Stochastic completeness of Markov processes

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Contents

0 Introduction

Let $\{X_t\}_{t\geq 0}$ be a reversible Markov process on state space M with a stationary measure μ. This process is called *stochastically complete* if its lifetime is almost surely ∞. In terms of the associated transition semigroup $\{P_t\}_{t\geq 0}$ the stochastic completeness means that $P_t 1 \equiv 1$ for all $t \geq 0$. If the process has no interior killing component (which is assumed to be the case) then the only way the stochastic incompleteness can occur is when the process leaves the state space in finite time due to a fast escape rate to ∞. Easy examples are diffusions in bounded domains with the Dirichlet boundary condition.

By far less trivial example was discovered by R.Azencott [1] in 1974: he showed that Brownian motion on a geodesically complete non-compact manifold can be stochastically incomplete. In his example, the manifold has negative sectional curvature that grows to $-\infty$ very fast with the distance to an origin. The stochastic incompleteness occurs because negative curvature plays the role of a drift towards infinity, and a very high negative curvature produces an extremely fast drift that sweeps the Brownian particle to infinity in a finite time.

The first sufficient condition for stochastic completeness of geodesically complete manifolds in terms of lower bound of Ricci curvature was proved by S.-T. Yau [19]. A general sufficient condition in terms of the volume growth was proved in [4]: if $V(r)$ denotes the volume of the geodesic ball of radius r centered at the origin and

$$\int^{\infty} \frac{r dr}{\log V(r)} = \infty \tag{0.1}$$

then the manifold in question is stochastically complete.

In the first part of this course we will give the proof of the stochastic completeness under the condition (0.1), following [6] and [7]. Then we will show the sharpness of (0.1).

On any stochastically complete manifold the notion of an *upper radius* makes sense. This is a function $R(t)$ such that the distance from the Brownian particle at time t to the origin is at most $R(t)$. Given the condition (0.1), function $R(t)$ can be determined via the volume function $V(r)$ ([10], [5], [8]). For example, under a polynomial volume growth $V(r) \leq Cr^N$ one obtains

$$R(t) = \text{const}\sqrt{t \log t}. \tag{0.2}$$

Note that the sharp upper radius in $\mathbb{R}^n$ is by Khinchine's theorem $R(t) = \sqrt{ct \log \log t}$ where $c > 4$. However, there are examples of manifolds with polynomial volume growth where the upper radius (0.2) cannot be improved ([11]). If $V(r) \leq \exp(Cr^\alpha)$ where $0 < \alpha < 2$ then the upper radius is

$$R(t) = \text{const}\, t^{\frac{1}{2-\alpha}}$$

whereas under the assumption $V(r) \leq \exp(Cr^2)$ one obtains $R(t) = \exp(\text{const}\, t)$. We plan to give in lectures a brief account of the above results on upper radius.

Consider now a symmetric jump process $\{X_t\}$ on a metric measure space (M, d, μ) that is determined by a jump kernel $J(x, y)$ (a rigorous definition of such a process is given by means a Dirichlet form in the spirit of [3]). If a certain condition that relates the metric d with the kernel $J(x, y)$ is satisfied then we call the metric d *adapted.* If d is adapted then the stochastic completeness of such a process can also be stated in terms of the volume growth: if $V(r) \leq \exp(Cr)$ then the process is stochastically complete ([9]). This theorem has nice applications to jump processes on fractal spaces, in particular, to those considered in [2]. We plan to give the proof of this theorem and discuss possibilities for relaxing the hypotheses.

A particular case of a jump process is a continuous time random walk in a graph, generated by an unnormalized Laplace operator, that is symmetric with respect to the counting measure. In this case the natural graph distance d is not adapted to the process. However, an additional argument with introduction of an adapted distance allows to prove the following: if $V(r)$ is the counting measure of the ball of radius r with respect to the graph distance and $V(r) \le Cr^3$ then the random walk is stochastically complete ([9]). Surprisingly enough, the cubic rate is volume growth is optimal: there are examples of stochastically incomplete graphs with $V(r) \le Cr^{3+\varepsilon}$ for any $\varepsilon > 0$ ([14], [18]). Other criteria for stochastic completeness can be obtained in terms of the degree growth ([14]).

1 Brownian motion on Riemannian manifolds

1.1 Laplace-Beltrami operator

Let (M, g) be a Riemannian manifold and μ be the Riemannian measure on M. The Laplace operator (or Laplace-Beltrami operator) Δ is defined to satisfy the Green formula: for all $u, v \in C_0^\infty(M)$

$$\int_M \Delta u \; v d\mu = -\int_M \langle \nabla u, \nabla v \rangle d\mu, \tag{1.1}$$

where ∇u and ∇v is the Riemannian gradients and $\langle \cdot, \cdot \rangle$ denotes the Riemannian inner product. In the local coordinates $x^1, ..., x^n$ we have

$$\langle \nabla u, \nabla v \rangle = g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j}$$

and

$$d\mu = \sqrt{\det g} dx^1 ... dx^n,$$

which implies the following expression for Δ:

$$\Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \frac{\partial}{\partial x^j} \right).$$

For example, in $\mathbb{R}^n$ with the Euclidean metric $(g_{ij}) = \mathrm{id}$, we obtain the classical Laplace operator

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{(\partial x^i)^2}$$

Initially Δ is defined as an operator on functions from $C_0^\infty(M)$, but its symmetry with respect to μ (that follows from (1.1)) allows to extend it to a self-adjoint operator in $L^2(M, \mu)$. In general, this extension may not be unique, but if M is geodesically complete (which will be assumed in the main results) then this extension is unique. With some abuse of notation, the self-adjoint extension of Δ will be denoted by the same letter.

1.2 Heat semigroup and heat kernel

As one can see from (1.1), the operator Δ is non-positive definite, which implies that the following operator $P_t := e^{t\Delta}$ is a bounded self-adjoint operator for any $t \ge 0$. The family $\{P_t\}_{t \ge 0}$ is called the *heat semigroup* of Δ for the reason that it resolves the heat equation. More precisely, the following is true:

- for any $f \in L^2$, the function $u(t,x) = P_t f(x)$ is a C^∞ smooth in $(t,x) \in (0,+\infty) \times M$ and satisfies the heat equation

$$\frac{\partial u}{\partial t} = \Delta u. \tag{1.2}$$

and the initial condition

$$u(t,\cdot) \to f \quad \text{as } t \to 0+ \tag{1.3}$$

where the convergence is understood in L^2-norm.

- If $f \in C_0^\infty(M)$ then the convergence in (1.3) can be understood in C^∞-sense.
- If $f \geq 0$ then $P_t f \geq 0$; if $f \leq 1$ then $P_t f \leq 1$.
- The semigroup property: $P_t P_s = P_{t+s}$.

Furthermore, there is a function $p_t(x,y)$ of $t > 0$ and $x, y \in M$ such that

$$P_t f(x) = \int_M p_t(x,y) f(y) \, d\mu(y). \tag{1.4}$$

The function $p_t(x,y)$ is called the *heat kernel* of Δ or of M. Alternatively, $p_t(x,y)$ is referred to as the transition density of Brownian motion on M.

For example, if $M = \mathbb{R}^n$ then

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

For general manifolds there is no explicit formula for the heat kernel. As it follows from the properties of the heat semigroup, the heat kernel satisfies the following properties:

- $p_t(x,y)$ is C^∞ smooth in $(t,x,y) \in (0,+\infty) \times M \times M$
- $p_t(x,y) \geq 0$ (and $p_t(x,y) > 0$ on connected manifolds) and

$$\int_M p_t(x,y) \, d\mu(y) \leq 1, \tag{1.5}$$

- $p_t(x,y) = p_t(y,x)$
- The semigroup identity: for all $x, y \in M$ and $t, s > 0$,

$$p_{t+s}(x,y) = \int_M p_t(x,z) \, p_s(z,y) \, d\mu(z). \tag{1.6}$$

The existence of the heat kernel allows to extend the domain of the operator P_t from L^2 to other spaces. For that, use the identity (1.4) now as the definition of P_t where f is any function such that the integral converges. In particular, P_t extends to a bounded operator on all spaces L^q, $q \in [0, \infty]$. We will need the following:

- If $f \in C_b(M)$ then $u(t,x) = P_t f$ is a C^∞ function of (t,x) that solves the heat equation (1.2) with the initial condition (1.3) where the convergence in the latter is understood locally uniformly. Besides, we have
$$\inf f \leq P_t f \leq \sup f.$$
- Let $u(t,x)$ be a non-negative solution to the heat equation in $(0,\infty) \times M$ such that
$$u(t,\cdot) \overset{L^2_{loc}}{\longrightarrow} f \text{ as } t \to 0, \tag{1.7}$$
for some $f \in L^2_{loc}(M)$. Then $P_t f(x)$ is finite, smooth, solves the heat equation in $(0,+\infty) \times M$, satisfies the initial condition (1.7), and
$$u(t,x) \geq P_t f(x), \tag{1.8}$$
for all $t > 0$ and $x \in M$. (the minimality of $P_t f$).

1.3 General conditions for stochastic completeness

Definition. A weighted manifold $(M, \mathbf{g}, \mu)$ is called *stochastically complete* if $P_t 1 \equiv 1$, that is, if the heat kernel $p_t(x,y)$ satisfies the identity
$$\int_M p_t(x,y)\, d\mu(y) = 1, \tag{1.9}$$
for all $t > 0$ and $x \in M$.

Note that in general we have $0 \leq P_t 1 \leq 1$. If the condition (1.9) fails, that is, $P_t 1 \not\equiv 1$ then the manifold M is called *stochastically incomplete.*

Our purpose here is to provide conditions for the stochastic completeness (or incompleteness) in various terms.

Fix $0 < T \leq \infty$, set $I = (0,T)$ and consider the Cauchy problem in $I \times M$
$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u, & \text{in } I \times M, \\ u|_{t=0} = f, \end{cases} \tag{1.10}$$
where f is a given function from $C_b(M)$. The problem (1.10) is understood in the classical sense, that is, $u \in C^\infty(I \times M)$ and $u(t,x) \to f(x)$ locally uniformly in $x \in M$ as $t \to 0$. Here we consider the question of the *uniqueness* of a bounded solution of (1.10).

Theorem 1.1 *Fix $\alpha > 0$ and $T \in (0, \infty]$. The following conditions are equivalent.*

(a) *M is stochastically complete.*

(b) *The equation $\Delta v = \alpha v$ in M has the only bounded non-negative solution $v = 0$.*

(c) *The Cauchy problem in $(0, T) \times M$ has at most one bounded solution.*

Remark. As we will see from the proof, in condition (b) the assumption that v is non-negative can be dropped without violating the statement.

Lemma 1.2 *Let f be a non-negative bounded smooth function on M. Set*

$$R_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x)\, dt,$$

where $\alpha > 0$ is a constant. Then $u = R_\alpha f$ is finite, smooth, and solves the equation

$$-\Delta u + \alpha u = f.$$

Proof. The integral is finite because $P_t f$ is bounded. Assume in addition that $f \in L^2$. Then we have by the spectral theory

$$R_\alpha f = \int_0^\infty e^{-\alpha t} e^{\Delta t} f dt = \int_0^\infty e^{-(-\Delta+\alpha)t} f dt = (-\Delta + \alpha)^{-1} f,$$

so that $(-\Delta + \alpha) u = f$. Initially this identity is true in the sense of operators in L^2 but then, using the local elliptic regularity and the hypothesis $f \in C^\infty$ we conclude that $u \in C^\infty$ and that the equation is understood in the classical sense.

A general f can be approximated by an increasing sequence $\{f_k\}$ of functions from L^2 so that $R_\alpha f_k \uparrow R_\alpha f$ whence the equation for $R_\alpha f$ follows by the local convergence properties of sequences of solutions to elliptic equations. ■

Proof of Theorem 1.1. We first assume $T < \infty$ and prove the following sequence of implications

$$\neg (a) \Longrightarrow \neg (b) \Longrightarrow \neg (c) \Longrightarrow \neg (a),$$

where $\neg$ means the negation of the statement.

Proof of $\neg (a) \Rightarrow \neg (b)$. So, we assume that M is stochastically incomplete and prove that there exists a non-zero non-negative bounded solution to the equation $-\Delta v + \alpha v = 0$. Consider the function

$$w(x) = R_\alpha 1(x) = \int_0^\infty e^{-\alpha t} P_t 1(x)\, dt, \tag{1.11}$$

which is by Lemma 1.2 C^∞-smooth and satisfies the equation

$$-\Delta w + \alpha w = 1 \tag{1.12}$$

Since $0 \le P_t 1 \le 1$, we obtain from (1.11) that

$$0 \le w \le \alpha^{-1}. \tag{1.13}$$

By the stochastic incompleteness, there exist $x \in M$ and $t > 0$ such that $P_t 1(x) < 1$. Then (1.11) implies that, for this value of x, we have a *strict* inequality $w(x) < \alpha^{-1}$. Hence, $w \not\equiv \alpha^{-1}$.

Finally, consider the function $v = 1 - \alpha w$, which by (1.12) satisfies the equation $\Delta v = \alpha v$. It follows from (1.13) that $0 \le v \le 1$, and $w \not\equiv \alpha^{-1}$ implies $v \not\equiv 0$. Hence, we have constructed a non-zero non-negative bounded solution to $\Delta v = \alpha v$, which finishes the proof.

Proof of $\neg(b) \Rightarrow \neg(c)$. Let v be a bounded non-zero solution to equation $\Delta v = \alpha v$. Then the function

$$u(t,x) = e^{\alpha t} v(x) \tag{1.14}$$

satisfies the heat equation because

$$\Delta u = e^{\alpha t} \Delta v = \alpha e^{\alpha t} v = \frac{\partial u}{\partial t}.$$

Hence, u solves the Cauchy problem in $\mathbb{R}_+ \times M$ with the initial condition $u(0,x) = v(x)$, and this solution u is bounded on $(0,T) \times M$ (note that T is *finite*). Let us compare $u(t,x)$ with another bounded solution to the same Cauchy problem, namely with $P_t v(x)$. We have

$$\sup |P_t v| \le \sup |v|,$$

whereas by (1.14)

$$\sup |u(t,\cdot)| = e^{\alpha t} \sup |v| > \sup |v|.$$

Therefore, $u \not\equiv P_t v$, and the bounded Cauchy problem in $(0,T) \times M$ has two different solutions with the same initial function v.

Proof of $\neg(c) \Rightarrow \neg(a)$. Assume that the problem (1.10) has two different bounded solutions with the same initial function. Subtracting these solutions, we obtain a non-zero bounded solution $u(t,x)$ to (1.10) with the initial function $f = 0$. Without loss of generality, we can assume that $0 < \sup u \le 1$. Consider the function $w = 1 - u$, for which we have $0 \le \inf w < 1$. The function w is a non-negative solution to the Cauchy problem (1.10) with the initial function $f = 1$. By the minimality property of the heat semigroup, we conclude that $w(t,\cdot) \ge P_t 1$. Hence, $\inf P_t 1 < 1$ and M is stochastically incomplete.

Finally, let us prove the equivalence of (a), (b), (c) in the case $T = \infty$. Since the condition (c) with $T = \infty$ is weaker than that for $T < \infty$, it suffices to show that (c) with $T = \infty$ implies (a). Assume from the contrary that M is stochastically incomplete, that is, $P_t 1 \not\equiv 1$. Then the functions $u_1 \equiv 1$ and $u_2 = P_t 1$ are two different bounded solutions to the Cauchy problem (1.10) in $\mathbb{R}_+ \times M$ with the same initial function $f \equiv 1$, so that (a) fails, which was to be proved. ∎

Two more useful results.

Theorem 1.3 *Let M be a connected manifold and $K \subset M$ be a compact set. Assume that, for some $\alpha \ge 0$, there exists an α-superharmonic function v in $M \setminus K$ (that is $-\Delta u + \alpha u \ge 0$) such that $v(x) \to +\infty$ as $x \to \infty$. Then M is stochastically complete.*

Idea of proof. Note that a positive constant c is α-superharmonic and the minimum of two α-superharmonic functions is again α-superharmonic although in a generalized sense, as a continuous function. It follows that the function $w = \min(v, c)$ is α-superharmonic in $M \setminus K$. Taking c large enough and using the condition $v \to +\infty$ we obtain that $w \equiv c$ in a neighborhood of K. Hence, w is α-superharmonic on entire M.

Now assume that M is stochastically incomplete. By Theorem 1.1 M admits a non-trivial α-harmonic function u, such that $0 \le u \le 1$. Fix $\varepsilon > 0$. Taking large enough precompact open subset $\Omega \subset M$, we obtain that $\varepsilon w \ge u$ on $\partial\Omega$. By the comparison principle it follows that $\varepsilon w \ge u$ in Ω. By exhausting M by such sets Ω we obtain $\varepsilon w \ge u$ on M. Letting $\varepsilon \to 0$ we obtain $u \equiv 0$, which contradicts the hypothesis. ■

Theorem 1.4 *Let M be a connected manifold. Assume that there exists a non-negative superharmonic function u on M (that is, $-\Delta u \ge 0$) such that $u \not\equiv \mathrm{const}$ and $u \in L^1(M)$. Then M is stochastically incomplete.*

Idea of proof. Consider the Green function

$$g(x, y) = \int_0^\infty p_t(x, y)\, dt.$$

The existence of a non-constant non-negative superharmonic function u implies (and is even equivalent to) the finiteness of $g(x, y)$ off-diagonal. Then the Green function is a fundamental solution of the Laplace equation $\Delta v = 0$. In particular, as $x \to y$ it has the same singularity as in $\mathbb{R}^n$, which implies its local integrability as a function of x (with fixed y). If in addition $u \in L^1(M)$ then the comparison with $g(x, y)$ shows that also $g(x, y) \in L^1(M)$ with respect to the variable x. However, we have

$$\int_M g(x, y)\, d\mu(x) = \int_0^\infty \int_M p_t(x, y)\, d\mu(y)\, dt = \int_0^\infty P_t 1(x)\, dt.$$

If M is stochastically complete then $P_t 1 \equiv 1$ and the above integral diverges. This contradiction shows that M is stochastically incomplete. ■

1.4 Model manifolds

By a model manifold we mean $\mathbb{R}^n$ with the metric

$$ds^2 = dr^2 + \psi^2(r)\, d\theta^2 \tag{1.15}$$

where (r, θ) are the polar coordinates in $\mathbb{R}^n$, where $r > 0$ and $\theta \in \mathbb{S}^{n-1}$, $d\theta^2$ is the standard spherical metric in $\mathbb{S}^{n-1}$, and $\psi(r)$ some positive smooth function on $(0, +\infty)$. In fact, in order to be able to extend this metric to the origin, ψ must satisfy probability space $\psi(0) = 0$ and $\psi'(0) = 1$. For example, $\psi(r) = r$ gives the Euclidean metric in $\mathbb{R}^n$, $\psi(r) = \sinh r$ gives the hyperbolic metric, which makes $\mathbb{R}^n$ into $\mathbb{H}^n$. The function $\psi(r) = \sin r$, $0 < r < \pi$, gives the spherical metric in $\mathbb{S}^n$.

Denote $\mathbb{R}^n$ with the metric (1.15) by M_ψ. Consider the ball

$$B_r = \{|x| < r\}$$

and its boundary sphere

$$S_r = \{|x| = r\}.$$

The area of S_r is equal to

$$S(r) = \omega_n \psi(r)^{n-1}$$

and the volume of B_r is

$$V(r) = \int_0^r S(t)\,dt.$$

The Laplace operator on M_ψ is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{S'(r)}{S(r)}\frac{\partial}{\partial r} + \frac{1}{\psi^2(r)}\Delta_{\mathbb{S}^{n-1}}. \tag{1.16}$$

For example, the Euclidean Laplacian is

$$\Delta_{\mathbb{R}^n} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}}. \tag{1.17}$$

the hyperbolic Laplacian is

$$\Delta_{\mathbb{H}^n} = \frac{\partial^2}{\partial r^2} + (n-1)\coth r\frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}}, \tag{1.18}$$

the spherical Laplacian is

$$\Delta_{\mathbb{S}^n} = \frac{\partial^2}{\partial r^2} + (n-1)\cot r\frac{\partial}{\partial r} + \frac{1}{\sin^2 r}\Delta_{\mathbb{S}^{n-1}}. \tag{1.19}$$

Remark. The following radial function

$$u(r) = \int_{r_0}^r \frac{dr}{S(r)}$$

is harmonic, that is, satisfies the Laplace equation $\Delta u = 0$.

Theorem 1.5 *The model M_ψ is stochastically complete if and only if*

$$\int^\infty \frac{V(r)}{S(r)}dr = \infty. \tag{1.20}$$

Proof. Let us show that (1.20) implies the stochastic completeness of M. By Theorem 1.3, it suffices to construct a 1-superharmonic function $v = v(r)$ in the domain $\{r > 1\}$ such that $v(r) \to +\infty$ as $r \to \infty$.

In fact, we construct v as a solution to the equation $\Delta v = v$, which in the polar coordinates has the form

$$v'' + \frac{S'}{S}v' - v = 0. \tag{1.21}$$

So, let v be the solution of the ordinary differential equation (1.21) on $[1,+\infty)$ with the initial values $v(1)=1$ and $v'(1)=0$. The function $v(r)$ is monotone increasing because the equation (1.21) after multiplying by Sv and integrating from 1 to R, amounts to

$$Svv'(R)=\int_1^R S\left(v'^2+v^2\right)dr\geq 0.$$

Hence, we have $v\geq 1$.

Multiplying (1.21) by S, we obtain

$$(Sv')'=Sv,$$

which implies by two integrations

$$v(R)=1+\int_1^R\frac{dr}{S(r)}\int_1^r S(t)v(t)dt.$$

Using $v\geq 1$ in the right hand side, we obtain, for $R>2$,

$$v(R)\geq\int_1^R\frac{dr}{S(r)}\int_1^R S(t)dt=\int_1^R\frac{(V(r)-V(1))dr}{S(r)}\geq c\int_2^R\frac{V(r)dr}{S(r)},$$

where $c=1-\frac{V(1)}{V(2)}>0$. Finally, (1.20) implies $v\left(R\right)\to\infty$ as $R\to\infty$.

Now we assume that

$$\int^\infty\frac{V\left(r\right)}{S\left(r\right)}dr<\infty, \tag{1.22}$$

and prove that M is stochastically incomplete. By Theorem 1.4, it suffices to construct on M a non-negative function $u\in L^1\left(M\right)$ such that

$$-\Delta u=f, \tag{1.23}$$

where where $f\in C_0^\infty\left(M\right)$, $f\geq 0$ and $f\not\equiv 0$. Both functions u and f will depend only on r so that (1.23) in the domain of the polar coordinates becomes

$$u''+\frac{S'}{S}u'=-f. \tag{1.24}$$

Choose $f\left(r\right)$ to be any non-negative non-zero function from $C_0^\infty\left(1,2\right)$, and set, for any $R>0$,

$$u\left(R\right)=\int_R^\infty\frac{dr}{S\left(r\right)}\int_0^r S\left(t\right)f\left(t\right)dt. \tag{1.25}$$

Since f is bounded, the condition (1.20) implies that u is finite. It is easy to see that u satisfies the equation

$$\left(Su'\right)'=-Sf,$$

which is equivalent to (1.24). The function $u\left(R\right)$ is constant on the interval $0<R<1$ because $f\left(t\right)\equiv 0$ for $0<t<1$. Hence, u extends by continuity to the origin and satisfies (1.23) on the whole manifold.

We are left to verify that $u \in L^1(M)$. Since $f(t) \equiv 0$ for $t > 2$, we have for $R > 2$

$$u(R) = C \int_R^\infty \frac{dr}{S(r)}$$

where $C = \int_0^2 S(t) f(t)\, dt$. Therefore,

$$\begin{aligned}
\int_{\{R>2\}} u d\mu &= \int_2^\infty u(R) S(R)\, dR \\
&= C \int_2^\infty \left(\int_R^\infty \frac{dr}{S(r)} \right) S(R)\, dR \\
&= C \int_2^\infty \left(\int_2^r S(R)\, dR \right) \frac{dr}{S(r)} \\
&\leq C \int_2^\infty \frac{V(r)}{S(r)} dr < \infty,
\end{aligned}$$

which gives $u \in L^1(M)$. ■

1.5 The log-volume test

Define the *volume function* $V(x,r)$ of a weighted manifold $(M, \mathbf{g}, \mu)$ by

$$V(x,r) := \mu(B(x,r)),$$

where $B(x,r)$ is the geodesic ball. Note that $V(x,r) < \infty$ for all $x \in M$ and $r > 0$ provided M is complete.

Recall that a manifold M is *stochastically* complete, if the heat kernel $p_t(x,y)$ satisfies the identity

$$\int_M p_t(x,y)\, d\mu(y) = 1,$$

for all $x \in M$ and $t > 0$ (see Section 1.3). The result of this section is the following volume test for the stochastic completeness.

Theorem 1.6 *Let $(M, \mathbf{g}, \mu)$ be a complete connected weighted manifold. If, for some point $x_0 \in M$,*

$$\int^\infty \frac{r dr}{\log V(x_0, r)} = \infty, \tag{1.26}$$

then M is stochastically complete.

Condition (1.26) holds, in particular, if

$$V(x_0, r) \leq \exp\left(Cr^2\right) \tag{1.27}$$

for all r large enough or even if

$$V(x_0, r_k) \leq \exp\left(Cr_k^2\right), \tag{1.28}$$

for a sequence $r_k \to \infty$ as $k \to \infty$. This provides yet another proof of the stochastic completeness of $\mathbb{R}^n$ and $\mathbb{H}^n$.

Fix $0 < T \leq \infty$, set $I = (0, T)$ and consider the following Cauchy problem in $I \times M$

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } I \times M, \\ u|_{t=0} = 0. \end{cases} \tag{1.29}$$

A solution is sought in the class $u \in C^\infty(I \times M)$, and the initial condition means that $u(t, x) \to 0$ locally uniformly in $x \in M$ as $t \to 0$ (cf. Section 1.3). By Theorem 1.1, the stochastic completeness of M is equivalent to the uniqueness property of the Cauchy problem in the class of bounded solutions. In other words, in order to prove Theorem 1.6, it suffices to verify that the only bounded solution to (1.29) is $u \equiv 0$.

The assertion will follow from the following more general fact.

Theorem 1.7 *Let $(M, \mathbf{g}, \mu)$ be a complete connected weighted manifold, and let $u(x, t)$ be a solution to the Cauchy problem (1.29). Assume that, for some $x_0 \in M$ and for all $R > 0$,*

$$\int_0^T \int_{B(x_0,R)} u^2(x, t)\, d\mu(x) dt \leq \exp\left(f(R)\right), \tag{1.30}$$

where $f(r)$ is a positive increasing function on $(0, +\infty)$ such that

$$\int^\infty \frac{r dr}{f(r)} = \infty. \tag{1.31}$$

Then $u \equiv 0$ in $I \times M$.

Theorem 1.7 provides the uniqueness class (1.30) for the Cauchy problem. The condition (1.31) holds if, for example, $f(r) = Cr^2$, but fails for $f(r) = Cr^{2+\varepsilon}$ when $\varepsilon > 0$.

Before we embark on the proof, let us mention the following consequence.

Corollary 1.8 *If $M = \mathbb{R}^n$ and $u(t, x)$ be a solution to (1.29) satisfying the condition*

$$|u(t, x)| \leq C \exp\left(C |x|^2\right) \quad \textit{for all}\, t \in I,\ x \in \mathbb{R}^n, \tag{1.32}$$

then $u \equiv 0$. Moreover, the same is true if u satisfies instead of (1.32) the condition

$$|u(t, x)| \leq C \exp\left(f(|x|)\right) \quad \textit{for all}\, t \in I,\ x \in \mathbb{R}^n, \tag{1.33}$$

where $f(r)$ is a convex increasing function on $(0, +\infty)$ satisfying (1.31).

Proof. Since (1.32) is a particular case of (1.33) for the function $f(r) = Cr^2$, it suffices to treat the condition (1.33). In $\mathbb{R}^n$ we have $V(x, r) = cr^n$. Therefore, (1.33) implies that

$$\int_0^T \int_{B(0,R)} u^2(x, t)\, d\mu(x) dt \leq CR^n \exp\left(f(R)\right) = C \exp(\widetilde{f}(R)),$$

where $\widetilde{f}(r) := f(r) + n \log r$. The convexity of f implies that $\log r \leq Cf(r)$ for large enough r. Hence, $\widetilde{f}(r) \leq Cf(r)$ and function $\widetilde{f}$ also satisfies the condition (1.31). By Theorem 1.7, we conclude $u \equiv 0$. ∎

The class of functions u satisfying (1.32) is called the *Tikhonov class*, and the conditions (1.33) and (1.31) define the *Täcklind class*. The uniqueness of the Cauchy problem in $\mathbb{R}^n$ in each of these classes is a classical result of Tikhonov and Täcklind, respectively.

Proof of Theorem 1.6. By Theorem 1.1, it suffices to verify that the only bounded solution to the Cauchy value problem (1.29) is $u \equiv 0$. Indeed, if u is a bounded solution of (1.29), then setting

$$S := \sup |u| < \infty$$

we obtain

$$\int_0^T \int_{B(x_0,R)} u^2(t,x)d\mu(x) \leq S^2 T V(x_0, R) = \exp(f(R)),$$

where

$$f(r) := \log\left(S^2 T V(x_0, r)\right).$$

It follows from the hypothesis (1.26) that the function f satisfies (1.31). Hence, by Theorem 1.7, we obtain $u \equiv 0$. ∎

Proof of Theorem 1.7. Denote for simplicity $B_r = B(x_0, r)$. The main technical part of the proof is the following claim.

Claim. *Let $u(t,x)$ solve the heat equation in $(b,a) \times M$ where $b < a$ are reals, and assume that $u(t,x)$ extends to a continuous function in $[b,a] \times M$. Assume also that, for all $R > 0$,*

$$\int_a^b \int_{B_R} u^2(x,t)\, d\mu(x)dt \leq \exp(f(R)),$$

where f is a function as in Theorem 1.6. *Then, for any $R > 0$ satisfying the condition*

$$a - b \leq \frac{R^2}{8f(4R)}, \tag{1.34}$$

the following inequality holds:

$$\int_{B_R} u^2(a,\cdot)d\mu \leq \int_{B_{4R}} u^2(b,\cdot)d\mu + \frac{4}{R^2}. \tag{1.35}$$

Let us first show how this Claim allows to prove that any solution u to (1.29), satisfying (1.30), is identical 0. Extend $u(t,x)$ to $t = 0$ by setting $u(0,x) = 0$ so that u is continuous in $[0,T) \times M$. Fix $R > 0$ and $t \in (0,T)$. For any non-negative integer k, set

$$R_k = 4^k R$$

and, for any $k \geq 1$, choose (so far arbitrarily) a number τ_k to satisfy the condition

$$0 < \tau_k \leq c \frac{R_k^2}{f(R_k)}, \tag{1.36}$$

where $c = \frac{1}{128}$. Then define a decreasing sequence of times $\{t_k\}$ inductively by $t_0 = t$ and $t_k = t_{k-1} - \tau_k$ (see Fig. 1).

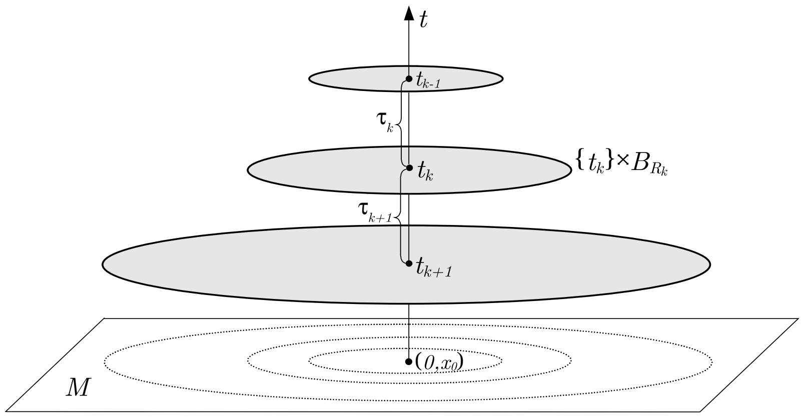

Figure 1: The sequence of the balls B_{R_k} and the time moments t_k.

If $t_k \geq 0$ then function u satisfies all the conditions of the Claim with $a = t_{k-1}$ and $b = t_k$, and we obtain from (1.35)

$$\int_{B_{R_{k-1}}} u^2(t_{k-1}, \cdot) d\mu \leq \int_{B_{R_k}} u^2(t_k, \cdot) d\mu + \frac{4}{R_{k-1}^2}, \tag{1.37}$$

which implies by induction that

$$\int_{B_R} u^2(t, \cdot) d\mu \leq \int_{B_{R_k}} u^2(t_k, \cdot) d\mu + \sum_{i=1}^{k} \frac{4}{R_{i-1}^2}. \tag{1.38}$$

If it happens that $t_k = 0$ for some k then, by the initial condition in (1.29),

$$\int_{B_{R_k}} u^2(t_k, \cdot) d\mu = 0.$$

In this case, it follows from (1.38) that

$$\int_{B_R} u^2(t, \cdot) d\mu \leq \sum_{i=1}^{\infty} \frac{4}{R_{i-1}^2} = \frac{C}{R^2},$$

which implies by letting $R \to \infty$ that $u(\cdot, t) \equiv 0$ (here we use the connectedness of M).

Hence, to finish the proof, it suffices to construct, for any $R>0$ and $t\in(0,T)$, a sequence $\{t_k\}$ as above that vanishes at a finite k. The condition $t_k=0$ is equivalent to

$$t=\tau_1+\tau_2+...+\tau_k\,. \tag{1.39}$$

The only restriction on τ_k is the inequality (1.36). The hypothesis that $f(r)$ is an increasing function implies that

$$\int_R^\infty \frac{rdr}{f(r)} \le \sum_{k=0}^\infty \int_{R_k}^{R_{k+1}} \frac{rdr}{f(r)} \le \sum_{k=0}^\infty \frac{R_{k+1}^2}{f(R_k)},$$

which together with (1.31) yields

$$\sum_{k=1}^\infty \frac{R_k^2}{f(R_k)}=\infty.$$

Therefore, the sequence $\{\tau_k\}_{k=1}^\infty$ can be chosen to satisfy simultaneously (1.36) and

$$\sum_{k=1}^\infty \tau_k=\infty.$$

By diminishing some of τ_k, we can achieve (1.39) for any finite t, which finishes the proof.

Now we prove the above Claim. Since the both integrals in (1.35) are continuous with respect to a and b, we can slightly reduce a and slightly increase b; hence, we can assume that $u(t,x)$ is not only continuous in $[b,a]\times M$ but also smooth.

Let $\rho(x)$ be a Lipschitz function on M (to be specified below) with the Lipschitz constant 1. Fix a real $s\notin[b,a]$ (also to be specified below) and consider the following the function

$$\xi(t,x):=\frac{\rho^2(x)}{4(t-s)},$$

which is defined on $\mathbb{R}\times M$ except for $t=s$, in particular, on $[b,a]\times M$. The distributional gradient $\nabla\rho$ is in $L^\infty(M)$ and satisfies the inequality $|\nabla\rho|\le 1$, which implies, for any $t\ne s$,

$$|\nabla\xi(t,x)|\le\frac{\rho(x)}{2(t-s)}.$$

Since

$$\frac{\partial\xi}{\partial t}=-\frac{\rho^2(x)}{4(t-s)^2},$$

we obtain

$$\frac{\partial\xi}{\partial t}+|\nabla\xi|^2\le 0. \tag{1.40}$$

For a given $R>0$, define a function $\varphi(x)$ by

$$\varphi(x)=\min\left(\left(3-\tfrac{d(x,x_0)}{R}\right)_+,1\right)$$

(see Fig. 2). Obviously, we have $0 \leq \varphi \leq 1$ on M, $\varphi \equiv 1$ in B_{2R}, and $\varphi \equiv 0$ outside B_{3R}. Since the function $d(\cdot, x_0)$ is Lipschitz with the Lipschitz constant 1, we obtain that φ is Lipschitz with the Lipschitz constant $1/R$. Then we have $|\nabla\varphi| \leq 1/R$. By the completeness of M, all the balls in M are relatively compact sets, which implies $\varphi \in Lip_0(M)$.

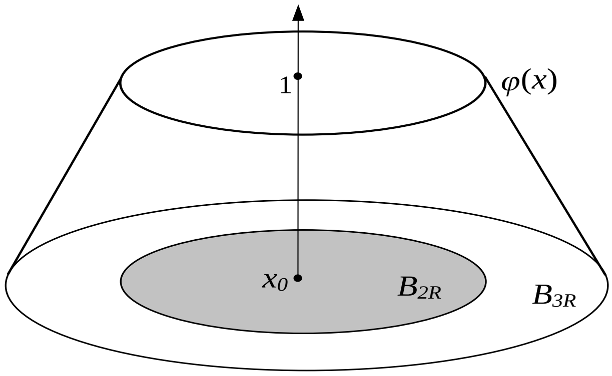

Figure 2: Function $\varphi(x)$

Consider the function $u\varphi^2 e^\xi$ as a function of x for any fixed $t \in [b, a]$. Since it is obtained from locally Lipschitz functions by taking product and composition, this function is locally Lipschitz on M. Since this function has a compact support, it belongs to $Lip_0(M)$, whence

$$u\varphi^2 e^\xi \in W_c^1(M).$$

Multiplying the heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

by $u\varphi^2 e^\xi$ and integrating it over $[b, a] \times M$, we obtain

$$\int_b^a \int_M \frac{\partial u}{\partial t} u\varphi^2 e^\xi d\mu dt = \int_b^a \int_M (\Delta u) u\varphi^2 e^\xi d\mu dt. \tag{1.41}$$

Since both functions u and ξ are smooth in $t \in [b, a]$, the time integral on the left hand side can be computed as follows:

$$\frac{1}{2}\int_b^a \frac{\partial(u^2)}{\partial t}\varphi^2 e^\xi dt = \frac{1}{2}\left[u^2\varphi^2 e^\xi\right]_b^a - \frac{1}{2}\int_b^a \frac{\partial \xi}{\partial t} u^2 \varphi^2 e^\xi dt. \tag{1.42}$$

Using the Green formula to evaluate the spatial integral on the right hand side of (1.41), we obtain

$$\int_M (\Delta u) u\varphi^2 e^\xi d\mu = -\int_M \langle \nabla u, \nabla(u\varphi^2 e^\xi)\rangle d\mu.$$

Applying the product rule and the chain rule to compute $\nabla(u\varphi^2 e^\xi)$, we obtain

$$\begin{aligned}
-\langle \nabla u, \nabla(u\varphi^2 e^\xi)\rangle &= -\left|\nabla u\right|^2 \varphi^2 e^\xi - \langle \nabla u, \nabla \xi\rangle u\varphi^2 e^\xi - 2\langle \nabla u, \nabla \varphi\rangle u\varphi e^\xi \\
&\le -\left|\nabla u\right|^2 \varphi^2 e^\xi + \left|\nabla u\right| \left|\nabla \xi\right| \left|u\right| \varphi^2 e^\xi \\
&\quad + \left(\frac{1}{2}\left|\nabla u\right|^2 \varphi^2 + 2\left|\nabla \varphi\right|^2 u^2\right) e^\xi \\
&= \left(-\frac{1}{2}\left|\nabla u\right|^2 + \left|\nabla u\right| \left|\nabla \xi\right| \left|u\right|\right) \varphi^2 e^\xi + 2\left|\nabla \varphi\right|^2 u^2 e^\xi .
\end{aligned}$$

Combining with (1.41), (1.42), and using (1.40), we obtain

$$\begin{aligned}
\left[\int_M u^2\varphi^2 e^\xi d\mu\right]_b^a &= \int_b^a \int_M \frac{\partial \xi}{\partial t} u^2\varphi^2 e^\xi \, d\mu dt + 2\int_b^a \int_M (\Delta u)\, u\varphi^2 e^\xi d\mu dt \\
&\le \int_b^a \int_M \left(-\left|\nabla \xi\right|^2 u^2 - \left|\nabla u\right|^2 + 2\left|\nabla u\right| \left|\nabla \xi\right| \left|u\right|\right) \varphi^2 e^\xi d\mu dt \\
&\quad +4\int_b^a \int_M \left|\nabla \varphi\right|^2 u^2 e^\xi d\mu dt \\
&= -\int_b^a \int_M \left(\left|\nabla \xi\right| \left|u\right| - \left|\nabla u\right|\right)^2 \varphi^2 e^\xi d\mu dt \\
&\quad +4\int_b^a \int_M \left|\nabla \varphi\right|^2 u^2 e^\xi d\mu dt
\end{aligned}$$

whence

$$\left[\int_M u^2\varphi^2 e^\xi d\mu\right]_b^a \le 4\int_b^a \int_M \left|\nabla \varphi\right|^2 u^2 e^\xi d\mu dt. \tag{1.43}$$

Using the properties of function $\varphi(x)$, in particular, $|\nabla \varphi| \le 1/R$, we obtain from (1.43)

$$\int_{B_R} u^2(a,\cdot)e^{\xi(a,\cdot)}d\mu \le \int_{B_{4R}} u^2(b,\cdot)e^{\xi(b,\cdot)}d\mu + \frac{4}{R^2}\int_b^a \int_{B_{4R}\setminus B_{2R}} u^2 e^\xi d\mu dt. \tag{1.44}$$

Let us now specify $\rho(x)$ and s. Set $\rho(x)$ to be the distance function from the ball B_R, that is,

$$\rho(x) = (d(x,x_0) - R)_+$$

(see Fig. 3).

Set $s = 2a - b$ so that, for all $t \in [b,a]$,

$$a - b \le s - t \le 2(a - b),$$

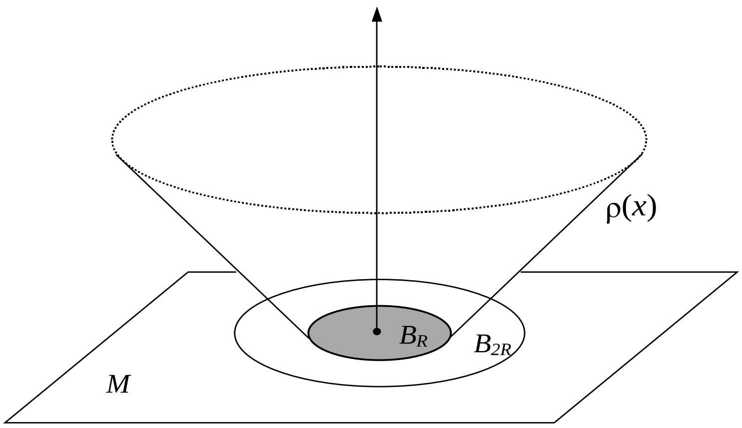

Figure 3: Function $\rho(x)$.

whence

$$\xi(t,x) = -\frac{\rho^2(x)}{4(s-t)} \leq -\frac{\rho^2(x)}{8\ (a-b)} \leq 0. \tag{1.45}$$

Consequently, we can drop the factor e^{ξ} on the left hand side of (1.44) because $\xi = 0$ in B_R, and drop the factor e^{ξ} in the first integral on the right hand side of (1.44) because $\xi \leq 0$. Clearly, if $x \in B_{4R} \backslash B_{2R}$ then $\rho(x) \geq R$, which together with (1.45) implies that

$$\xi\,(t,x) \leq -\frac{R^2}{8\ (a-b)} \quad \text{in} \quad [b,a] \times B_{4R} \backslash B_{2R}.$$

Hence, we obtain from (1.44)

$$\int_{B_R} u^2(a,\cdot) d\mu \leq \int_{B_{4R}} u^2(b,\cdot) d\mu + \frac{4}{R^2} \exp\left(-\frac{R^2}{8\ (a-b)}\right) \int_b^a \int_{B_{4R}} u^2 d\mu dt.$$

By (1.30) we have

$$\int_b^a \int_{B_{4R}} u^2 d\mu dt \leq \exp\left(f(4R)\right)$$

whence

$$\int_{B_R} u^2(a,\cdot) d\mu \leq \int_{B_{4R}} u^2(b,\cdot) d\mu + \frac{4}{R^2} \exp\left(-\frac{R^2}{8\ (a-b)} + f(4R)\right).$$

Finally, applying the hypothesis (1.34), we obtain (1.35). ■

Example. The hypothesis

$$\int^{\infty} \frac{rdr}{\log V(x_0,r)} = \infty \tag{1.46}$$

of Theorem 1.6 is sufficient for the stochastic completeness of M but not necessary. Nevertheless, let us show that the condition (1.46) is sharp in the following sense: if $f(r)$ is a smooth positive convex function on $(0,+\infty)$ with $f'(r) > 0$ and such that

$$\int^{\infty} \frac{rdr}{f(r)} < \infty, \tag{1.47}$$

then there exists a complete but stochastically incomplete weighted manifold M such that

$$\log V(x_0, r) = f(r),$$

for some $x_0 \in M$ and large enough r. Indeed, let M be a model manifold. Note that M is geodesically complete. Define its volume function $V(r)$ for large r by

$$V(r) = \exp(f(r))$$

so that

$$\frac{V(r)}{V'(r)} = \frac{1}{f'(r)}. \tag{1.48}$$

Let us show that, for all $r \geq 1$,

$$\frac{1}{f'(r)} \leq c\frac{r}{f(r)}, \tag{1.49}$$

where

$$c = \min\left(\frac{f'(1)}{f(1)}, 1\right) > 0.$$

Indeed, the function

$$h(r) = rf'(r) - cf(r)$$

is non-negative for $r = 1$ and its derivative is

$$h'(r) = rf''(r) + (1-c) f'(r) \geq 0.$$

Hence, h is increasing and $h(r) \geq 0$ for $r \geq 1$, whence (1.49) follows.

Combining (1.48), (1.49), and (1.47), we obtain

$$\int^{\infty} \frac{V(r)}{V'(r)} dr < \infty,$$

which implies by Theorem 1.5 the stochastic incompleteness of M.

Example. We say that a weighted manifold $(M, \mathbf{g}, \mu)$ has *bounded geometry* if there exists $\varepsilon > 0$ such that all the geodesic balls $B(x, \varepsilon)$ are *uniformly* quasi-isometric to the Euclidean ball B_ε; that is, there is a constant C and, for any $x \in M$, a diffeomorphism $\varphi_x : B(x, \varepsilon) \to B_\varepsilon$ such that φ_x changes the Riemannian metric and the measure at most by the factor C (see Fig. 4).

For example, $\mathbb{R}^n$ and $\mathbb{H}^n$ have bounded geometry. Any manifold of bounded geometry is stochastically complete, which follows from the fact that it is complete and its volume function satisfies the estimate

$$V(x, r) \leq \exp(Cr),$$

for all $x \in M$ and large r.

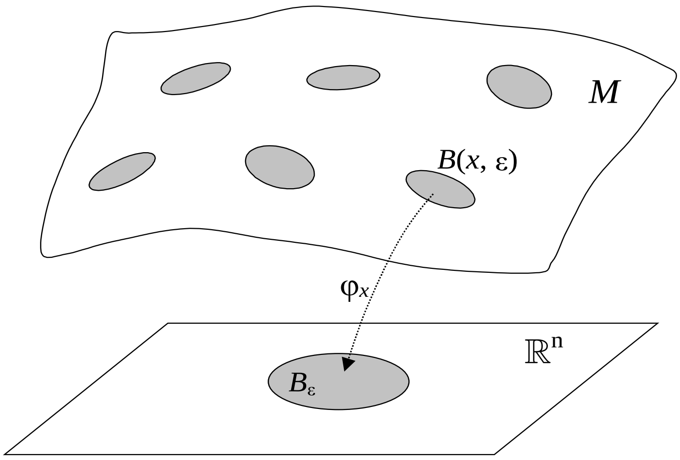

Figure 4: A manifold of bounded geometry is "patched" by uniformly distorted Euclidean balls.

2 Escape rate of Brownian motion

2.1 Upper radius in terms of volume function

Let M be a geodesically complete noncompact Riemannian manifold. As before let $\left(\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M}\right)$ be Brownian motion on M generated by the Laplace operator. Fix a reference point $x_0 \in M$ and let $\rho(x) = d(x, x_0)$. We say that a function $R(t)$ is an *upper rate function* for Brownian motion on M if

$$\mathbb{P}_{x_0}\{\rho(X_t) \leq R(t) \text{ for all sufficiently large } t\} = 1.$$

Let us first point out that the notion of an upper rate function makes sense only if the lifetime of Brownian motion is infinite, that is, when the manifold M is stochastically complete.

Recall that M is stochastically complete provided

$$\int^{\infty} \frac{r\,dr}{\log V(x_0, r)} = \infty. \tag{2.1}$$

The integral in (2.1) will be used here to construct an upper rate function.

Before we state the result, let us recall the classical Khinchin's law of the iterated logarithm that says that for a Brownian motion in $\mathbb{R}^n$

$$\limsup_{t\to\infty} \frac{\rho(X_t)}{\sqrt{4t\log\log t}} = 1 \quad \text{a.s.}$$

(the factor 4 instead of the classical 2 appears because in our setting a Brownian motion is generated by Δ rather than $\frac{1}{2}\Delta$). It follows that, for any $\varepsilon > 0$,

$$R(t) = \sqrt{(4+\varepsilon)\,t\log\log t} \tag{2.2}$$

is an upper rate function.

We will construct an upper rate function under the most general condition (2.1). However, we assume in addition that M is a Cartan-Hadamard manifold, that is, a geodesically complete simply connected Riemannian manifold of non-positive sectional curvature. This assumption is not essential for the result as the latter can be proved for arbitrary geodesically complete manifolds. However, the proof for Cartan-Hadamard manifolds is somewhat simpler. The property of Cartan-Hadamard manifolds that we use is the Sobolev inequality: if $N = \dim M$ then, for any function $f \in C_0^\infty(M)$,

$$\left(\int_M |f|^{\frac{N}{N-1}} d\mu\right)^{\frac{N-1}{N}} \leq C_N \int_M |\nabla f| \, d\mu, \tag{2.3}$$

where C_N is a constant depending only on N – see [12]. The Sobolev inequality allows us to carry through the Moser iteration argument in [16] and prove a mean value estimate for solutions of the heat equation on M, which is one of the ingredients of our proof.

Now we state our main result.

Theorem 2.1 *Let M be a Cartan-Hadamard manifold. Assume that the following volume estimate holds for a fixed point $x_0 \in M$ and all sufficiently large large R:*

$$V(x_0, R) \leq \exp(f(R)), \tag{2.4}$$

where $f(R)$ is a positive, strictly increasing, and continuous function on $[0, +\infty)$ such that

$$\int^\infty \frac{r\,dr}{f(r)} = \infty. \tag{2.5}$$

Let $\phi(t)$ be the function on $\mathbb{R}_+$ defined by

$$t = \int_0^{\phi(t)} \frac{r\,dr}{f(r)}. \tag{2.6}$$

Then $R(t) = \phi(C\,t)$ is an upper rate function for Brownian motion on M for some absolute constant C (for example, for any $C > 128$).

If we set $f(R) = \log V(x_0, R)$ for large R then the condition (2.5) becomes identical to (2.1).

Let us show some examples.

1. If

$$V(x_0, R) \leq CR^D \tag{2.7}$$

for some constant C and D then (2.4) holds with

$$f(R) = D \log R + \text{const},$$

and (2.6) yields

$$t \simeq \frac{\phi^2}{2D \log \phi}.$$

It follows that $\log t \simeq \log \phi^2$ and

$$\phi(t) \simeq \sqrt{Dt \log t}.$$

Hence, the following function

$$R(t) = \sqrt{CDt \log t}$$

is a upper rate function.

2. If $V(x_0, R) \leq \exp(Cr^\alpha)$ for some $0 < \alpha < 2$ then (2.4) holds with $f(R) = Cr^\alpha$, and (2.6) yields $t \simeq \phi(t)^{2-\alpha}$ whence we obtain the upper rate function
$$R(t) = Ct^{\frac{1}{2-a}}.$$

3. If
$$V(x_0, R) \leq \exp\left(CR^2\right),$$
then $f(R) = CR^2$. Then (2.6) yields $t \simeq \ln \phi(t)$. Hence, we obtain the upper rate function
$$R(t) = \exp(Ct).$$
This result is new. Similarly, if
$$V(x_0, R) \leq \exp\left(CR^2 \log R\right)$$
then (2.6) yields $t \simeq \log \log \phi$ whence
$$R(t) = \exp(\exp(Ct)).$$

Proof of Theorem 2.1. We first explain the main idea of the proof. For any open set $\Omega \subset M$, denote by τ_Ω the first exit time from Ω, that is,

$$\tau_\Omega = \inf\{t > 0 : X_t \notin \Omega\}.$$

Fix a reference point $x_0 \in M$ and set $\rho(x) = d(x, x_0)$ .Let $\{R_n\}_{n=1}^\infty$ be a sequence of strictly increasing radii to be fixed later such that $\lim_{n\to\infty} R_n = \infty$ and consider the following sequence of stopping times

$$\tau_n = \tau_{B(x_0, R_n)}.$$

Then $\tau_n - \tau_{n-1}$ is the amount of time the Brownian motion X_t takes to cross from $\partial B(x_0, R_{n-1})$ to $\partial B(x_0, R_n)$ for the first time (if $n = 0$ then set $R_0 = 0$ and $\tau_0 = 0$). Let $\{c_n\}_{n=1}^\infty$ be a sequence of positive numbers to be fixed later. Suppose that we can show that

$$\sum_{n=1}^\infty \mathbb{P}_{x_0}\{\tau_n - \tau_{n-1} \leq c_n\} < \infty. \tag{2.8}$$

Then, by the Borel-Cantelli lemma, with $\mathbb{P}_{x_0}$-probability 1 we have

$$\tau_n - \tau_{n-1} > c_n, \quad \text{for all large enough } n. \tag{2.9}$$

For any $n \geq 1$, set

$$T_n = \sum_{k=1}^{n} c_k.$$

It follows from (2.9) that, for all sufficiently large n,

$$\tau_n > T_n - T_0,$$

where T_0 is a large enough (random) number. In other words, we have the implication

$$t \leq T_n - T_0 \;\Rightarrow\; \rho(X_t) \leq R_n, \text{ if } n \text{ is large enough.} \tag{2.10}$$

Let ψ be an increasing bijection of $\mathbb{R}_+$ onto itself such that

$$T_{n-1} - \psi(R_n) \to +\infty \text{ as } n \to \infty. \tag{2.11}$$

We claim that ψ^{-1} is an upper rate function. Indeed, for large enough t, choose n such that

$$T_{n-1} - T_0 < t \leq T_n - T_0.$$

If t is large enough then also n is large enough so that by (2.10)

$$\rho(X_t) \leq R_n$$

and by (2.11)

$$T_{n-1} - \psi(R_n) > T_0.$$

It follows that

$$t > T_{n-1} - T_0 > \psi(R_n),$$

whence

$$\rho(X_t) \leq R_n < \psi^{-1}(t),$$

which proves that ψ^{-1} is an upper rate function.

Now let us find c_n such that (2.8) is true. By the strong Markov property of Brownian motion we have

$$\mathbb{P}_{x_0}\{\tau_n - \tau_{n-1} \leq c_n\} = \mathbb{E}_{x_0}\mathbb{P}_{X_{\tau_{n-1}}}\{\tau_n \leq c_n\}. \tag{2.12}$$

Note that $X_{\tau_{n-1}} \in \partial B(x_0, R_{n-1})$. If a Brownian motion starts from a point $y \in \partial B(x_0, R_{n-1})$, then it has to travel no less than distance

$$r_n = R_n - R_{n-1}$$

before it reaches $\partial B(x_0, R_n)$ (see Fig. 5), hence

$$\mathbb{P}_y\{\tau_n \leq c_n\} \leq \mathbb{P}_y\left\{\tau_{B(y,r_n)} \leq c_n\right\}, \quad y \in \partial B(x_0, R_{n-1}).$$

From the above inequality and (2.12) we obtain

$$\mathbb{P}_{x_0}\{\tau_n - \tau_{n-1} \leq c_n\} \leq \sup_{y\in\partial B(x_0,R_{n-1})} \mathbb{P}_y\left\{\tau_{B(y,r_n)} \leq c_n\right\}. \tag{2.13}$$

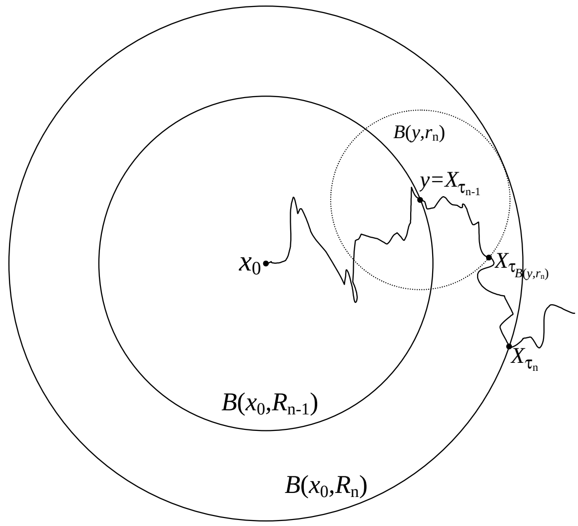

Figure 5: Brownian motion X_t exits the ball $B(y, r_n)$ before $B(x_0, R_n)$

For a fixed $y \in \partial B(x_0, R_{n-1})$, consider the function

$$u(x,t) = \mathbb{P}_x \left\{ \tau_{B(y,r_n)} \leq t \right\}.$$

Clearly, $u(x,t)$ is the solution of the heat equation in the cylinder $B(y, r_n) \times \mathbb{R}_+$. Furthermore, $0 \leq u \leq 1$ and

$$u(x,0) = 0 \quad \text{for } x \in B(y, r_n).$$

We use the following estimate of the solution at the center of the ball:

$$u(y,t) \leq C\mu(B(y,r_n))^{1/2} \frac{r_n}{\sqrt{t}^{1+N/2}} \exp\left(-\frac{r_n^2}{16t}\right) \tag{2.14}$$

provided $t < r_n^2$. The proof of (2.14) uses the Sobolev inequality and will be skipped.

The probability we want to estimate is the value of the solution u at the center of the ball:

$$\mathbb{P}_y \left\{ \tau_{B(y,r_n)} \leq c_n \right\} = u(y, c_n). \tag{2.15}$$

Applying the estimate (2.14) and noting that $B(y, r_n) \subset B(x_0, R_n)$ so that

$$\mu(B(y,r_n)) \leq \exp(f(R_n)),$$

we obtain

$$u(y,c_n) \leq C \exp(f(R_n)/2) \frac{r_n}{\sqrt{c_n}^{1+N/2}} \exp\left(-\frac{r_n^2}{16c_n}\right), \tag{2.16}$$

provided $c_n < r_n^2$. Now we choose c_n to satisfy the identity

$$\frac{r_n^2}{16c_n} = f(R_n)$$

that is,

$$c_n = \frac{1}{16} \frac{r_n^2}{f(R_n)}.$$

Since $f(R) \to \infty$ as $R \to \infty$, we have $c_n < r_n^2$ for large enough n. Hence, we obtain from (2.16)

$$\begin{aligned} u(y, c_n) &\leq C_N \frac{r_n}{\sqrt{c_n}^{1+N/2}} \exp(-f(R_n)/2). \\ &= C_N r_n^{-N/2} f(R_n)^{\frac{2+N}{4}} \exp(-f(R_n)/2) \\ &\leq C_N r_n^{-N/2}. \end{aligned}$$

Set now $R_n = 2^n$ so that $r_n = 2^{n-1}$. The above estimate together with (2.13) and (2.15) yields

$$\sum_{n=1}^{\infty} \mathbb{P}_{x_0} \{\tau_n - \tau_{n-1} \leq c_n\} \leq C_N \sum_{n=1}^{\infty} r_n^{-N/2} < \infty,$$

that is (2.8).

Knowing the sequences $\{R_n\}$ and $\{c_n\}$, we can now determine a function ψ that satisfies (2.11). Indeed, we have

$$\begin{aligned} T_n &= c_1 + ... + c_n \\ &= \frac{1}{16} \sum_{k=1}^{n} \frac{r_k^2}{f(R_k)} \\ &= \frac{1}{128} \sum_{k=1}^{n} \frac{R_{k+1}(R_{k+1} - R_k)}{f(R_k)} \\ &\geq \frac{1}{128} \sum_{k=1}^{n} \int_{R_k}^{R_{k+1}} \frac{rdr}{f(r)} \\ &= \frac{1}{128} \int_{R_1}^{R_{n+1}} \frac{rdr}{f(r)}. \end{aligned}$$

Setting

$$\psi(r) = c \int_0^r \frac{rdr}{f(r)},$$

where $c < \frac{1}{128}$, and using (2.5), we obtain that

$$T_n - \psi(R_{n+1}) \to \infty \text{ as } n \to \infty,$$

which is equivalent to (2.11). Therefore, ψ^{-1} is an upper rate function. Clearly, $\psi^{-1}(t) = \phi(Ct)$ where ϕ is defined by (2.6) and $C = c^{-1}$, which finishes the proof of our main result Theorem 2.1. ■

2.2 Upper radius on model manifolds

Let M be a model manifold, that is, $\mathbb{R}^n$ with the metric

$$ds^2 = dr^2 + \psi(r)^2 d\theta^2. \tag{2.17}$$

As before, let $S(r) = \omega_n \psi(r)^{n-1}$ be the boundary area function and $V(r) = \int_0^r S(t)\,dt$ be the volume function of M. The Laplace operator of the metric (2.17) is represented in the polar coordinates as follows:

$$\Delta = \frac{\partial^2}{\partial r^2} + m(r)\frac{\partial}{\partial r} + \frac{1}{\psi^2(r)}\Delta_{\mathbb{S}^{n-1}},$$

where

$$m(r) = (n-1)\frac{\psi'}{\psi} = \frac{S'}{S} = \frac{V''}{V'}.$$

The function $m(r)$ plays an important role in what follows. Clearly, m satisfies the identity

$$S(r) = S(r_0)\exp\left(\int_{r_0}^r m(s)\,ds\right) \tag{2.18}$$

for all $r > r_0 > 0$. We assume in the sequel that

$$m(r) > 0 \text{ and } m'(r) \geq 0 \text{ for large enough } r \tag{2.19}$$

and

$$\int^\infty \frac{dr}{m(r)} = \infty. \tag{2.20}$$

For example, we have $m(r) = \frac{n-1}{r}$ in $\mathbb{R}^n$ and $m(r) = (n-1)K\coth Kr$ in $\mathbb{H}_K^n$. In neither case is the hypothesis (2.19) satisfied. On the other hand, if $S(r) = \exp(r^\alpha)$ then $m(r) = \alpha r^{\alpha-1}$, and both (2.19) and (2.20) are satisfied provided $1 \leq \alpha \leq 2$. If $S(r) = \exp\left(r^2 \log^\beta r\right)$ then (2.19) and (2.20) hold for all $0 \leq \beta \leq 1$.

Claim. *Under the condition* (2.19) *(Brownian motion on) M is transient, and under the conditions* (2.19)-(2.20) *M is stochastically complete.*

We use the following well-known results (cf. [6]) that for model manifolds the recurrence is equivalent to

$$\int^\infty \frac{dr}{S(r)} = \infty, \tag{2.21}$$

and the stochastic completeness is equivalent to

$$\int^\infty \frac{V(r)}{S(r)}dr = \infty. \tag{2.22}$$

Clearly, (2.19) implies that $m(r) \geq c$ for some positive constant c and for all large enough r. It follows from (2.18) that $S(r)$ grows at least exponentially as $r \to \infty$, which implies (2.21). To prove (2.22), observe that, for large enough $r > r_0$ we have

$$\begin{aligned} V(r) - V(r_0) &= \int_{r_0}^{r} S(t)\, dt = \int_{r_0}^{r} \frac{S'(t)}{m(t)} dt \\ &\geq \frac{1}{m(r)} \int_{r_0}^{r} S'(t)\, dt = \frac{1}{m(r)} \left(S(r) - S(r_0)\right), \end{aligned}$$

whence

$$\frac{1}{m(r)} \leq \frac{V(r) - V(r_0)}{S(r) - S(r_0)} \sim \frac{V(r)}{S(r)} \text{ as } r \to \infty.$$

Hence, (2.22) follows from (2.20).

Let us define the function $r(t)$ by the identity

$$t = \int_{0}^{r(t)} \frac{ds}{m(s)}. \tag{2.23}$$

Our main result in this section is as follows.

Theorem 2.2 *Under the above assumptions, the function $r((1+\varepsilon)t)$ is the upper rate function for Brownian motion on M for any $\varepsilon > 0$, and is not for any $\varepsilon < 0$.*

Let us compare the function $r(t)$ with the upper rate function $R(t)$ given by Theorem 2.1, which is defined by the identity

$$\int_{0}^{R(t)} \frac{r dr}{\log V(r)} = Ct.$$

For "nice" functions $V(r)$, one has

$$\frac{V''}{V'} \simeq \frac{V'}{V} = (\log V)' \simeq \frac{\log V(r)}{r}, \tag{2.24}$$

which means that the functions $r(t)$ and $R(t)$ are comparable up to a constant multiple in front of t. For example, (2.24) holds for functions like $V(r) = \exp(r^{\alpha})$ and $V(t) = \exp\left(r^{\alpha} \log^{\beta} r\right)$, where $\alpha > 0$, etc. On the other hand, it is easy to construct an example of $V(r)$ when $r(t)$ may be significantly less that $R(t)$, because one can modify a "nice" function $V(r)$ to make the second derivative $V''(r)$ very small in some intervals without affecting too much the values of V' and V. Then the function $r(t)$ in (2.23) will drop significantly, while $R(t)$ will not change very much.

Proof of Theorem 2.2. By the Ito decomposition, the radial process $r_t = \rho(X_t)$ satisfies the identity

$$r_t = \sqrt{2} W_t + \int_0^t m(r_s)\, ds, \tag{2.25}$$

where W_t is a one-dimensional Brownian motion (see [13]). Since the process X_t is transient, $r_t \to \infty$ as $t \to \infty$ with probability 1. Hence, $m(r_t) \geq$ const for large enough t so that the second term in the right hand side of (2.25) grows at least linearly in t. Since $W_t = o(t)$ as $t \to \infty$, we have with probability 1,

$$r_t \sim \int_0^t m(r_s)\, ds \text{ as } t \to \infty. \tag{2.26}$$

Consider the function

$$u(t) = \int_0^t m(r_s)\, ds$$

It follows from (2.26) that, for any $c > 1$ and for large enough t,

$$r_t \leq cu(t) \tag{2.27}$$

whence by the monotonicity of m,

$$m(r_t) \leq m(cu(t)).$$

Since $\frac{du}{dt}(t) = m(r_t)$, we obtain the differential inequality for $u(t)$:

$$\frac{du}{dt} \leq m(cu(t)).$$

Solving it by separation of variables, we obtain, for large enough t_0 and for all $t > t_0$,

$$\int_{cu(t_0)}^{cu(t)} \frac{d\xi}{m(\xi)} \leq c(t - t_0),$$

whence

$$\int_0^{cu(t)} \frac{d\xi}{m(\xi)} \leq ct + c_0, \tag{2.28}$$

where c_0 is a large enough (random) constant. Comparing (2.28) with (2.23) and using again (2.27), we obtain

$$r_t \leq cu(t) \leq r(ct + c_0) \leq r\left(c^2 t\right)$$

for large enough r with probability 1. Since $c > 1$ was arbitrary, this proves that $r((1+\varepsilon)t)$ is an upper rate function for any $\varepsilon > 0$. In the same way one proves that $r_t \geq r(c^{-2}t)$ for large enough t so that $r((1-\varepsilon)t)$ is not an upper rate function. ■

3 Jump processes

Let (M, d) be a metric space such that all closed metric balls

$$B(x, r) = \{y \in M : d(x, y) \leq r\} \tag{3.1}$$

are compact. In particular, (M, d) is locally compact and separable. Let μ be a Radon measure with full support on M.

Recall that a *Dirichlet form* $(\mathcal{E}, \mathcal{F})$ in $L^2(M, \mu)$ is a symmetric, non-negative definite, bilinear form $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ defined on a dense subspace $\mathcal{F}$ of $L^2(M, \mu)$, which satisfies in addition the following properties:

- Closedness: $\mathcal{F}$ is a Hilbert space with respect to the following inner product:

$$\mathcal{E}_1(f,g) := \mathcal{E}(f,g) + (f,g)\,. \tag{3.2}$$

- The Markov property: if $f \in \mathcal{F}$ then also $\widetilde{f} := (f \wedge 1)_+$ belongs to $\mathcal{F}$ and $\mathcal{E}(\widetilde{f}) \leq \mathcal{E}\left(f\right)$, where $\mathcal{E}\left(f\right) := \mathcal{E}\left(f,f\right)$.

Then $(\mathcal{E},\mathcal{F})$ has the *generator* Δ that is a non-positive definite, self-adjoint operator on $L^2\left(M,\mu\right)$ with domain $\mathcal{D} \subset \mathcal{F}$ such that

$$\mathcal{E}\left(f,g\right) = \left(-\Delta f, g\right)$$

for all $f \in \mathcal{D}$ and $g \in \mathcal{F}$. The generator Δ determines the *heat semigroup* $\{P_t\}_{t\geq 0}$ by $P_t = e^{t\Delta}$ in the sense of functional calculus of self-adjoint operators. It is known that $\{P_t\}_{t\geq 0}$ is strongly continuous, contractive, symmetric semigroup in L^2, and is *Markovian*, that is, $0 \leq P_t f \leq 1$ for any $t > 0$ if $0 \leq f \leq 1$.

The Markovian property of the heat semigroup implies that the operator P_t preserves the inequalities between functions, which allows to use monotone limits to extend P_t from L^2 to L^∞ (in fact, P_t extends to any L^q, $1 \leq q \leq \infty$ as a contraction). In particular, $P_t 1$ is defined.

Definition. The form $(\mathcal{E},\mathcal{F})$ is called *conservative* or *stochastically complete* if $P_t 1 = 1$ for every $t > 0$.

Assume in addition that $(\mathcal{E},\mathcal{F})$ is *regular*, that is, the set $\mathcal{F} \cap C_0\left(M\right)$ is dense both in $\mathcal{F}$ with respect to the norm (3.2) and in $C_0\left(M\right)$ with respect to the sup-norm. By a theory of Fukushima [3], for any regular Dirichlet form there exists a Hunt process $\{X_t\}_{t\geq 0}$ such that, for all bounded Borel functions f on M,

$$\mathbb{E}_x f(X_t) = P_t f(x) \tag{3.3}$$

for all $t > 0$ and almost all $x \in M$, where $\mathbb{E}_x$ is expectation associated with the law of $\{X_t\}$ started at x.

Using the identity (3.3), one can show that the lifetime of X_t is almost surely ∞ if and only if $P_t 1 = 1$ for all $t > 0$, which motivates the term "stochastic completeness" in the above definition.

One distinguishes local and non-local Dirichlet forms. The Dirichlet form $(\mathcal{E},\mathcal{F})$ is called *local* if $\mathcal{E}\left(f,g\right) = 0$ for all functions $f, g \in \mathcal{F}$ with disjoint compact support. It is called *strongly local* if the same is true under a milder assumption that $f =$ const on a neighborhood of $\operatorname{supp} g$. For example, the classical Dirichlet form on a Riemannian manifold

$$\mathcal{E}\left(f,g\right) = \int_M \nabla f \cdot \nabla g d\mu$$

is strongly local.

It is known that any regular Dirichlet form can be represented in the form

$$\mathcal{E} = \mathcal{E}^{(c)} + \mathcal{E}^{(j)} + \mathcal{E}^{(k)}$$

where $\mathcal{E}^{(c)}$ is a strongly local part, that has the form

$$\mathcal{E}^{(c)}(f,g)=\int_M \Gamma(f,g)\,d\mu,$$

where $\Gamma(f,g)$ is a so called *energy density* (generalizing $\nabla f\cdot\nabla g$ on manifolds); $\mathcal{E}^{(j)}$ is a jump part that has the form

$$\mathcal{E}^{(j)}(f,g)=\frac{1}{2}\int\int_{X\times X}(f(x)-f(y))(g(x)-g(y))\,dJ(x,y)$$

with some measure J on $X\times X$ that is called a *jump measure*; and $\mathcal{E}^{(k)}$ is a killing part that has the form

$$\mathcal{E}^{(k)}(f,g)=\int_X fgdk$$

where k is a measure on X that is called a *killing measure.*

In terms of the associated process this means that X_t is in some sense a mixture of a diffusion process, jump process and a killing condition.

The log-volume test of Theorem 1.6 can be extended to strongly local Dirichlet forms as follows. Set as before $V(x,r)=\mu(B(x,r))$

Theorem 3.1 *Let $(\mathcal{E},\mathcal{F})$ be a regular strongly local Dirichlet form. Assume that the distance function $\rho(x)=d(x,x_0)$ on M satisfies the condition*

$$\Gamma(\rho,\rho)\le C, \tag{3.4}$$

for some constant C. If

$$\int^\infty\frac{rdr}{\log V(x_0,r)}=\infty$$

then the Dirichlet form $(\mathcal{E},\mathcal{F})$ is stochastically complete.

Basically, the method of the proof is the same as for Theorem 1.6 because for strongly local forms the same chain rule and product rules are available, that were used in the proof of Theorem 1.6, and the condition (3.4) replaces the condition $|\nabla d|\le 1$ that was repeatedly used in the proof. However, still a lot of technical details has to be filled (see [17]).

Now let us turn to jump processes. For simplicity let us assume that the jump measure J has a density $j(x,y)$. Namely, let $j(x,y)$ be is a non-negative Borel function on $M\times M$ that satisfies the following two conditions:

(a) $j(x,y)$ is symmetric:

$$j(x,y)=j(y,x). \tag{3.5}$$

(b) there is a positive constant C such that

$$\sup_{x\in M}\int_M(1\wedge d(x,y)^2)j(x,y)\mu(dy)\le C. \tag{3.6}$$

We say that a distance function d is *adapted* to a kernel $j(x,y)$ (or j is adapted to d) if (3.6) is satisfied. For the purpose of investigation of stochastic completeness the condition (3.6) plays the same role as (3.4) does for diffusion.

Consider the following bilinear functional

$$\mathcal{E}(f,g)=\frac{1}{2}\int_M\int_M (f(x)-f(y))(g(x)-g(y))j(x,y)\mu(dx) \tag{3.7}$$

that is defined on Borel functions f and g whenever the integral makes sense. Define the maximal domain of $\mathcal{E}$ by

$$\mathcal{F}_{\max}=\left\{f\in L^2:\mathcal{E}(f,f)<\infty\right\},$$

where $L^2=L^2(M,\mu)$. By the polarization identity, $\mathcal{E}(f,g)$ is finite for all $f,g\in\mathcal{F}_{\max}$. Moreover, $\mathcal{F}_{\max}$ is a Hilbert space with the following norm:

$$\|f\|^2_{\mathcal{F}_{\max}}=\mathcal{E}_1(f,f):=\|f\|^2_{L^2}+\mathcal{E}(f,f).$$

Denote by $\mathrm{Lip}_0(M)$ the class of Lipschitz functions on M with compact support. It follows from (3.6) that $\mathrm{Lip}_0(M)\subset\mathcal{F}_{\max}$. Indeed, for any $f\in\mathrm{Lip}_0(M)$ we have

$$|f(x)-f(y)|\leq L\wedge(Ld(x,y))$$

where $L=\max\left(\|f\|_{\mathrm{Lip}},2\sup|f|\right)$ and $\|f\|_{\mathrm{Lip}}$ is the Lipschitz constant of f. Denoting $K=\operatorname{supp}f$, we obtain using (3.6)

$$\begin{aligned}\mathcal{E}(f,f) &= \frac{1}{2}\int_{M'}\int_M (f(x)-f(y))^2 j(x,y)\,d\mu(x)\,d\mu(y)\\ &\leq \int_{K\times M}(f(x)-f(y))^2 j(x,y)\,d\mu(y)\,d\mu(x)\\ &\leq L\int_K\int_M (1\wedge d(x,y)^2)j(x,y)\,d\mu(y)\,d\mu(x)\\ &\leq LC\mu(K)<\infty,\end{aligned}$$

which proves that $f\in\mathcal{F}_{\max}$.

Define the space $\mathcal{F}$ as the closure of $\mathrm{Lip}_0(M)$ in $(\mathcal{F}_{\max},\|\cdot\|_{\mathcal{F}_{\max}})$. Under the above hypothesis, $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form in $L^2(M,\mu)$. The associated Hunt process $\{X_t\}$ is a pure jump process with the jump density $j(x,y)$.

Many examples of jump processes are provided by Lévy-Khintchine theorem where the Lévy measure corresponds to $j(x,y)\,d\mu(y)$. The condition (3.6) appears also in Lévy-Khintchine theorem, so that the Euclidean distance in $\mathbb{R}^n$ is adapted to any Lévy measure. An explicit example of a jump density in $\mathbb{R}^n$ is

$$j(x,y)=\frac{\mathrm{const}}{|x-y|^{n+\alpha}},$$

where $\alpha\in(0,2)$. The corresponding Levy process (=the Hunt process of $(\mathcal{E},\mathcal{F})$) is the symmetric α-stable process with the generator $-(-\Delta)^{\alpha/2}$, where Δ is the Laplace operator in $\mathbb{R}^n$.

Sufficient condition for stochastic completeness of the Dirichlet form of jump type is given in the following theorem that was proved in [9].

Theorem 3.2 *Assume that j satisfies (3.5) and (3.6) and $(\mathcal{E}, \mathcal{F})$ be the jump form defined as above. There is a constant b such that if*

$$V(x_0, r) \leq \exp(br \log r) \tag{3.8}$$

for some $x_0 \in M$ and all large enough[1] r then the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is stochastically complete.

It comes from the proof that any value $b < \frac{1}{2}$ will do but it is not known if $\frac{1}{2}$ is sharp. For example, (3.8) is satisfied if, for some constant C and all r,

$$V(x_0, r) \leq \exp(Cr) \tag{3.9}$$

In the proof of Theorem 3.2 we split the jump kernel $j(x,y)$ into the sum of two parts:

$$j'(x,y) = j(x,y)\mathbf{1}_{\{d(x,y)\leq 1\}} \text{ and } j''(x,y) = j(x,y)\mathbf{1}_{\{d(x,y)>1\}} \tag{3.10}$$

and show first the stochastic completeness of the Dirichlet form $(\mathcal{E}', \mathcal{F})$ associated with j'. For that we adapt the methods used for stochastic completeness for the local form. The bounded range of j' allows to treat the Dirichlet form $(\mathcal{E}', \mathcal{F})$ as "almost" local: if f, g are two functions from $\mathcal{F}$ such that $d(\operatorname{supp} f, \operatorname{supp} g) > 1$ then $\mathcal{E}(f, g) = 0$. The condition (3.6) plays in the proof the same role as the condition (3.4) in the local case. However, the lack of locality brings up in the estimates various additional terms that have to be compensated by a stronger hypothesis of the volume growth (3.8), instead of the quadratic exponential growth in Theorem 3.1.

The tail j'' can regarded as a small perturbation of j' in the following sense: $(\mathcal{E}, \mathcal{F})$ is stochastically complete if and only if $(\mathcal{E}', \mathcal{F})$ is so. The proof is based on the fact that the integral operator with the kernel j'' is a bounded operator in $L^2(M, \mu)$, because by (3.6)

$$\int_M j''(x,y)\, d\mu(y) \leq C.$$

It is not clear if the volume growth condition (3.8) in Theorem 3.2 is sharp. We conjecture that it is sharp in the following sense: if b is too large then the statement of Theorem 3.2 is no longer true. If this conjecture is true then a new interesting question arrises: what is the optimal value of b?

To explain the motivation behind this conjecture, let us briefly mention a recent result of Xueping Huang [15], that is analogous of Theorem 1.7 about the uniqueness class for the Cauchy problem on a geodesically complete manifold. Recall that this theorem states the following: the Cauchy problem for the heat equation on a manifold has the following uniqueness class:

$$\int_0^T \int_{B(x,R)} u^2(t,x)\, d\mu(x)\, dt \leq \exp(f(R))$$

[1]In fact it suffices to have (3.8) for $r = r_k$ where $\{r_k\}$ is any sequence such that $r_k \to \infty$ as $k \to \infty$.

where $\int^{\infty} \frac{rdr}{f(r)} = \infty$. X.Huang proved a similar theorem for the heat equation associated with the jump Dirichlet form on graphs satisfying (a) and (b): the associated heat equation has the following uniqueness class

$$\int_0^T \int_{B(x,R)} u^2(t,x)\, d\mu(x)\, dt \le \exp(br \log r)$$

for some constant b. Moreover, he has shown on an example that for large enough values of b this statement fails. However, the function u in this example is unbounded, so that it cannot serve to show the sharpness of the condition (3.8) in Theorem 3.2.

4 Random walks on graphs

Let us now turn to random walks on graphs. Let (X, E) be a locally finite, infinite, connected graph, where X is the set of vertices and E is the set of edges. We assume that the graph is undirected. Let μ be the counting measure on X. Define the jump kernel by $j(x,y) = 1_{\{x \sim y\}}$, where $x \sim y$ means that x, y are neighbors, that is, $(x,y) \in E$. The corresponding Dirichlet form is

$$\mathcal{E}(f) = \frac{1}{2} \sum_{\{x,y:x\sim y\}} (f(x) - f(y))^2,$$

and its generator is

$$\Delta f(x) = \sum_{y\sim x} (f(y) - f(x)).$$

The operator Δ is called *unnormalized* (or *physical)* Laplace operator on (X, E). This is to distinguish from the *normalized* or *combinatorial* Laplace operator

$$\hat{\Delta} f(x) = \frac{1}{\deg(x)} \sum_{y\sim x} (f(y) - f(x)),$$

where $\deg(x)$ is the number of neighbors of x. The normalized Laplacian $\hat{\Delta}$ is the generator of the same Dirichlet form but with respect to the degree measure $\deg(x)$.

Both Δ and $\hat{\Delta}$ generate the heat semigroups $e^{t\Delta}$ and $e^{t\hat{\Delta}}$ and, hence, associated continuous time random walks on X. It is easy to prove that $\hat{\Delta}$ is a bounded operator in $L^2(X, \deg)$, which then implies that the associated random walk is always stochastically complete. On the contrary, the random walk associated with the unnormalized Laplace operator can be stochastically incomplete.

We say that the graph (X, E) is stochastically complete if the heat semigroup $e^{t\Delta}$ is stochastically complete.

Denote by $\rho(x,y)$ the graph distance on X, that is the minimal number of edges in an edge chain connecting x and y. Let $B_\rho(x,r)$ be closed metric balls with respect to this distance ρ and let $V_\rho(x,r) = |B_\rho(x,r)|$ where and $|\cdot| := \mu(\cdot)$ denotes the number of vertices in the given set.

Stochastic completeness can be stated in terms of the volume growth as follows.

Theorem 4.1 *If there is a point $x_0 \in X$ and a constant $c > 0$ such that*

$$V_\rho(x_0, r) \leq cr^3 \tag{4.1}$$

for all large enough r, then the graph (X, E) is stochastically complete.

Note that the cubic rate of the volume growth here is sharp. Indeed, Wojciechowski [18] has shown that, for any $\varepsilon > 0$ there is a stochastically incomplete graph that satisfies $V_\rho(x_0, r) \leq cr^{3+\varepsilon}$. For any non-negative integer r, set

$$S_r = \{x \in X : \rho(x_0, x) = r\}. \tag{4.2}$$

In the example of Wojciechowski every vertex on S_r is connected to all vertices on S_{r-1} and S_r (see Fig. 6).

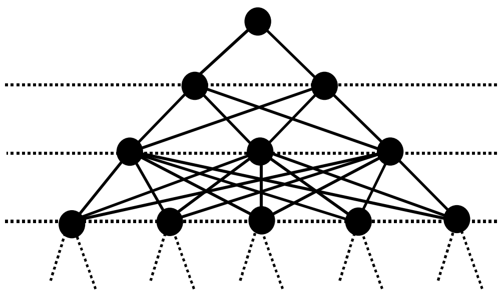

Figure 6: Anti-tree of Wojciechowski

For this type of graphs, that are called *anti-trees*, the stochastic incompleteness is equivalent to the following condition ([18]):

$$\sum_{r=1}^{\infty} \frac{V_\rho(x_0, r)}{|S_{r+1}|\,|S_r|} < \infty. \tag{4.3}$$

Indeed, assuming (4.3), one constructs a non-trivial bounded solution to the equation $\Delta u - u = 0$, which is enough to ensure the stochastic incompleteness. For a radial function $u = u(r)$ this equation acquires the form

$$u(r+1) = u(r) + \frac{1}{|S_{r+1}|\,|S_r|} \sum_{i=0}^{r} |S_i|\, u(i). \tag{4.4}$$

Setting $u(0) = 1$ and solving this equation inductively in r, we obtain a positive solution $u(r)$ that increases in r. It follows that

$$u(r+1) \leq \left(1 + \frac{1}{|S_{r+1}|\,|S_r|} \sum_{i=0}^{r} |S_i|\right) u(r)$$

whence by induction

$$u(R) \leq \prod_{r=0}^{R-1} \left(1 + \frac{V_\rho(x_0, r)}{|S_{r+1}| |S_r|}\right).$$

The condition (4.3) implies that the product in the right hand side is bounded so that u is a bounded function.

If $|S_r| \simeq r^{2+\varepsilon}$ then $V_\rho(x_0, r) \simeq r^{3+\varepsilon}$ and the condition (4.3) is satisfied so that the graph is stochastically incomplete (the relation $f \simeq g$ means that the ratio of functions f and g is bounded from above and below by positive constants).

The proof of Theorem 4.1 is based on the following ideas. Firstly, the graph distance ρ is in general not adapted. Indeed, the integral in (3.6) is equal to

$$\sum_y \left(1 \wedge \rho^2(x, y)\right) j(x, y) = \sum_y j(x, y) = \deg(x)$$

so that (3.6) holds if and only if the graph has uniformly bounded degree, which is not interesting. In general, we can construct an *adapted* distance d as follows. For all $x \sim y$ set

$$\sigma(x, y) = \frac{1}{\sqrt{\deg(x)}} \wedge \frac{1}{\sqrt{\deg(y)}} \tag{4.5}$$

and regard $\sigma(x, y)$ as the length for the edge $x \sim y$. Then for all $x, y \in X$ define $d(x, y)$ as the smallest total length of all edges in an edge chain connecting x and y. It is easy to verify that d satisfies (3.6):

$$\sum_y \left(1 \wedge d^2(x, y)\right) j(x, y) \leq \sum_y \left(\frac{1}{\deg(x)} \wedge \frac{1}{\deg(y)}\right) j(x, y) \leq \sum_{y \sim x} \frac{1}{\deg(x)} = 1.$$

Then one proves that (4.1) for ρ-balls implies that the d-balls have at most exponential volume growth, so that the stochastic completeness follows by Theorem 3.2.

To see why the cubic volume growth for the graph distance is related to the exponential volume growth for the adapted distance, let us consider a more restrictive hypothesis

$$|S_r| \leq C r^2 \quad \text{for } r \geq 1, \tag{4.6}$$

where S_r is defined by (4.2) (clearly, (4.6) is a stronger hypothesis than (4.1)). Any point $x \in S_r$ admits the estimate of the degree as follows (see Fig. 7):

$$\deg(x) \leq |S_{r-1}| + |S_r| + |S_{r+1}| \leq C_1 r^2. \tag{4.7}$$

Therefore, if x, y are two neighboring vertices in $B_\rho(x_0, r)$, then by (4.5) and (4.7)

$$\sigma(x, y) = \frac{1}{\sqrt{\deg(x)}} \wedge \frac{1}{\sqrt{\deg(y)}} \geq \frac{c_1}{r}, \tag{4.8}$$

with some constant $c_1 > 0$.

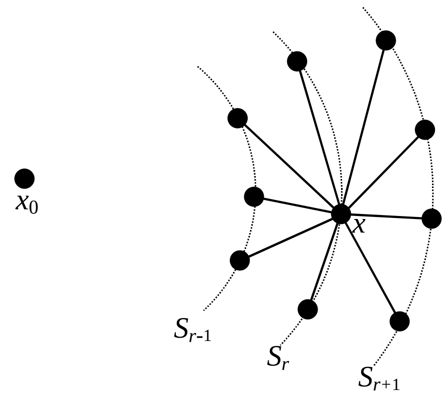

Figure 7: A vertex $x \in S_\rho(r)$ can be connected only to the vertices on $S_\rho(r-1)$, $S_\rho(r)$, and $S_\rho(r+1)$

Fix a vertex $x \in S_R$ and let $\{x_i\}_{i=0}^N$ be a path connecting x_0 to x with the minimal σ-length (see Fig. 8). Clearly, we have $\rho(x_0, x_i) \le i$ whence it follows from (4.8) that $\sigma(x_{i-1}, x_i) \ge \frac{c_1}{i}$ and, hence for some $c_2 > 0$,

$$d(x_0, x) = \sum_{i=1}^{N} \sigma(x_{i-1}, x_i) \ge c_1 \sum_{i=1}^{R} \frac{1}{i} > c_2 \log R.$$

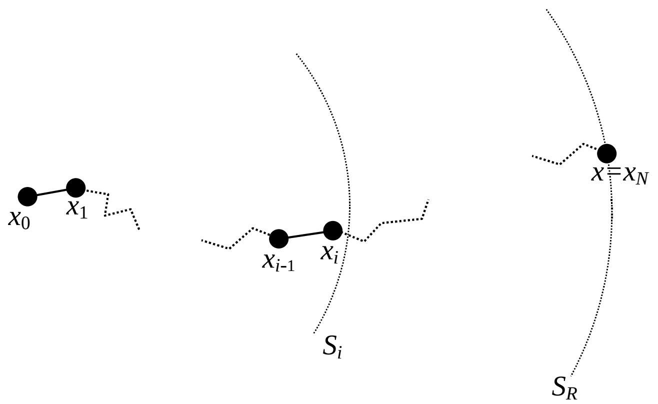

Figure 8: For any path $\{x_i\}_{i=0}^N$ connecting x_0 and $x \in S_\rho(R)$, we have $N \ge R$ and $\sigma(x_{i-1}, x_i) \ge \frac{c_1}{i}$.

Denoting by B_d the d-balls, we obtain

$$B_d(x_0, c_2 \log R) \subset B_\rho(x_0, R).$$

Changing variables $r = c_2 \log R$ and denoting by V_d the volume of B_d, we obtain using (4.1) that

$$V_d(x_0, r) \le V_\rho(x_0, e^{r/c_2}) \le \exp(c_3 r)$$

for some $c_3 > 0$ and all large enough r, which was claimed.

In the general case (4.1) does not imply (4.6) for all $r \geq 1$, but nevertheless (4.6) holds for sufficiently many values of r, which can be used to prove the estimates of d as above.

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