mathcal{D}}(x, y) := \delta^{k_{x,y}}. \tag{5.3}$$

In particular, if $x = y$ then $d_{\mathcal{D}}(x, x) = 0$, and if $x, y$ do not belong to one quadrant then $d_{\mathcal{D}}(x, y) = \infty$.

**Lemma 5.4.** *$d_{\mathcal{D}}$ is an ultra-metric on $\mathcal{X}$.*

*Proof.* It is easy to see that $d_{\mathcal{D}}$ is non-negative and symmetric. Let us verify that $d_{\mathcal{D}}$ satisfies (5.2). By (5.3) it suffices to prove that

$$k_{x,y} \geq \min(k_{x,z}, k_{y,z}). \tag{5.4}$$

If either $k_{x,z} = -\infty$ or $k_{y,z} = -\infty$, then (5.2) is trivially satisfied. Otherwise each of the pairs $x, y$ and $y, z$ belongs to one quadrant, that is, all the points $x, y, z$ belong to the same quadrant, which will be assumed in the sequel.

The cubes $Q_{y,z}$ and $Q_{x,z}$ contain the point $z$, that is, they have non-empty intersection. By Theorem 5.1, one of the term contains the other, for example, let $Q_{y,z} \subseteq Q_{x,z}$, so that $k_{y,z} \geq k_{x,z}$. Then $y$ belong to $Q_{x,z}$, which implies $Q_{x,y} \subseteq Q_{x,z}$ and, hence, $k_{x,y} \geq k_{x,z}$, which proves (5.4). □

**Remark 5.5.** Let us discuss some properties of the ultra-metric $d_{\mathcal{D}}$.

(i) It follows from Theorem 5.1(iii) and the definition (5.3) of $d_{\mathcal{D}}$ that

$$d(x, y) \lesssim d_{\mathcal{D}}(x, y), \tag{5.5}$$

because $Q_{x,y} \subseteq B(z, C_1\delta^{k_{x,y}})$ and, hence, $d(x, y) \leq 2C_1\delta^{k_{x,y}}$, while $d_{\mathcal{D}}(x, y) = \delta^{k_{x,y}}$. In general, the converse inequality may fail because that it can happen that two points in different quadrants are very close to each other so that $d(x, y)$ is very small but $d_{\mathcal{D}}(x, y) = \infty$.

(ii) For any $x \in \mathcal{X}$ and $r \in (0, \infty)$, let

$$\overline{B}_{\mathcal{D}}(x, r) := \{y \in \mathcal{X} : \ d_{\mathcal{D}}(y, x) \leq r\} \tag{5.6}$$

be a closed metric ball of $d_{\mathcal{D}}$. Choose $k \in \mathbb{Z}$ to satisfy $\delta^k \leq r < \delta^{k-1}$ and let $Q_{x,r} \in \mathcal{D}_k$ be the unique dyadic cube of $k$-th generation that contains $x$. By (5.3), we obtain that

$$y \in \overline{B}_{\mathcal{D}}(x, r) \quad \Leftrightarrow \quad \delta^{k_{x,y}} \leq r \quad \Leftrightarrow \quad k_{x,y} \geq k \quad \Leftrightarrow \quad y \in Q_{x,r}$$

that is,

$$\overline{B}_{\mathcal{D}}(x, r) = Q_{x,r}. \tag{5.7}$$

Hence, the dyadic cubes coincide with closed ultra-metric balls. Besides, applying Theorem 5.1(iii) and the doubling property (VD), we obtain from (5.7) that

$$\mu(\overline{B}_{\mathcal{D}}(x, r)) = \mu(Q_{x,r}) \simeq \mu(B(z_\alpha^k, \delta^k)) \simeq \mu(B(x, r)) = V(x, r). \tag{5.8}$$

(iii) Suppose that the metric $d$ itself is an ultra-metric. Then, by the property of ultra-metric, any two balls of the same radius are either disjoint or coincide. Hence, for each $k \in \mathbb{Z}$, there exists a family of balls $\{B(z_\alpha^k, \delta^k)\}_{\alpha \in \mathscr{A}_k}$ which forms a partition of $\mathcal{X}$, where $\mathscr{A}_k$ is an at most countable index set. It is easy to check that the family

$$\mathscr{D} := \left\{Q_\alpha^k := B(z_\alpha^k, \delta^k) : \ k \in \mathbb{Z}, \ \alpha \in \mathscr{A}_k\right\}$$

satisfies all the claims of Theorem 5.1. Thus, $\mathscr{D}$ forms a dyadic system on $(\mathcal{X}, d, \mu)$, which further generates an ultra-metric $d_{\mathscr{D}}$ by (5.3). It is easy to see that in this case $\delta d_{\mathscr{D}} \leq d \leq d_{\mathscr{D}}$.

## 5.3 Heat kernel generated by the ultra-metric $d_{\mathcal{D}}$

In this subsection we prove Theorem 1.6, which, in fact, is covered by the following theorem.

**Theorem 5.6.** *Let $(\mathcal{X}, d, \mu)$ be a metric measure space satisfying (VD). Let $d_{\mathcal{D}}$ be the ultra-metric generated by a dyadic system $\mathcal{D}$ on $(\mathcal{X}, d, \mu)$ as in (5.3). Then, for any $\beta \in (0, \infty)$, there exists a stochastically complete heat kernel $\{p_t^{\mathcal{D}}\}_{t>0}$ on $\mathcal{X}$ such that, for all $t \in (0, \infty)$ and $x, y \in \mathcal{X}$,*

$$p_t^{\mathcal{D}}(x, y) \simeq \frac{1}{V(x, \ t^{1/\beta} + d_{\mathcal{D}}(x, y)))} \left(1 + \frac{d_{\mathcal{D}}(x, y)}{t^{1/\beta}}\right)^{-\beta}. \tag{5.9}$$

*Consequently, for all $t \in (0, \infty)$ and $x, y \in \mathcal{X}$,*

$$0 \leq p_t^{\mathcal{D}}(x, y) \leq \frac{C}{V(x, \ t^{1/\beta} + d(x, y)))} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-\beta} \tag{5.10}$$

*and*

$$p_t^{\mathcal{D}}(x, x) \simeq \frac{1}{V(x, \ t^{1/\beta})}. \tag{5.11}$$

Let us emphasize that the volume function $V$ in (5.9)-(5.10)-(5.11) is taken with respect to the original metric $d$, that is, as in (1.1). Since $d \lesssim d_{\mathcal{D}}$ by (5.5), the estimates (5.10) and (5.11) are easy consequences of (5.9).

*Proof of Theorem 5.6.* Now we prove the existence of a heat kernel satisfying (5.9). For that, we use a construction from [10] of heat kernels on ultra-metric measure spaces. Since the ultra-metric in [10] has to be finite, we can apply the results of [10] in any quadrant $C$, endowed by the metric $d_{\mathcal{D}}$.

Let $\sigma : [0,\infty] \to [0,1]$ be a *distance distribution function*, that is, $\sigma$ is a strictly monotone increasing continuous function such that $\sigma(0) = 0$ and $\sigma(\infty) = 1$. It follows from [10, Theorem 2.2] that the following formula determines in a quadrant $C$ a stochastically complete heat kernel:

$$p_t^{\mathcal{D}}(x,y) := \int_{d_{\mathcal{D}}(x,y)}^{\infty} \frac{d\sigma(r)^t}{\mu(\overline{B}_{\mathcal{D}}(x,r))} \tag{5.12}$$

for all $t \in (0,\infty)$ and $x, y \in C$, where $\overline{B}_{\mathcal{D}}(x,r)$ is a closed ultra-metric ball as defined in (5.6).

In particular, let us take here, for all $r \in (0,\infty)$,

$$\sigma(r) = \exp(-1/r^{\beta}), \tag{5.13}$$

and prove that the resulting heat kernel $\{p_t^{\mathcal{D}}\}_{t>0}$ satisfies (5.9). For that we need the following Lemma 5.7. In addition, we say that a function $V : (0,\infty) \to (0,\infty)$ is *doubling* provided that $V(2r) \leq V(r)$ for all $r \in (0,\infty)$.

**Lemma 5.7.** *Let $V$ and $\Psi$ be positive monotone increasing functions on $(0,\infty)$. Assume that $\Psi$ has the inverse $\Psi^{-1}$ that is also defined on $(0,\infty)$, and let the functions $V$ and $\Psi^{-1}$ be doubling. Then, for all $R, t > 0$, we have*

$$\int_R^{\infty} \frac{d\exp\left(-t/\Psi(r)\right)}{V(r)} \simeq \frac{1}{V(\Psi^{-1}(\tau))}\frac{t}{\tau}, \tag{5.14}$$

*where $\tau := \max\{t, \Psi(R)\}$.*

Assuming Lemma 5.7 for the moment, we apply it to continue the proof of Theorem 5.6. Substituting the value of $\sigma$ from (5.13) into (5.12) and using (5.8), we obtain that

$$p_t^{\mathcal{D}}(x,y) \simeq \int_{d_{\mathcal{D}}(x,y)}^{\infty} \frac{d\exp(-t/r^{\beta})}{V(x,r)}$$

Applying Lemma 5.7 with $\Psi(r) = r^{\beta}$ and $V(r) = V(x,r)$ and observing that $\tau \simeq t + d_{\mathcal{D}}(x,y)^{\beta}$ we obtain

$$p_t^{\mathcal{D}}(x,y) \simeq \frac{1}{V(x,(t + d_{\mathcal{D}}(x,y)^{\beta})^{1/\beta})}\frac{t}{t + d_{\mathcal{D}}(x,y)^{\beta}}, \tag{5.15}$$

which is equivalent to (5.9) provided that $x, y$ in the same quadrant $C$.

Now let us extend this heat kernel to all $x, y \in \mathcal{X}$ by setting

$$p_t^{\mathcal{D}}(x,y) := 0$$

provided that $x$ and $y$ are not in the same quadrant $C$. If one of the points $x, y$ lies in $C$ and the other is not in $C$ then $d_{\mathcal{D}}(x,y) = \infty$, so that the estimate (5.15) is trivially satisfied.

Finally, we construct $p_t^{\mathcal{D}}(x,y)$ for all points $x,y \in \mathcal{X}$. The resulting function is a stochastically complete heat kernel that satisfies (5.15) for all $x,y \in \mathcal{X}$, which completes the proof of Theorem 5.6. □

*Proof of Lemma 5.7.* Let us first make change $s = \Psi(r)$ in the integral and obtain

$$\int_R^\infty \frac{d\exp(-t/\Psi(r))}{V(r)} = \int_{\Psi(R)}^\infty \frac{d\exp(-t/s)}{V(\Psi^{-1}(s))} = \int_T^\infty \frac{t\exp(-t/s)}{sF(s)}ds,$$

where

$$T = \Psi(R) \ \text{ and } \ F(s) = sV(\Psi^{-1}(s)).$$

Observe also that $\tau = \max\{t, T\}$ and

$$\frac{1}{V(\Psi^{-1}(\tau))}\frac{t}{\tau} = \frac{t}{F(\tau)}.$$

Hence, it suffices to prove that

$$\int_T^\infty \frac{\exp(-t/s)}{sF(s)}ds \simeq \frac{1}{F(\tau)}. \tag{5.16}$$

The lower bound here is easy: using that $F(s)$ is monotone increasing and doubling, we obtain

$$\int_T^\infty \frac{\exp(-t/s)}{sF(s)}ds \geq \int_\tau^{2\tau} \frac{\exp(-t/s)}{sF(s)}ds \geq \frac{\exp(-t/\tau)}{2\tau F(2\tau)}\tau \simeq \frac{1}{F(\tau)}.$$

To prove the upper bound in (5.16), consider first the case $T \geq \frac{1}{2}\tau$. Using that $F(s)/s$ is monotone increasing, we obtain

$$\begin{aligned}\int_T^\infty \frac{\exp(-t/s)}{sF(s)}ds &\leq \int_{\frac{1}{2}\tau}^\infty \frac{\exp(-t/s)}{sF(s)}ds \leq \int_{\frac{1}{2}\tau}^\infty \frac{ds}{s^2(F(s)/s)} \\ &\leq \frac{1}{F(\frac{1}{2}\tau)/(\frac{1}{2}\tau)}\int_{\frac{1}{2}\tau}^\infty \frac{ds}{s^2} \simeq \frac{1}{F(\tau)}.\end{aligned} \tag{5.17}$$

Let $T < \frac{1}{2}\tau$. Then $t = \tau$ and we obtain

$$\int_T^\infty \frac{\exp(-t/s)}{sF(s)}ds \leq \int_0^\infty \frac{\exp(-t/s)}{sF(s)}ds \leq \left(\int_0^t + \int_{\frac{1}{2}\tau}^\infty\right)\frac{\exp(-t/s)}{sF(s)}ds.$$

The second integral here is estimated as in (5.17). In the first integral we make a change $\frac{s}{t} = \xi$ and obtain

$$\int_0^t \frac{\exp(-t/s)}{sF(s)}ds = \int_0^1 \frac{\exp(-1/\xi)}{t\xi F(t\xi)}td\xi = \int_0^1 \frac{\exp(-1/\xi)}{\xi F(t)}\frac{F(t)}{F(t\xi)}d\xi.$$

Since by the doubling properties of $V$ and $\Psi^{-1}$,

$$\frac{F(t)}{F(t\xi)} \leq \frac{C}{\xi^N}$$

for some positive constants $C$ and $N$, it follows that

$$\int_0^t \frac{\exp(-t/s)}{sF(s)}ds \leq \frac{C}{F(t)}\int_0^1 \frac{\exp(-1/\xi)}{\xi^{N+1}}d\xi \simeq \frac{1}{F(\tau)},$$

which completes the proof of (5.16) and, hence, (5.14). This ends the proof of Lemma 5.7. □

### 5.4 Adjacent dyadic cubes

Concerning the system of dyadic cubes $\mathcal{D}$ constructed in Theorem 5.1, let us observe that even if two points $x, y \in \mathcal{X}$ are very close to each other, there may exist no dyadic cube containing them both. This is the reason for why there is no stable-like lower estimate (with respect to $d$) of $\{p_t^{\mathcal{D}}\}_{t>0}$ in Theorem 5.6. To overcome this defect, we use the adjacent dyadic systems from [33, Theorem 4.1 and Proposition 4.3] (see Figure 6).

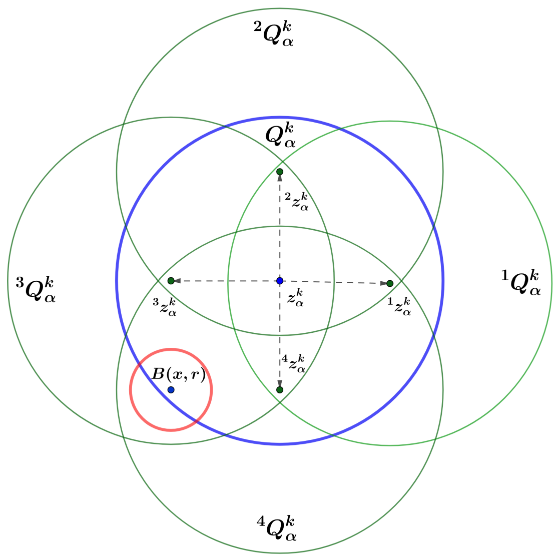


Figure 6: Adjacent dyadic cubes

**Theorem 5.8** ([33])**.** *Let* $(\mathcal{X}, d, \mu)$ *be a metric measure space satisfying* (VD)*. Suppose that* $0 < c_0 \le C_0 < \infty$ *and* $\delta \in (0,1)$ *such that*

$$96C_0\delta \le c_0.$$

*Given a fixed point* $x_o \in \mathcal{X}$*, there exists a finite collection of families* $\{\mathcal{D}^\tau : \tau = 1, 2, \ldots, K\}$*, where* $K \in \mathbb{N}$ *depends only on the doubling constant* $C_D$ *in* (VD)*, and each* $\mathcal{D}^\tau$ *is a collection of dyadic cubes*

$$\mathcal{D}^\tau := \{{}^\tau Q_\alpha^k : k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\}$$

*satisfies the following:*

(i′) *for any* $k \in \mathbb{Z}$*, the dyadic cubes* $\{{}^\tau Q_\alpha^k\}_{\alpha \in \mathcal{A}_k}$ *are disjoint, and* $\mathcal{X} = \bigcup_{\alpha \in \mathcal{A}_k} {}^\tau Q_\alpha^k$*;*

(ii′) *any two dyadic cubes* ${}^\tau Q_\alpha^k$ *and* ${}^\tau Q_\beta^j$ *with* $j \le k$ *are either disjoint or* ${}^\tau Q_\alpha^k \subseteq {}^\tau Q_\beta^j$*;*

(iii′) *for any* $k \in \mathbb{Z}$ *and* $\alpha \in \mathcal{A}_k$*, there exists* ${}^\tau z_\alpha^k \in Q_\alpha^k$ *such that*

$$B\left({}^\tau z_\alpha^k,\, 6^{-1} c_0 \delta^k\right) \subseteq {}^\tau Q_\alpha^k \subseteq B\left({}^\tau z_\alpha^k,\, 4C_0 \delta^k\right).$$

*Moreover, the following hold:*

(a) *for any* $\tau \in \{1, 2, \ldots, K\}$ *and* $k \in \mathbb{Z}$, *there exists an* $\alpha \in \mathcal{A}_k$ *and such that* $x_o = {}^\tau z_\alpha^k$;

(b) *for any ball* $B \subseteq \mathcal{X}$ *of radius* $r \in (0, \infty)$, *there exist some* $\tau \in \{1, 2, \ldots, K\}$ *and* $Q \in \mathcal{D}^\tau$ *such that* $B \subseteq Q$ *and* $\operatorname{diam}(Q) \leq Cr$, *where* $C$ *is a positive constant depending only on* $c_0, C_0$, $\delta$ *and* $C_D$.

To have a better understanding of the adjacent dyadic systems on $\mathcal{X}$, let us take a look at the case when $\mathcal{X}$ is the Euclidean space $\mathbb{R}^n$. Then, for the standard dyadic system

$$\mathcal{D} := \left\{2^{-k}\left([0, 1)^n + \alpha\right) : \ k \in \mathbb{Z}, \ \alpha \in \mathbb{Z}^n\right\},$$

its $3^n$-translations form the adjacent dyadic systems

$$\left\{\mathcal{D}^\tau : \ \tau \in \left\{0, \ \frac{1}{3}, -\frac{1}{3}\right\}^n\right\},$$

where each $\mathcal{D}^\tau$ is defined by

$$\mathcal{D}^\tau := \left\{2^{-k}\left([0, 1)^n + \alpha + \tau\right) : \ k \in \mathbb{Z}, \ \alpha \in \mathbb{Z}^n\right\}.$$

One may easily verify that all properties in Theorem 5.8 are satisfied.

## 5.5 The sum of adjacent heat kernels

In this subsection we prove Theorem 1.7. For that, we refine the heat kernel construction in Subsection 5.3 to obtain a family of heat kernels with the required lower bound for their sum. The main method is to "shift" the dyadic system and get a family of adjacent dyadic systems

$$\{\mathcal{D}^\tau : \ \tau \in \{1, 2, \ldots, K\}\},$$

where $K$ is a constant determined by $(\mathcal{X}, d, \mu)$ and each $\mathcal{D}^\tau$ is a collection of dyadic systems on $\mathcal{X}$. In this way, every ball of a metric measure space is contained in a dyadic cube of comparable size from one of the adjacent dyadic cube systems.

By the result from Subsection 5.3, we obtain an adjacent family of stochastically complete heat kernels

$$\left\{\{p_t^{\mathcal{D}^\tau}\}_{t>0} : \tau = 1, 2, \ldots, K\right\}, \tag{5.18}$$

where each $\{p_t^{\mathcal{D}^\tau}\}_{t>0}$ is the heat kernel associated with the dyadic structure $\mathcal{D}^\tau$ as in Theorem 5.6. The next theorem shows that the sum of adjacent heat kernels $\sum_{\tau=1}^K p_t^{\mathcal{D}^\tau}$ satisfies the desired two-sided stable-like estimate and, hence, contains Theorem 1.7.

**Theorem 5.9.** *Let* $(\mathcal{X}, d, \mu)$ *be a metric measure space satisfying* (VD). *For any* $\beta \in (0, \infty)$, *there is a family* (5.18) *of stochastically complete heat kernels such that*

$$\sum_{\tau=1}^K p_t^{\mathcal{D}^\tau}(x, y) \simeq \frac{C}{V(x, \ t^{1/\beta} + d(x, y))}\left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-\beta} \tag{5.19}$$

*for all* $t \in (0, \infty)$ *and* $x, y \in \mathcal{X}$.

*Proof.* Fix a parameter $\beta \in (0,\infty)$. For any $\tau \in \{1,2,\dots,K\}$, let $\mathcal{D}^\tau$ be the collection of adjacent dyadic cubes as in Theorem 5.8. For any $k \in \mathbb{Z}$, the $k$-th generation $\mathcal{D}_k^\tau$ is defined by

$$\mathcal{D}_k^\tau := \{{}^\tau Q_\alpha^k : \alpha \in \mathcal{A}_k\}.$$

For each dyadic system $\mathcal{D}^\tau$, we have by Theorems 5.6 that there is a stochastically complete heat kernel $\{p_t^{\mathcal{D}^\tau}\}_{t>0}$ satisfying that for all $t \in (0,\infty)$ and $x, y \in \mathcal{X}$,

$$p_t^{\mathcal{D}^\tau}(x,y) \simeq \frac{1}{V(x,\, t^{1/\beta} + d_{\mathcal{D}^\tau}(x,y))}\left(1 + \frac{d_{\mathcal{D}^\tau}(x,y)}{t^{1/\beta}}\right)^{-\beta}, \tag{5.20}$$

where $d_{\mathcal{D}^\tau}$ is the ultra-metric induced by $\mathcal{D}^\tau$ in the same way as in Definition 5.3.

Now let us prove the required estimate (5.19) for the sum of these heat kernels. The upper bound in (5.19) follows from (5.10) in Theorem 5.6. Let us prove the lower bound in (5.19). Given any two points $x, y \in \mathcal{X}$, by Theorem 5.8(b), there exist some $\tau_0 \in \{1,2,\dots,K\}$ and a dyadic cube $Q_{x,y} \in \mathcal{D}^{\tau_0}$ such that $\{x,y\} \subseteq Q_{x,y}$ and $\operatorname{diam}(Q_{x,y}) \lesssim d(x,y)$. By this and Definition 5.3, we have

$$d_{\mathcal{D}^{\tau_0}}(x,y) \lesssim \operatorname{diam}(Q_{x,y}) \lesssim d(x,y).$$

For such $\tau_0$, we again apply (5.20) and (VD) to derive

$$\begin{aligned}\sum_{\tau=1}^K p_t^{\mathcal{D}^\tau}(x,y) &\geq {}^{\tau_0}p_t(x,y)\\ &\simeq \frac{1}{V(x,\, t^{1/\beta} + d_{\mathcal{D}^{\tau_0}}(x,y))}\left(1 + \frac{d_{\mathcal{D}^{\tau_0}}(x,y)}{t^{1/\beta}}\right)^{-\beta}\\ &\gtrsim \frac{1}{V(x,\, t^{1/\beta} + d(x,y))}\left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta},\end{aligned}$$

which was to be proved. □

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