Path homology theory of multigraphs and quivers

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Abstract

We construct a new homology theory for the categories of quivers and multigraphs and describe the basic properties of introduced homology groups. We introduce a conception of homotopy in the category of quivers and we prove the homotopy invariance of homology groups.

Contents

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1 Introduction

There are several approaches to construct a (co)homology theory for graphs, multigraphs or digraphs: using of cliques of graph (see [22] and [7]), the Hochschild homology of the path algebra (see [21], [20], [9], and [13]), singular graph homolgy (see [26] and [5]), and the path homology. The comparison of these approaches is shortly described in [19, Introduction]. The path cohomology for digraphs was introduced by Dimakis and Müller-Hoissen in [10], [4], [11]. This approach was developed in [15], [16], [17], [18], and [19], where deep relations between path homology groups, Hochschild homology, simplicial homology, and Atkins theory were obtained (see also [1], [2], [3], and [6]). The path homology theory has good functorial properties, it is compatible with the homotopy theory on the graphs (digraphs), and respects the basic graph-theoretical operations: the Cartesian product and the join of two digraphs. Additionally, the path homology theory can be used for topological data analysis and investigation of various networks (cf. [12], [8], [25]).

In the present paper we construct a homotopy invariant homology theory for quivers and multigraphs, that is a natural generalization of the path homology theory for simple digraphs and non-directed graphs that was introduced and developed in [15], [16], [17], and [18]. Then we discuss possible applications of the results and provide several examples of computations.

In Section 2, we give a preliminary material about the category of quivers and path algebras of quivers.

In Section 3 and 4, we construct a homology theory on the category of quivers, including chain complexes that arise naturally from a quiver structure.

In Sections 5, we introduce the concept of homotopy between two morphisms of quivers and we prove the homotopy invariance of the homology groups under a mild assumption on the ring of coefficients.

In Section 6, transfer obtained results from the category of quivers to that of multigraph, discuss the results, and we present several examples of computation.

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2 The category of quivers and path algebras

In this section we recall a category of quivers and describe the path algebras arising naturally on quivers.

Definition 2.1 A finite *quiver* is a quadruple $Q = (V, E, s, t)$ where V is a finite set of *vertices*, E is a finite set of *arrows*, and $s, t\colon E \to V$ are two maps. For an arrow $a \in E$ we refer to the point $s(a) \in V$ as the *start vertex* of a and to the point $t(a)$ as the *target vertex* of a.

In what follows we shall consider only finite quivers. Usually the elements of V are denoted by $0, 1, 2, \dots, n$.

Definition 2.2 Given a positive integer r, an *elementary r-path in a quiver Q* is a non-empty sequence $a_0, a_1, \dots, a_{r-1}$ of arrows in Q such that $t(a_i) = s(a_{i+1})$ for $i = 0, 1, \dots, r-2$. Denote this r-path by $p = a_0a_1 \dots a_{r-1}$. Define *the start vertex of p* by $s(p) = s(a_0)$ and *the target vertex of p* by $t(p) = t(a_{r-1})$.

For $r = 0$ define *an elementary* 0*-path* p *by* $p := v$ where $v \in V$ is any vertex. For this path set $s(p) = t(p) = v$.

The number r is called the length of arbitrary r-path p and is denoted by $|p|$.

The set of all elementary r-paths of Q is denoted by P_rQ and the union of all P_rQ for all $r \geq 0$ is denoted by PQ.

Definition 2.3 Let $Q = (V, E, s, t)$, $Q' = (V', E', s', t')$ be two finite quivers. A *morphism* of quivers $f : Q \to Q'$ is defined as a pair of maps (f_V, f_E), where $f_V : V \to V'$ is a map of vertices and $f_E : E \to E'$ is a map of arrows, such that the following conditions are satisfied for any $a \in E$:

$$f_V(s(a)) = s'(f_E(a)) \text{ and } f_V(t(a)) = t'(f_E(a)).$$

It follows immediately from Definitions 2.1 and 2.3, that the quivers with the introduced morphisms form a category that we denote by $\mathcal{Q}$.

Definition 2.4 Let K be a commutative ring with a unity and such that no positive integer in K is a zero divisor. *The graded path algebra* $\Lambda_*(Q) = K[PQ]$ is the free K-module spanned by all elementary paths in Q, and the multiplication in $\Lambda_*(Q)$ is defined as a K-linear extension of *concatenation* of any two elementary paths p, q on Q.

The concatenation is defined as follows: for the paths $p = a_0a_1 \dots a_n$ and $q = b_0b_1 \dots b_m$ with $n, m \geq 0$ set

$$p \cdot q = \begin{cases} a_0a_1 \dots a_nb_0b_1 \dots b_m, & \text{if } t(a_n) = s(b_0), \\ 0, & \text{otherwise,} \end{cases}$$

for the paths $p = v \in V$ and $q = b_0b_1 \dots b_m$, set

$$p \cdot q = \begin{cases} q, & \text{if } v = s(b_0), \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad q \cdot p = \begin{cases} q, & \text{if } v = t(b_m), \\ 0, & \text{otherwise,} \end{cases}$$

and for the paths $p = v$, $q = w$ where $v, w \in V$, set

$$p \cdot q = \begin{cases} v, & \text{if } v = w, \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that the formal path $\sum\limits_{v \in V} v \in \Lambda_0(Q)$ is the left and right unity of $\Lambda_*(Q)$.

Let $f : Q \to Q'$ be a morphism as above. For any path $p \in PQ$ define the path $f_*(p) \in PQ'$ by the following way:

- for $|p| = 0$ and, hence, $p = v \in V$ we put $f_*(v) = f_V(v) \in V'$;
- for $|p| \geq 1$ and $p = a_0a_1 \dots a_n$ where $a_i \in E$, we put

$$f_*(p) = f_E(a_0)f_E(a_1) \dots f_E(a_n) \text{ where } f_E(a_i) \in E'.$$

It is clear that $|f_*(p)| = |p|$. Thus, a morphism $f\colon Q \to Q'$ induces linear maps $f_*\colon \Lambda_n(Q) \to \Lambda_n(Q')$ for any $n \geq 0$.

Simple examples show that it may happen that $f_*(p \cdot q) \neq f_*(p) \cdot f_*(q)$.

Example 2.5 Let $Q_i = (V_i, E_i, s_i, t_i)(i = 1, 2)$ be two quivers given on the next diagrams

$$\begin{array}{ccc} v_1 & \xrightarrow{a_1} & v_2 \\ \downarrow b_1 & & \downarrow a_2 \\ v_4 & \xrightarrow{b_2} & v_3 \end{array} \quad \text{and} \quad w_1 \xrightarrow{c_1} w_2 \xrightarrow{c_2} w_3$$

correspondingly. Define a morphism $f\colon Q_1 \to Q_2$ putting

$$f_V(v_1) = w_1,\ \ f_V(v_4) = f_V(v_2) = w_2,\ \ f_V(v_3) = w_3 \text{ and } f_E(a_i) = f_E(b_i) = c_i\ (i = 1, 2).$$

Then for the paths $p = a_1$ and $q = b_2$, we have $f_*(p \cdot q) = f_*(0) = 0$ and $f_*(p) \cdot f_*(q) = c_1 c_2 \neq 0$.

Let $Q = (V, E, s, t)$ be a quiver. For any ordered pair of vertices $(v, w) \in V \times V$ define $\mu(v, w)$ as a number of arrows from v to w (this includes also the case $v = w$ when $\mu(v, v)$ is the number of loops at the vertex v). Set

$$N_0 := \max_{v,w \in V} \mu(v, w). \tag{2.1}$$

The number N_0 will be referred to as the *power* of Q.

Definition 2.6 A quiver Q is called *complete of power* N if, for any two vertices v, w there is exactly N arrows with the start vertex v and the target vertex w.

Let us describe the procedure of completion of an arbitrary quiver Q of power N_0. Fix an integer N such that

$$N \geq N_0 \tag{2.2}$$

Definition 2.7 Define a quiver $\widetilde{Q} = (\widetilde{V}, \widetilde{E}, \widetilde{s}, \widetilde{t})$ as follows. We put $\tilde{V} = V$ and, for any ordered pair of vertices (v, w) (including the case $v = w$) we add $(N - \mu(v, w))$ new arrows from v to w, that obtaining $\widetilde{E}$. Clearly, $\widetilde{Q}$ is a complete quiver of power N. We shall refer to $\widetilde{Q}$ as *the completion of* Q *of power* N. We will denote $\widetilde{Q}$ also by $\widetilde{Q}^N$ when the dependence on N should be emphasized.

Note that we have a natural inclusion of quivers $\tau\colon Q \hookrightarrow \widetilde{Q}$ that induces an inclusion of K-modules

$$\tau_* : \Lambda_n(Q) \hookrightarrow \Lambda_n(\widetilde{Q}), \quad \text{for any} \quad n \geq 0.$$

3 Homology groups of complete quivers

In this section we construct a chain complex and homology groups on a complete quiver. Let us recall the following standard definition.

Definition 3.1 [27] *A* Δ*-set consists of a sequence of sets* X_n $(n = 0, 1, 2, \dots)$ *and maps* $\partial_i \colon X_{n+1} \to X_n$ *for each* $n \geq 0$ *and* $0 \leq i \leq n+1$*, such that* $\partial_i \partial_j = \partial_{j-1} \partial_i$ *whenever* $i < j$*.*

Consider a complete quiver $Q = (V, E, s, t)$ of the power $N \geq 1$. Define a product

$$\Lambda_1(Q) \times \Lambda_1(Q) \to \Lambda_1(Q), \ (p, q) \to [pq]$$

first on the arrows $a, b \in E$ by

$$[ab] \colon = \begin{cases} \sum c, & \text{for } t(a) = s(b), s(c) = s(a), t(c) = t(b) \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

and then extend it by linearity in each argument on $\Lambda_1(Q) \times \Lambda_1(Q)$. Note that the sum in (3.1) contains all arrows starting at $s(a)$ and ending at $t(b)$. It follows directly from the definition that

$$[a[bc]] = [[ab]c] = \begin{cases} N \sum d, & \text{for } \begin{bmatrix} t(a) = s(b), t(b) = s(c), \\ s(d) = s(a), t(d) = t(c), \end{bmatrix} \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Now we introduce homomorphisms

$$\partial_i \colon \Lambda_{n+1}(Q) \to \Lambda_n(Q)$$

for all $n \geq 0$ and $0 \leq i \leq n+1$. It suffices to define $\partial_i p$ for any elementary $(n+1)$-paths $p = a_0 a_1 \dots a_n$ and then extend ∂_i by linearity.
For $n = 0$, $i = 0, 1$, we put

$$\partial_0 p = N t(p), \qquad \partial_1 p = N s(p). \tag{3.3}$$

For $n \geq 1$, $i = 0, n+1$, we put

$$\partial_0 p = N(a_1 a_2 \dots a_n), \qquad \partial_{n+1} p = N(a_0 a_1 \dots a_{n-1}). \tag{3.4}$$

For $n \geq 1$, $1 \leq i \leq n$, we put

$$\partial_i p = \sum_{c \in E : s(c) = s(a_{i-1}), t(c) = t(a_i)} a_0 \dots a_{i-2} c a_{i+1} \dots a_n. \tag{3.5}$$

Using the notation (3.1), we can rewrite (3.5) shortly as follows:

$$\partial_i (a_0 \dots a_n) = a_0 \dots a_{i-2} [a_{i-1} a_i] a_{i+1} \dots a_n. \tag{3.6}$$

Lemma 3.2 *Let $p = (a_0 a_1 \dots a_n)$ with $n \geq 2$ and $1 \leq i \leq n-1$, we have the following relations*

$$a_0 \dots a_{i-2}[[a_{i-1}a_i]a_{i+1}]a_{i+2} \dots a_n = a_0 \dots a_{i-2}[a_{i-1}[a_i a_{i+1}]]a_{i+2} \dots a_n.$$

Proof. Follows from definition (3.6) of ∂_i and (3.2). ■

We put $\Lambda_{-1}(Q) = \{0\}$ and define $\partial_0 : \Lambda_0(Q) \to \Lambda_{-1}(Q)$ by $\partial_0 = 0$.

Theorem 3.3 *For all $n \geq 0$, $0 \leq i < j \leq n+1$ we have*

$$\partial_i \partial_j p = \partial_{j-1} \partial_i p \tag{3.7}$$

for any $p \in \Lambda_{n+1}(Q)$. Hence, the sequence $\Lambda_i(Q), i \geq 0$, with the differentials ∂_i is a Δ-set.

Proof. In the case $n = 0$ we have necessarily $i = 0$ and $j = 1$. Then we have trivially $\partial_0 \partial_1 = \partial_0 \partial_0 = 0$.

Assume $n \geq 1$ and consider various cases. It suffices to prove (3.7) for $p = (a_0 a_1 a_2 \dots a_n)$.

i) Let $i = 0$ and $j = 1$. For $n = 1$, we have

$$\partial_0 \partial_1 (a_0 a_1) = \partial_0([a_0 a_1]) = N^2 t(a_1),$$

and

$$\partial_0 \partial_0 (a_0 a_1) = \partial_0(N(a_1)) = N^2 t(a_1).$$

For $n \geq 2$ we have

$$\partial_0 \partial_1 (a_0 a_1 a_2 \dots a_n) = \partial_0([a_0 a_1]a_2 \dots a_n) = N^2(a_2 \dots a_n),$$

and

$$\partial_0 \partial_0 (a_0 a_1 a_2 \dots a_n) = N^2(a_2 \dots a_n).$$

In the both cases, we have $\partial_0 \partial_1 p = \partial_0 \partial_0 p$.

ii) Let $i = 0$ and $2 \leq j \leq n$. For $n = 2$ and, hence $j = 2$, we have

$$\partial_0 \partial_2 (a_0 a_1 a_2) = \partial_0(a_0[a_1 a_2]) = N([a_1 a_2])$$

and

$$\partial_1 \partial_0 (a_0 a_1 a_2) = N \partial_1 (a_1 a_2) = N([a_1 a_2]).$$

For $n \geq 3$, we have

$$\partial_0 \partial_j (a_0 a_1 \dots a_n) = \partial_0 (a_0 \dots a_{j-2}[a_{j-1} a_j] a_{j+1} \dots a_n) = N(a_1 \dots a_{j-2}[a_{j-1} a_j] a_{j+1} \dots a_n)$$

and

$$\partial_{j-1} \partial_0 (a_0 a_1 \dots a_n) = N \partial_{j-1} (a_1 a_2 \dots a_n) = N(a_1 \dots a_{j-2}[a_{j-1} a_j] a_{j+1} \dots a_n).$$

Hence, $\partial_0 \partial_j p = \partial_{j-1} \partial_0 p$.

iii) Let $i = 0$ and $j = n + 1$. For $n = 1$ and hence $j = 2$, we have

$$\partial_0 \partial_2 (a_0 a_1) = N\partial_0 (a_0) = N^2 t(a_0),$$

and

$$\partial_1 \partial_0 (a_0 a_1) = N\partial_1 (a_1) = N^2 (s(a_1) = N^2 t(a_0).$$

For $n \geq 2$, we have

$$\partial_0 \partial_{n+1} (a_0 \dots a_n) = N\partial_0 (a_0 \dots a_{n-1}) = N^2 (a_1 \dots a_{n-1})$$

and

$$\partial_n \partial_0 (a_0 \dots a_n) = N\partial_n (a_1 \dots a_n) = N^2 (a_1 \dots a_{n-1}).$$

Hence, $\partial_0 \partial_j p = \partial_{j-1} \partial_0 p$.

iv) Let $j = n + 1$. This case is treated exactly the same way as the case $i = 0$ considered in i) – iii).

v) Let $i \geq 1$, $j \leq n$ and $j = i + 1$. We have

$$\begin{aligned} \partial_i \partial_{i+1} (a_0 \dots a_n) &= \partial_i (a_0 \dots a_{i-1} [a_i a_{i+1}] a_{i+2} \dots a_n) \\ &= a_0 \dots a_{i-2} [a_{i-1} [a_i a_{i+1}]] a_{i+2} \dots a_n \end{aligned}$$

and

$$\begin{aligned} \partial_i \partial_i (a_0 \dots a_n) &= \partial_i (a_0 \dots a_{i-2} [a_{i-1} a_i] a_{i+1} \dots a_n) \\ &= a_0 \dots [[a_{i-1} a_i] a_{i+1}] \dots a_n \\ &= a_0 \dots [a_{i-1} [a_i a_{i+1}]] \dots a_n \end{aligned}$$

by Lemma 3.2. Hence, $\partial_i \partial_{i+1} p = \partial_i \partial_i p$.

vi) Finally, let $i \geq 1$ and $i + 2 \leq j \leq n$. We have

$$\begin{aligned} \partial_i \partial_j (a_0 \dots a_n) &= \partial_i (a_0 \dots a_{j-2} [a_{j-1} a_j] \dots a_n) \\ &= a_0 \dots a_{i-2} [a_{i-1} a_i] \dots [a_{j-1} a_j] a_{j+1} \dots a_n \end{aligned}$$

and

$$\begin{aligned} \partial_{j-1} \partial_i (a_0 \dots a_n) &= \partial_{j-1} (a_0 \dots a_{i-2} [a_{i-1} a_i] a_{i+1} \dots a_n) \\ &= a_0 \dots a_{i-2} [a_{i-1} a_i] \dots [a_{j-1} a_j] a_{j+1} \dots a_n. \end{aligned}$$

Hence, $\partial_i \partial_j p = \partial_{j-1} \partial_i p$. ■

Definition 3.4 Let Q be a complete quiver of power N. For all $n \geq -1$, define homomorphisms $\partial \colon \Lambda_{n+1}(Q) \to \Lambda_n(Q)$ by

$$\partial = \sum_{i=0}^{n+1} (-1)^i \partial_i.$$

Consequently, for an elementary path $a_0...a_n$, we have

$$\begin{aligned} \partial\left(a_0...a_n\right) &= Na_1...a_n - \left[a_0a_1\right]a_2...a_n + a_0\left[a_1a_2\right]...a_n \\ &\quad +... + (-1)^n a_0...a_{n-2}\left[a_{n-1}a_n\right] + (-1)^{n+1} Na_0...a_{n-1}. \end{aligned} \tag{3.8}$$

By a standard result about Δ-sets (see [27]), we obtain the following.

Corollary 3.5 *We have $\partial^2 = 0$ which yields a chain complex*

$$0 \overset{\partial}{\leftarrow} \Lambda_0(Q) \overset{\partial}{\leftarrow} \Lambda_1(Q) \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Lambda_n(Q) \overset{\partial}{\leftarrow} \dots \tag{3.9}$$

The chain complex (3.9) is called a *path chain complex* of a complete quiver Q.

Previously we have defined $\Lambda_{-1}(Q) = \{0\}$. Alternatively, we can set $\Lambda_{-1}(Q) = K$, and define $\partial : \Lambda_0(Q) \to \Lambda_{-1}(Q)$ by $\partial = \varepsilon$ where

$$\varepsilon\left(\sum_v k_v v\right) = \sum_v k_v,\ v \in V, k_v \in K,$$

is an augmentation. Then we obtain a *chain complex with the augmentation* of a complete quiver Q

$$0 \leftarrow K \overset{\varepsilon}{\leftarrow} \Lambda_0(Q) \overset{\partial}{\leftarrow} \Lambda_1(Q) \leftarrow \dots \overset{\partial}{\leftarrow} \Lambda_n(Q) \leftarrow \dots \tag{3.10}$$

4 Homology of arbitrary quivers

Now we define a chain complex and homology groups of an arbitrary finite quiver $Q = (V, E, s, t)$. Fix a positive integer N as in (2.2). Let $\widetilde{Q}$ be a completion of Q of the power N (see Section 2). We have a natural inclusion $\tau \colon Q \to \widetilde{Q}$ that is a morphism of quivers. It induces isomorphisms $\Lambda_n(Q) \to \Lambda_n(\widetilde{Q})$ of K-modules for $n = -1, 0$, and monomorphisms of K-modules

$$\tau_* \colon \Lambda_n(Q) \to \Lambda_n(\widetilde{Q}) \text{ for } n \geq 1$$

defined on the elementary n-paths $p = a_1 \dots a_n$ by

$$\tau_*(a_1 \dots a_n) = \tau_E(a_1) \dots \tau_E(a_n). \tag{4.1}$$

Since τ_* is an inclusion, we shall identify any elementary path $p \in PQ$ with its image $\tau_*(p) \in P\widetilde{Q}$ and we shall consider $\Lambda_n(Q)$ as a submodule of $\Lambda_n(\widetilde{Q})$ for any $n \geq -1$.

Definition 4.1 Any elementary n-path $p \in P\widetilde{Q}$ is called *allowed* if $p \in PQ$, and *non-allowed* otherwise. The elements of $\Lambda_n(Q)$ are called *formal allowed* n-paths.

Note that the submodules $\Lambda_n(Q) \subset \Lambda_n(\widetilde{Q})$ are in general *not* invariant for ∂ as defined by (3.5) in $\Lambda_*(\widetilde{Q})$. For $n \geq 0$, consider the following submodules of $\Lambda_n(Q)$

$$\Omega_n(Q) := \{v \in \Lambda_n(Q) : \partial v \in \Lambda_{n-1}(Q)\}. \tag{4.2}$$

It is clear that $\Omega_n(Q)$ are ∂-invariant, that is,

$$\partial\left(\Omega_n(Q)\right) \subset \Omega_{n-1}(Q),$$

which follows directly from the identity $\partial^2 = 0$ in $\Lambda_*(\widetilde{Q})$. Hence, we obtain a chain complex $\Omega_* = \Omega_*(Q)$:

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_n \overset{\partial}{\leftarrow} \dots \tag{4.3}$$

Note that $\Omega_0(Q) = \Lambda_0(Q) = \Lambda_0(\widetilde{Q})$, $\Omega_1(Q) = \Lambda_1(Q) \subset \Lambda_1(\widetilde{Q})$, and $\Omega_n(Q) \subset \Lambda_n(Q) \subset \Lambda_n(\widetilde{Q})$ for $n \geq 2$. Note also, that $\Omega_*(\widetilde{Q}) = \Lambda_*(\widetilde{Q})$ as follows trivially from (4.2).

Note that the definition of $\Omega_*(Q)$ depends on the choice of the parameter N as $\widetilde{Q}$ was defined as the completion of Q of power N. In order to emphasize this, we may use an extended notation $\widetilde{Q}^N$ for the completion of Q of power N and $\Omega_*^N(Q)$ for the chain complex $\Omega_*(Q)$.

Definition 4.2 Define for any $n \geq 0$ the *homologies of the quiver Q with coefficients from K* by

$$H_n^N(Q, K) = H_n(\Omega_*^N(Q)).$$

If N is fixed then we may use also the shorter notation $H_n(Q, K)$.

Using the augmentation homomorphisms $\varepsilon\colon \Omega_0(Q) = \Lambda_0(\widetilde{Q}) \to K$ defined above, we obtain the *reduced homology* $\widetilde{H}_n(Q, K)$ as the homology of the chain complex with the augmentation

$$0 \longleftarrow K \overset{\varepsilon}{\longleftarrow} \Omega_0 \overset{\partial}{\longleftarrow} \Omega_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\longleftarrow} \Omega_n \overset{\partial}{\longleftarrow} \dots$$

In the case of quivers of power $N = 1$ (digraphs) without loops the homology theory was constructed in the papers [15], [16], [17]. It is an easy exercise to transfer results of the present paper to the case of simple digraphs and to check that the obtained homology theories are isomorphic. One of advantages of the construction of the present paper is that it provides a homology theory for quivers of power $N = 1$ allowing loops, that contains as a particular case the theory [15], [16], [17].

As an example of computation of homology groups, let us prove the following statement. We say a quiver $Q = (V, E, s, t)$ is *connected* if, for any two vertices $v, w \in V$ there is a sequence of vertices $v = v_0, v_1, \dots, v_n = w$ such that for any pair of vertices (v_i, v_{i+1}) $(i = 0, 1, 2, \dots, n-1)$ there is at least one arrow $a \in E$ such that $s(a) = v_i, t(a) = v_{i+1}$ or $s(a) = v_{i+1}, t(a) = v_i$.

Proposition 4.3 *Let Q be a connected quiver. Then*

$$H_0^N(Q,\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{for } N=1 \\ \mathbb{Z}\oplus\left(\bigoplus_n (\mathbb{Z}/N\mathbb{Z})\right), & \text{for } N\geq 2, n=|V|-1. \end{cases} \tag{4.4}$$

In particular, for $N \geq 2$ in the group $H_0^N(Q,\mathbb{Z})$ there is an element of order N.

Proof. We can write directly the basic elements of $\Omega_i(Q)$ for $i=0,1$. We have

$$\Omega_0(Q) = \langle v_0 \dots, v_n | v_i \in V\rangle,$$

$$\Omega_1(Q) = \langle a_1, \dots a_k, b_1, \dots, b_m | a_i, b_j \in E; s(a_i) \neq t(a_i), s(b_j) = t(b_j)\rangle .$$

The differential $\partial\colon \Omega_1(Q) \to \Omega_0(Q)$ is defined by

$$\partial a_i = Nt(a_i) - Ns(a_i),\ \partial b_j = 0.$$

Hence $\varepsilon \circ \partial = 0$ for the augmentation $\varepsilon\colon \Omega_0(Q) \to \mathbb{Z}$. Since $\varepsilon(v_0) = 1 \neq 0$, we conclude that $v_0 \notin \operatorname{Im}\partial$. For $N=1$, the same line of arguments as in [14, Proposition 2.12] shows that $v_i - v_0 \in \operatorname{Im}\partial$, which proves (4.4) in the case $N=1$.

For $N \geq 2$, the $\mathbb{Z}$-modul $\Omega_0(Q)$ is generated by $\langle v_0, v_1 - v_0, \dots, v_n - v_0\rangle$ and, hence, is isomorphic to $\mathbb{Z}\oplus\left(\bigoplus_n \mathbb{Z}\right)$. Again, the same line of arguments as in [14, Proposition 2.12] shows that $\operatorname{Im}\partial$ coincides with the subgroup of $\Omega_0(Q)$ generated by $\langle N(v_1 - v_0), \dots, N(v_n - v_0)\rangle$. Clearly, this subgroup is isomorphic to $\bigoplus_n N\mathbb{Z}$, whence the result follows. ■

Let $Q = (V, E, s, t)$ be a quiver. As before, let N_0 be defined by (2.1) and let $N \geq N_0$. In the next statement we are concerned with the dependence of the complex $\Omega_*^N(Q)$ on N.

Theorem 4.4 *Let $Q = (V, E, s, t)$ be a connected quiver. Let N_0 be the power of Q. Then the K-modules $\Omega_n^N(Q)$ are naturally isomorphic for all $N \geq N_0 + 1$.*

Proof. Clearly, $\Omega_0^N(Q)$ does not depend on any $N > N_0$. Hence, in what follows we assume $n \geq 1$. Let

$$p = \sum_{I=(i_1,\dots,i_n)} c_I(a_{i_1}\dots a_{i_n}) \in \Omega_n^N(Q),$$

where $c_I \in K$ and $N > N_0$. Recall that $p \in \Omega_n^N(Q)$ if and only if $p \in \Lambda_n(Q)$ and $\partial^N p \in \Lambda_{n-1}(Q)$. For the operator ∂^N, we have

$$\partial^N p = \partial_0^N p + \sum_{k=1}^{n-1} (-1)^k \partial_k^N p + (-1)^n \partial_n^N p, \tag{4.5}$$

where

$$\partial_0^N p = N\sum_I c_I(a_{i_2}\dots a_{i_n}), \quad \partial_n^N p = N\sum_I c_I(a_{i_1}\dots a_{i_{n-1}}),$$

and, for $1 \leq k \leq n-1$,

$$\partial_k^N p = \sum_I c_I (a_{i_1} \ldots a_{i_{k-1}} [a_{i_k} a_{i_{k+1}}] a_{i_{k+2}} \ldots a_{i_n}).$$

Since $p \in \Lambda_n(Q)$, it is clear that $\partial_0^N p$ and $\partial_n^N p$ lie in $\Lambda_{n-1}(Q)$. Since $\partial^N p \in \Lambda_{n-1}(Q)$, it follows that

$$\sum_{k=1}^{n-1} (-1)^k \partial_k^N p = \partial^N p - \partial_0^N p - (-1)^n \partial_n^N p \in \Lambda_{n-1}(Q).$$

The key observation is that $[a_{i_k} a_{i_{k+1}}]$ is the sum of N arrows in $\widetilde{Q}^N$ with the same start and target vertices. Since in Q the maximal number of arrows with the same start and target vertices is at most $N_0 < N$, we see that $[a_{i_k} a_{i_{k+1}}]$ is the sum of at most N_0 allowed arrows in Q and at least $N - N_0$ non-allowed arrows from $\widetilde{Q}^N$. Therefore, the sum $(a_{i_1} \ldots a_{i_{k-1}} [a_{i_k} a_{i_{k+1}}] a_{i_{k+2}} \ldots a_n)$ contains at least $N - N_0$ elementary paths that are not allowed in Q. Therefore, all such terms in the sum (4.5) must cancel out in order to ensure that $\partial^N p \in \Lambda_{n-1}(Q)$, that is,

$$\sum_{k=1}^{n-1} (-1)^k \partial_k^N p = 0.$$

Note also, that if the cancellation of all the terms in this sum occurs for some $N > N_0$ then it will take place also for any other $N' > N_0$. Hence, we obtain

$$\partial^{N'} p = \partial_0^{N'} p + (-1)^n \partial_n^{N'} p \in \Lambda_{n-1}(Q), \tag{4.6}$$

which implies $p \in \Omega_n^{N'}(Q)$. ■

Remark 4.5 Let us emphasize that although the K-modules $\Omega_*^N(Q)$ do not depend on $N > N_0$, the differentials in $\Omega_*^N(Q)$ do depend on N and, in fact, the homology groups $H_*^N(Q)$ may actually depend on N. For example, Proposition 4.3 shows that $H_0^N(Q)$ depends on N.

Example 4.6 Consider the following quiver Q:

$$\begin{tikzcd} v_1 \arrow[r, "b"] & v_2 \\ v_0 \arrow[u, "a"] \arrow[ur, "c"'] & \end{tikzcd}$$

Here

$$\Omega_0^1(Q) = \langle v_0, v_1, v_2 \rangle, \quad \Omega_1^1(Q) = \langle a, b, c \rangle, \quad \Omega_2^1(Q) = \langle ab \rangle, \quad \text{and } \Omega_i^1(Q) = 0 \quad \text{for } i \geq 3.$$

It is easy to see that for any $N \geq 2$ we have

$$\Omega_0^N(Q) = \langle v_0, v_1, v_2 \rangle, \quad \Omega_1^N(Q) = \langle a, b, c \rangle, \quad \text{and } \Omega_i^N(Q) = 0 \quad \text{for } i \geq 2.$$

Hence, the chain complexes $\Omega_*^1(Q)$ and $\Omega_*^N(Q)$ are not isomorphic for $N \geq 2$. In this case we also have

$$0 = H_1^1(Q, \mathbb{Z}) \ncong H_1^N(Q, \mathbb{Z}) \cong \mathbb{Z} \quad \text{for } N \geq 2.$$

Now we construct homomorphisms of homology groups that are induced by a morphism of quivers.

Let $Q = (V, E, s, t)$ and $Q' = (V', E', s', t')$ be quivers of power N_0 and N_0', respectively, and $f = (f_V, f_E) \colon Q \to Q'$ be a morphism of quivers. Consider quivers $\widetilde{Q}^N$ and $\widetilde{Q'}^N$ that are the completions of power

$$N \geq \max\{N_0 + 1, N_0'\} \tag{4.7}$$

of the quivers Q and Q', respectively. For any $n \geq 0$, consider a diagram

$$\begin{array}{ccccc} \Omega_n^N(Q) & \subset & \Lambda_n(Q) & \subset & \Lambda_n(\widetilde{Q}^N) \\ & & \downarrow f_* & & \\ \Omega_n^N(Q') & \subset & \Lambda_n(Q') & \subset & \Lambda_n(\widetilde{Q'}^N) \end{array}$$

where the map f_* is a homomorphism induced by f. Our first aim is to restrict f_* to $\Omega_n^N(Q)$.

Proposition 4.7 *For $N \geq \max\{N_0 + 1, N_0'\}$ the restriction of the homomorphism f_* to $\Omega_n^N(Q)$ induces a morphism of chain complexes*

$$f_* \colon \Omega_*^N(Q) \longrightarrow \Omega_*^N(Q'),$$

and, hence, a homomorphism

$$f_* \colon H_*^N(Q, K) \to H_*^N(Q', K)$$

of homology groups.

Proof. We need to prove that

$$f_* \left(\Omega_n^N(Q)\right) \subset \Omega_n^N(Q')$$

and that f_* commutes with ∂^N. The case $n = 0$ is obvious, so let us assume $n \geq 1$.

Recall that $p \in \Omega_n^N(Q)$ if and only if $p \in \Lambda_n(Q)$ and $\partial^N p \in \Lambda_{n-1}(Q)$. Let

$$p = \sum_{I=(i_1,\ldots,i_n)} c_I (a_{i_1} \ldots a_{i_n}) \in \Omega_n^N(Q), \qquad c_I \in K.$$

We have

$$\begin{aligned} \partial^N p &= \underbrace{\partial_0^N p + (-1)^n \partial_n^N p}_{\in \Lambda_{n-1}(Q)} \\ &\quad + \sum_{k=1}^{n-1} (-1)^k \sum_I c_I (a_{i_1} \ldots a_{i_{k-1}} [a_{i_k} a_{i_{k+1}}] a_{i_{k+2}} \ldots a_{i_n}). \end{aligned} \tag{4.8}$$

Since $N > N_0$, by the same argument as in the proof of Theorem 4.4, all the terms

$$a_{i_1} \ldots a_{i_{k-1}} [a_{i_k} a_{i_{k+1}}] a_{i_{k+2}} \ldots a_n$$

from (4.8) cancel out in order to ensure that $\partial^N p \in \Lambda_{n-1}(Q)$. Hence,

$$\partial^N p = \partial_0^N p + (-1)^n \partial_n^N p.$$

Let us show that $f_* p \in \Omega_n^N(Q')$. For that, we need to verify that $\partial^N (f_* p) \in \Lambda_{n-1}(Q')$. We have

$$f_* p = \sum_{I=(i_1,\dots,i_n)} c_I (b_{i_1} \dots b_{i_n}),$$

where $b_j = f_*(a_j) \in E'$, and

$$\begin{aligned} \partial^N (f_* p) &= \underbrace{\partial_0^N (f_* p) + (-1)^n \partial_n^N (f_* p)}_{\in \Lambda_{n-1}(Q')} \\ &+ \sum_{k=1}^{n-1} (-1)^k \sum_I c_I (b_{i_1} \dots b_{i_{k-1}} [b_{i_k} b_{i_{k+1}}] b_{i_{k+2}} \dots b_{i_n}), \end{aligned} \tag{4.9}$$

where $b_j = f_*(a_j) \in E'$. Because of the cancellation of all the terms in the sum (4.8), we see that all the terms in the sum (4.9) cancel out. Therefore, we have

$$\partial^N (f_* p) = \partial_0^N (f_* p) + (-1)^n \partial_n^N (f_* p) = f_*(\partial_0^N p + (-1)^n \partial_n^N p) = f_*(\partial^N p) \in \Lambda_{n-1}(Q').$$

It follows that $f_* p \in \Omega_n^N(Q')$ and that f_* commutes with ∂^N, which finishes the proof. ■

Now assume that instead of (4.7) we have $N = \max\{N_0, N_0'\}$ and investigate the induced morphisms of the chain complexes $\Omega_*^N(Q)$ and $\Omega_*^N(Q')$. In this case we impose an additional condition.

Definition 4.8 A morphism $f : Q \to Q'$ is called *strong* if, for any two distinct arrows $a, b \in E$ with $s(a) = s(b)$ and $t(a) = t(b)$ we have $f_E(a) \neq f_E(b)$.

The quivers with strong morphisms define a subcategory $\mathcal{QI}$ of the category $\mathcal{Q}$. Any strong morphism $f : Q \to Q'$ can be extended to a strong morphism $\widetilde{f} : \widetilde{Q}^N \to \widetilde{Q'}^N$ as on the following diagram (that is defined up to isomorphism):

$$\begin{array}{ccc} Q & \xrightarrow{f} & Q' \\ \downarrow \tau & & \downarrow \tau' \\ \widetilde{Q}^N & \xrightarrow{\widetilde{f}} & \widetilde{Q'}^N \end{array} \tag{4.10}$$

Here τ and τ' are natural inclusions, and the map $\widetilde{f} = (\widetilde{f}_{\widetilde{V}}, \widetilde{f}_{\widetilde{E}})$ is defined as follows:

i) $\widetilde{f}_{\widetilde{V}}$ coincides with f_V (recall that $V = \widetilde{V}, V' = \widetilde{V'}$).

ii) The restriction $\widetilde{f}_{\widetilde{E}}|_E$ coincides with f_E (recall that $E \subset \widetilde{E}$, $E' \subset \widetilde{E'}$).

iii) For any two vertices $v, w \in V$, denote by $E_{v,w}$ the set of arrows in $\widetilde{E}$ that does not lie in E and have the start vertex v and target vertex w. Denote by $E'_{f(v),f(w)}$ the set of arrows in $\widetilde{E'}$ that does not lie in $f_E(E)$ and have the start vertex $f(v)$ and target vertex $f(w)$. By the injectivity of f_E, we have $|E_{v,w}| = |E'_{f(v),f(w)}|$. Then we extend f_E to $\widetilde{E}$ by an isomorphism of sets $E_{v,w} \to E'_{f(v),f(w)}$ thus obtaining $\widetilde{f}_{\widetilde{E}}$.

Hence, $\widetilde{f}$ is a strong morphism.

Remark 4.9 Note that, for any $v, w \in V$, the strong morphism $\widetilde{f}$ provides a bijection between the set of arrows with start vertex v and target vertex w and the set of vertices with start vertex $f(v)$ and target vertex $f(w)$.

Proposition 4.10 *If $f: Q \to Q'$ is a strong morphism, then the morphism $\widetilde{f}: \widetilde{Q} \longrightarrow \widetilde{Q'}$, defined by (4.10), induces a morphism of chain complexes*

$$\widetilde{f}_*: \Lambda_*(\widetilde{Q}) \longrightarrow \Lambda_*(\widetilde{Q'}) \tag{4.11}$$

and, hence, a homomorphism $H_*(\widetilde{Q}, K) \to H_*(\widetilde{Q'}, K)$ of homology groups.

Proof. It is sufficient to check that $\partial_i(\widetilde{f}_*(p)) = \widetilde{f}_*(\partial_i p)$ for any elementary $(n+1)$-path $p = a_0 \dots a_n$ and $0 \le i \le n+1$. By definition, we have

$$\widetilde{f}_*(p) = \widetilde{f}_{\widetilde{E}}(a_0) \dots \widetilde{f}_{\widetilde{E}}(a_n), \quad a_i \in \widetilde{E}.$$

The cases $i = 0$ and $i = n+1$ follow from relation between $\widetilde{f}_{\widetilde{E}}$ and $\widetilde{f}_{\widetilde{V}}$ for $n = 0$ and from Remark 4.9 for $n \ge 1$. For the rest cases it is sufficient to check that

$$\widetilde{f}_{\widetilde{E}}[a_{i-1}a_i] = [\widetilde{f}_{\widetilde{E}}(a_{i-1})\widetilde{f}_{\widetilde{E}}(a_i)],$$

which follows from Remark 4.9, as well. ■

Proposition 4.11 *Let $Q = (V, E, s, t)$ and $Q' = (V', E', s', t')$ be quivers of power N_0 and N'_0, respectively, and $f: Q \to Q'$ be a strong morphism. Let $N = \max\{N_0, N'_0\}$ and let $p \in \Omega_n^N(Q)$. Then $f_*(p) \in \Omega_n^N(Q')$ and the morphism f induces a morphism of chain complexes*

$$\Omega_*^N(Q) \longrightarrow \Omega_*^N(Q')$$

and hence a homomorphism of homology groups

$$H_*^N(Q, K) \longrightarrow H_*^N(Q', K)$$

in all dimensions.

Proof. By Proposition 4.10 we have a morphism (4.11) of chain complexes. For any n, consider the restriction of this morphism to $\Lambda_n(Q)$, that is, we have a commutative diagram:

$$\begin{array}{ccc} \Lambda_n(Q) & \xrightarrow{f_*} & \Lambda_n(Q') \\ \downarrow \tau & & \downarrow \tau' \\ \Lambda_n(\widetilde{Q}) & \xrightarrow{\widetilde{f}_*} & \Lambda_n(\widetilde{Q'}) \end{array}$$

For any $p \in \Lambda_n(Q)$ we have

$$\partial^N p = \partial^N \tau(p) \in \Lambda_{n-1}(Q).$$

Then $f_*(p) \in \Lambda_n(Q')$ and

$$\partial^N f_*(p) = \partial^N(\tau' f_*(p)) = \partial^N(\widetilde{f}_* \tau(p)) = \widetilde{f}_*(\partial^N \tau(p)) = f_*(\partial^N \tau(p)) \in \Lambda_{n-1}(Q'),$$

whence $f_*(p) \in \Omega_n(Q')$. ■

5 Homotopy invariance of path homology groups of quivers

In this section we define the notion of homotopy between two quiver morphisms and give conditions when homotopic maps induce the same homomorphism of homology groups.

Let $I_n = (V_n, E_n, s_n, t_n)$ $(n \geq 1)$ be a quiver with the set of vertices $V_n = \{0, 1, \ldots, n\}$ and the set of arrows E_n that contains exactly one of the two arrows $(i \to (i+1))$ and $((i+1) \to i)$ for $i = 0, 1, \ldots, n-1$, and no other arrow. We denote by I_0 the quiver which has one vertex 0 and has no arrows.

Any quiver I_n is called a *line quiver* of the length n. Denote also by $\mathcal{I} = \cup_{n\geq 0} I_n$ the set of all line quivers. The length of a line quiver J will be also denoted by $|J|$.

Definition 5.1 $Q = (V, E, s, t)$ be a quiver and $I_n = (V_n, E_n, s_n, t_n)$ be a line quiver. Define the *Cartesian-product*

$$\Pi = Q \Box I_n = (V_\Pi, E_\Pi, s_\Pi, t_\Pi)$$

as a quiver with the set of vertices $V_\Pi = V \times V_n$, the set of arrows

$$E_\Pi = \{E \times V_n\} \sqcup \{V \times E_n\},$$

and the maps s_Π, t_Π as follows:

$$s_\Pi(a, i) = (s(a), i), \quad t_\Pi(a, i) = (t(a), i) \ \text{ for } a \in E, i \in V_n,$$

$$s_\Pi(v, b) = (v, s_n(b)), \quad t_\Pi(v, b) = (v, t_n(b)) \ \text{ for } v \in V, b \in E_n.$$

The product $Q \Box I_n$ can be considered as a *cylinder* over the quiver Q. We have identifications Q with the bottom of $Q \Box \{0\}$ and with the top $Q \Box \{n\}$ of the cylinder by using natural inclusions.

Let I be the line quiver of length 1 with two vertices $\{0, 1\}$ and exactly one arrow $(0 \to 1)$.

Definition 5.2 Let Q and R be two quivers.

i) We call two morphisms $f, g \colon Q \to R$ *one-step homotopic* and write $f \simeq_1 g$ if there exists a morphism $F \colon Q \Box I \to R$ such that at least one of the two following conditions is satisfied:

1. $F|_{Q\Box\{0\}} = f, \quad F|_{Q\Box\{1\}} = g;$
2. $F|_{Q\Box\{0\}} = g, \quad F|_{Q\Box\{0\}} = f.$

ii) We call two (strong) morphisms $f, g \colon Q \to R$ *homotopic* and write $f \simeq g$ if there exists a sequence of (strong) morphisms

$$f_i \colon Q \to R, \quad i = 0, ..., n,$$

such that $f = f_0 \simeq_1 f_1 \simeq_1 \cdots \simeq_1 f_n = g$.

iii) Two quivers Q and R are called *(strong) homotopy equivalent* if there exist (strong) morphisms

$$f\colon Q \to R, \quad g\colon R \to Q$$

such that

$$fg \simeq \mathrm{Id}_R, \quad gf \simeq \mathrm{Id}_Q .$$

In this case, we shall write $Q \simeq R$ (or $Q \overset{s}{\simeq} R$ in the case of strong homotopy) and shall call the morphisms f, g *(strong) homotopy inverses* of each other.

In order to state and prove the main result, let us introduce some notations. For any quiver $Q = (V, E, s, t)$ set

$$\widehat{Q} = Q \Box I.$$

We shall put the hat " $\widehat{}$ " over all notations related to $\widehat{Q}$ that are similar to corresponding notations for Q. For example, $\widehat{V}$ is the set of vertices of $\widehat{Q}$, $\widehat{E}$ is the set of arrows of $\widehat{Q}$, $\widehat{\Lambda}_n = \Lambda_n(\widehat{Q})$ and $\widehat{\Omega}_n^N = \Omega_n(\widehat{Q}^N)$. Write also $P = PQ$ and $\widehat{P} = P(\widehat{Q})$.

Any vertex $v \in V$ is identified with the vertex $(v, 0) \in \widehat{V}$. Set also $v' = (v, 1) \in \widehat{V}$. Similarly, any arrow $a \in E$ is identified with $(a, 0) \in \widehat{E}$. Set also $a' = (a, 1) \in \widehat{E}$. For any path $p \in P$ define the path $p' \in \widehat{P}$ as follows: if $p = v \in V$ then $p' = v'$ and if $p = a_0 \dots a_n$ then

$$p' = a_0' \dots a_n'.$$

For any vertex $v \in V$, denote by b_v the arrow $(v \to v') \in \widehat{E}$. For a path $p \in P$, define the path $\widehat{p} \in \widehat{P}$ that is called *lifting* of p as follows. For any 0-path $p = v \in V = P_0$ set

$$\widehat{p} = b_v \in \widehat{P}_1.$$

For any path $p = a_0 a_1 a_2 \dots a_n \in P_{n+1}$ $(n \geq 0)$ set

$$\widehat{p} = b_{s(a_0)} a_0' a_1' \dots a_n' + \sum_{i=0}^{n} (-1)^{i+1} \left(a_0 \dots a_i b_{t(a_i)} a_{i+1}' \dots a_n' \right), \tag{5.1}$$

so that $\widehat{p} \in \widehat{P}_{n+2}$. By K-linearity this definition extends to all $p \in \Lambda_{n+1}$ $(n \geq -1)$ thus giving $\widehat{p} \in \widehat{\Lambda}_{n+2}$.

Let N_0 be the power of Q. Fix some $N \geq N_0$ and write for simplicity $\partial^N \equiv \partial$.

Lemma 5.3 *For any $p \in \Lambda_n$ with $n \geq 0$, we have*

$$\partial \widehat{p} = -\widehat{\partial p} + N(p' - p). \tag{5.2}$$

Proof. It suffices to prove (5.2) for any $p \in P_n$. Let us first prove (5.2) for $p = v \in V = P_0$. In this case we have $\partial p = 0$, $\widehat{\partial p} = 0$, and $\widehat{p} = b_v = (v \to v')$ whence

$$\partial \widehat{p} = N(p' - p) = -\widehat{\partial p} + N(p' - p).$$

Then it suffices to prove (5.2) for $p = a_0...a_n$ where $n \geq 0$, which will be done by induction in n.

For $n = 1$, we have $p = a_0 =: a$. Set $a = (v \to w)$. Then we have

$$\partial p = N(w - v),\ \widehat{\partial p} = N(b_w - b_v),\ \ \widehat{p} = b_v a' - a b_w, \tag{5.3}$$

whence

$$\begin{aligned} \partial \widehat{p} &= Na' - [b_v a'] + N b_v - (N b_w - [a b_w] + N a) \\ &= N(a' - a) - [b_v a'] + [a b_v] + N(b_v - b_w). \end{aligned} \tag{5.4}$$

Note that $[b_v a']$ is the sum of all arrows from $s(b_v) = v$ to $t(a') = w'$, while $[a b_w]$ is the sum of all arrows from $s(a) = v$ to $t(b_w) = w'$, whence we see that

$$[b_v a'] = [a b_v]. \tag{5.5}$$

Combining (5.3) and (5.4) we obtain (5.2).

In the inductive step we shall use the following identity. For any path $a_0....a_n \in P_{n+1}$ with $n \geq 1$ set $\beta = a_0...a_{n-1}$ and

$$\gamma = \begin{cases} a_0...a_{n-2}, & n \geq 2, \\ s(a_0), & n = 1. \end{cases}$$

Then it follows from (3.8) that

$$\begin{aligned} \partial(\beta a_n) &= (\partial\beta - (-1)^n N\gamma) a_n + (-1)^n \gamma [a_{n-1} a_n] + (-1)^{n+1} N\beta \\ &= (\partial\beta) a_n + (-1)^{n+1} N\gamma a_n + (-1)^n \gamma [a_{n-1} a_n] + (-1)^{n+1} N\beta. \end{aligned} \tag{5.6}$$

For the inductive step from n to $n+1$, consider $p = a_0...a_n \in P_{n+1}$ and set

$$u = a_0...a_{n-1} \quad \text{and} \quad w = \begin{cases} a_0...a_{n-2}, & n \geq 2, \\ s(a_0), & n = 1. \end{cases}$$

Set also

$$j = t(a_{n-1}) = s(a_n) \quad \text{and} \quad k = t(a_n)$$

as on the following diagram:

$$\begin{tikzcd} {} \arrow[r] & \bullet \arrow[r, "a'_{n-1}"] & \bullet_{j'} \arrow[r, "a'_n"] & \bullet_{k'} \\ {} \arrow[r] & \bullet \arrow[u] \arrow[r, "a_{n-1}"'] & \bullet_j \arrow[u, "b_j"'] \arrow[r, "a_n"'] & \bullet_k \arrow[u, "b_k"'] \end{tikzcd}$$

We obtain from (5.1) that

$$\widehat{p} = \widehat{ua} = \widehat{u} a'_n + (-1)^{n+1} u a_n b_k, \tag{5.7}$$

whence

$$\partial\widehat{p} = \partial(\widehat{u} a'_n) + (-1)^{n+1} \partial(u a_n b_k). \tag{5.8}$$

Since $u = wa_{n-1}$, it follows from (5.1) that

$$\widehat{u} = \widehat{w}a'_{n-1} + (-1)^n wa_{n-1}b_j.$$

In order to compute $\partial(\widehat{u}a'_n)$ observe that every elementary path in $\widehat{u}$ has the end vertex j', while the last arrow can be of two kinds: a'_{n-1} or b_j. Hence, applying (5.6) in order to compute ∂ of elementary paths of these two kinds, we obtain

$$\begin{aligned}\partial(\widehat{u}a'_n) &= (\partial\widehat{u})\,a'_n + (-1)^{n+2}N(\gamma' + \gamma'')\,a'_n \\ &\quad + (-1)^{n+1}\gamma'\left[a'_{n-1}a'_n\right] + (-1)^{n+1}\gamma''\left[b_j a'_n\right] + (-1)^{n+2}N\widehat{u}\end{aligned} \tag{5.9}$$

where

$$\gamma' = \widehat{w} \text{ and } \gamma'' = (-1)^n wa_{n-1} = (-1)^n u.$$

Observe also that in (5.9) $\gamma'' a'_n = (-1)^n ua'_n = 0$.

Next, using again (5.6), we obtain

$$\partial(ua_nb_k) = \partial(ua_n)\,b_k + (-1)^{n+2}Nub_k + (-1)^{n+1}u\left[a_nb_k\right] + (-1)^{n+2}Nua_n. \tag{5.10}$$

Combining (5.9) and (5.10), we obtain

$$\begin{aligned}\partial\widehat{p} &= (\partial\widehat{u})\,a'_n + (-1)^{n+2}N\widehat{w}a'_n \\ &\quad + (-1)^{n+1}\widehat{w}\left[a'_{n-1}a'_n\right] - u\left[b_ja'_n\right] + (-1)^{n+2}N\widehat{u} \\ &\quad + (-1)^{n+1}\partial(ua_n)\,b_k + (-1)^{2n+3}Nub_k + u\left[a_nb_k\right] - Nua_n\end{aligned}$$

Using

$$\partial(ua_n) = (\partial u)\,a_n + (-1)^{n+1}Nwa_n + (-1)^n w\left[a_{n-1}a_n\right] + (-1)^{n+1}Nu \tag{5.11}$$

and observing that, similarly to (5.5), $[b_ja'_n] = [a_nb_k]$, we obtain

$$\begin{aligned}\partial\widehat{p} &= (\partial\widehat{u})\,a'_n + (-1)^{n+2}N\widehat{w}a'_n \\ &\quad + (-1)^{n+1}\widehat{w}\left[a'_{n-1}a'_n\right] - u\left[b_ja'_n\right] + (-1)^{n+2}N\widehat{u} \\ &\quad + (-1)^{n+1}(\partial u)\,a_nb_k + Nwa_nb_k - w\left[a_{n-1}a_n\right]b_k + Nub_k \\ &\quad - Nub_k + u\left[a_nb_k\right] - Nua_n \\ &= (\partial\widehat{u})\,a'_n + (-1)^{n+2}N\widehat{w}a'_n \\ &\quad + (-1)^{n+1}\widehat{w}\left[a'_{n-1}a'_n\right] + (-1)^{n+2}N\widehat{u} \\ &\quad + (-1)^{n+1}(\partial u)\,a_nb_k + Nwa_nb_k - w\left[a_{n-1}a_n\right]b_k - Nua_n.\end{aligned}$$

By the inductive hypothesis, we have

$$\partial\widehat{u} = -\widehat{\partial u} + N(u' - u)$$

and, hence,

$$\begin{aligned}\partial\widehat{p} &= -(\widehat{\partial u})a'_n + Nu'a'_n + (-1)^{n+2}N\widehat{w}a'_n \\ &\quad + (-1)^{n+1}\widehat{w}\left[a'_{n-1}a'_n\right] + (-1)^{n+2}N\widehat{u} \\ &\quad + (-1)^{n+1}(\partial u)\,a_nb_k + Nwa_nb_k - w\left[a_{n-1}a_n\right]b_k - Nua_n,\end{aligned}$$

where we have used again $ua_n' = 0$.

On the other hand, using (5.11) and (5.7), we have

$$\begin{aligned}
\widehat{\partial p} &= \widehat{\partial (ua_n)} = \widehat{(\partial u) a_n} + (-1)^{n+1} N \widehat{w a_n} + (-1)^n \, \widehat{wc} + (-1)^{n+1} N \widehat{u} \\
&= (\widehat{\partial u}) a_n' + (-1)^n \, (\partial u) \, a_n b_k + \\
&+ \; (-1)^{n+1} N \widehat{w} a_n' + (-1)^{n+1} (-1)^n N w a_n b_k \\
&+ \; (-1)^n \, \widehat{w} \left[a_{n-1}' a_n' \right] + (-1)^n \, (-1)^n \, w \left[a_{n-1} a_n \right] b_k + (-1)^{n+1} \, N \widehat{u}
\end{aligned}$$

where $c = [a_{n-1} a_n]$. Adding up the two identities, we see that most of the terms cancel out, and we obtain

$$\partial \widehat{p} + \widehat{\partial p} = N u' a_n' - N u a_n = N (p' - p),$$

which finishes the proof of (5.2). ∎

Proposition 5.4 *Let Q be a quiver of power N_0 and $N \geq N_0$. If $p \in \Omega_n^N$ then $\widehat{p} \in \widehat{\Omega}_{n+1}^N$.*

Proof. The condition $p \in \Omega_n^N$ means that $p \in \Lambda_n$ and $\partial p \in \Lambda_{n-1}$. Since $\widehat{p} \in \widehat{\Lambda}_{n+1}$ and $\widehat{\partial p} \in \widehat{\Lambda}_n$, we obtain by (5.2) that also $\partial \widehat{p} \in \widehat{\Lambda}_n$. Hence, $\widehat{p} \in \widehat{\Omega}_{n+1}^N$. ∎

Now we can prove the main result about connection between homotopy and the homology groups of quivers.

Theorem 5.5 *Let Q, R be two quivers of power N_0 and N_0', respectively. Let K be a commutative ring with unity. Fix an integer $N \geq \max\{N_0, N_0'\}$ and assume that the element $N \in K$ is invertible. Let $f \simeq g \colon Q \to R$ be two homotopic morphisms of quivers. Assume that either $N > \max\{N_0, N_0'\}$ or $N \geq \max\{N_0, N_0'\}$ and f, g are strong morphisms.*

(*i*) *Then f and g induce the identical homomorphisms*

$$f_* = g_* \colon H_*^N(Q, K) \to H_*^N(R, K).$$

(*ii*) *Let the quivers Q and R be homotopy equivalent by mutually inverse morphisms $f \colon Q \to R$ and $g \colon R \to Q$. Then the induced maps f_* and g_* provide mutually inverse isomorphisms of the homology groups $H_*^N(Q, K)$ and $H_*^N(R, K)$.*

Proof. (*i*) Let F be a homotopy between f and g as in Definition 5.2. It suffices to prove the statement for the one-step homotopy using the line quiver $I = (0 \to 1)$. By Propositions 4.7 and 4.11, the maps f and g induce morphisms of chain complexes

$$f_*, g_* \colon \Omega_*^N(Q) \to \Omega_*^N(R),$$

and F induces a morphism of chain complexes

$$F_* \colon \Omega_*^N(\widehat{Q}) \to \Omega_*^N(R).$$

where $\widehat{Q} = Q\Box I$. Note that, for any path $p \in P(\widehat{Q})$ that lies in $P(Q)$, we have $F_*(p) = f_*(p)$, and for any path $p' \in P(\widehat{Q})$ that lies in $P(Q')$, we have $F_*(p') = g_*(p)$.

In order to prove that f_* and g_* induce the identical homomorphisms $H_*^N(Q) \to H_*^N(R)$, it suffices by [23, page 40, Theorem 2.1] to construct a chain homotopy between the chain complexes $\Omega_*^N(Q)$ and $\Omega_*^N(R)$, that is, the K-linear maps

$$L_n \colon \Omega_n^N(Q) \to \Omega_{n+1}^N(R)$$

such that

$$\partial L_n + L_{n-1}\partial = g_* - f_*,$$

where $\partial \equiv \partial^N$. Let us define the mapping L_n as follows

$$L_n(p) = \frac{F_*(\widehat{p})}{N},$$

for any $p \in \Omega_n^N(Q)$. Here $\widehat{p} \in \Omega_{n+1}^N(\widehat{Q})$ is lifting of $p \in \Omega_n^N(Q)$ as above.

Since F_* is a morphism of chain complexes we have $\partial F_* = F_*\partial$. Now using (5.2) we obtain

$$\begin{aligned}
(\partial L_n + L_{n-1}\partial)(p) &= \frac{1}{N}\partial(F_*(\widehat{p})) + \frac{1}{N}F_*(\widehat{\partial p}) \\
&= \frac{1}{N}F_*(\partial\widehat{p}) + \frac{1}{N}F_*(\widehat{\partial p}) \\
&= \frac{1}{N}F_*(\partial\widehat{p} + \widehat{\partial p}) = \frac{1}{N}F_*(N(p' - p)) \\
&= F_*(p') - F_*(p) = g_*(p) - f_*(p).
\end{aligned}$$

(ii) Note that morphisms f, g induce the following homomorphisms

$$H_n^N(Q,K) \xrightarrow{f_*} H_n^N(R) \xrightarrow{g_*} H_n^N(Q,K) \xrightarrow{f_*} H_n^N(R,K),$$

where by (i), $f_* \circ g_* = \mathrm{Id}$ and $g_* \circ f_* = \mathrm{Id}$, which implies that f_* and g_* are mutually inverse isomorphisms of $H_n^N(Q,K)$ and $H_n^N(R,K)$. ■

6 Homology of multigraphs and examples

In this section we transfer the homology theory to a category of multigraphs, discuss possible applications, and give several examples of computations.

At first we describe how to transfer the homology theory from the category of quivers to that of multigraphs. To denote a multigraph and morphisms of multigraphs we shall use a bold font (similarly to [15, §6]), while for quivers we use a normal font.

Definition 6.1 i) A finite *multigraph* is a triple $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{r})$ where $\mathbf{V}$ is a finite set of *vertices*, $\mathbf{E}$ is a finite set of *edges*, and $\mathbf{r} \colon \mathbf{E} \to \mathbf{V} \times \mathbf{V}$ is a map which assigns to each edge an unordered pair of endpoint vertices. An edge $\mathbf{a} \in \mathbf{E}$ with $\mathbf{r}(\mathbf{a}) = (\mathbf{x}, \mathbf{x}), \mathbf{x} \in \mathbf{V}$ is called a *loop*.

ii) A *morphism* from a multigraph $\mathbf{G} = (\mathbf{V_G}, \mathbf{E_G}, \mathbf{r_G})$ to a multigraph $\mathbf{H} = (\mathbf{V_H}, \mathbf{E_H}, \mathbf{r_H})$ is a pair of maps $\mathbf{f_V}\colon \mathbf{V_G} \to \mathbf{V_H}$ and $\mathbf{f_E}\colon \mathbf{E_G} \to \mathbf{E_H}$ such that for any edge $\mathbf{a} \in \mathbf{E_G}$ with $\mathbf{r_G}(\mathbf{a}) = (\mathbf{x}, \mathbf{y})$ we have

$$\mathbf{r_H}(\mathbf{f_E}(\mathbf{a})) = (\mathbf{f_V}(\mathbf{x}), \mathbf{f_V}(\mathbf{y})).$$

We will refer to morphisms of multigraphs as *graph maps*.

For a multigraph $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{r})$ and any nonordered pair of vertices $(\mathbf{x}, \mathbf{y}) \in \mathbf{V} \times \mathbf{V}$ define $\mu(\mathbf{x}, \mathbf{y})$ as the number of edges $\mathbf{a} \in \mathbf{E}$ for which $\mathbf{r}(\mathbf{a}) = (\mathbf{x}, \mathbf{y})$. Set

$$N_0 := \max_{(\mathbf{x},\mathbf{y})\in\mathbf{V}\times\mathbf{V}} \left\{ \{\mu(\mathbf{x}, \mathbf{y}) | \mathbf{x} \neq \mathbf{y}\}, 2\mu((\mathbf{x}, \mathbf{x})) \right\}. \tag{6.12}$$

The set of finite multigraphs with graph maps forms a category $\mathcal{G}$. We can associate to each multigraph $\mathbf{G} = (\mathbf{V_G}, \mathbf{E_G}, \mathbf{r_G})$ a symmetric quiver

$$G = \mathcal{O}(\mathbf{G}) = (V_G, E_G, s_G, t_G)$$

where $V_G = \mathbf{V_G}$ and E_G, s_G, t_G are defined as follows. For any $\mathbf{a} \in \mathbf{E_G}$ with $\mathbf{r_G}(\mathbf{a}) = (\mathbf{x}, \mathbf{y})$ we have two arrows $a_1, a_2 \in E_G$ with

$$s(a_1) = x, t(a_1) = y \text{ and } \ s(a_2) = y, t(a_2) = x.$$

Thus we obtain a functor $\mathcal{O}$ that provides an isomorphism of the category $\mathcal{G}$ and a subcategory of symmetric quivers of the category $\mathcal{Q}$.

Definition 6.2 For any multigraph $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{r})$ and $N \geq N_0$ define the homologies with coefficient from K by

$$H_n^N(\mathbf{G}) = H_n^N(\mathcal{O}(\mathbf{G})).$$

It follows directly from the Definition, that the homology groups of a multigraph are well defined. It is an easy exercise to obtain the basic properties of these homology groups from the corresponding results about quivers.

Now we generalize the notion of the Atkins connectivity graph that was defined for simplicial complexes in [1], [2], [6]. Namely, we define a connectivity multigraph of a CW-complex, so that the path homology theory of multigraphs can be applied for investigation of connectivity properties of CW-complexes. Recall the definition of CW-complex (cf. [24]).

Definition 6.3 A CW-complex X is a topological space X which is the union of a sequence of subspaces X_n such that, inductively, X_0 is a discrete set of points (called vertices) and X_n is the pushout obtained from X_{n-1} by attaching disks D^n along attaching maps $\partial(D^n) \to X_{n-1}$. Each resulting map $j : D^n \to X$ is called a *n-cell*.

Given a finite CW-complex X, let us fix two integers $0 \le s < n$, enumerate all n-cells of X by integers and define the *connectivity quiver* $G_s^n = (V, E, r)$ as follows. The vertices of G_s^n are given by all n-cells of X, and the arrows of G_s^n are given by s-cells of X by the following rule. A s-cell $j^0\colon D^s \to X$ is an arrow from the vertex $j^1\colon D^n \to X$ to the vertex $j^2\colon D^n \to X$ if the number of j^1 is smaller than that of j^2 and $j^0(D^s) \subset j^i(D^n)$ for $i = 1, 2$.

Similarly, the connectivity *multigraph* $\mathbf{G_s^n} = (\mathbf{V}, \mathbf{E}, \mathbf{r})$ of X is defined as follows. The vertices of $\mathbf{G_s^n}$ are n-cells of X, and a s-cell $j^0\colon D^s \to X$ determines an edge between two vertices $j^i\colon D^n \to X$ $(i = 1, 2)$ if $j^0(D^s) \subset j^i(D^n)$ for $i = 1, 2$.

Now we give several examples of computations of homology groups of quivers in small dimensions.

Example 6.4 Consider a cell complex with three 2-cell that are enumerated by v_0, v_1, v_2 as on Fig. 1.

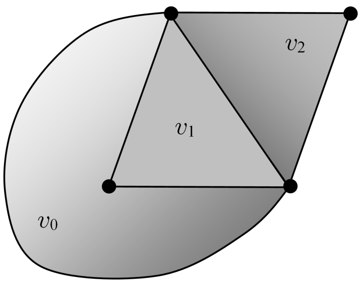

Figure 1: A CW-complex.

The corresponding connectivity quiver $G_1^2 = Q = (V, E, s, t)$ is given on the following diagram

$$\begin{array}{ccc} v_1 & \xrightarrow{b} & v_2 \\ {\scriptstyle a}\uparrow\uparrow{\scriptstyle a_1} & & \\ v_0 & & \end{array}$$

Here $V = \{v_0, v_1, v_2\}$ and $E = \{a, a_1, b\}$. Let us take $N = 2$ and set $\Omega_* \equiv \Omega_*^2$. Clearly, $\Omega_n(Q) = 0$ for $n \ge 3$ and

$$\Omega_0(Q) = \langle v_0, v_1, v_2\rangle, \quad \Omega_1(Q) = \langle a, a_1, b\rangle, \quad \Omega_2(Q) = \langle ab - a_1b\rangle.$$

The boundaries are given by

$$\partial a = 2(v_1 - v_0), \ \partial a_1 = 2(v_1 - v_0), \ \partial b = 2(v_2 - v_1),$$

and

$$\partial(ab - a_1b) = 2(a_1 - a).$$

Let us change the bases in $\Omega_*(Q)$ as follows:

$$\Omega_0(Q) = \langle v_0, v_1 - v_0, v_2 - v_1\rangle, \quad \Omega_1(Q) = \langle a, a_1 - a, b\rangle, \quad \Omega_2(Q) = \langle ab - a_1b\rangle.$$

Then

$$\partial a = 2(v_1 - v_0),\ \partial(a_1 - a) = 0,\ \partial b = 2(v_2 - v_1),$$

and

$$\partial(ab - a_1 b) = 2(a_1 - a).$$

Hence, $H_1(Q,\mathbb{Z}) = \mathbb{Z}_2$ is generated by $(a_1 - a_0) \bmod 2$, $H_2(Q,\mathbb{Z}) = 0$, and $H_0(Q,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ where $\mathbb{Z}$ is generated by v_0 and the summands $\mathbb{Z}_2$ are generated by $(v_1 - v_0) \bmod 2$ and $(v_2 - v_1) \bmod 2$, respectively.

Example 6.5 Let X consist of two identical n-gons ($n \geq 3$) with identified boundaries. Then its connectivity quiver $G_1^2 = Q = (V,E,s,t)$ has two vertices $V = \{v_0, v_1\}$ and n arrows $E = \{a_1, \dots a_n\}$, such that $s(a_i) = v_0, t(a_i) = v_1$, for $i = 1, \dots, n$. Let $N = n$ and $\Omega_* \equiv \Omega_*^n$. Then $\Omega_k(Q) = 0$ for $k \geq 2$ and

$$\Omega_0(Q) = \langle v_0, v_1 \rangle \ \text{ and } \ \Omega_1(Q) = \langle a_1, \dots a_n \rangle.$$

The only nontrivial differential is $\partial\colon \Omega_1(Q) \to \Omega_0(Q)$ given by $\partial a_i = n(v_1 - v_0)$ for $i = 1, \dots, n$. Changing the basis in $\Omega_*(Q)$ we can write

$$\Omega_0(Q) = \langle v_0, v_1 - v_0 \rangle \ \text{ and } \ \Omega_1(Q) = \langle a_1, a_1 - a_2, \dots, a_1 - a_n \rangle.$$

Clearly, $\partial a_1 = n(v_1 - v_0)$ and $\partial(a_1 - a_i) = 0$ for $i = 2, \dots n$. Hence $H_k(Q,\mathbb{Z}) = 0$ for $k \geq 2$ and

$$H_0(Q,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_n \ \text{ and } \ H_1(Q,\mathbb{Z}) \cong \mathbb{Z}^{n-1}.$$

Example 6.6 Now we give an exampe of quiver Q for which $H_2^2(Q,\mathbb{Z})$ is nontrivial. Consider at first a quiver Q_1 ("cycle") as on the diagram:

$$\begin{array}{ccc} v_2 & \xrightarrow{a_2} & v_3 \\ {\scriptstyle a_1} \uparrow & \swarrow {\scriptstyle a_3} & \\ v_1 & & \end{array}$$

with $V_1 = \{v_1, v_2, v_3\}$ and $E_1 = \{a_1, a_2, a_3\}$. Now construct the quiver $Q = (V,E,s,t)$ that is a "double suspension" over Q_1 as follows. We add two new vertices v_4 and v_5 and arrows $b_1, b_1', b_2, b_2', b_3, b_3', c_1, c_1', c_2, c_2', c_3, c_3'$ such that

$$s(b_i) = s(b_i') = s(c_i) = s(c_i') = v_i, \quad i = 1,2,3,$$

and

$$t(b_i) = t(b_i') = v_4,\ t(c_i) = t(c_i') = v_5, \quad i = 1,2,3.$$

Let $N = 2$ and $\Omega_* \equiv \Omega_*^2$. We have $\Omega_n(Q) = 0$ for $n \geq 3$. Consider a nontrivial element $\omega \in \Omega_2(Q)$ as follows:

$$\omega = a_3 b_1 + a_3 b_1' + a_1 b_2 + a_1 b_2' + a_2 b_3 + a_2 b_3' - a_3 c_1 - a_3 c_1' - a_1 c_2 - a_1 c_2' - a_2 c_3 - a_2 c_3'.$$

It is easy to check, that $\partial\omega = 0$, and hence $H_2(Q,\mathbb{Z}) \neq 0$.

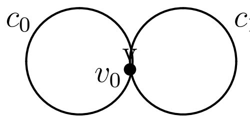

Figure 2: A quiver with two loops.

Using the iteration of the suspension it is relatively easy to construct quivers with nontrivial homology group in any dimension similarly to [16].

Example 6.7 Now we give an example of quiver with nontrivial finite one-dimensionsl homology group. Consider a quiver $Q = (V, E, s, t)$ as in Fig. 2. with one vertex $\{v_0\}$ and two arrows c_0, c_1, such that $s(c_i) = t(c_i) = v_0$. Set $N = 2$ and $\Omega_* = \Omega^2_*$. We have

$$\Omega_0(Q) = \langle v_0 \rangle,\ \Omega_1(Q) = \langle c_0, c_1 \rangle,\ \text{and } \Omega_2 = \langle c_0c_0, c_0c_1, c_1c_0, c_1c_1 \rangle.$$

The differentials are trivial in dimensions zero and one. We have

$$\begin{aligned} \partial(c_0c_0) &= 2c_0 - c_0 - c_1 + 2c_0 = 3c_0 - c_1, \\ \partial(c_0c_1) &= 2c_1 - c_0 - c_1 + 2c_0 = c_0 + c_1, \\ \partial(c_1c_0) &= 2c_0 - c_0 - c_1 + 2c_1 = c_0 + c_1, \\ \partial(c_1c_1) &= 2c_1 - c_0 - c_1 + 2c_1 = 3c_1 - c_0. \end{aligned} \tag{6.13}$$

and, hence, $\partial(c_0c_0 + c_0c_1) = 4c_0$. It is easy to check directly, that $2c_0 \notin \mathrm{Im}\{\partial\colon \Omega_2 \to \Omega_1\}$. Changing the basis we have

$$\Omega_1(Q) = \langle c_0, c_0 + c_1 \rangle.$$

Thus we obtain

$$H_i(Q, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{for } i = 0, \\ \mathbb{Z}_4, & \text{for } i = 1. \end{cases}$$

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