

Leibenson's equation on Riemannian manifolds

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1 Introduction

We are concerned with an evolution equation

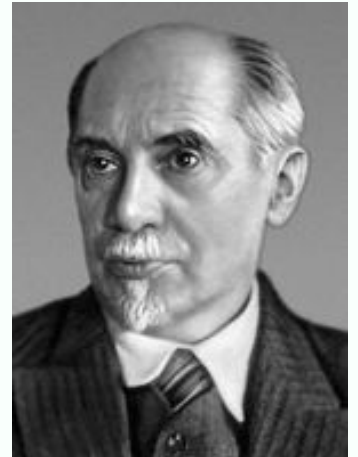
$$\partial_t u = \Delta_p u^q \quad (1)$$

where $u(x, t)$ is an unknown non-negative function, p, q are positive constants and Δ_p is the p -Laplacian:

$$\Delta_p v = \operatorname{div} (|\nabla v|^{p-2} \nabla v).$$

Equation (1) was introduced by L. S. Leibenson in 1945 in order to describe filtration of turbulent compressible fluid through a porous medium. The physical meaning of u is the *volumetric moisture content*, i.e. the (infinitesimal) fraction of volume of the medium taken by the liquid.

Parameter p characterizes the turbulence of a flow whereas q determines the compressibility of the liquid.



Leonid Samuilovich Leibenson



Caspian Sea and Baku



Oil fields

The physically meaningful values of p and q are as follows: $\frac{3}{2} \leq p \leq 2$ and $q \geq 1$.

The case $p = 2$ corresponds to laminar flow (=absence of turbulence). In this case (1) becomes a *porous medium* equation $\partial_t u = \Delta u^q$, if $q > 1$, and the classical heat equation $\partial_t u = \Delta u$ if $q = 1$.

In what follows, let us always assume that $p > 1$ and $q > 0$.

G.I.Barenblatt constructed in 1952 spherically symmetric self-similar solutions of (1) in \mathbb{R}^n that are nowadays called *Barenblatt solutions*. Let us first assume that

$$\delta := q(p-1) - 1 > 0.$$

In this case the Barenblatt solution is as follows:

$$u(x, t) = \frac{1}{t^{n/\beta}} \left(C - \kappa \left(\frac{|x|}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)_+^\gamma,$$

where $C > 0$ is any constant, and

$$\beta = p + n\delta, \quad \gamma = \frac{p-1}{\delta}, \quad \kappa = \frac{\delta}{pq} \beta^{-\frac{1}{p-1}}. \quad (2)$$



Grigory Isaakovich Barenblatt

Parameter β determines the space/time scaling and is analogous to the *walk dimension*.

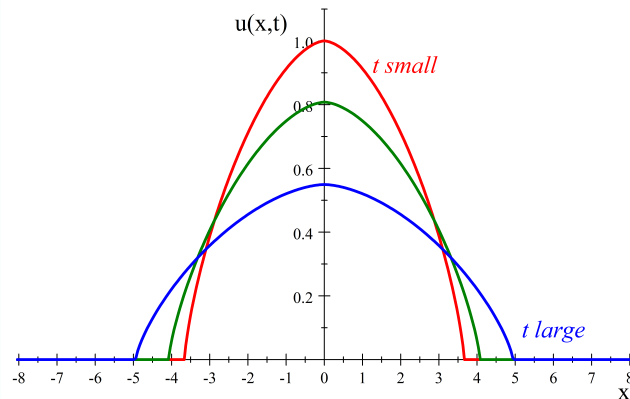
It is obvious that for the Barenblatt solution

$$u(x, t) = 0 \quad \text{for } |x| > ct^{1/\beta}$$

so that $u(\cdot, t)$ has a *compact support* for any t .

One says that u has a *finite propagation speed*.

Here are the graphs of function $x \mapsto u(x, t)$ for different values of t in the case $n = 1$:



In the opposite case $\delta < 0$, that is, $q(p - 1) < 1$, we have $\gamma, \kappa < 0$, and the Barenblatt solution

$$u(x, t) = \frac{1}{t^{n/\beta}} \left(C + |\kappa| \left(\frac{|x|}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)^{-|\gamma|}$$

is positive for all x, t . In the borderline case $q(p - 1) = 1$, the Barenblatt solution is

$$u(x, t) = \frac{1}{t^{n/p}} \exp \left(-c \left(\frac{|x|}{t^{1/p}} \right)^{\frac{p}{p-1}} \right),$$

where $c = (p - 1)^2 p^{-\frac{p}{p-1}}$. Hence, if $\delta \leq 0$ then u has infinite propagation speed.

2 Propagation speed inside a ball

From now on let M be a geodesically complete Riemannian manifold of dimension n . Consider on M the Leibenson equation

$$\partial_t u = \Delta_p u^q, \quad (3)$$

where we assume that

$$q > \frac{1}{p-1}, \quad (4)$$

that is, $\delta := q(p-1) - 1 > 0$. Solutions of (3) are understood in a certain weak sense.

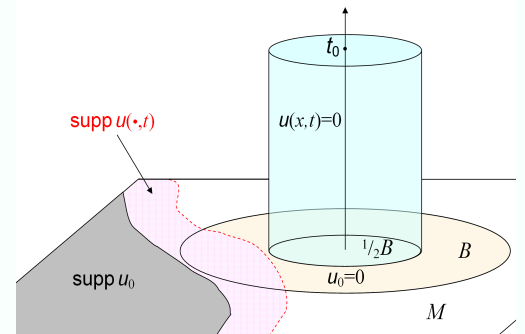
Theorem 1 (AG and Ph.Sürig, *Comm. Anal. Geom.* **32**, 2024) *Let $u(x, t)$ be a bounded non-negative subsolution of (3) in $M \times \mathbb{R}_+$. Let B be a ball in M of radius R such that $u_0 := u(\cdot, 0) = 0$ in B . Then*

$$u(\cdot, t) = 0 \text{ in } \frac{1}{2}B \text{ for all } t \leq t_0,$$

where

$$t_0 = \eta R^p \|u_0\|_{L^\infty(M)}^{-\delta}$$

and $\eta > 0$ depends on intrinsic geometry of B .



Note that the range (4) of parameters p, q is the same as that in the Barenblatt solutions with a finite propagation speed.

The only previously known case of Theorem 1 was when $p > 2$ and $q = 1$, that is, when (3) is the equation $\partial_t u = \Delta_p u$. In this case a finite propagation speed was proved by S. Dekkers in *Comm. Anal. Geom.* **14** (2005).

Another interesting case is when $p = 2$ and $q > 1$, that is, when (3) is a *porous medium* equation $\partial_t u = \Delta u^q$. Theorem 1 is new in this case.

Remark. The constant η depends on p, q, n as well as on the *normalized Sobolev constant* c_B in B : for any $u \in W_0^{1,p}(B)$

$$\left(\int_B |\nabla u|^p \right)^{1/p} \geq \frac{c_B}{R} \left(\int_B |u|^{p\kappa} \right)^{1/p\kappa} \quad (5)$$

where κ is the Sobolev exponent: $\kappa = \frac{n}{n-p}$ if $n > p$ and $\kappa > 1$ is any if $n \leq p$.

Remark. The Leibenson equation (3), that is, $\partial_t u = \Delta_p u^q$ can be equivalently rewritten in the form

$$\partial_t u = \operatorname{div} \left(u^{m-1} |\nabla u|^{p-2} \nabla u \right),$$

where $m = 1 + (q - 1)(p - 1) = \delta + 3 - p$. The condition $\delta > 0$ is, hence, equivalent to $m + p > 3$. Therefore, Theorem 1 holds for this equation when $p > 1$ and $m + p > 3$.

3 Finite propagation speed of support

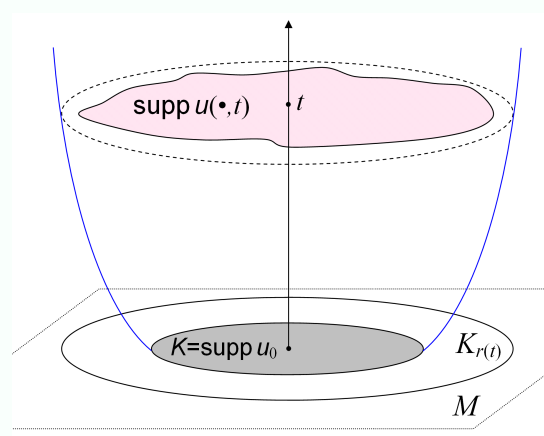
Let u be a bounded non-negative subsolution of (3) with $u(\cdot, 0) = u_0$. For any set $K \subset M$ and any $r > 0$, denote by K_r the closed r -neighborhood of K .

Corollary 2 *Let $K := \text{supp } u_0$ be a compact set. Then there an increasing positive function $r : (0, T) \rightarrow \mathbb{R}_+$ with some $T \in (0, \infty]$, such that*

$$\text{supp } u(\cdot, t) \subset K_{r(t)}$$

for all $t \in (0, T)$.

Function $r(t)$ is referred to as a *propagation rate* of solution u .



Problem 3 Is it true that one can always have $T = \infty$? Either prove it or give a counterexample: a manifold and a solution u such that $\text{supp } u_0$ is compact, while $\text{supp } u(\cdot, t)$ is unbounded for large enough t .

Let M have non-negative Ricci curvature. Then the normalized Sobolev constant c_B in (5) can be taken the same for all balls. Hence, the constant η from Theorem 1 is also the same for all balls, which allows to obtain the following.

Corollary 4 *If $\text{Ricci}_M \geq 0$ then any bounded non-negative subsolution u with compactly supported u_0 has a propagation rate $r(t) = Ct^{1/p}$ for all $t > 0$.*

Recall that in \mathbb{R}^n the propagation rate of the Barenblatt solution is

$$r(t) = Ct^{1/\beta},$$

where

$$\beta = p + n\delta. \tag{6}$$

This implies that, for any bounded non-negative solution u in \mathbb{R}^n with compactly supported u_0 , the propagation rate is also $r(t) = Ct^{1/\beta}$ for large t .

Since $p < \beta$, we see that the propagation rate of Corollary 4 is not sharp in \mathbb{R}^n .

4 Sharp propagation rate

Instead of the previous conditions (4), we assume here more restrictive hypotheses:

$$\frac{1}{p-1} < q \leq 1.$$

Theorem 5 (AG and Ph.Sürig, *Ann. Scuola Norm. Sup. Pisa*, 2024) *Let u be a bounded non-negative subsolution of (3) in $M \times \mathbb{R}_+$, with initial function $u_0 := u(\cdot, 0) \in L^1$. Let B be a ball in M of radius R such that $u_0 = 0$ in B . Then*

$$u(\cdot, t) = 0 \text{ in } \frac{1}{2}B \text{ for all } t \leq t_0$$

where

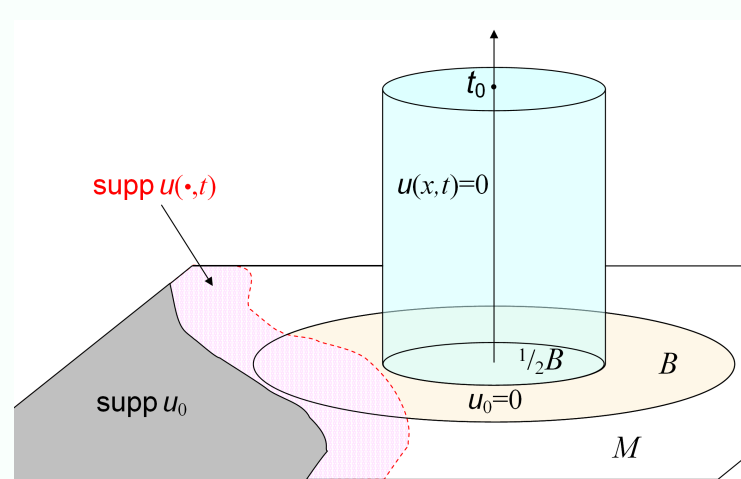
$$t_0 = \eta R^p \mu(B)^{\frac{\delta}{\sigma}} \|u_0\|_{L^\sigma(M)}^{-\delta}.$$

Here σ is any real number such that

$$\sigma \geq 1 \text{ and } \sigma > \delta, \quad (*)$$

where $\delta = q(p-1) - 1 > 0$,

and $\eta = \eta(p, q, n, \sigma, c_B) > 0$.



Corollary 6 Assume that $\text{Ricci}_M \geq 0$. Fix a point $x_0 \in M$ and assume that

$$\mu(B(x_0, r)) \geq cr^\alpha \quad \text{for all } r \geq r_0, \quad (7)$$

with some $\alpha, c > 0$. Then u has a propagation rate $r(t) = Ct^{1/\beta}$ for large t , where

$$\beta = p + \alpha \frac{\delta}{\sigma} \quad (8)$$

and σ is as in (*).

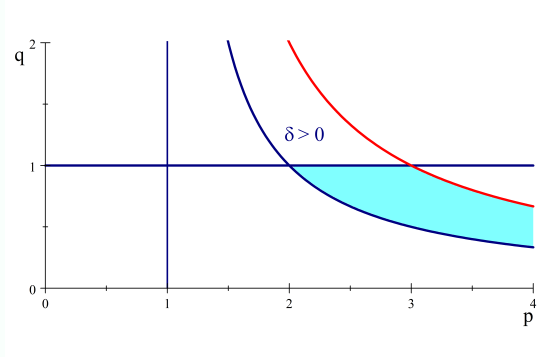
In \mathbb{R}^n we have $\alpha = n$. Setting $\sigma = 1$, we obtain $\beta = p + n\delta$ that matches propagation rate (6) in \mathbb{R}^n . However, we can take $\sigma = 1$ in (*) only if $\delta < 1$, that is, if $q(p-1) < 2$.

The next diagram shows the following range of p, q :

$$q \leq 1 \quad \text{and} \quad 1 < q(p-1) < 2.$$

For such p, q , we obtain in Corollary 6 a sharp propagation rate $r(t) = Ct^{1/\beta}$ with

$$\beta = p + \alpha\delta.$$



Conjecture 7 The result of Theorem 5 holds with $\sigma = 1$ if $q > \frac{1}{p-1}$ (that is, $\delta > 0$).

5 The case $\delta := q(p-1) - 1 < 0$

In this case we have the following upper bound of solutions.

Theorem 8 (AG, Jin Sun and Ph.Sürig, preprint 2026) *Assume that $\text{Ricci}_M \geq 0$ and*

$$\mu(B(x, r)) \geq cr^\alpha \quad \text{for all } x \in M \text{ and } r \geq r_0,$$

Assume that $\delta < 0$ and

$$\beta := p + \alpha\delta > 0.$$

Let u be a bounded non-negative solution of (3) with the initial function $u_0 \in L^1(M)$. Set $K = \text{supp } u_0$ and denote $|x| = d(x, K)$. Then, for all $t > 0$ and $x \in M$,

$$u(x, t) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{|x|}{t^{1/\beta}} \right)^{\frac{p}{\delta}}. \quad (9)$$

The estimate (9) is optimal for the Barenblatt solution in \mathbb{R}^n with $\alpha = n$:

$$u(x, t) = \frac{1}{t^{n/\beta}} \left(C + c \left(\frac{|x|}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)^{\frac{p-1}{\delta}}.$$

6 Linear non-local evolution equation

Let $v(x, t)$ be a positive solution of the non-local heat equation

$$\partial_t v + (-\Delta)^{\beta/2} v = 0$$

where $\beta \in (0, 2)$ is arbitrary. Let $K = \text{supp } v_0$ and $|x| = d(x, K)$. One can prove that, under the hypotheses of Theorem 8,

$$v(x, t) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{|x|}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \quad (10)$$

Indeed, by the Li-Yau theorem, the heat kernel of the heat equation $\partial_t u - \Delta u = 0$ admits the estimate

$$p_t(x, y) \leq \frac{C}{t^{\alpha/2}} \exp\left(-\frac{d(x, y)^2}{ct}\right),$$

whence (10) follows by well-known subordination techniques.

Problem 9 *Is the similarity of (9) and (10) - a coincidence or it has a deeper reason?*

7 Mean value inequality

The main ingredient of the proof of Theorem 1 is the following mean value inequality. We assume here that $\delta \geq 0$.

Lemma 10 *Let $B = B(x_0, R)$ be a ball in M . Let $u(x, t)$ be a non-negative bounded subsolution of (3) in the cylinder*

$$Q = B \times [0, T]$$

such that $u_0 \equiv u(\cdot, 0) = 0$ in B .

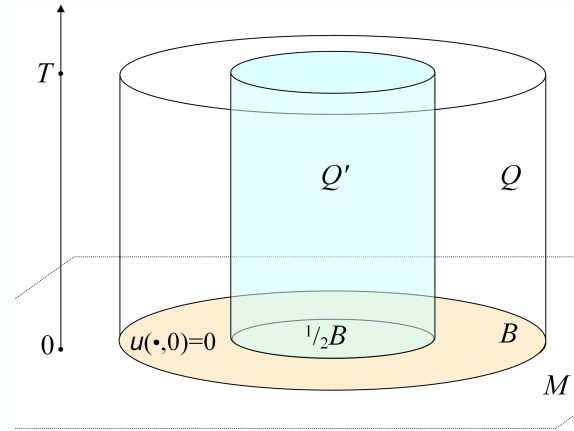
Then, for the cylinder

$$Q' = \frac{1}{2}B \times [0, T]$$

and for any

$$\lambda \geq \max(p, pq),$$

the following inequality holds:



$$\|u\|_{L^\infty(Q')} \leq C \left(\frac{T}{R^p}\right)^{1/\lambda} \|u\|_{L^\infty(Q)}^{\delta/\lambda} \left(\int_Q u^\lambda\right)^{1/\lambda}, \quad (11)$$

where $C = C(p, q, n, \lambda, c_B)$.

8 From mean value to finite propagation speed

Sketch of proof of Theorem 1. Set $r = \frac{1}{2}R$ and fix for a while a point $x \in \frac{1}{2}B$.

Hence, we have $B(x, r) \subset B$.

Fix also some $t > 0$ and set

$$Q_k = B(x, 2^{-k}r) \times [0, t],$$

$$J_k = \|u\|_{L^\infty(Q_k)}.$$

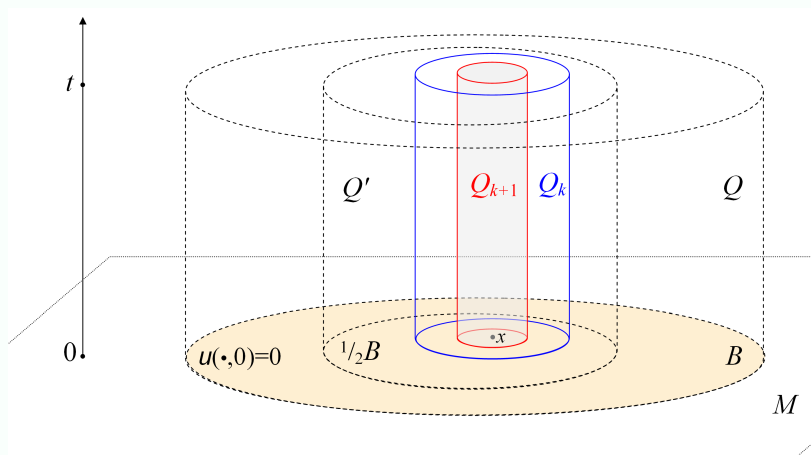
Let λ be as it is needed for

Lemma 10. Then by (12)

$$J_{k+1} \leq C \left(\frac{t}{(2^{-k}R)^p} \right)^{1/\lambda} J_k^{1+\frac{\delta}{\lambda}} = C 2^{k/\lambda} \left(\frac{t}{R^p} \right)^{1/\lambda} J_k^{1+\frac{\delta}{\lambda}}.$$

Iterating this inequality, we obtain an upper bound of J_k via J_0 that implies the following:
if

$$C \left(\frac{t}{R^p} \right)^{1/\lambda} \leq 2^{-1/\delta} J_0^{-\delta/\lambda} \tag{13}$$



then, for all k ,

$$J_k \leq 2^{-k/\delta} J_0. \quad (14)$$

The condition (13) is equivalent to

$$t \leq \eta R^p J_0^{-\delta}. \quad (15)$$

Since $J_0 = \|u\|_{L^\infty(Q)} \leq \|u_0\|_{L^\infty(M)}$ and, hence,

$$t_0 = \eta R^p \|u_0\|_{L^\infty(M)}^{-\delta} \leq \eta R^p J_0^{-\delta}$$

we see that (15) is satisfied for $t \leq t_0$. For such t , we obtain from (14) that, for any k ,

$$\|u\|_{L^\infty(B(x, 2^{-k}r) \times [0, t])} \leq 2^{-k/\delta} \|u_0\|_{L^\infty}.$$

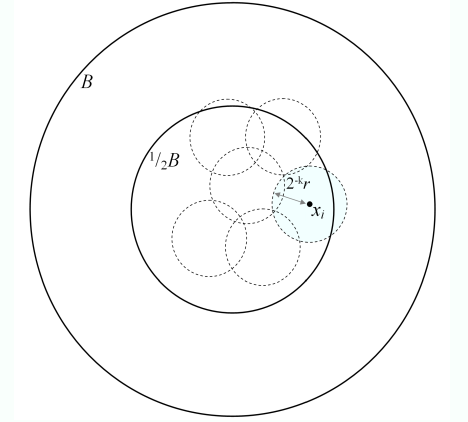
For any k , we cover the ball $\frac{1}{2}B$ by a finite sequence of balls $B(x_i, 2^{-k}r)$ with $x_i \in \frac{1}{2}B$. Since for all i

$$\|u\|_{L^\infty(B(x_i, 2^{-k}r) \times [0, t])} \leq 2^{-k/\delta} \|u_0\|_{L^\infty}$$

we obtain that

$$\|u\|_{L^\infty(\frac{1}{2}B \times [0, t])} \leq 2^{-k/\delta} \|u_0\|_{L^\infty}.$$

As $k \rightarrow \infty$, we obtain that $u = 0$ in $\frac{1}{2}B \times [0, t]$, which was to be proved. ■



9 Mean value inequality 2

The main ingredient in the proof of Theorem 5 is the following version of the mean value inequality. We assume here that $p > 2$ and $\frac{1}{p-1} < q \leq 1$.

Lemma 11 *Let $B = B(x_0, R)$ be a precompact ball in M .*

Let u be a non-negative bounded subsolution of (3) in the cylinder

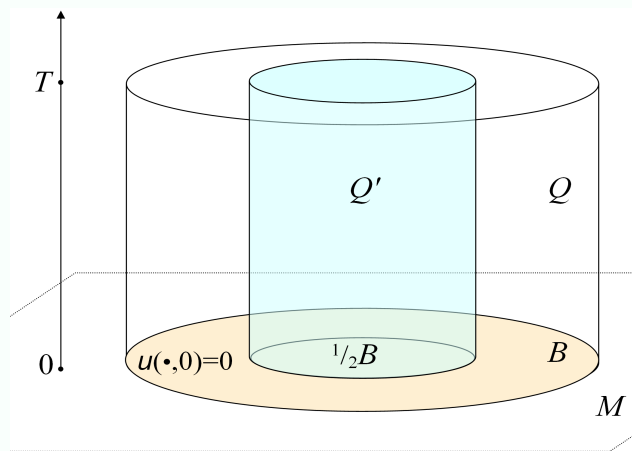
$$Q = B \times [0, T],$$

and let $u(\cdot, 0) = 0$ in B . Then,

for the cylinder

$$Q' = \frac{1}{2}B \times [0, T],$$

the following inequality holds:



$$\|u\|_{L^\infty(Q')} \leq C \left(\frac{T}{R^p}\right)^{1/\lambda} \left(\int_Q u^{\lambda+\delta}\right)^{1/\lambda}, \quad (16)$$

where $\lambda > 0$ is any, $\delta = q(p-1) - 1$, and $C = C(p, q, n, \lambda, c_B)$.

In the proof of Lemma 11 we use the following lemma.

Lemma 12 *Let u be a non-negative subsolution of (3).*

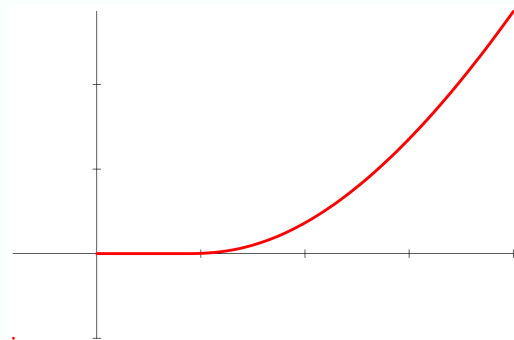
Set

$$a = \frac{q(p-1) - 1}{p-2}.$$

If $0 < a \leq 1$ then the function

$$v = (u^a - \theta)_+^{1/a}$$

is a subsolution for any $\theta > 0$.



Function $f_\theta(s) = (s^a - \theta)_+^{1/a}$
It satisfies $f_{\theta_1} \circ f_{\theta_2} = f_{\theta_1 + \theta_2}$

The condition $0 < a \leq 1$ holds, in particular, in the case when

$$p > 2 \quad \text{and} \quad \frac{1}{p-1} < q \leq 1$$

For the p -Laplacian case, that is, when $q = 1$, we have $a = 1$. In this case it is well known that $v = (u - \theta)_+$ is a subsolution. If also $p = 2$ that is, if (3) is the heat equation, then $v = f(u)$ is a subsolution for any convex f .

Sketch of proof of Lemma 11. Fix some $\theta > 0$ and define a sequence $\{u_k\}_{k=0}^\infty$ of functions:

$$u_0 = u, \quad u_k = \left(u_{k-1}^a - 2^{-k}\theta\right)_+^{1/a} \text{ for } k \geq 1$$

It is easy to see that $u_k = \left(u^a - (1 - 2^{-k})\theta\right)_+^{1/a}$.

Consider a decreasing sequence of radii

$$r_k = \left(\frac{1}{2} + 2^{-k-1}\right) R$$

so that $r_0 = r \geq r_k \searrow \frac{1}{2}R$, and cylinders

$$Q_k = B(x_0, r_k) \times [0, t]$$

so that

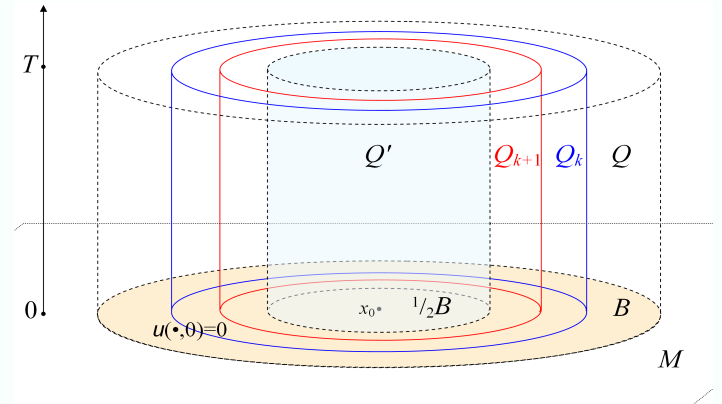
$$Q_0 = Q \supset Q_k \searrow Q'$$

as $k \rightarrow \infty$.

Set

$$J_k = \int_{Q_k} u_k^{\lambda+\delta}.$$

Clearly, $J_{k+1} \leq J_k$. Using a Caccioppoli type inequality for u_k and u_{k+1} as well as a certain Faber-Krahn type inequality for Δ_p in B (which reflects the intrinsic geometry of B), we prove that



$$J_{k+1} \leq \frac{CA^k}{\left(\mu(B)\theta^{\frac{\lambda}{a}}r^p\right)^\nu} J_k^{1+\nu},$$

where $\nu > 0$ is the Faber-Krahn exponent for Δ_p , and C, A are some constants.

Analyzing this recursive inequality, we show that if

$$\theta \geq \left(\frac{CJ_0}{\mu(B)r^p}\right)^{\frac{a}{\lambda}}, \quad (17)$$

then $J_k \rightarrow 0$ as $k \rightarrow \infty$, which implies

$$\int_{Q'} \left[(u^a - \theta)_+^{1/a} \right]^{\lambda+\delta} = 0,$$

that is, $u^a \leq \theta$ in Q' . Choosing the minimal value of θ in (17), we obtain

$$u \leq \left(\frac{CJ_0}{\mu(B)r^p}\right)^{\frac{1}{\lambda}} = \left(\frac{C}{\mu(B)r^p} \int_Q u^{\lambda+\delta}\right)^{\frac{1}{\lambda}} \quad \text{in } Q'$$

which proves (16).

This method works for $\lambda \geq 2$. The case $0 < \lambda < 2$ is obtained from $\lambda = 2$ using an additional iteration procedure. ■