Leibenson’s equation on Riemannian manifolds

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1 Introduction

We are concerned with an evolution equation

$$\partial_t u = \Delta_p u^q$$

(1)

where $p, q > 0$, $u(x, t)$ is an unknown non-negative function, and $\Delta_p$ is the $p$-Laplacian:

$$\Delta_p v = \text{div} \left( |\nabla v|^{p-2} \nabla v \right).$$

Equation (1) was introduced by L. S. Leibenson in 1945 in order to describe filtration of turbulent compressible fluid through a porous medium. The physical meaning of $u$ is the volumetric moisture content, i.e. the (infinitesimal) fraction of volume of the medium taken by the liquid.

Parameter $p$ characterizes the turbulence of a flow while $q - 1$ is the index of polytropy of the liquid, that determines relation $PV^{q-1} = \text{const}$ between volume $V$ and pressure $P$. 

Leonid Samuilovich Leibenson
The physically interesting values of $p$ and $q$ are as follows: $\frac{3}{2} \leq p \leq 2$ and $q \geq 1$.

The case $p = 2$ corresponds to laminar flow (=absence of turbulence). In this case (1) becomes a porous medium equation $\partial_t u = \Delta u^q$, if $q > 1$, and the classical heat equation $\partial_t u = \Delta u$ if $q = 1$.

From mathematical point of view, the entire range $p > 1$, $q > 0$ is interesting.

G.I.Barenblatt constructed in 1952 spherically symmetric self-similar solutions of (1) in $\mathbb{R}^n$ that are nowadays called Barenblatt solutions. Let us assume that

$$q(p - 1) > 1.$$ 

In this case the Barenblatt solution is as follows:

$$u(x, t) = \frac{1}{t^{n/\beta}} \left( C - \kappa \left( \frac{|x|}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)^\gamma,$$

where $C > 0$ is any constant, and

$$\beta = p + n [q(p - 1) - 1] , \quad \gamma = \frac{p-1}{q(p-1)-1} , \quad \kappa = \frac{q(p-1)-1}{pq} \beta^{-\frac{1}{p-1}} .$$

(2)
Parameter $\beta$ determines the space/time scaling and is analogous to the walk dimension.

It is obvious that for the Barenblatt solution

$$u(x, t) = 0 \quad \text{for} \quad |x| > ct^{1/\beta}$$

so that $u(\cdot, t)$ has a compact support for any $t$. One says that $u$ has a finite propagation speed.

Here are the graphs of function $x \mapsto u(x, t)$ for different values of $t$ in the case $n = 1$.

In the case $q(p - 1) < 1$, we have $\gamma, \kappa < 0$, and the Barenblatt solution

$$u(x, t) = \frac{1}{t^{n/\beta}} \left( C + |\kappa| \left( \frac{r}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)^{-|\gamma|}$$

is positive for all $x, t$. In the borderline case $q(p - 1) = 1$, the Barenblatt solution is

$$u(x, t) = \frac{1}{t^{n/p}} \exp \left( -c \left( \frac{r}{t^{1/p}} \right)^{\frac{p}{p-1}} \right),$$

where $c = (p - 1)^2 p^{-\frac{p}{p-1}}$. Hence, if $q(p - 1) \leq 1$ then $u$ has infinite propagation speed.
2  Propagation speed inside a ball

On an arbitrary manifold $M$ of dimension $n$, consider Leibenson’s equation

$$\partial_t u = \Delta_p u^q,$$

(3)

where we assume that

$$p > 1 \text{ and } q > \frac{1}{p-1},$$

(4)

that is, $\delta := q(p - 1) - 1 > 0$. Solutions of (3) are understood in a certain weak sense.

**Theorem 1** Let $u$ be a bounded non-negative subsolution of (3) in $M \times \mathbb{R}_+$. Let $B$ be a precompact ball in $M$ of radius $R$, such that $u_0 := u(\cdot, 0) = 0$ in $B$. Then

$$u(\cdot, t) = 0 \text{ in } \frac{1}{2} B \text{ for all } t \leq t_0,$$

where

$$t_0 = \eta R^p \|u_0\|^{-\delta}_{L^\infty(M)}$$

and $\eta > 0$ depends on intrinsic geometry of $B$. 

\[4\]
Note that the range (4) of parameters \( p, q \) is the same as that in the Barenblatt solutions with a finite propagation speed.

The only previously known case of Theorem 1 was when \( p > 2 \) and \( q = 1 \), that is, when (3) is the equation \( \partial_t u = \Delta_p u \). In this case a finite propagation speed was proved by S. Dekkers in *Comm. Anal. Geom.* 14 (2005).

Another interesting case is when \( p = 2 \) and \( q > 1 \), that is, when (3) is a *porous medium* equation \( \partial_t u = \Delta u^q \). Theorem 1 is new in this case.

**Remark.** The constant \( \eta \) depends on the *normalized Sobolev constant* \( c_B \) in \( B \): for any \( u \in W^{1,p}_0(B) \)

\[
\left( \int_B |\nabla u|^p \right)^{1/p} \geq \frac{c_B}{R} \left( \int_B |u|^\kappa \right)^{1/p\kappa}
\]

where \( \kappa \) is the Sobolev exponent: \( \kappa = \frac{n}{n-p} \) if \( n > p \) and \( \kappa > 1 \) is any if \( n \leq p \).

**Remark.** The Leibenson equation (3), that is, \( \partial_t u = \Delta_p u^q \) can be equivalently rewritten in the form

\[ \partial_t u = \text{div} \left( u^{m-1} |\nabla u|^{p-2} \nabla u \right), \]

where \( m = 1 + (q-1)(p-1) = \delta + 3 - p \). The condition \( \delta > 0 \) is, hence, equivalent to \( m + p > 3 \). Therefore, Theorem 1 holds for this equation when \( p > 1 \) and \( m + p > 3 \).
3 Finite propagation speed of support

Let $M$ be complete. Let $u$ be a bounded non-negative subsolution of (3) with $u(\cdot, 0) = u_0$. For any set $K \subset M$ and any $r > 0$, denote by $K_r$ the closed $r$-neighborhood of $K$.

**Corollary 2** Let $K := \text{supp } u_0$ be a compact set. Then there an increasing positive function $r : (0, T) \rightarrow \mathbb{R}_+$ with some $T \in (0, \infty]$ such that

$$\text{supp } u(\cdot, t) \subset K_{r(t)}$$

for all $t \in (0, T)$.

Function $r(t)$ is referred to as a *propagation rate* of solution $u$.

**Problem 3** Is it true that one can always have $T = \infty$? Either prove it or give a counterexample: a complete manifold and a solution $u$ such that $\text{supp } u_0$ is compact, while $\text{supp } u(\cdot, t)$ is unbounded for large enough $t$. 
Let $M$ have non-negative Ricci curvature. Then the normalized Sobolev constant $c_B$ in (5) can be taken the same for all balls and, hence, the constant $\eta$ from Theorem 1 is also the same for all balls, which allows to obtain the following.

**Corollary 4** If $\text{Ricci}_M \geq 0$ then any subsolution $u$ with compactly supported $u_0$ has a propagation rate $r(t) = Ct^{1/p}$ for all $t > 0$.

Recall that in $\mathbb{R}^n$ the propagation rate of the Barenblatt solution is $r(t) = Ct^{1/\beta}$ where

$$\beta = p + n \left[ q(p - 1) - 1 \right] = p + n\delta. \quad (6)$$

This implies that, for any bounded non-negative solution $u$ in $\mathbb{R}^n$ with compactly supported $u_0$, the propagation rate is also $r(t) = Ct^{1/\beta}$ for large $t$.

Since $p < \beta$, we see that the propagation rate of the above Corollary is not sharp in $\mathbb{R}^n$. 

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4 Sharp propagation rate

We assume here that

\[ p > 2 \text{ and } \frac{1}{p-1} < q \leq 1. \]

**Theorem 5** Let \( u \) be a bounded non-negative subsolution of (3) in \( M \times \mathbb{R}_+ \), with initial function \( u_0 := u(\cdot, 0) \in L^1 \). Let \( B \) be a precompact ball of radius \( R \) s.t. \( u_0 = 0 \) in \( B \). Then

\[ u(\cdot, t) = 0 \text{ in } \frac{1}{2}B \text{ for all } t \leq t_0 \]

where

\[ t_0 = \eta R^p \mu(B) \frac{\delta}{\sigma} \| u_0 \|_{L^\sigma(M)}^{-\delta}. \]

Here \( \sigma \) is any real number such that

\[ \sigma \geq 1 \text{ and } \sigma > \delta, \quad (\dagger) \]

and \( \eta = \eta(B, p, q, n, \sigma) > 0 \).
**Corollary 6** Assume that $M$ is complete and $\text{Ricci}_M \geq 0$. Fix a point $x_0 \in \text{supp} \ u_0$ and assume that

$$\mu(B(x_0, r)) \geq cr^\alpha \text{ for all } r \geq r_0,$$

with some $c, \alpha > 0$. Then $u$ has propagation rate $r(t) = Ct^{1/\beta}$ for large $t$, where

$$\beta = p + \alpha \frac{\delta}{\sigma}$$

(7)

and $\sigma$ is as in $\left(\frac{7}{7}\right)$.

In $\mathbb{R}^n$ we have $\alpha = n$. Setting $\sigma = 1$, we obtain $\beta = p + n \delta$ that matches (6). However, we can take $\sigma = 1$ in $\left(\frac{7}{7}\right)$ only if $\delta < 1$, that is, if $q(p - 1) < 2$.

The next diagram shows the following range of $p, q$:

$$p > 2 \text{ and } 1 < q(p - 1) < 2.$$

For these $p, q$, we obtain a sharp propagation rate not only in $\mathbb{R}^n$, but also in a large class of model manifolds with $\text{Ricci}_M \geq 0$ and with any $\alpha \in (0, n]$.

**Conjecture 7** The result of Theorem 5 holds for all $p > 1$, $q > \frac{1}{p-1}$ and for $\sigma = 1$. 

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5 Mean value inequality

The main ingredient of the proof of Theorem 1 is the following mean value inequality. We assume here that $p > 1$ and $\delta \geq 0$.

**Lemma 8** Let a ball $B = B(x_0, R)$ be precompact. Let $u$ be a non-negative bounded subsolution of (3) in cylinder

$$Q = B \times [0, T],$$

such that $u(\cdot, 0) = 0$ in $B$. Then, for the cylinder

$$Q' = \frac{1}{2} B \times [0, T],$$

and for any

$$\lambda \geq \max (2 + \delta, p),$$

the following inequality holds:

$$\|u\|_{L^\infty(Q')} \leq C \left( \frac{T}{R^p} \right)^{1/\lambda} \|u\|_{L^\infty(Q)}^{\delta/\lambda} \left( \int_Q u^\lambda \right)^{1/\lambda},$$

(8)

where $C = C(B, p, n, \lambda)$. 

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[Diagram of cylinders $Q$ and $Q'$ with notation and conditions for $u$ and $\lambda$.]

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For the proof we use the Sobolev inequality inside $B$ and Moser’s iteration argument.

For that consider a shrinking sequence of cylinders $\{Q_k\}_{k=0}^\infty$ interpolating between $Q_0 = Q$ and $Q_\infty = Q'$, and prove that

$$\int_{Q_{k+1}} u^{\sigma(1+\nu)} \leq C(\cdots) \left(\int_{Q_k} u^{\sigma}\right)^{1+\nu} \quad (*)$$

for $\sigma \gg 1$ and $\nu > 0$ that comes from the Sobolev inequality.

In the classical Moser argument, one proves $(*)$ first for $\sigma = 2$ and then applies this inequality also to $u^{\sigma/2}$ with any $\sigma > 2$ because $u^{\sigma/2}$ is also subsolution. This allows to set in $(*)$ $\sigma = \lambda (1 + \nu)^k$ and to reach $\|u\|_{L^\infty(Q')}^*$ in the left hand side as $k \to \infty$.

In our case this trick is not possible: no power of subsolution is again a subsolution. Hence, we need to prove $(*)$ directly for any $\sigma$ and to compute carefully the constant $C = C(\sigma)$ in $(*)$. It turns out that $C \simeq \sigma^A$ for some $A$ and, surprisingly, this moderate growth of $C$ with $\sigma$ still allows to complete the iteration argument and to obtain (8).

Using $\left(\int_Q u^{\lambda}\right)^{1/\lambda} \leq \|u\|_{L^\infty(Q)}$, we obtain from (8)

$$\|u\|_{L^\infty(Q')} \leq C \left(\frac{T}{R^p}\right)^{1/\lambda} \|u\|_{L^\infty(Q)}^{1+\delta/\lambda}. \quad (9)$$
6 From mean value to finite propagation speed

Sketch of proof of Theorem 1. Set \( r = \frac{1}{2}R \) and fix for a while a point \( x \in \frac{1}{2}B \).

Hence, we have \( B(x, r) \subset B \).

Fix also some \( t > 0 \) and set

\[
Q_k = B(x, 2^{-k}r) \times [0, t],
\]

\[
J_k = \|u\|_{L^\infty(Q_k)}.
\]

Let \( \lambda \) be as it is needed for Lemma 8. Then by (9)

\[
J_{k+1} \leq C \left( \frac{t}{(2^{-k}R)^p} \right)^{1/\lambda} J_k^{1 + \frac{\delta}{\lambda}} = C 2^{k/\lambda} \left( \frac{t}{R^p} \right)^{1/\lambda} J_k^{1 + \frac{\delta}{\lambda}}.
\]

Iterating this inequality, we obtain an upper bound of \( J_k \) via \( J_0 \) that implies the following: if

\[
C \left( \frac{t}{R^p} \right)^{1/\lambda} \leq 2^{-1/\delta} J_0^{-\delta/\lambda}
\]
then, for all $k$,
\[ J_k \leq 2^{-k/\delta} J_0. \] (11)

The condition (10) is equivalent to
\[ t \leq \eta R^p J_0^{-\delta}. \] (12)

Since $J_0 = \|u\|_{L^\infty(Q)} \leq \|u_0\|_{L^\infty(M)}$ and, hence,
\[ t_0 = \eta R^p \|u_0\|^{-\delta}_{L^\infty(M)} \leq \eta R^p J_0^{-\delta}, \]
we see that (12) is satisfied for $t = t_0$. For this $t$, we obtain from (11) that, for any $k$,
\[ \|u\|_{L^\infty(B(x,2^{-k}r) \times [0,t])} \leq 2^{-k/\delta} \|u_0\|_{L^\infty}. \]

For any $k$, we cover the ball $\frac{1}{2} B$ by a countable (or even finite) sequence of balls $B(x_i, 2^{-k}r)$ with $x_i \in \frac{1}{2} B$. Since for all $i$
\[ \|u\|_{L^\infty(B(x_i,2^{-k}r) \times [0,t])} \leq 2^{-k/\delta} \|u_0\|_{L^\infty}, \]
we obtain that
\[ \|u\|_{L^\infty(\frac{1}{2} B \times [0,t])} \leq 2^{-k/\delta} \|u_0\|_{L^\infty}. \]

Finally, letting $k \to \infty$, we obtain that $u = 0$ in $\frac{1}{2} B \times [0, t]$, which was to be proved. □
7 Mean value inequality 2

The main ingredient in the proof of Theorem 5 is the following version of the mean value inequality. We assume here that \( p > 2 \) and \( \frac{1}{p-1} < q \leq 1 \).

Lemma 9 Let \( B = B(x_0, R) \) be a precompact ball in \( M \).

Let \( u \) be a non-negative bounded subsolution of (3) in the cylinder
\[
Q = B \times [0, T],
\]
and let \( u(\cdot, 0) = 0 \) in \( B \). Then,

for the cylinder
\[
Q' = \frac{1}{2} B \times [0, T],
\]

the following inequality holds:

\[
\|u\|_{L^\infty(Q')} \leq C \left( \frac{T}{R^p} \right)^{1/\lambda} \left( \int_Q u^{\lambda + \delta} \right)^{1/\lambda},
\]

where \( \lambda > 0 \) is any, \( \delta = q(p-1) - 1 \), and \( C = C(B, p, \delta, \lambda) \).
In the proof of Lemma 9 we use the following lemma.

**Lemma 10** Let $u$ be a non-negative subsolution of (3).

Set

$$a = \frac{q(p - 1) - 1}{p - 2}.$$ 

If $0 < a \leq 1$ then the function

$$v = (u^a - \theta)^{1/a}$$

is a subsolution for any $\theta > 0$.

The condition $0 < a \leq 1$ holds, in particular, in the case when

$$p > 2 \quad \text{and} \quad \frac{1}{p - 1} < q \leq 1$$

For the $p$-Laplacian case, that is, when $q = 1$, we have $a = 1$. In this case it is well known that $v = (u - \theta)_+$ is a subsolution. If also $p = 2$ that is, if (3) is the heat equation, then $v = f(u)$ is a subsolution for any convex $f$. 

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Sketch of proof of Lemma 9. Fix some $\theta > 0$ and define a sequence \( \{u_k\}_{k=0}^{\infty} \) of functions:

\[
    u_0 = u, \quad u_k = (u_{k-1}^a - 2^{-k}\theta)^{1/a} \quad \text{for} \quad k \geq 1
\]

It is easy to see that \( u_k = (u^a - (1 - 2^{-k})\theta)^{1/a} \).

Consider a decreasing sequence of radii \( r_k = (\frac{1}{2} + 2^{-k-1}) R \) so that \( r_0 = r \geq r_k \searrow \frac{1}{2} R \), and cylinders \( Q_k = B(x_0, r_k) \times [0, t] \) so that \( Q_0 = Q \supset Q_k \searrow Q' \) as \( k \to \infty \).

Set

\[
    J_k = \int_{Q_k} u_{k+1}^{\lambda+\delta}. \quad \text{(16)}
\]

Clearly, \( J_{k+1} \leq J_k \). Using a Caccioppoli type inequality for \( u_k \) and \( u_{k+1} \) as well as a certain Faber-Krahn type inequality for \( \Delta_p \) in \( B \) (which reflects the intrinsic geometry of \( B \)), we prove that
\[ J_{k+1} \leq \frac{CA^k}{\left(\frac{\mu(B)\theta^\alpha}{\lambda r^p}\right)^\nu} J_k^{1+\nu}, \]

where \( \nu > 0 \) is the Faber-Krahn exponent for \( \Delta_p \), and \( C, A \) are some constants.

Analyzing this recursive inequality, we show that if

\[ \theta \geq \left( \frac{C J_0}{\mu(B) r^p} \right)^\frac{\alpha}{\lambda}, \quad (14) \]

then \( J_k \to 0 \) as \( k \to \infty \), which implies

\[ \int_{Q'} \left[ (u^a - \theta)^{1/a^+} \right]^\lambda + \delta = 0, \]

that is, \( u^a \leq \theta \) in \( Q' \). Choosing the minimal value of \( \theta \) in (14), we obtain

\[ u \leq \left( \frac{C J_0}{\mu(B) r^p} \right)^\frac{1}{\lambda} = \left( \frac{C}{\mu(B) r^p} \int_Q u^{\lambda + \delta} \right)^\frac{1}{\lambda} \text{ in } Q' \]

which proves (13).

This method works for \( \lambda \geq 2 \). The case \( 0 < \lambda < 2 \) is obtained from \( \lambda = 2 \) using an additional iteration procedure.