Propagation speed of non-linear parabolic equations on Riemannian manifolds

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1 Introduction

We are concerned with an evolution equation

$$\partial_t u = \Delta_p u^q$$  \hspace{1cm} (1.1)

where \(p, q > 0\), \(u(x,t)\) is an unknown non-negative function, and \(\Delta_p\) is the \(p\)-Laplacian:

$$\Delta_p v = \text{div} \left( |\nabla v|^{p-2} \nabla v \right).$$

Equation (1.1) was introduced by L. S. Leibenson in 1945 in order to describe filtration of turbulent compressible fluid through a porous medium. The physical meaning of \(u\) is the \textit{volumetric moisture content}, i.e. the (infinitesimal) fraction of volume of the medium taken by the liquid. Parameter \(p\) characterizes the turbulence of a flow while \(q - 1\) is the index of \textit{polytropy} of the liquid, that determines relation \(PV^{q-1} = \text{const}\) between volume \(V\) and pressure \(P\).

The physically interesting values of \(p\) and \(q\) are as follows: \(\frac{3}{2} \leq p \leq 2\) and \(q \geq 1\). The case \(p = 2\) corresponds to laminar flow (=absence of turbulence). In this case (1.1) becomes a \textit{porous medium} equation \(\partial_t u = \Delta u^q\), if \(q > 1\), and the classical heat equation \(\partial_t u = \Delta u\) if \(q = 1\).

From mathematical point of view, the entire range \(p > 1, q > 0\) is interesting. G.I. Barenblatt constructed in 1952 spherically symmetric self-similar solutions of (1.1) in \(\mathbb{R}^n\) that are nowadays called \textit{Barenblatt solutions}. Let us assume that

$$q(p - 1) > 1.$$  \hspace{1cm} (1.2)

In this case the Barenblatt solution is as follows:

$$u(x,t) = \frac{1}{|x|^{n/\beta}} \left( C - \kappa \left( \frac{|x|}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)^\gamma,$$

where \(C > 0\) is any constant, and

$$\beta = p + n \left[ q(p-1) - 1 \right], \gamma = \frac{p-1}{q(p-1)-1}, \kappa = \frac{q(p-1)-1}{pq} \beta^{-\frac{1}{p-1}}.$$  \hspace{1cm} (1.2)

Parameter \(\beta\) determines the space/time scaling and is analogous to the \textit{walk dimension}.

It is obvious that, for the Barenblatt solution,

$$u(x,t) = 0 \text{ for } |x| > ct^{1/\beta}$$

so that \(u(\cdot, t)\) has a \textit{compact support} for any \(t\). One says that \(u\) has a \textit{finite propagation speed}.

Here are the graphs of function \(x \mapsto u(x,t)\) for different values of \(t\) in the case \(n = 1\).
Assume now that \( q(p - 1) < 1 \).

Then we have \( \gamma, \kappa < 0 \), and the Barenblatt solution is

\[
u(x, t) = \frac{1}{t^{n/\beta}} \left( C + |\kappa| \left( \frac{|x|}{t^{1/\beta}} \right)^{p-1} \right)^{-|\gamma|},\]

that is, it is positive for all \( x, t \).

In the borderline case \( q(p - 1) = 1 \),

the Barenblatt solution is

\[
u(x, t) = \frac{1}{t^{n/p}} \exp \left( -c \left( \frac{|x|}{t^{1/p}} \right)^{\frac{p}{p-1}} \right),\]

where \( c = (p - 1)^2 p^{-\frac{p}{p-1}} \). Hence, if \( q(p - 1) \leq 1 \) then \( u \) has infinite propagation speed.

Of course, if here \( p = 2 \) then \( q = 1 \), and we obtain the fundamental solution of the heat equation \( \partial_t u = \Delta u \):

\[
u(x, t) = \frac{1}{t^{n/2}} \exp \left( -\frac{1}{4} \left( \frac{|x|}{t^{1/2}} \right)^{2} \right),\]

2 Leibenson’s equation on manifolds

Consider on an arbitrary Riemannian manifold the operator

\[
Lv = \Delta_p (v^q)
\]

where

\[ p > 1 \quad \text{and} \quad q > 0, \]

and

\[
\Delta_p u = \text{div} \left( |\nabla u|^{p-2} \nabla u \right).
\]

We will be concerned with the associated evolution equation

\[
\partial_t v = \Delta_p (v^q)
\]

that is called the Leibenson equation. Our aim is to prove the following theorem.

**Theorem 1.** If \( q(p - 1) > 1 \) the any bounded non-negative solution to the Leibenson equation has a finite propagation speed.

The exact meaning of “finite propagation speed” will be explained later on. The proof will also be given later on.
Now we show how to obtain the Barenblatt solutions in $\mathbb{R}^n$. We start with deriving a chain rule for the $p$-Laplacian. Consider on an arbitrary manifold the $p$-Laplacian

$$\Delta_p v = \text{div} \left( |\nabla v|^{p-2} \nabla v \right), \quad (2.1)$$

where $p > 1$. Let us compute $\Delta_p f(u)$ assuming that $f$ is smooth enough and $f \geq 0$ and $f' \leq 0$.

We have $\nabla f(u) = f'(u) \nabla u$ and

$$\Delta_p f(u) = \text{div} \left( |f'(u)|^{p-2} f'(u) \nabla u \right)$$

$$= \text{div} \left( |f'(u)|^{p-2} f' \nabla u \right)$$

$$= -\text{div} \left( |f'(u)|^{p-1} |\nabla u|^{p-2} \nabla u \right)$$

$$= -|f'(u)|^{p-1} \text{div} \left( |\nabla u|^{p-2} \nabla u \right) - \nabla \left( |f'(u)|^{p-1} \right) |\nabla u|^{p-2} \nabla u$$

$$= -|f'(u)|^{p-1} \Delta_p u - \nabla \left( (-f'(u))^{p-1} \right) |\nabla u|^{p-2} \nabla u$$

$$= -|f'(u)|^{p-1} \Delta_p u + (p-1)(-f'(u))^{p-2} f''(u) \nabla u |\nabla u|^{p-2} \nabla u$$

$$= -|f'(u)|^{p-1} \Delta_p u + (p-1)|f'(u)|^{p-2} f''(u) |\nabla u|^p.$$

Hence, we obtain

$$\Delta_p f(u) = -(-f'(u))^{p-1} \Delta_p u + (p-1)(-f'(u))^{p-2} f''(u) |\nabla u|^p. \quad (2.2)$$

### 3 Solutions on models

#### 3.1 Model manifolds

Let $M$ be a model manifold $\mathbb{R}_+ \times \mathbb{S}^{n-1}$ with the polar coordinates $(r, \theta)$ (where $r \in \mathbb{R}_+$ and $\theta \in \mathbb{S}^{n-1}$) and with the Riemannian metric

$$ds^2 = dr^2 + \psi^2(r) \, d\theta^2.$$ 

Here $d\theta^2$ is the standard Riemannian metric on $\mathbb{S}^{n-1}$ and $\psi$ is a smooth positive function on $\mathbb{R}_+$. For example, $\mathbb{R}^n \setminus \{0\}$ can be considered as a model manifold with $\psi(r) = r$.

Denote by $S(r)$ the boundary area function

$$S(r) = \omega_n \psi(r)^{n-1}.$$ 

For example, in $\mathbb{R}^n$ we have $S(r) = \omega_n r^{n-1}$. 

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It is known that the Laplace-Beltrami operator $\Delta$ on $M$ admits the following representation in the polar coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{S'}{S} \frac{\partial}{\partial r} + \frac{1}{\psi^2(r)} \Delta_\theta,$$

where $\Delta_\theta$ is the Laplace-Beltrami operator on $S^{n-1}$. In particular, considering the polar radius $r$ as a function in $M$, we obtain that

$$\Delta r = \frac{S'}{S}. \quad (3.1)$$

For example, in $\mathbb{R}^n$ we have $\Delta r = \frac{n-1}{r}$. Using (3.1) and $|\nabla r| = 1$, we obtain that

$$\Delta_p r = \text{div} \left( |\nabla r|^{p-2} \nabla r \right) = \text{div} (\nabla r) = \Delta r = \frac{S'}{S}.$$

Setting in (2.2) $u = r$, we obtain

$$\Delta_p f (r) = - (f' (r))^{p-1} \Delta_p r + (p - 1) (-f' (r))^{p-2} f'' (r)$$

$$= - (f' (r))^{p-1} \frac{S'}{S} + (p - 1) (-f' (r))^{p-2} f'' (r). \quad (3.2)$$

Note that

$$\left( (-f' (r))^{p-1} S \right)' = (-f' (r))^{p-1} S' - (p - 1) (-f' (r))^{p-2} f'' (r) S.$$

Hence, (3.2) can be rewritten in the form

$$\Delta_p f (r) = - \frac{1}{S} \left( S (-f' (r))^{p-1} \right)'.$$

The parabolic equation $\partial_t u = \Delta_p u$ for a function $u = u (r, t)$ (such that $u \geq 0$ and $\partial_r u \leq 0$) becomes therefore

$$\partial_t u = - \frac{1}{S} \partial_r \left( S (-\partial_r u)^{p-1} \right), \quad (3.3)$$

and the Leibenson equation $\partial_t u = \Delta_p (u^q)$ becomes

$$\partial_t u = - \frac{1}{S} \partial_r \left( S (-\partial_r u^q)^{p-1} \right). \quad (3.4)$$
3.2 Barenblatt solutions

We solve here (3.4) assuming \( p > 1, q > 0 \) and

\[ S(r) = r^{\alpha-1}, \]

where \( \alpha \) is a positive real. In particular, for \( \alpha = n \in \mathbb{N} \), this will give us the Barenblatt solution in \( \mathbb{R}^n \).

The equation (3.4) becomes with this \( S(r) \)

\[ \partial_t u = -\frac{1}{r^{\alpha-1}} \partial_r \left( r^{\alpha-1} (-\partial_r u^q)^{p-1} \right), \tag{3.5} \]

and we look for a solution in the form

\[ u(x,t) = t^a f(rt^b), \]

where \( f \) is a decreasing non-negative function and \( a, b \) are (negative) reals, yet to be determined.

Let us require in addition that the solution \( u(x,t) \) has a constant \( L^1 \)-norm, that is,

\[ \int_M t^a f(rt^b) d\mu = \text{const} < \infty, \]

where \( \mu \) is the Riemannian measure. Computing the integral in the polar coordinates and using

\[ d\mu = \psi(r)^{n-1} dr d\theta = \frac{1}{\omega_n} S(r) dr d\theta, \]

we obtain that

\[ \int_0^\infty t^a f(rt^b) r^{\alpha-1} dr = \text{const} < \infty. \]

A change \( s = rt^b \) in the integral gives

\[ \int_0^\infty t^a f(rt^b) r^{\alpha-1} dr = \int_0^\infty t^a f(s) (st^{-b})^{\alpha-1} t^{-b} ds = t^{a-b\alpha} \int_0^\infty f(s) s^{\alpha-1} ds. \]

Hence, we must have

\[ \int_0^\infty f(s) s^{\alpha-1} ds < \infty \tag{3.6} \]

and

\[ a = \alpha b. \]

Using again the variable \( s = rt^b \), we obtain

\[ \partial_t u = \partial_t \left( t^a f(rt^b) \right) \]
\[ = at^{a-1} f(rt^b) + t^a f'(rt^b) rt^{-b} \]
\[ = b\alpha t^{a-1} f(rt^b) + bt^{a-1} t^{-b} f'(rt^b) \]
\[ = bt^{a-1} (\alpha f(s) + s f'(s)) \]
\[ bt^{a-1}s^{a-1} (s^a f(s))' \]

and

\[
\partial_s u^q = q u^{q-1} \partial_s u \\
= q (t^a f(r t^b))^{q-1} \partial_s (t^a f(r t^b)) \\
= q t^{a(q-1)} f(r t^b)^{q-1} t^{a+b} f'(r t^b) \\
= q t^{aq+b} f(s)^{q-1} f'(s). 
\]

Hence, (3.5) is equivalent to

\[
\frac{bt^{a-1}}{s^{a-1}} (s^a f(s))' = -\frac{1}{r^{a-1}} \partial_r \left( r^{a-1} (-qt^{aq+b} f(s)^{q-1} f'(s))^{p-1} \right) \\
= -\frac{q^{p-1} t^{(aq+b)(p-1)}}{(st-b)^{a-1}} \partial_r \left( (st-b)^{a-1} (-f(s)^{q-1} f'(s))^{p-1} \right) \\
= -\frac{q^{p-1} t^{(aq+b)(p-1)}}{s^{a-1}} t^{b} \partial_s \left( s^{a-1} (-f(s)^{q-1} f'(s))^{p-1} \right). 
\]

We require that the powers of \( t \) in the both sides to match, that is,

\[(aq + b)(p - 1) + b = a - 1,\]

which together with \( a = b \alpha \) yields

\[ [(\alpha q + 1)(p - 1) + 1 - \alpha] b = -1, \]

\[ [\alpha (q (p - 1) - 1) + p] b = -1. \]

Setting

\[ \delta = q(p - 1) - 1 \]

we obtain

\[ (\alpha \delta + p) b = -1 \]

whence

\[ b = -\frac{1}{\alpha \delta + p}. \]

In particular, we see that

\[ b < 0 \Leftrightarrow \delta > -\frac{p}{\alpha} \Leftrightarrow q > \frac{1 - p/\alpha}{p - 1}. \]

In what follows, we always assume that (3.8) is satisfied.

With this choice of \( b \) and \( a = \alpha b \), the powers of \( t \) and \( s \) in (3.7) cancel out, and we obtain an ODE for \( f \):

\[ b (s^a f(s))' = -q^{p-1} \left( s^{a-1} (-f(s)^{q-1} f'(s))^{p-1} \right)'. \]
Hence, we have

\[ bs^\alpha f(s) = -q^{p-1}s^{\alpha-1} (-f(s)^{q-1} f'(s))^{p-1} \]  

(ignoring a constant of integration). Since \( b < 0 \), we obtain

\[ |b| s f = q^{p-1} (-f(q-1))^{p-1}, \]
\[ f(q-1)^{(p-1)-1} (-f')^{p-1} \]
\[ f(q-1)^{p-1} = \frac{|b| s}{pq}, \]
\[ f(q-1)^{-\frac{1}{p-1}} f' = -\left(\frac{|b| s}{pq}\right)^{\frac{1}{p-1}}. \]

Set \( \gamma := q - \frac{1}{p-1} = q(p-1) - 1 = \frac{\delta}{p-1} \) and rewrite the above ODE in the form

\[ f^{\gamma-1} f' = -\frac{|b|^{\frac{1}{p-1}}}{q} s^{\frac{1}{p-1}}. \]  

Assume first that \( \delta \neq 0 \), that is, \( \gamma \neq 0 \). Then (3.10) is equivalent to

\[ (f^{\gamma})' = \gamma f^{\gamma-1} f' = -\gamma \frac{|b|^{\frac{1}{p-1}}}{q} s^{\frac{1}{p-1}}, \]

and integrating it, we obtain

\[ f^{\gamma} = C - \kappa s^{\frac{\gamma}{p-1}} \]

where

\[ \kappa = \frac{p-1}{p} \frac{\gamma}{q} = \frac{\delta}{pq} |b|^{\frac{1}{p-1}}. \]

Hence,

\[ f(s) = \left( C - \kappa s^{\frac{\gamma}{p-1}} \right)^{1/\gamma}, \]

with a positive constant \( C \).

**Case 1.** Let \( \delta > 0 \) that is, \( q(p-1) > 1 \),

(which implies also that \( b < 0 \)).

Then \( \kappa > 0 \) and we see that \( f(s) \) is well defined for \( s \in [0, s_0] \) where \( s_0 \) is determined by

\[ C = \kappa s_0^{\frac{\gamma}{p-1}}. \]

Let us extend \( f(s) \) for all \( s \in [0, \infty) \) by setting \( f(s) = 0 \) for \( s > s_0 \), that is,

\[ f(s) = \left( C - \kappa s^{\frac{\gamma}{p-1}} \right)^{1/\gamma}. \]
Then this function $f$ is a weak solution of the ODE (3.10) in $[0, \infty]$ because $f$ is continuous in $[0, \infty]$ and solves (3.10) in the both intervals $[0, s_0]$ and $[s_0, \infty)$. Consequently, we obtain in this case a (weak) solution of (3.5)

$$u(x, t) = t^{\alpha b} f \left( rt^b \right) = \frac{1}{t^{\alpha/\beta}} \left( C - \kappa \left( \frac{r}{t^{1/\beta}} \right)^{p-1} \right)^{1/\gamma}$$

where

$$\beta = -\frac{1}{b} = \alpha \delta + p > 0.$$  

Clearly, this solution has a finite propagation speed. Note that in this case $\beta > p$.

Case 2. Let $\delta < 0$ that is,

$$q (p - 1) < 1.$$

Since $\kappa < 0$, the solution

$$f (s) = \left( C + |\kappa| s^{p-1} \right)^{-1/\gamma}$$

is defined and positive for all $s \geq 0$. Note that by (3.8)

$$\frac{p}{p - 1} = \frac{p}{(p-1) \left( \frac{1}{p-1} - q \right)} = \frac{p}{1 - q (p-1)} = \frac{p}{-\delta} > \alpha,$$

that is,

$$f (s) \simeq s^{-(\alpha + \varepsilon)} \text{ as } s \to \infty$$

where $\varepsilon > 0$. Since also

$$f(s) \simeq \text{const} \text{ as } s \to 0$$

we obtain the finiteness of the integral (3.6).

We obtain in this case a solution

$$u(x, t) = \frac{1}{t^{\alpha/\beta}} \left( C + |\kappa| \left( \frac{r}{t^{1/\beta}} \right)^{p-1} \right)^{-1/\gamma}$$

that is defined for all $x$ and $t > 0$ and belongs to $L^1 (M)$ for any $t > 0$. Hence, this solution has infinite propagation speed. Note that in this case $\beta < p$.

Case 3. Let $\delta = 0$ that is,

$$q (p - 1) = 1,$$

In this case $\gamma = 0$ and

$$b = -\frac{1}{\alpha \delta + p} = -\frac{1}{p}.$$

Then (3.10) becomes

$$\frac{f'}{f} = -\frac{s^{1/p-1}}{p^{p-1} q}$$
whence
\[ \ln f = -\frac{1}{p^{\alpha/p}} \frac{s^{p}}{p^{\alpha}} = -\kappa s^{p}, \]
where
\[ \kappa = \frac{1}{p^{\alpha/p}} \frac{p}{p^{\alpha}} = \frac{(p - 1)^{2}}{p^{\alpha}} > 0. \]
It follows that
\[ f(s) = \exp \left( -\kappa s^{p} \right) \]
whence
\[ u(x, t) = \frac{1}{t^{\alpha/p}} \exp \left( -\kappa \left( \frac{r}{t^{1/p}} \right)^{p} \right). \]
For example, in the case \( p = 2 \) and, hence, \( q = 1 \) we obtain \( \kappa = \frac{1}{4} \) and
\[ u(x, t) = \frac{1}{t^{\alpha/2}} \exp \left( -\frac{1}{4} \frac{r^{2}}{t} \right). \]
Hence, the finite propagation speed for the above solutions occurs if and only if \( \delta > 0 \), that is, \( q(p - 1) > 1 \).

4 Weak solutions

Let \( \Omega \) be an open subset of \( M \) and \( I \) be an interval in \([0, \infty)\). By a subsolution of the equation
\[ \partial_{t}v = \Delta_{p}(v^{q}) \quad (4.1) \]

in the cylinder \( \Omega \times I \) we mean a non-negative function \( v \) of an appropriate class satisfying
\[ \partial_{t}v \leq \Delta_{p}(v^{q}). \quad (4.2) \]
In fact, this equation is understood in a certain weak sense, and a function \( v \) is taken from the following class:
\[ v \in C\left(I; L^{2} (\Omega)\right) \quad \text{and} \quad v^{q} \in L^{p}_{\text{loc}} \left(I; W^{1,p}(\Omega)\right). \]
That is, for any \( t \in I \),
\[ v(\cdot, t) \in L^{2}(\Omega), \quad v^{q}(\cdot, t) \in W^{1,p}(\Omega), \]
the function \( t \mapsto v(\cdot, t) \) is continuous in \( L^{2}(\Omega) \), and, for any compact subinterval \( J \subset I \),
\[ \int_{J} \|v^{q}(\cdot, t)\|_{W^{1,p}(\Omega)}^{p} \, dt < \infty, \]
that is,
\[ \int_{J} \int_{\Omega} (v^{qp} + |\nabla v^{q}|^{p}) \, d\mu dt < \infty. \quad (4.3) \]
Let us first show that the Leibenson operator $Lv = \Delta_p (v^q)$ can be rewritten in the form

$$Lv = c \operatorname{div} \left( v^m |\nabla v|^{p-2} \nabla v \right)$$

for some $c, m$, that is,

$$\operatorname{div} \left( |\nabla v|^p \nabla v \right) = c \operatorname{div} \left( v^m |\nabla v|^{p-2} \nabla v \right).$$

Indeed, we have

$$\nabla v^q = q v^{q-1} \nabla v$$

and

$$\operatorname{div} \left( |\nabla v|^p \nabla v \right) = q^{p-1} \operatorname{div} \left( v^{(q-1)(p-1)} |\nabla v|^{p-2} \nabla v \right).$$

Hence, (4.5) holds provided

$$m = (q - 1) (p - 1)$$

and $c = q^{p-1}$. The Leibenson equation becomes

$$\partial_t v = q^{p-1} \operatorname{div} \left( v^m |\nabla v|^{p-2} \nabla v \right),$$

and (4.2) becomes

$$\partial_t v \leq q^{p-1} \operatorname{div} \left( v^m |\nabla v|^{p-2} \nabla v \right).$$

(4.7)

5 Caccioppoli type inequality

We start here the proof of Theorem 1. The first step is obtaining a Caccioppoli type inequality.

For simplicity of notation, we omit in all integrations the notation of measure. All integration in $M$ is done with respect to $d\mu$, and in $M \times \mathbb{R}$ – with respect to $d\mu dt$. We assume that

$$p > 1, \quad q > 0$$

and use the notation

$$\delta = q(p - 1) - 1.$$

Let $\Omega$ be an open subset of $M$ and $I$ be an interval in $\mathbb{R}$.

**Lemma 2.** Let $v = v(x, t)$ be a bounded non-negative subsolution to (4.1) in a cylinder $\Omega \times I$. Let $\eta(x, t)$ be a Lipschitz non-negative bounded function in $\Omega \times (0, T)$ such that $\eta(\cdot, t)$ has compact support in $\Omega$ for all $t \in I$. Fix some real $\sigma$ such that

$$\sigma \geq \max(p, pq).$$

Set

$$\lambda = \sigma - \delta \quad \text{and} \quad \alpha = \frac{\sigma}{p}.$$  

(5.2)

Then, for all $t_1, t_2 \in I$ such that $t_1 < t_2$, we have

$$\left[ \int_{\Omega} v^\lambda \eta^p \right]_{t_1}^{t_2} + c_1 \int_{t_1}^{t_2} \int_{\Omega} |\nabla (v^\alpha \eta)|^p \leq \int_{t_1}^{t_2} \int_{\Omega} \left[ pv^\lambda \eta^{p-1} \eta_t + c_2 v^\sigma |\nabla \eta|^p \right] ,$$

(5.3)

where $c_1, c_2$ are positive constants depending on $p, q, \lambda$ (see below (5.13) and (5.14)).
In particular, if $\eta$ does not depend on $t$ then
\[
\left[ \int_\Omega v^\lambda \eta^p \right]_{t_1}^{t_2} + c_1 \int_{t_1}^{t_2} \int_\Omega |\nabla (v^\alpha \eta)|^p \leq c_2 \int_{t_1}^{t_2} \int_\Omega v^\sigma |\nabla \eta|^p.
\] (5.4)

Let us explain why all the integrals in (5.3) are well defined. Observe that always $\lambda \geq 2$. Indeed, if $q \geq 1$ then, using $\sigma \geq pq$, we obtain
\[
\lambda = \sigma - \delta \geq pq - (q(p - 1) - 1) = q + 1 \geq 2,
\] (5.5)
and if $q < 1$ then, using $\sigma \geq p$, we obtain
\[
\lambda = \sigma - \delta \geq p - \delta = p - (q(p - 1) - 1) = (p + 1) - (p - 1) q > (p + 1) - (p - 1) = 2.
\]

Since $v(\cdot, t) \in L^2(\Omega)$ and $v$ is bounded, it follows that, for any $t \in I$,
\[
\int_\Omega v^\lambda(\cdot, t) \leq \|v\|_{L^\infty}^{\lambda - 2} \int_\Omega v^2(\cdot, t) < \infty.
\]

Consequently, the expression
\[
\left[ \int_\Omega v^\lambda \eta^p \right]_{t_1}^{t_2}
\]
is well-defined. It also follows that
\[
\int_{t_1}^{t_2} \int_\Omega v^\lambda \eta^{p-1} |\eta_t| \leq \text{const} \int_{t_1}^{t_2} \int_\Omega v^2(\cdot, t) < \infty.
\]

Since $\nabla \eta(\cdot, t)$ and $v$ are bounded and $\sigma \geq pq$, we obtain
\[
\int_{t_1}^{t_2} \int_\Omega v^\sigma |\nabla \eta|^p \leq \text{const} \int_{t_1}^{t_2} \int_\Omega v^\sigma \leq \|v\|_{L^\infty}^{\sigma - pq} \int_{t_1}^{t_2} \int_\Omega v^{pq} < \infty, \quad (5.6)
\]
where we have used (4.3). The hypothesis $\sigma \geq pq$ implies that $\alpha \geq q$. Hence, the function $\Phi(s) = s^q$ is Lipschitz on any bounded interval in $[0, \infty)$. Since
\[
\nabla v^\alpha = \nabla \Phi(v^q) = \Phi'(v^q) \nabla v^q
\]
and $v^q$ is bounded, it follows that
\[
|\nabla v^\alpha| \leq C |\nabla v^q|.
\]

We obtain that
\[
\int_{t_1}^{t_2} \int_\Omega |\nabla (v^\alpha \eta)|^p \leq C \int_{t_1}^{t_2} \int_\Omega |\nabla v^\alpha|^p \eta^p + v^{\alpha p} |\nabla \eta|^p \\
\leq C \int_{t_1}^{t_2} \int_\Omega |\nabla v^q|^p + v^\sigma |\nabla \eta|^p \\
< \infty,
\]
where we have used (4.3) and (5.6). Hence, all the integrals in (5.3) are well-defined. Let us record for a later usage that
\[
v^\alpha \eta \in L^p_{loc}(I; W^{1,p}_0(\Omega)) \quad (5.7)
\]
because
\[ \int_{t_1}^{t_2} \int_{\Omega} (v^\alpha \eta)^p + |\nabla (v^\alpha \eta)|^p \leq \text{const} \int_{t_1}^{t_2} \int_{\Omega} v^\sigma + \int_{t_1}^{t_2} \int_{\Omega} |\nabla (v^\alpha \eta)|^p < \infty. \]

**Proof of Lemma 2.** Let us rewrite (4.7) in the form
\[ q^{1-p} \partial_t v \leq \text{div} (v^m |\nabla v|^{p-2} \nabla v) \]
where
\[ m = (q - 1) (p - 1). \]

Multiplying (5.8) by \( v^{\lambda - 1} \eta^p \) and integrating it over the cylinder \( Q = \Omega \times [t_1, t_2] \), we obtain
\[ q^{1-p} \int_Q v_i v^{\lambda - 1} \eta^p \leq \int_Q \text{div} \left( v^m |\nabla v|^{p-2} \nabla v \right) v^{\lambda - 1} \eta^p \]
\[ = - \int_Q v^m |\nabla v|^{p-2} \nabla v \nabla \left( v^{\lambda - 1} \eta^p \right) \]
\[ = - \int_Q v^m |\nabla v|^{p-2} \nabla v \left[ (\lambda - 1) v^{\lambda - 2} \eta^p \nabla v + pv^{\lambda - 1} \eta^{p-1} \nabla \eta \right] \]
\[ = - (\lambda - 1) \int_Q v^{\lambda + m - 2} |\nabla v|^p \eta^p - p \int_Q v^{\lambda + m - 1} |\nabla v|^{p-2} \eta^{p-1} \nabla (v, \nabla \eta) \]
\[ \leq - (\lambda - 1) \int_Q v^{\lambda + m - 2} |\nabla v|^p \eta^p + p \int_Q v^{\lambda + m - 1} |\nabla v|^{p-1} \eta^{p-1} |\nabla \eta|. \]
(5.10)

Observe that, for any fixed \( t \), the function \( v^{\lambda - 1} \eta^p \) belongs to \( W^{1,p}_0(\Omega) \) which allows to use the integration-by-part formula without the boundary term. Indeed, we have \( v^q \in W^{1,p}(\Omega) \) by the definition of a weak solution, which implies \( v^{\lambda - 1} \in W^{1,p}(\Omega) \) because \( v \) is bounded and \( \lambda - 1 \geq q \) by (5.5), whence the inclusion \( v^{\lambda - 1} \eta^p \in W^{1,p}_0(\Omega) \) follows because \( \eta \) is compactly supported in \( \Omega \).

Since
\[ \lambda + m - 2 = \lambda + (q - 1) (p - 1) - 2 = \lambda + (p - 1) q - 1 - p = \lambda + \delta - p = \sigma - p, \]
we rewrite (5.10) as follows:
\[ q^{1-p} \int_Q v_i v^{\lambda - 1} \eta^p \leq - (\lambda - 1) \int_Q v^{\sigma - p} |\nabla v|^p \eta^p + p \int_Q v^{\sigma - p + 1} |\nabla v|^{p-1} \eta^{p-1} |\nabla \eta|. \]
(5.11)

Since \( \sigma \geq p \), the function \( v \) enters all the integrals in (5.11) in non-negative powers; hence, the integrals are finite.

Next, let us use the following inequality for all \( X, Y \geq 0 \) and \( \varepsilon > 0 \):
\[ XY \leq \varepsilon^p X^p + \frac{1}{\varepsilon^p} Y^p \]

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where \( p' = \frac{p}{p-1} \) is the Hölder conjugate of \( p \) (here we use that \( p > 1 \)). Applying this inequality with

\[
X = v_{\xi} |\nabla v|^{p-1} \eta^{p-1} \quad \text{and} \quad Y = v^{(\sigma-p+1-\xi)} |\nabla \eta|
\]

(where \( \varepsilon \) and \( \xi \) are yet to be determined) we obtain

\[
v^{\sigma-p+1} |\nabla v|^{p-1} \eta^{p-1} |\nabla \eta| = XY \leq \varepsilon \varepsilon' \left( v^{\xi} |\nabla v|^{p-1} \eta^{p-1} \right)^{p'} + \frac{1}{\varepsilon} \left( v^{\sigma-p+1-\xi} |\nabla \eta| \right)^{p}.
\]

We would like to have

\[
\xi p' = \sigma - p
\]

whence

\[
\xi := \frac{\sigma - p}{p'}.
\]

With this \( \xi \) we have

\[
(\sigma - p + 1 - \xi) p = \left( \sigma - p + 1 - \frac{(\sigma - p)}{p'} \right) p = \left( \frac{\sigma - p}{p} + 1 \right) p = \sigma
\]

and

\[
v^{\sigma-p+1} |\nabla v|^{p-1} \eta^{p-1} |\nabla \eta| \leq \varepsilon \varepsilon' v^{\sigma-p} |\nabla v|^{p} \eta^{p} + \frac{1}{\varepsilon} v^{\sigma} |\nabla \eta|^{p}.
\]

It follows that

\[
q^{1-p} \int_{Q} v_{v} \nu^{\lambda-1} \eta^{p} \leq - (\lambda - 1) \int_{Q} v^{\sigma-p} |\nabla v|^{p} \eta^{p} + p \int_{Q} \left[ \varepsilon \varepsilon' v^{\sigma-p} |\nabla v|^{p} \eta^{p} + \frac{1}{\varepsilon} v^{\sigma} |\nabla \eta|^{p} \right]
\]

\[
= - \left( \lambda - 1 - p \varepsilon \varepsilon' \right) \int_{Q} v^{\sigma-p} |\nabla v|^{p} \eta^{p} + \frac{p}{\varepsilon} \int_{Q} v^{\sigma} |\nabla \eta|^{p}. \quad (5.12)
\]

On the other hand, we have

\[
|\nabla (v^{\alpha} \eta)|^{p} = |\alpha v^{\alpha-1} \eta \nabla v + v^{\alpha} \nabla \eta|^{p}
\]

\[
\leq 2^{p-1} \alpha^{p} v^{p(\alpha-1)} |\nabla v|^{p} \eta^{p} + 2^{p-1} v^{\alpha p} |\nabla \eta|^{p}
\]

\[
= 2^{p-1} \alpha^{p} v^{\sigma-p} |\nabla v|^{p} \eta^{p} + 2^{p-1} v^{\sigma} |\nabla \eta|^{p},
\]

where we have used that, by (5.2),

\[
p(\alpha - 1) = \sigma - p.
\]

It follows that

\[
v^{\sigma-p} |\nabla v|^{p} \eta^{p} \geq 2^{1-p} \alpha^{-p} |\nabla (v^{\alpha} \eta)|^{p} - \alpha^{-p} v^{\sigma} |\nabla \eta|^{p}.
\]

Substituting into (5.12) yields

\[
q^{1-p} \int_{Q} v_{v} \nu^{\lambda-1} \eta^{p} \leq - \left( \lambda - 1 - p \varepsilon \varepsilon' \right) 2^{1-p} \alpha^{-p} \int_{Q} |\nabla (v^{\alpha} \eta)|^{p}
\]
+ \left( (\lambda - 1 - p\varepsilon^{p'}) \alpha^{-p} + \frac{p}{\varepsilon^p} \right) \int_Q v^\sigma |\nabla \eta|^p.

Hence,
\[ \lambda \int_Q v_t v^{\lambda - 1} \eta^p \leq -c_1 \int_Q |\nabla (v^\alpha \eta)|^p + c_2 \int_Q v^\sigma |\nabla \eta|^p \]
where
\[ c_1 = \lambda q^{p-1} (\lambda - 1 - p\varepsilon^{p'}) 2^{1-p} \alpha^{-p} \]
and
\[ c_2 = \lambda q^{p-1} \left( (\lambda - 1 - p\varepsilon^{p'}) \alpha^{-p} + \frac{p}{\varepsilon^p} \right) . \]

We choose \( \varepsilon \) so small that \( c_1 > 0 \), that is,
\[ p\varepsilon^{p'} < \lambda - 1. \]

Since
\[ \lambda \int_Q v_t v^{\lambda - 1} \eta^p = \int_Q \partial_t v^{\lambda \eta^p} = \left[ \int_\Omega v^{\lambda \eta^p} \right]_t^{t_2} - p \int_Q v^{\lambda \eta^{p-1}} \eta_t \]
we obtain
\[ \left[ \int_\Omega v^{\lambda \eta^p} \right]_t^{t_2} = \lambda \int_Q v_t v^{\lambda - 1} \eta^p + p \int_Q v^{\lambda \eta^{p-1}} \eta_t \]
\[ \leq -c_1 \int_Q |\nabla (v^\alpha \eta)|^p + c_2 \int_Q v^\sigma |\nabla \eta|^p + p \int_Q v^{\lambda \eta^{p-1}} \eta_t \]
and, hence,
\[ \left[ \int_\Omega v^{\lambda \eta^p} \right]_t^{t_2} + c_1 \int_Q |\nabla (v^\alpha \eta)|^p \leq \int_Q \left[ p v^{\lambda \eta^{p-1}} \eta_t + c_2 v^\sigma |\nabla \eta|^p \right] . \]

Finally, let us specify \( c_1 \) and \( c_2 \). Let us choose \( \varepsilon \) so that
\[ p\varepsilon^{p'} = \frac{1}{2} (\lambda - 1) \]
that is
\[ c_1 = \lambda (\lambda - 1) 2^{-p} q^{p-1} \alpha^{-p} \] (5.13)

It follows that
\[ c_2 = \lambda q^{p-1} \left( \frac{1}{2} (\lambda - 1) \alpha^{-p} + \frac{p}{\varepsilon^p} \right) \]
\[ = \lambda q^{p-1} \left( \frac{1}{2} (\lambda - 1) \alpha^{-p} + \frac{p}{(\frac{1}{2} (\lambda - 1) / p)^{p/p'}} \right) \]
\[ = \lambda q^{p-1} \left( \frac{1}{2} (\lambda - 1) \alpha^{-p} + \frac{2^{p/p'} p^{1+p/p'}}{(\lambda - 1)^{p/p'}} \right) . \]
Since \[ \frac{p}{p'} + 1 = \frac{p}{p/(p-1)} + 1 = p \]
we have

\[
c_2 = \frac{1}{2} \lambda (\lambda - 1) q^{p-1} \alpha^{-p} + \frac{\lambda 2^{p-1} p q^{p-1}}{(\lambda - 1)^{p-1}}. \tag{5.14}
\]

\[\blacksquare\]

\textit{Remark.} For the future we need the ratio \( \frac{c_2}{c_1} \). It follows from (5.13) and (5.14) that

\[
\frac{c_2}{c_1} = 2^{p-1} + \frac{2^{p-1} p}{\lambda (\lambda - 1)^{p-1} \lambda (\lambda - 1) 2^{-p} \alpha^{-p}}
= 2^{p-1} + \frac{2^{2p-1} \sigma^p}{(\lambda - 1)^p},
\]

where we have used that \( \alpha p = \sigma \). Since \( \sigma = \lambda + \delta \), we obtain

\[
\frac{c_2}{c_1} = 2^{p-1} + \frac{2^{2p-1} (\lambda + \delta)^p}{(\lambda - 1)^p}.
\]

It follows that, for all \( \lambda \geq 2 \),

\[
\frac{c_2}{c_1} \leq C_{p,q}, \tag{5.15}
\]

where \( C_{p,q} \) depend only on \( p \) and \( q \) but does not depend on \( \lambda \).

\textit{Remark.} Let obtain an upper bound of \( c_2 \). Using

\[
\alpha = \frac{\sigma}{p} = \frac{\lambda + \delta}{p}
\]

we obtain

\[
c_2 = \frac{1}{2} \lambda (\lambda - 1) \frac{q^{p-1} p^p}{(\lambda + \delta)^p} + \frac{\lambda 2^{p-1} p q^{p-1}}{(\lambda - 1)^{p-1}}.
\]

As \( \lambda \geq 2 \) and \( \lambda + \delta \geq p > 1 \), it follows that

\[
c_2 \leq C_{p,q} \lambda^{2-p}. \tag{5.16}
\]

Of course, if \( p \geq 2 \) then \( c_2 \) is uniformly bounded by a constant \( C_{p,q} \) independently of \( \lambda \), but if \( p < 2 \) then \( c_2 \) may grow with \( \lambda \) as \( \lambda^{2-p} \).

\textbf{Lemma 3}. Let \( M \) be geodesically complete. Let \( v = v(x,t) \) be a bounded non-negative subsolution to (4.7) in \( M \times I \). Then, for all large enough \( \lambda \), including \( \lambda = \infty \), the function

\[ t \mapsto \|v(\cdot,t)\|_{L^\lambda(M)} \]

is monotone decreasing. Consequently, if \( I = [a,b] \) then

\[
\|v\|_{L^\infty(M \times I)} \leq \|v(\cdot,a)\|_{L^\infty(M)}.
\]
Proof. If $M$ is geodesically complete, then $W_{0}^{1,p}(M) = W^{1,p}(M)$. Hence, $v^\lambda - 1, t \in I$, and we can use the argument in the proof of Lemma 2 with $\eta \equiv 1$ (without assumption that $\eta(\cdot, t)$ is compactly supported). Assuming that $\lambda$ is large enough so that $\sigma := \lambda + \delta$ satisfies (5.1), we obtain from (5.3) that, for all $t_1, t_2 \in I$, $t_1 < t_2$,

$$\left[ \int_{M} v^\lambda \right]_{t_1}^{t_2} \leq 0,$$
which proved the claim for a finite $\lambda$. The case of an infinite $\lambda$ is obtained then by letting $\lambda \rightarrow \infty$. ■

6 Sobolev and Moser inequalities

Let $B$ be a precompact ball in a manifold $M$ of dimension $n$. The Sobolev inequality in $B$ of order $p$ says the following: for any non-negative function $w \in W_{0}^{1,p}(B)$

$$\left( \int_{B} w^{p\nu} \right)^{1/\kappa} \leq S_B \int_{B} |\nabla w|^p,$$

(6.1)

where $\kappa > 1$ is some constant and $S_B$ is called the Sobolev constant in $B$. The value of $\kappa$ is independent of $B$ and can be chosen as follows:

$$\kappa = \frac{n}{n - p} \text{ if } n > p,$$

and $\kappa$ is an arbitrary real number $> 1$ if $n \leq p$.

We always assume that $S_B$ is chosen to be minimal possible. In this case the function

$$B \mapsto S_B$$

is clearly monotone increasing with respect to inclusion of balls.

Fix a precompact ball $B \subset M$ and set $Q = B \times I$, where $I \subset \mathbb{R}$ is an interval Assume that the Sobolev inequality (10.5) holds in $B$ with exponent $\kappa > 1$. Let $\kappa'$ be its Hölder conjugate. Set

$$\nu = \frac{1}{\kappa'} = \frac{\kappa - 1}{\kappa}.$$

Lemma 4. Let $w \in L^p(I;W_{0}^{1,p}(B))$ be a non-negative function. Then

$$\int_{Q} w^{p(1+\nu)} \leq S_B \left( \int_{Q} |\nabla w|^p \right) \sup_{t \in I} \left( \int_{B} w^p \right)^\nu$$

(6.2)

Proof. By the Hölder inequality, we have, for any fixed $t \in I$,

$$\int_{B} w^{p(1+\nu)} \leq \int_{B} w^{p} w^{\nu} \leq \left( \int_{B} w^{p\nu} \right)^{1/\kappa} \left( \int_{B} w^{p\kappa} \right)^{1/\kappa'}$$
By the Sobolev inequality (6.1) we have, for any \( t \in I \),
\[
\left( \int_B u^{p\kappa} \right)^{1/\kappa} \leq S_B \int_B |\nabla w|^p.
\]
It follows that
\[
\int_B u^{p(1+\nu)} \leq S_B \left( \int_B |\nabla w|^p \right) \sup_{t \in I} \left( \int_B u^\nu \right).
\]
Integrating this inequality in \( t \in I \) gives (6.2).

7 Comparison in two cylinders

Here we assume that
\[
\delta := q(p - 1) - 1 \geq 0.
\]

**Lemma 5.** Consider two balls \( B = B(x, r) \) and \( B' = B(x, r') \) with \( 0 < r' < r \), and two cylinders
\[
Q = B \times [0, T], \quad Q' = B' \times [0, T].
\]
Assume that \( B \) is precompact. Let \( \sigma \) be any real such that
\[
\sigma \geq \max(p, pq).
\]
Let \( v \) be non-negative bounded subsolution of (4.1) in \( Q \) such that
\[
v(\cdot, 0) = 0 \text{ in } B.
\]
Then
\[
\int_{Q'} v^{\sigma(1+\nu)} \leq C S_B \sigma^{(2-p)\nu} \left( \int_Q v^\sigma \right) \left( \int_Q v^{\sigma+\delta} \right)^\nu,
\]
where \( C \) depends only on \( p, q \) and \( \nu \), while it is independent of \( \sigma \).
Proof. As in Lemma 2, set \( \lambda = \sigma - \delta \) and \( \alpha = \frac{\sigma}{p} \) and recall that \( \alpha \geq 1 \) and \( \lambda \geq 2 \). Let \( \eta = \eta(x) \) be a bump function of \( B' \) in \( B \). By (5.7), we have
\[
w := v^\sigma \eta \in L^p \left([0, T]; W_0^{1,p}(B) \right)
\]
Applying (6.2) with this function \( w \) and using
\[
w^p = v^\sigma \eta^p,
\]
we obtain that
\[
\int_Q v^{\sigma(1+\nu)} \eta^{p(1+\nu)} \leq S_B \left( \int_Q |\nabla (v^\sigma \eta)|^p \right) \sup_{t \in [0,T]} \left( \int_B v^\sigma \eta^p \right)^\nu.
\]
By (5.4) we have
\[
\int_Q |\nabla (v^\sigma \eta)|^p \leq \frac{c_2}{c_1} \int_Q v^\sigma |\nabla \eta|^p.
\]
and
\[
\sup_{t \in [0,T]} \left( \int_B v^{\lambda \eta^p} \right) \leq c_2 \int_Q v^\sigma |\nabla \eta|^p.
\]
Let us use the latter inequality also for other values of the parameters as follows:
\[
\sup_{t \in [0,T]} \left( \int_B v^{\lambda' \eta^p} \right) \leq c'_2 \int_Q v^{\sigma'} |\nabla \eta|^p,
\]
where \( \sigma' = \sigma + \delta \) and \( \lambda' = \sigma' - \delta = \sigma \).
Then we have
\[
\sup_{t \in [0,T]} \left( \int_B v^{\sigma' \eta^p} \right) \leq c'_2 \int_Q v^{\sigma'} |\nabla \eta|^p.
\]
It follows that
\[
\int_Q v^{\sigma(1+\nu)} \eta^{p(1+\nu)} \leq S_B \frac{c_2}{c_1} \int_Q v^\sigma |\nabla \eta|^p \left( c'_2 \int_Q v^{\sigma'} |\nabla \eta|^p \right)^\nu.
\]
Using that \( \eta = 1 \) in \( B' \) and \( |\nabla \eta| \leq \frac{1}{r^{-p}} \) we obtain
\[
\int_{Q'} v^{\sigma(1+\nu)} \leq S_B \frac{c_2}{c_1} \frac{(c'_2)^\nu}{(r_1 - r_2)^p(1+\nu)} \left( \int_Q v^\sigma \right) \left( \int_Q v^{\sigma'} \right)^\nu.
\]
By (5.15) we have
\[
\frac{c_2}{c_1} \leq C_{p,q},
\]
and (5.16)
\[
c'_2 \leq C_{p,q} (\lambda')^{2-p} = C_{p,q} \sigma^{2-p}.
\]
Hence, (7.2) follows. 

Corollary 6. Under the hypotheses of Lemma 5, we have
\[
\int_{Q'} v^{\sigma(1+\nu)} \leq \frac{C S_B \sigma^{(2-p)\nu} ||v||_{L^\infty(Q)}^{\delta \nu}}{(r - r')^{p(1+\nu)}} \left( \int_Q v^\sigma \right)^{1+\nu},
\]
where \( C = C(p, q, \nu) \).
8 Auxiliary lemma about sequences

Lemma 7. Let a sequence \( \{J_k\}_{k=0}^{\infty} \) of non-negative reals satisfies

\[
J_{k+1} \leq \frac{A^k}{D} J_k^{1+\omega} \quad \text{for all } k \geq 0.
\]  
(8.1)

where \( A, D, \omega > 0 \). Then, for all \( k \geq 0 \),

\[
J_k \leq \left( \left( \frac{A^{1/\omega} D}{D} \right)^{1/\omega} J_0 \right)^{(1+\omega)k} \left( A^{-k-1/\omega} D \right)^{1/\omega}.
\]  
(8.2)

In particular, if

\[
D \geq A^{1/\omega} J_0^\omega,
\]  
(8.3)

then, for all \( k \geq 0 \),

\[
J_k \leq A^{-k/\omega} J_0.
\]  
(8.4)

Proof. Consider the sequence

\[
X_k = \left( \left( \frac{A^{1/\omega} D}{D} \right)^{1/\omega} J_0 \right)^{(1+\omega)k} \left( A^{-k-1/\omega} D \right)^{1/\omega}.
\]

Then we have

\[
X_0 = \left( A^{1/\omega} D^{-1} \right)^{1/\omega} J_0 \left( A^{-1/\omega} D \right)^{1/\omega} = J_0
\]

and

\[
\frac{A^k}{D} X_k^{1+\omega} = \frac{A^k}{D} \left( \left( \frac{A^{1/\omega} D}{D} \right)^{1/\omega} J_0 \right)^{(1+\omega)k+1} \left( A^{-k-1/\omega} D \right)^{1+\omega} = \left( \left( \frac{A^{1/\omega} D}{D} \right)^{1/\omega} J_0 \right)^{(1+\omega)k+1} A^k D^{-1} \left( A^{-k-1/\omega} D \right) \left( A^{-k-1/\omega} D \right)^{1/\omega} = \left( \left( \frac{A^{1/\omega} D}{D} \right)^{1/\omega} J_0 \right)^{(1+\omega)k+1} A^{-1/\omega} \left( A^{-k-1/\omega} D \right)^{1/\omega} = \left( \left( \frac{A^{1/\omega} D}{D} \right)^{1/\omega} J_0 \right)^{(1+\omega)k+1} \left( A^{-k-1/\omega} D \right)^{1/\omega} = X_{k+1}.
\]

Hence, by comparison we obtain \( J_k \leq X_k \), which was to be proved.

For the second statement, if (8.3) holds then we can assume without loss of generality that

\[
D = A^{1/\omega} J_0^\omega.
\]

Substituting this value of \( D \) into (8.2) we obtain

\[
J_k \leq \left( A^{-k/\omega} J_0^\omega \right)^{1/\omega}
\]

which is equivalent to (8.4). ■
9 Mean value inequality

We assume here that
\[ \delta = q (p - 1) - 1 \geq 0. \]

**Lemma 8.** Let \( B = B(x_0, R) \) be a precompact ball. Let \( u \) be a non-negative bounded subsolution of (4.1) in
\[ Q = B \times [0, t] \]
such that
\[ u(\cdot, 0) = 0 \text{ in } B. \]

Let \( \sigma \) be as in (7.1). Then, for the cylinder
\[ Q' = \frac{1}{2} B \times [0, t], \]
we have
\[ \| u \|_{L^\infty(Q')} \leq \left( \frac{C S_B}{R^{p(1 + \nu)}} \right)^{\frac{1}{\nu}} \| u \|_{L^\infty(Q)} \| u \|_{L^\sigma(Q)}, \]
where \( C = C(p, q, \nu, \sigma). \)

**Remark.** Since
\[ \| u \|_{L^\sigma(Q)} = \left( \int_0^t \int_\Omega u^\sigma \right)^{1/\sigma} \leq (t \mu(\Omega))^{\frac{1}{\sigma}} \| u \|_{L^\infty(Q)}, \]
we obtain from (9.1) that
\[ \| u \|_{L^\infty(Q')} \leq \left( \frac{C S_B}{R^{p(1 + \nu)}} \right)^{\frac{1}{\sigma}} t \mu(B)^{\frac{1}{\sigma}} \| u \|_{L^\infty(Q)}^{1 + \frac{\delta}{\sigma}}. \]

**Remark.** Unlike Lemma 5 where we have explicitly traced the dependence of the constants on \( \sigma \), in (9.1) and (9.2) the dependence of \( C \) on \( \sigma \) is unimportant because these inequalities will be applied only with a fixed \( \sigma \).
Proof. Consider a sequence of radii

\[ r_k = \left( \frac{1}{2} + 2^{-k-1} \right) R \]

so that \( r_0 = R \) and \( r_k \downarrow \frac{1}{2} R \) as \( k \to \infty \). Set

\[ B_k = B(x_0, r_k), \quad Q_k = B_k \times [0, t] \]

so that \( B_0 = B, \quad Q_0 = Q \) and \( Q_\infty := \lim_{k \to \infty} Q_k = Q' \).

Set also

\[ \sigma_k = \sigma (1 + \nu)^k \]

and

\[ J_k = \int_{Q_k} u^{\sigma_k}. \]

Applying (7.3) to the cylinders \( Q_k \) and \( Q_{k+1} \), we obtain

\[
J_{k+1} = \int_{Q_{k+1}} u^{\sigma_{k+1}(1+\nu)} \leq \frac{CS_B \sigma_k^{(2-p)\nu} \|u\|_{L^\infty(Q_k)}^\delta }{(r_k - r_{k+1})^{p(1+\nu)}} J_k^{1+\nu}
\]

\[
= \frac{CS_B \sigma_k^{(2-p)\nu} \|u\|_{L^\infty(Q_k)}^\delta }{(r_k - r_{k+1})^{p(1+\nu)}} J_k^{1+\nu}
\]

\[
\leq \frac{C 2^{k(p(1+\nu)) (1 + \nu)^k (2-p)^{-p} \nu S_B \|u\|_{L^\infty(Q)}^\delta }{R^{p(1+\nu)}} J_k^{1+\nu}
\]

\[
\leq A^k D^{-1} J_k^{1+\nu},
\]

where

\[ A = 2^{p(1+\nu)} (1 + \nu)^{(2-p)\nu} \geq 1 \]

and

\[ D^{-1} = \frac{CS_B \|u\|_{L^\infty(Q)}^\delta }{R^{p(1+\nu)}}, \]

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where we have absorbed $\sigma^{(2-p)\nu}$ into $C$. By Lemma 7 we conclude that

$$J_k \leq \left( A^{1/\nu}D^{-1} \right)^{1/\nu} J_0^{(1+\nu)^k} \left( A^{1-1/\nu}D \right)^{1/\nu} = A^{(1+\nu)^k-\frac{1}{\nu}} D^{-(1+\nu)^k-\frac{1}{\nu}} J_0^{(1+\nu)^k}.$$ 

It follows that

$$\left( \int_{Q_k} u^{\sigma k} \right)^{1/\sigma} = J_k^{\sigma^{(1+\nu)^k}} \leq A^{\frac{1-(1+\nu)^{-k}}{\sigma^2}} D^{-\frac{1-(1+\nu)^{-k}}{\sigma^2}} \left( \int_Q u^\sigma \right)^{1/\sigma}.$$ 

As $k \to \infty$, we obtain

$$\|u\|_{L^\infty(Q')} \leq A^{\frac{1}{\sigma^2}} D^{-\frac{1}{\sigma^2}} \|u\|_{L^\sigma(Q)} = \left( A^{\frac{1}{\sigma}} D^{-1} \right)^{\frac{1}{\sigma^2}} \|u\|_{L^\sigma(Q)} \leq \left( A^{\frac{1}{\sigma}} \frac{CS_B \|u\|_{L^\infty(Q)}}{R^{p(1+\nu)}} \right)^{\frac{1}{\sigma^2}} \|u\|_{L^\sigma(Q)} = \left( \frac{CS_B}{R^{p(1+\nu)}} \right)^{\frac{1}{\sigma^2}} \|u\|_{L^\infty(Q)} \|u\|_{L^\sigma(Q)},$$

where $A^{\frac{1}{\sigma}}$ was absorbed into $C$. ■

10 Normalized Sobolev constant

Let $B$ be a precompact ball in $M$ and $w \in W_0^{1,p}(B)$. Dividing the Sobolev inequality (6.1) by $\mu(B)^{1/\kappa}$, we obtain

$$\left( \int_B w^{\mu \kappa} \right)^{1/\kappa} \leq \mu(B)^\nu S_B \int_B |\nabla w|^p$$

where

$$\nu = \frac{\kappa - 1}{\kappa} = \frac{1}{\kappa^*},$$

and

$$\left( \int_B w^{\mu \kappa} \right)^{1/(p \nu)} \leq (\mu(B)^\nu S_B)^{1/p} \left( \int_B |\nabla w|^p \right)^{1/p}, \quad (10.3)$$

Denoting by $r(B)$ the radius of $B$, let us define a new quantity

$$\iota(B) = \frac{1}{\mu(B)} \left( \frac{r(B)^p}{S_B} \right)^{1/\nu}$$

so that

$$S_B = \frac{r(B)^p}{(\iota(B)\mu(B))^{p \nu}} \quad (10.4)$$

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and
\[(\mu(B)^\nu S_B)^{1/p} = \frac{r(B)}{\nu(B)^{1/p}}.\]
Hence, \((10.3)\) can be rewritten in the form
\[\left(\int_B |\nabla w|^p \right)^{1/p} \geq \frac{\nu(B)^{1/p}}{r(B)} \left(\int_B w^p \right)^{1/p}.\] \((10.5)\)

It is clear from \((10.5)\) that the value of \(\kappa\) can be always reduced (by modifying the
value of \(\nu(B)\)). It is only important that \(\kappa > 1\). In fact, the exact value of \(\kappa\) does not
affect the results, although various constants depend on \(\kappa\).

The constant \(\nu(B)\) is called the normalized Sobolev constant in \(B\). It is known that if
\(M\) is complete and \(\text{Ricci}_M \geq 0\) then, for all balls \(B\), the normalized Sobolev constant
\(\nu(B)\) is bounded below by a positive constant.

11 Propagation speed inside a ball

We assume here that \(M\) is geodesically complete and
\[\delta = q(p - 1) - 1 > 0.\]

**Theorem 9.** Let \(u\) be a bounded non-negative subsolution of \((4.1)\) in \(M \times [0, T]\) with
the initial condition \(u(\cdot, 0) = u_0\). Let \(B_0 = B(z_0, R)\) be a ball in \(M\) such that
\[u_0 = 0 \text{ in } B_0.\]

Set
\[t_0 = \eta \nu(B_0) R^p \|u_0\|_{L^\infty(M)}^{-\delta} \wedge T,\] \((11.1)\)
where \(\eta\) is a sufficiently small positive constant depending only on \(p, q, \nu\). Then
\[u = 0 \text{ in } \frac{1}{2} B_0 \times [0, t_0].\]
Proof. Set \( r = \frac{1}{2} R \) and fix for a while a point \( x \in \frac{1}{2} B_0 \) so that \( B := B (x, r) \subset B_0 \). Fix also some \( t \in (0, T] \) and set

\[
Q_k = 2^{-k} B \times [0, t] \quad \text{and} \quad J_k = \| u \|_{L^\infty (Q_k)}.
\]

Our purpose is to obtain an upper bound for \( J_k (x) = \| u \|_{L^\infty (Q_k)} \) that ensures that \( J_k (x) \to 0 \) as \( k \to \infty \) uniformly in \( x \in \frac{1}{2} B_0 \).

Let us fix \( \sigma \) satisfying (7.1), for example, \( \sigma = \max (p, pq) \). Applying inequality (9.2) of Lemma 8 in the cylinders \( Q_k, Q_{k+1} \), we obtain

\[
J_{k+1} = \| u \|_{L^\infty (Q_{k+1})} \leq \left( \frac{C S_{2^{-k} B}}{t \mu (2^{-k} B)} \right)^\frac{1}{\nu} t \mu (2^{-k} B) \| u \|_{L^\infty (Q_k)}^{1 + \frac{\omega}{\sigma}} J_k^{1 + \frac{\omega}{\sigma}},
\]

where we have used that \( S_{2^{-k} B} \leq S_{B_0} \) and \( \mu (2^{-k} B) \leq \mu (B_0) \). By (10.4) we have

\[
\left( \frac{S_{B_0}}{R^p (1+\nu)} \right)^{1/\nu} = \left( \frac{R^p}{(\mu (B_0) \mu (B_0))^{1/\nu}} \right)^{1/\nu} = \frac{R^{p/\nu}}{\mu (B_0) \mu (B_0)},
\]

whence

\[
\left( \frac{S_{B_0}}{R^p (1+\nu)} \right)^{1/2} \mu (B_0) = \frac{R^{p/\nu}}{R^{p/\nu} \mu (B_0) \mu (B_0)} \frac{1}{\mu (B_0) R^p}.
\]

It follows that

\[
J_{k+1} \leq 2^{(k+1) \frac{p}{\sigma \nu}} \left( \frac{C t}{\mu (B_0) R^p} \right)^\frac{1}{\nu} J_k^{1 + \frac{\omega}{\sigma}} = A^k D^{-1} J_k^{1 + \omega},
\]

where

\[
\omega = \frac{\delta}{\sigma}, \quad A = 2^{\frac{p}{\sigma \nu}}.
\]
and
\[ D^{-1} = A \left( \frac{Ct}{\iota(B_0) R^p} \right)^{\frac{1}{\sigma}}. \]
By Lemma 7, if
\[ D^{-1} \leq A^{-1/\omega} J_0^{-\omega} \] (11.2)
then, for all \( k \geq 0 \),
\[ J_k \leq A^{-k/\omega} J_0. \] (11.3)
The condition (11.2) is equivalent to
\[ A \left( \frac{Ct}{\iota(B_0) R^p} \right)^{\frac{1}{\sigma}} \leq A^{-1/\omega} J_0^{-\omega} \]
that is, to
\[ t \leq C^{-1} \iota(B_0) R^p J_0^{-\delta}, \] (11.4)
where \( A \) is absorbed to \( C \). Since by Lemma 3
\[ J_0 = \| u \|_{L^\infty(Q)} \leq \| u_0 \|_{L^\infty(M)}, \]
the condition (11.4) is satisfied for \( t = t_0 \), where \( t_0 \) is determined by (11.1) with \( \eta = C^{-1} \).
Hence, for \( t = t_0 \) we obtain from (11.3) that, for any \( k \),
\[ \| u \|_{L^\infty(2^{-k} B \times [0, t])} \leq A^{-k/\omega} \| u_0 \|_{L^\infty} \].
For any \( k \), we cover the ball \( \frac{1}{2} B_0 \) by a finite sequence of balls \( B(x_i, 2^{-k} r) \) with \( x_i \in \frac{1}{2} B_0 \).

Since for all \( i \)
\[ \| u \|_{L^\infty(B(x_i, 2^{-k} r) \times [0, t])} \leq A^{-k/\omega} \| u_0 \|_{L^\infty}, \]
we obtain that
\[ \| u \|_{L^\infty(\frac{1}{4} B_0 \times [0, t])} \leq A^{-k/\omega} \| u_0 \|_{L^\infty} \].
Finally, letting \( k \to \infty \), we obtain that \( u = 0 \) in \( \frac{1}{2} B_0 \times [0, t] \), which was to be proved.
\[ \blacksquare \]
12 Propagation speed of support

In this section we assume $M$ is geodesically complete, that is, all geodesic balls are precompact. Let also \[ \delta = q(p - 1) - 1 > 0. \]

For any set $K \subset M$ and any $r > 0$, denote by $K_r$ a closed $r$-neighborhood of $K$.

**Theorem 10.** Let $u(x, t)$ be a non-negative bounded subsolution of (4.1) in $M \times \mathbb{R}_+$ with the initial function $u_0 = u(\cdot, 0)$. Assume that the support $K = \text{supp} u_0$ is compact. Then there exists $T > 0$ and an increasing continuous function $\rho : (0, T) \to \mathbb{R}_+$ such that

\[ \text{supp } u(\cdot, t) \subset K_{\rho(t)} \]

for all $t \in (0, T)$.

Here both $T$ and $\rho(t)$ may depend on $u$. The function $\rho(t)$ is called a propagation rate of $u$.

**Proof.** Let us fix a reference point $x_0 \in K$ and define the following function for all $r > 0$:

\[ \varphi(r) = \frac{\eta}{4^{p+p/\nu}} (B(x_0, r))r^p \| u_0 \|_{L^\infty(M)}^{-\delta}. \] (12.1)

Denote

\[ r_0 = \text{diam } K. \]

Let us prove that that, for any $r \geq r_0$,

\[ t \leq \varphi(3r + r_0) \Rightarrow \text{supp } u(\cdot, t) \subset K_r, \]

that is,

\[ u(\cdot, t) = 0 \text{ in } M \setminus K_r. \]

Let us first prove that

\[ u(\cdot, t) = 0 \text{ in } K_{2r} \setminus K_r. \]

Fix a point $x \in K_{2r} \setminus K_r$. We have

\[ d(x, K) \leq 2r \Rightarrow d(x, x_0) \leq 2r + r_0. \]
It follows that
\[ B(x, r) \subset B(x_0, 3r + r_0) = B(x_0, R) \]
where
\[ R := 3r + r_0. \]
The condition \( r \geq r_0 \) implies \( R \leq 4r \). Since \( B(x, r) \subset B(x_0, R) \), we have by the monotonicity of function (10.4) that
\[
\frac{\iota(B(x, r))\mu(B(x, r))}{r^{p/\nu}} \geq \frac{\iota(B(x_0, R))\mu(B(x_0, R))}{R^{p/\nu}}.
\]
It follows that
\[
\frac{\iota(B(x, r))r^p}{\iota(B(x_0, R))R^p} \geq \frac{(r^p/R^p)\mu(B(x_0, R))}{\mu(B(x, r))} \geq \frac{1}{4^{p/p/\nu}}\iota(B(x, r))R^p.
\]
Therefore, the hypothesis
\[
t \leq \varphi(R) = \frac{\eta}{4^{p/p/\nu}}\iota(B(x_0, R))R^p \|u_0\|_{L^\infty(M)}^{-\delta}
\]
implies that
\[
t \leq \eta\iota(B(x, r))r^p \|u_0\|_{L^\infty(M)}^{-\delta}.
\]
Since \( u(\cdot, 0) = 0 \) in \( B(x, r) \), we conclude by Theorem 9 that
\[
u(\cdot, t) = 0 \text{ in } B(x, r/2).
\]
Since this is true for any \( x \in K_{2r} \setminus K_r \), we obtain that
\[
u(\cdot, t) = 0 \text{ in } K_{2r} \setminus K_r. \tag{12.2}
\]
Let us show that also
\[
u(\cdot, t) = 0 \text{ in } M \setminus K_{2r}. \tag{12.3}
\]
Fix some \( s \gg 2r \) and let \( \eta(x) \) be a bump function of \( K_s \setminus K_{2r} \) in \( K_{2s} \setminus K_r \); that is, \( \eta \) is the following function of \( |x| := d(x, K) \):

\[
\eta(x) = \begin{cases} 
\left( \frac{|x|}{r} - 1 \right)_+, & |x| \leq 2r, \\
1, & |x| \in [2r, s], \\
2 \left( 1 - \frac{|x|}{2s} \right)_+, & |x| \geq s.
\end{cases}
\]

Applying the inequality (5.4) of Lemma 2 with some fixed \( \lambda \), we obtain

\[
\left[ \int_M u^{\lambda} \eta^p \right]_0^t \leq c_2 \int_0^t \int_M u^\sigma |\nabla \eta|^p. \tag{12.4}
\]

Since \( u(\cdot, 0) = 0 \) on \( \text{supp} \eta \) and \( \eta = 1 \) on \( K_s \setminus K_{2r} \), the left hand side here is bounded below by

\[
\int_{K_s \setminus K_{2r}} u^\lambda(\cdot, t).
\]

Since \( \eta = 0 \) in \( K_s \), \( u(\cdot, \tau) = 0 \) in \( K_{2r} \setminus K_s \) for all \( \tau \leq t \) (by (12.2)), and \( \nabla \eta = 0 \) in \( K_s \setminus K_{2r} \), the right hand side in (12.4) is equal to

\[
c_2 \int_0^t \int_{M \setminus K_s} u^\sigma |\nabla \eta|^p.
\]

Since

\[
|\nabla \eta| \leq \frac{1}{s} \text{ in } M \setminus K_s,
\]

we obtain that

\[
\int_{K_s \setminus K_{2r}} u^\lambda(\cdot, t) \leq c_2 \int_0^t \int_{M \setminus K_s} u^\sigma |\nabla \eta|^p \leq \frac{c_2}{s^p} \int_0^t \int_{M \setminus K_s} u^\sigma.
\]

The right hand side goes to 0 as \( s \to \infty \), which implies that \( u(\cdot, t) = 0 \) in \( M \setminus K_{2r} \), thus proving (12.3).

Now let us define in \([r_0, \infty)\) a function

\[
\psi(r) = \frac{1}{2} \sup_{r_0 \leq s \leq r} \varphi(3s + r_0)
\]
so that \( \psi(r) \) is monotone increasing. If \( t \leq \psi(r) \) then \( t \leq \varphi(3s + r_0) \) for some \( s \in [r_0, r] \), which implies by the first part of the proof that
\[
u(\cdot, t) = 0 \quad \text{in} \quad M \setminus K_s
\]
and, hence,
\[
u(\cdot, t) = 0 \quad \text{in} \quad M \setminus K_r.
\]
It is unclear whether \( \psi \) is continuous or not. As a monotone function, \( \psi \) may have only jump discontinuities. By subtracting all these jumps, we obtain a continuous monotone function \( \tilde{\psi} \leq \psi \) with the same property:
\[
t \leq \tilde{\psi}(r) \Rightarrow \nu(\cdot, t) = 0 \quad \text{in} \quad M \setminus K_r. \tag{12.5}
\]
As a continuous monotone increasing function, \( \tilde{\psi} \) has an inverse \( \rho = \tilde{\psi}^{-1} \) on \([t_0, T)\) where
\[
t_0 = \tilde{\psi}(r_0) \quad \text{and} \quad T = \sup \tilde{\psi}.
\]
Let us extend \( \rho(t) \) to \( t < t_0 \) by setting \( \rho(t) = \rho(t_0) \). Then \( r = \rho(t) \) implies \( t \leq \tilde{\psi}(r) \), and by (12.5)
\[
u(\cdot, t) = 0 \quad \text{in} \quad M \setminus K_r,
\]
which was to be proved. \( \blacksquare \)

### 13 Curvature and propagation rate

In this section we assume again that \( M \) is geodesically complete and
\[
\delta = q(p - 1) - 1 > 0.
\]

**Theorem 11.** Let \( M \) be geodesically complete, non-compact, and let \( \text{Ricci}_M \geq 0 \). Let \( u \) be a bounded non-negative subsolution of \((4.1)\) in \( M \times \mathbb{R}_+ \) with the initial condition \( u(\cdot, 0) = u_0 \). Set \( K = \text{supp} u_0 \). Then, for any \( t \geq 0 \),
\[
\text{supp} u(\cdot, t) \subset K_{Ct^{1/p}},
\]
where \( C \) depends on \( \|u_0\|_{L^\infty}, p, q, n. \)

**Proof.** It is known that on such manifolds \( \iota(B) \geq \text{const} > 0 \) for all balls \( B \subset M \).

Let \( B = B(x, r) \) be any ball that is disjoint with \( K \). It follows from Theorem 9, that is
\[
t \leq cr^p \|u_0\|_{L^\infty(M)}^{-\delta},
\]
where \( c > 0 \) is a small enough constant, then
\[
u(\cdot, t) = 0 \quad \text{in} \quad \frac{1}{2} B.
\]
Hence, if
\[
r \geq Ct^{1/p}
\]
where $C = c^{-1/p} \|u_0\|_{L^\infty(M)}^{\delta/p}$, then

$$\text{supp } u(\cdot, t) \cap \frac{1}{2} B = \emptyset.$$  

It follows that

$$\text{supp } u(\cdot, t) \subset K_{\frac{1}{2}r},$$

whence the claim follows.  ■