Finite propagation speed for Leibenson’s equation on Riemannian manifolds

Alexander Grigor’yan
http://www.math.uni-bielefeld.de/~grigor

Based on a joint work with Philipp Sürig

August 2022
1 Introduction

We are concerned with an evolution equation

$$\partial_t u = \Delta_p u^q$$

(1)

where $p, q > 0$, $u(x, t)$ is an unknown non-negative function, and $\Delta_p$ is the $p$-Laplacian:

$$\Delta_p v = \text{div} \left( |\nabla v|^{p-2} \nabla v \right).$$

Equation (1) was introduced by L. S. Leibenson in 1945 in order to describe filtration of turbulent compressible fluid through a porous medium. The physical meaning of $u$ is the volumetric moisture content, i.e. the (infinitesimal) fraction of volume of the medium taken by the liquid.

Parameter $p$ characterizes the turbulence of a flow while $q - 1$ is the index of polytropy of the liquid, that determines relation $PV^{q-1} = \text{const}$ between volume $V$ and pressure $P$.

Leonid Samuilovich Leibenson
The physically interesting values of $p$ and $q$ are as follows: $\frac{3}{2} \leq p \leq 2$ and $q \geq 1$.

The case $p = 2$ corresponds to laminar flow (=absence of turbulence). In this case (1) becomes a *porous medium* equation $\partial_t u = \Delta u^q$, if $q > 1$, and the classical heat equation $\partial_t u = \Delta u$ if $q = 1$.

From mathematical point of view, the entire range $p > 1$, $q > 0$ is interesting.

G.I.Barenblatt constructed in 1952 spherically symmetric self-similar solutions of (1) in $\mathbb{R}^n$ that are nowadays called *Barenblatt solutions*. Let us assume that $q(p-1) > 1$.

In this case the Barenblatt solution is as follows:

$$u(x, t) = \frac{1}{t^{n/\beta}} \left( C - \kappa \left( \frac{|x|}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)^\gamma +,$$

where $C > 0$ is any constant, and

$$\beta = p + n [q(p-1) - 1] , \quad \gamma = \frac{p-1}{q(p-1)-1} , \quad \kappa = \frac{q(p-1)-1}{pq} \beta^{-\frac{1}{p-1}}.$$

(2)
Parameter $\beta$ determines the space/time scaling and is analogous to the walk dimension. It is obvious that for the Barenblatt solution

$$u(x, t) = 0 \text{ for } |x| > ct^{1/\beta}$$

so that $u(\cdot, t)$ has a compact support for any $t$. One says that $u$ has a finite propagation speed.

Here are the graphs of function $x \mapsto u(x, t)$ for different values of $t$ in the case $n = 1$.

In the case $q(p - 1) < 1$, we have $\gamma, \kappa < 0$, and the Barenblatt solution

$$u(x, t) = \frac{1}{t^{n/\beta}} \left( C + |\kappa| \left( \frac{r}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)^\gamma$$

is positive for all $x, t$. In the borderline case $q(p - 1) = 1$, the Barenblatt solution is

$$u(x, t) = \frac{1}{t^{n/p}} \exp \left( -c \left( \frac{r}{t^{1/p}} \right)^{\frac{p}{p-1}} \right),$$

where $c = (p - 1)^2 p^{-\frac{p}{p-1}}$. Hence, if $q(p - 1) \leq 1$ then $u$ has infinite propagation speed.
2 \( p \)-Laplacian on Riemannian manifolds

Our goal is to investigate finite propagation speed for Leibenson’s equation (1) on an arbitrary Riemannian manifold \( M \). Solutions are understood in a certain weak sense. Consider first the case \( q = 1 \), that is, the following evolution equation for \( p \)-Laplacian:

\[ \partial_t u = \Delta_p u. \]

For this equation the following result was known.

**Theorem 1** (S. Dekkers 2005) Let \( p > 2 \) and let \( u(x, t) \) be a bounded non-negative solution to \( \partial_t u = \Delta_p u \) on \( M \times \mathbb{R}_+ \) with initial function \( u_0 = u(\cdot, 0) \).

Let \( B_0 = B(x_0, R) \) be a precompact ball in \( M \) such that \( u_0 = 0 \) in \( B_0 \). Then

\[ u = 0 \text{ in } \frac{1}{2} B_0 \times [0, t_0], \]

where

\[ t_0 = \eta R^p \| u_0 \|_{L^\infty(M)}^{(p-2)} \]

and \( \eta > 0 \) depends on the intrinsic geometry of \( B_0 \).
Hence, solution $u$ has a finite propagation speed inside $B_0$, and the speed of propagation depends on the geometry of $B_0$ via the constant $\eta$.

For any set $K \subset M$, denote by $K_r$ the open $r$-neighborhood of $K$.

**Corollary 2** Let $M$ be complete and non-compact, $p > 2$, and $K = \text{supp } u_0$ be compact. Then there exists an increasing function

$$r : (0, T) \rightarrow \mathbb{R}_+$$

for some $T \in (0, \infty]$, such that

$$\text{supp } u(\cdot, t) \subset K_{r(t)}$$

for all $t \in (0, T)$.

If $\text{Ricci}_M \geq 0$ then $r(t) = Ct^{\frac{1}{p}}$ and $T = \infty$.

The function $r(t)$ is called a *propagation rate* of $u$. Using the Barenblatt solution in $\mathbb{R}^n$, one obtains that a propagation rate in $\mathbb{R}^n$ for large $t$ is

$$r(t) = C t^{\frac{1}{p+\frac{n}{p-2}}}$$

so that the result of Corollary 2 is not sharp in this case.
3 Main result

On an arbitrary manifold $M$ of dimension $n$, consider Leibenson’s equation

$$\partial_t u = \Delta_p u^q,$$  \hspace{1cm} (3)

where we assume that $p > 2$ and $\frac{1}{p-1} < q \leq 1$. In particular, $q(p-1) - 1 > 0$.

**Theorem 3** Let $u$ be a bounded non-negative subsolution of (3) in $M \times \mathbb{R}_+$, and assume that $u_0 := u(\cdot,0) \in L^1(M)$. Let $B_0$ be a precompact ball of radius $R$ s.t. $u_0 = 0$ in $B_0$. Then $u = 0$ in $\frac{1}{2}B_0 \times [0,t_0]$, where

$$t_0 = \eta R^p \mu(B_0) \frac{q(p-1)-1}{\sigma} \|u_0\|_{L^\sigma(M)}^{-[q(p-1)-1]}.$$  \hspace{1cm} (4)

Here $\sigma$ is any real such that $\sigma \geq 1$ and $\sigma > q(p-1) - 1$ and $\eta = \eta(B_0,p,q,n,\sigma) > 0$.

Besides, the value $\sigma = \infty$ is also included and in this case

$$t_0 = \eta R^p \|u_0\|_{L^\infty(M)}^{-[q(p-1)-1]}.$$
Theorem 1 is a particular case of Theorem 3 for $q = 1$ and $\sigma = \infty$.

In the next two statements $M$ is complete and non-compact, and $K = \text{supp } u_0$ is compact.

**Corollary 4** There exists $T \in (0, T]$ and an increasing function $r : (0, T) \to \mathbb{R}_+$ such that

$$\text{supp } u(\cdot, t) \subset K_{r(t)}$$

for all $t \in (0, T)$.

Let us refer to $r(t)$ as a propagation rate of solution $u$.

**Corollary 5** Assume that $\text{Ricci}_M \geq 0$. Fix $x_0 \in K$ and assume that

$$\mu(B(x_0, r)) \geq cr^\alpha \text{ for all } r \geq r_0,$$

with some $c, \alpha > 0$. Then a propagation rate is $r(t) = Ct^{1/\beta}$ for $t \geq t_0$, where

$$\beta = p + \alpha \frac{q(p - 1) - 1}{\sigma}.$$  \((5)\)
Recall that in $\mathbb{R}^n$ a propagation rate is $r(t) = Ct^{1/\beta}$ where
\[ \beta = p + n [q(p - 1) - 1]. \]
Since in $\mathbb{R}^n \alpha = n$, we see that the value of $\beta$ in (5) is sharp if $\sigma = 1$.

We can take $\sigma = 1$ in Theorem 3 provided $q(p - 1) - 1 < 1$, that is, when
\[ 1 < q(p - 1) < 2. \]

This range of $p, q$ is shown here:

For this range of $p, q$, we obtain a sharp propagation rate not only in $\mathbb{R}^n$ but also in a large class of models with $Ricci \geq 0$, with any $\alpha \in (0, n]$.

**Conjecture 6** The statement of Theorem 3 holds for $\sigma = 1$ and for all
\[ p > 1 \quad \text{and} \quad q > \frac{1}{p - 1}. \]
4 Mean value inequality

The main ingredient in the proof of Theorem is the following mean value inequality.

**Theorem 7** Let $B = B(x_0, r)$ be a precompact ball in $M$.

Let $u$ be a non-negative bounded subsolution of (3) in the cylinder $Q = B \times [0, T]$, and let $u(\cdot, 0) = 0$ in $B$. Then, for the cylinder $Q' = \frac{1}{2}B \times [0, T]$, the following inequality holds:

$$
\| u \|_{L^\infty(Q')} \leq \left( \frac{C}{\mu(B)r^p} \int_Q u^{\lambda + [q(p-1)-1]} \right)^{1/\lambda},
$$

where $\lambda > 0$ is any and $C$ depends on $p, q, \lambda$ and on the intrinsic geometry of $B$.  

\[ T \]
\[ u(\cdot,0)=0 \]
Proof of Theorem 7 starts with the following Lemma.

**Lemma 8**  Let $u$ be a non-negative subsolution of (3).

Set

$$a = \frac{q(p - 1) - 1}{p - 2}.$$

If $0 < a \leq 1$ then the function

$$v = (u^a - \theta)^{1/a}$$

is a subsolution for any $\theta > 0$.

The condition $0 < a \leq 1$ holds, in particular, in the case when

$$p > 2, \quad q(p - 1) > 1 \quad \text{and} \quad q \leq 1.$$

For the $p$-Laplacian case, that is, when $q = 1$, we have $a = 1$. In this case it is well known that $v = (u - \theta)_+$ is a subsolution. If also $p = 2$ that is, if (3) is the heat equation then $v = f(u)$ is a subsolution for any convex $f$. 
Sketch of proof of Theorem 7. Fix some \( \theta > 0 \) and define a sequence \( \{u_k\}_{k=0}^{\infty} \) of functions:

\[
u_0 = u, \quad u_k = (u^a_{k-1} - 2^{-k}\theta)^{1/a}_+ \quad \text{for} \quad k \geq 1
\]

It is easy to see that \( u_k = (u^a - (1 - 2^{-k}) \theta)^{1/a}_+ \).

Consider a decreasing sequence of radii

\[
r_k = \left( \frac{1}{2} + 2^{-k-1} \right) r
\]

so that \( r_0 = r \geq r_k \setminus \frac{1}{2} r \), and cylinders

\[
Q_k = B(x_0, r_k) \times [0, t]
\]

so that

\[
Q_0 = Q \supset Q_k \setminus Q'
\]

as \( k \to \infty \).

Set

\[
J_k = \int_{Q_k} u_{k}^{\lambda + [q(p-1)-1]}
\]

Clearly, \( J_{k+1} \leq J_k \). Using a Caccioppoli type inequality for \( u_k \) and \( u_{k+1} \) as well as a certain Faber-Krahn type inequality for \( \Delta_p \) in \( B \) (which reflects the intrinsic geometry of \( B \)), we prove that
\[ J_{k+1} \leq \frac{CA^k}{(\mu(B)\theta_{\frac{\lambda}{a}r^p})^\nu} J_k^{1+\nu}, \]

where \( \nu = p/n \) is the Faber-Krahn exponent for \( \Delta_p \) and \( C, A \) are some constants.

Analyzing this recursive inequality we show that if

\[ \theta \geq \left( \frac{CJ_0}{\mu(B)r^p} \right)^\frac{a}{\lambda}, \quad (7) \]

then \( J_k \to 0 \) as \( k \to \infty \), which implies

\[ \int_{Q'} \left[ (u^a - \theta)^{1/a}_+ \right]^{\lambda + q(p-1)-1} = 0, \]

that is, \( u^a \leq \theta \) in \( Q' \). Choosing the minimal value of \( \theta \) in (7), we obtain

\[ u \leq \left( \frac{CJ_0}{\mu(B)r^p} \right)^\frac{1}{\lambda} = \left( \frac{C}{\mu(B)r^p} \int_Q u^{\lambda + q(p-1)-1} \right)^\frac{1}{\lambda} \text{ in } Q' \]

which proves (6).

This method works for \( \lambda \geq 2 \). The case \( 0 < \lambda < 2 \) is obtained from \( \lambda = 2 \) using an additional iteration procedure. \( \blacksquare \)
5 From mean value to finite propagation speed

Sketch of proof of Theorem 3. Given a ball $B_0$ of radius $R$, set $r = \frac{1}{2} R$, fix some $t > 0$ and $x \in \frac{1}{2} B_0$, and set for any $k \in \mathbb{N}$

$$Q_k = B(x, 2^{-k} r) \times [0, t] \quad \text{and} \quad J_k = \int_{Q_k} u^\sigma.$$
Applying the mean value inequality (6) in $Q_k$ and $Q'_k = Q_{k+1}$ with

$$\lambda = \sigma - [q(p - 1) - 1] > 0$$

we obtain

$$\|u\|_{L^\infty(Q_{k+1})} \leq \left( \frac{C}{\mu(B(x, 2^{-k}r)) (2^{-k}r)^p} \int_{Q_k} u^\sigma \right)^{1/\lambda}.$$ 

Raising this to power $\sigma$ and integrating over $Q_{k+1}$, we obtain

$$J_{k+1} \leq t \mu(B_0) \left( \frac{C^k}{\mu(B_0) R^p J_k} \right)^{\sigma/\lambda}.$$ 

Since $\sigma/\lambda > 1$, iteration of this inequality allows to prove that $J_k$ decays double exponentially in $k$ provided $t \leq t_0$ (where $t_0$ is determined by (4)):

$$J_k = \int_{B(x, 2^{-k}r) \times [0,t]} u^\sigma \leq CA^{-(\sigma/\lambda)^k}, \quad (8)$$

and this is true for all $x \in \frac{1}{2}B_0$ and $k \geq 0$, with the same constants $C$ and $A > 1$. 

14
For any fixed $k$, let us cover $\frac{1}{2}B_0 = B(x_0, r)$ by a sequence of balls $B(x_i, 2^{-k}r)$ with some $x_i \in \frac{1}{2}B_0$. The minimal number of such balls is bounded by $D^k$ for some constant $D$.

Hence, adding up (8) for all $x = x_i$, we obtain

$$\int_{Q'} u^\sigma \leq CD^k A^{-(\sigma/\lambda)k}.$$ 

This inequality holds for any $k$. Letting $k \to \infty$ and noticing that the right hand side $\to 0$ thanks to $\sigma/\lambda > 1$ and $A > 1$, we obtain that $u = 0$ in $Q'$. □
References


