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Path chain complex and path homology on a digraph

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Abstract

In this paper, we give a brief overview of the theory of the path chain complexes and path homology on digraphs based on the previous work of the authors.

Contents

1 Introduction

There exists a number of ways to define the notion of homology for graphs and digraphs, for example, clique homology ([3], [11]) or singular homology ([1], [11], [12]). However, the notion of *path homology* has certain advantages as it enjoys the adequate functorial properties with respect to graph-theoretical operations, such as morphisms of digraphs, Cartesian products, joins, homotopy etc.

The concept of path homology is derived from the concept of a *path chain complex* that is non-trivial and highly interesting by itself as it encodes a lot of information about the underlying digraph. Based on the path chain complex, we define also the notion of *Hodge Laplacians* acting on the chain spaces. The study of spectra of Hodge Laplacians on digraphs is a new interesting area of research.

The notions of path homology and path chain complex have rich mathematical content, and we hope that they will become useful tools in various areas of pure and applied mathematics. This survey is based on the following papers: [5], [6], [7], [8], [9], [10].

2 Path chain complex

2.1 Boundary operator ∂

Let V be a finite set whose elements are called vertices. For any $p \geq 0$, an *elementary p-path* is any sequence $i_0, ..., i_p$ of $p+1$ vertices of V (allowing repetitions). Fix a field $\mathbb{K}$ and denote by $\Lambda_p = \Lambda_p(V, \mathbb{K})$ the $\mathbb{K}$-linear space consisting of all formal $\mathbb{K}$-linear combinations of elementary p-paths. Any element of Λ_p is called a p-path.

An elementary p-path $i_0, ..., i_p$ as an element of Λ_p will be denoted by $e_{i_0 \ldots i_p}$. For example, we have

$$\Lambda_0 = \langle e_i : i \in V \rangle, \quad \Lambda_1 = \langle e_{ij} : i, j \in V \rangle, \qquad \Lambda_2 = \langle e_{ijk} : i, j, k \in V \rangle.$$

Any p-path u has a form

$$u = \sum_{i_0, \ldots, i_p \in V} u^{i_0 \cdots i_p} e_{i_0 \ldots i_p},$$

where $u^{i_0 \cdots i_p} \in \mathbb{K}$.

Definition 2.1 For any $p \geq 1$ define a linear *boundary operator* $\partial : \Lambda_p \to \Lambda_{p-1}$ by

$$\partial e_{i_0 \ldots i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0 \ldots \widehat{i_q} \ldots i_p}, \tag{2.1}$$

where $\widehat{\ }$ means omission of the index, and then extend ∂ to all Λ_p by linearity. Set also $\Lambda_{-1} = \{0\}$ and define the operator $\partial : \Lambda_0 \to \Lambda_{-1}$ by $\partial = 0$.

For example, $\partial e_i = 0$, $\partial e_{ij} = e_j - e_i$ and $\partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}$.

Lemma 2.2 *We have $\partial^2 = 0$.*

Definition 2.3 An elementary p-path $e_{i_0 \ldots i_p}$ is called *regular* if $i_k \neq i_{k+1}$ for all $k = 0, ..., p-1$, and *irregular* otherwise.

Let $\mathcal{I}_p$ be the subspace of Λ_p spanned by irregular p-paths $e_{i_0 \ldots i_p}$. It is easy to see that $\partial \mathcal{I}_p \subset \mathcal{I}_{p-1}$, which allows to define ∂ on the quotient spaces $\mathcal{R}_p := \Lambda_p / \mathcal{I}_p$. Hence, we obtain a chain complex $\mathcal{R}_*(V)$:

$$0 \leftarrow \mathcal{R}_0 \overset{\partial}{\leftarrow} \mathcal{R}_1 \overset{\partial}{\leftarrow} \ldots \overset{\partial}{\leftarrow} \mathcal{R}_{p-1} \overset{\partial}{\leftarrow} \mathcal{R}_p \overset{\partial}{\leftarrow} \ldots \tag{2.2}$$

By setting all irregular p-paths to be equal to 0, we identify $\mathcal{R}_p$ with the subspace of Λ_p spanned by all regular paths. For example, if $i \neq j$ then $e_{iji} \in \mathcal{R}_2$ and

$$\partial e_{iji} = e_{ji} - e_{ii} + e_{ij} = e_{ji} + e_{ij} \in \mathcal{R}_1$$

because $e_{ii} = 0$ in $\mathcal{R}_1$.

2.2 Spaces of ∂-invariant paths

Definition 2.4 A *digraph (directed graph)* is a pair $G = (V, E)$ of a set V of vertices and a set $E \subset V \times V \setminus \operatorname{diag}$ of arrows (directed edges).

If $(i, j) \in E$ then we write $i \to j$. Here and in what follows, all digraphs are always finite.

Definition 2.5 An elementary p-path $i_0 \ldots i_p$ on V is called *allowed* if if $i_k \to i_{k+1}$ for any $k = 0, ..., p-1$, and *non-allowed* otherwise. A p-path u is called *allowed* if it is a $\mathbb{K}$-linear combination of allowed elementary p-paths.

The set of all allowed p-paths is denoted by $\mathcal{A}_p = \mathcal{A}_p(G)$. Clearly, $\mathcal{A}_p$ is a subspace of $\mathcal{R}_p$.

We would like to build a chain complex based on subspaces $\mathcal{A}_p$ of $\mathcal{R}_p$. However, the spaces $\mathcal{A}_p$ are in general *not* invariant for ∂. For example, in the digraph

$$\overset{a}{\bullet} \longrightarrow \overset{b}{\bullet} \longrightarrow \overset{c}{\bullet}$$

we have $e_{abc} \in \mathcal{A}_2$ but $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \notin \mathcal{A}_1$ because e_{ac} is non-allowed.

Definition 2.6 A regular path u is called ∂-*invariant* if u and ∂u are allowed.

Clearly, all 0-paths and 1-paths are ∂-invariant.

Let us give some examples of ∂-invariant 2-paths. By a *triangle* in a digraph G we mean a configuration of three distinct vertices a, b, c such that $a \to b \to c$ and $a \to c$. Then the 2-path e_{abc} is allowed, and its boundary $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab}$ is also allowed, so that e_{abc} is ∂-invariant.

The ∂-invariant 2-path e_{abc} will also be referred to as a triangle. A *square* is a configuration of four distinct vertices a, b, b', c such that $a \to b \to c$, $a \to b' \to c$ but $a \not\to c$. It determines a ∂-invariant 2-path $e_{abc} - e_{ab'c}$ that is also called a square.

A *double arrow* is a configuration of two distinct vertices a, b such that $a \to b \to a$. It determines a ∂-invariant 2-path e_{aba} that is also called a double arrow.

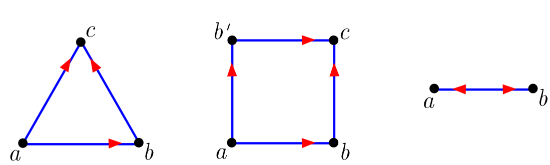

Figure 1: A triangle, a square and a double arrow

Denote by Ω_p the subspace of $\mathcal{A}_p$ that consists of ∂-invariant p-paths, that is,

$$\Omega_p \equiv \Omega_p(G) := \{u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1}\}.$$

It is easy to prove that $\partial\Omega_p \subset \Omega_{p-1}$ so that we obtain a chain complex $\Omega_* = \Omega_*(G)$:

$$0 \leftarrow \Omega_0 \stackrel{\partial}{\leftarrow} \Omega_1 \stackrel{\partial}{\leftarrow} \dots \stackrel{\partial}{\leftarrow} \Omega_{p-1} \stackrel{\partial}{\leftarrow} \Omega_p \stackrel{\partial}{\leftarrow} \dots \tag{2.3}$$

By construction we have $\Omega_0 = \mathcal{A}_0$ and $\Omega_1 = \mathcal{A}_1$, while in general $\Omega_p \subset \mathcal{A}_p$. One can prove that Ω_2 is spanned by all triangles, squares and double arrows. Note that squares can be linearly dependent.

Definition 2.7 The chain complex $\Omega_*(G)$ is called the *path chain complex* of the digraph G. The *path homologies* of G are defined as the homologies of the path chain complex $\Omega_*(G)$:

$$H_p = H_p(G) = \ker \partial|_{\Omega_p} \big/ \operatorname{Im} \partial|_{\Omega_{p+1}}. \tag{2.4}$$

The dimension $\beta_p := \dim H_p(G)$ is called the p-th Betti number of G.

One can show that β_0 is equal to the number of (undirected) connected components of G.

If the sequence $\{\Omega_p\}$ is finite in the sense that $\Omega_p = \{0\}$ for large enough p, then we can define the Euler characteristic of G by

$$\chi := \sum_{p=0}^{\infty} (-1)^p |\Omega_p| = \sum_{p=0}^{\infty} (-1)^p \beta_p.$$

Note that the chain complex (2.3) depends on the field $\mathbb{K}$ of coefficients although we have suppressed $\mathbb{K}$ from notation for simplicity. However, there are interesting non-trivial examples of digraphs where the Euler characteristic actually depends on $\mathbb{K}$. These examples, obtained in [4, Thm 5.4] and [2, Sect.6.2], show that the path homology theory cannot be reduced to the classical homology theory of topological spaces.

3 Cartesian product and Künneth formula

3.1 Cross product of regular paths

Given two finite sets X, Y, consider their product

$$Z = X \times Y = \{(a, b) : a \in X \text{ and } b \in Y\}.$$

Let $z = z_0 z_1 ... z_r$ be a regular elementary r-path on Z, where $z_k = (a_k, b_k)$ with $a_k \in X$ and $b_k \in Y$. We say that z is *stair-like* if, for any $k = 1, ..., r$, either $a_{k-1} = a_k$ or $b_{k-1} = b_k$ is satisfied. That is, any couple $z_{k-1} z_k$ of consecutive vertices is either vertical (when $a_{k-1} = a_k$) or horizontal (when $b_{k-1} = b_k$).

For any stair-like path z on Z, define its projection onto X as an elementary path x on X obtained from z by removing the Y-components in all the vertices of z and by collapsing in the resulting sequence of points of X consecutive repeated vertices to one vertex. In the same way we define projection of z onto Y and denote it by y. Then the projections $x = x_0...x_p$ and $y = y_0...y_q$ are regular elementary paths on X, resp. Y, and $p + q = r$.

Every vertex (x_i, y_j) of the path z can be represented as a point (i, j) of $\mathbb{Z}^2$ so that the path z is represented by the *staircase* $S(z)$ in $\mathbb{Z}^2$ connecting $(0, 0)$ and (p, q). The *elevation* $L(z)$ of z is defined as the number of cells in $\mathbb{Z}^2_+$ below $S(z)$.

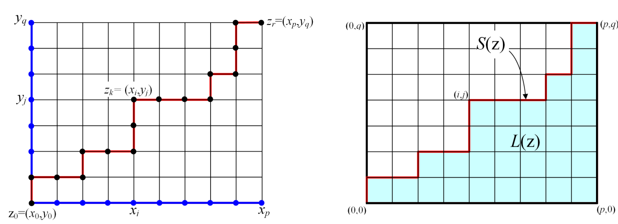

Figure 2: A stair-like path z, its staircase $S(z)$ and the elevation $L(z)$

For given elementary regular paths x on X and y on Y, denote by $\Pi_{x,y}$ the set of all stair-like paths z on Z whose projections on X and Y are respectively x and y.

Definition 3.1 Define the *cross product* of the elementary regular paths e_x and e_y as a path $e_x \times e_y$ on Z as follows:

$$e_x \times e_y = \sum_{z\in\Pi_{x,y}} (-1)^{L(z)} e_z \ , \tag{3.1}$$

and extend it by linearity to all $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ so that $u \times v \in \mathcal{R}_{p+q}(Z)$.

Lemma 3.2 (The product rule) *If $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ where $p, q \geq 0$, then*

$$\partial (u \times v) = (\partial u) \times v + (-1)^p u \times (\partial v) . \tag{3.2}$$

3.2 Cartesian product of digraphs

Denote a digraph and its set of vertices by the same letters to simplify notation. Given two digraphs X and Y, define their Cartesian product as a digraph $Z = X\Box Y$ as follows:

- the set of vertices of Z is $X \times Y$, that is, the vertices of Z are the couples (a, b) where $a \in X$ and $b \in Y$;

- the edges in Z are of two types: $(a,b) \to (a',b)$ where $a \to a'$ (a *horizontal* edge) and $(a,b) \to (a,b')$ where $b \to b'$ (a *vertical* edge):

$$
\begin{array}{ccccccc}
\vdots & & & \vdots & & \vdots & \\
b' \bullet & & \cdots & \overset{(a,b')}{\bullet} & \to & \overset{(a',b')}{\bullet} & \cdots \\
\uparrow & & & \uparrow & & \uparrow & \\
b \bullet & & \cdots & \overset{(a,b)}{\bullet} & \to & \overset{(a',b)}{\bullet} & \cdots \\
\vdots & & & \vdots & & \vdots & \\
Y & X & \cdots & \underset{a}{\bullet} & \to & \underset{a'}{\bullet} & \cdots
\end{array}
$$

It follows that any allowed elementary path in Z is stair-like. Moreover, any regular elementary path on Z is allowed if and only if it is stair-like and its projections onto X and Y are allowed.

Example 3.3 Let I be a digraph of two vertices 0 and 1 such that $0 \to 1$. Then $I^2 := I \Box I$ is a *square digraph* and $I^n := \underbrace{I \Box I \Box ... \Box I}_{n}$ is an *n-cube.*

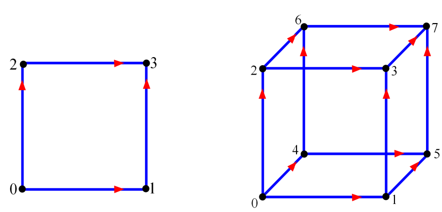

Figure 3: A square I^2 and a 3-cube I^3

Example 3.4 Let T be a digraph of 3 vertices 0, 1, 2 such that $0 \to 1 \to 2 \to 0$. The digraph $T^n = \underbrace{T \Box T \Box ... \Box T}_{n}$ is called an *n-torus.*

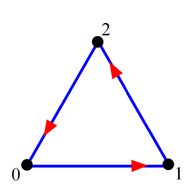

Figure 4: 1-torus T

It follows from the definition (3.1) of the cross product that

$$u \in \mathcal{A}_p(X) \text{ and } v \in \mathcal{A}_q(Y) \quad \Rightarrow \quad u \times v \in \mathcal{A}_{p+q}(Z). \tag{3.3}$$

Furthermore, the product rule of Lemma 3.2 implies that

$$u \in \Omega_p(X) \text{ and } v \in \Omega_q(Y) \Rightarrow u \times v \in \Omega_{p+q}(Z). \tag{3.4}$$

3.3 ∂-invariant paths on products

In all the statements below, X and Y are two digraphs, and $Z = X\Box Y$. The following theorem is the main result of this section. In some sense, it provides a converse statement to (3.4).

Theorem 3.5 *Let X, Y be two digraphs and $Z = X\Box Y$. Any path $w \in \Omega_r(Z)$ with $r \geq 0$ admits a representation in the form*

$$w = \sum_{i=1}^{k} u_i \times v_i \tag{3.5}$$

for some finite k and some $u_i \in \Omega_{p_i}(X)$ and $v_i \in \Omega_{q_i}(Y)$, where $p_i, q_i \geq 0$ and $p_i + q_i = r$.

Theorem 3.5 implies the following Künneth type formulas.

Theorem 3.6 (Künneth formula for product) *Let X, Y be two finite digraphs and $Z = X\Box Y$. Then we have an isomorphism of the chain complexes:*

$$\Omega_*(Z) \cong \Omega_*(X) \otimes \Omega_*(Y).$$

For any $r \geq 0$, we have

$$\Omega_r(Z) \cong \bigoplus_{\{p,q\geq 0: p+q=r\}} \Omega_p(X) \otimes \Omega_q(Y), \tag{3.6}$$

where the isomorphism is given by the map $u \otimes v \mapsto u \times v$ for $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$. Consequently, we have

$$H_r(Z) \cong \bigoplus_{\{p,q\geq 0: p+q=r\}} H_p(X) \otimes H_q(Y). \tag{3.7}$$

Example 3.7 It is easy to see that for the digraph $I = \{\bullet \to \bullet\}$ all homology groups are trivial, that is, $H_0 \cong \mathbb{K}$ and $H_p = \{0\}$ for all $p \geq 1$. It follows from (3.7) that also the n-cube I^n is homologically trivial.

For the 1-torus T as above we have $H_0 \cong H_1 \cong \mathbb{K}$ where H_1 is generated by the cycle $\{e_{01} + e_{12} + e_{20}\}$, and $H_p = \{0\}$ for all $p \geq 2$. It follows from (3.7) that the Betti numbers of the n-torus T^n are $\beta_p(T^n) = \binom{n}{p}$.

4 Reduced homology and join of digraphs

4.1 Augmented chain complex

In this section we use the augmented chain complex

$$0 \leftarrow \mathbb{K} \overset{\partial}{\leftarrow} \Omega_0 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots \tag{4.1}$$

with the added space $\Omega_{-1} = \mathbb{K}$. The operator $\partial : \Omega_0 \to \Omega_{-1}$ is now redefined by

$$\partial e_i = e := \text{the unity of } \mathbb{K}. \tag{4.2}$$

The homology groups of (4.1) are called the *reduced* homology groups of G and are denoted by $\widetilde{H}_p(G)$. The reduced Betti numbers of G are defined by

$$\widetilde{\beta}_p(G) = \dim \widetilde{H}_p(G).$$

Clearly, we have

$$\widetilde{H}_p(G) = H_p(G) \text{ for } p \geq 1 \text{ and } \widetilde{H}_0(G) = H_0(G)/\mathbb{K}.$$

4.2 Join of digraphs

Let X, Y be two digraphs.

Definition 4.1 The *join* $X * Y$ of two digraphs X, Y is a digraph whose set of vertices is a disjoint union of the sets of vertices of X and Y, and the set of arrows consists of all arrows of X and Y as well as from all arrows $x \to y$ where $x \in X$ and $y \in Y$.

Let $p, q \geq -1$. For a p-path u on X and a q-path v on Y, define the *join* $u * v$ as a $(p+q+1)$-path on $X * Y$ as follows: first define it for elementary paths by

$$e_{i_0 \dots i_p} * e_{j_0 \dots j_q} = e_{i_0 \dots i_p j_0 \dots j_q},$$

and then extend this definition by linearity to all u and v.

It is easy to see that the joint of two allowed elementary paths is allowed, which implies that

$$u \in \mathcal{A}_p(X) \text{ and } v \in \mathcal{A}_q(Y) \Rightarrow u * v \in \mathcal{A}_{p+q+1}(X * Y). \tag{4.3}$$

Lemma 4.2 (Product rule for the join) *For all $p, q \geq -1$ and $u \in \Lambda_p(X)$, $v \in \Lambda_q(Y)$ we have*

$$\partial (u * v) = (\partial u) * v + (-1)^{p+1} u * \partial v. \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$u \in \Omega_p(X) \text{ and } v \in \Omega_q(Y) \Rightarrow u * v \in \Omega_{p+q+1}(X * Y).$$

Theorem 4.3 *Let X, Y be two digraphs and $Z = X * Y$. Any path $w \in \Omega_r(Z)$ with $r \geq -1$ admits a representation in the form*

$$w = \sum_{j=1}^{k} u_j * v_j \tag{4.5}$$

for some finite k, with some $u_j \in \Omega_{p_j - 1}(X)$ and $v_j \in \Omega_{q_j}(Y)$, where $p_j \geq 0$, $q_j \geq -1$ and $p_j + q_j = r$.

On any digraph G, consider a shifted chain complex $\Omega'_*(G) = \{\Omega'_p\}_{p=0}^{\infty}$ where $\Omega'_p = \Omega_{p-1}$, with the same boundary operator ∂ as in (4.1).

Theorem 4.4 (Künneth formula for the join) *Let X, Y be two digraphs and $Z = X * Y$. Then we have an isomorphism of the chain complexes:*

$$\Omega'_*(Z) \cong \Omega'_*(X) \otimes \Omega'_*(Y).$$

In particular, for any $r \geq -1$, we have

$$\Omega_r(Z) \cong \bigoplus_{\{p,q \geq -1 : p+q=r-1\}} \Omega_p(X) \otimes \Omega_q(Y), \tag{4.6}$$

where the isomorphism is given by the map $u \otimes v \mapsto u * v$ with $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$. Consequently, for any $r \geq 0$,

$$\widetilde{H}_r(Z) \cong \bigoplus_{\{p,q \geq 0 : p+q=r-1\}} \widetilde{H}_p(X) \otimes \widetilde{H}_q(Y). \tag{4.7}$$

Example 4.5 Let Y consist of a single vertex. In this case the join $X * Y$ is called a *cone* over X. Since $\widetilde{\beta}_q(Y) = 0$ for all $q \geq 0$, the cone $X * Y$ is homologically trivial by (4.7). Let K_n be a complete digraph with vertices $\{1, ..., n\}$ and arrows

$$i \to j \Leftrightarrow i < j.$$

In other words, K_n is a directed $(n-1)$-simplex.

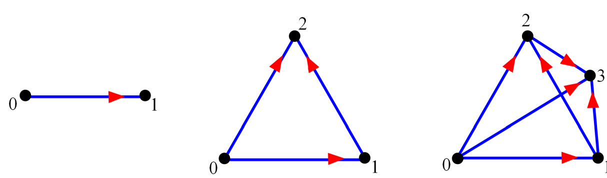

Figure 5: Simplices K_2 (interval), K_3 (triangle) and K_4 (tetrahedron)

It is easy to see that K_{n+1} is a cone over K_n, which implies that all K_n are homologically trivial.

Example 4.6 Let Y consist of two vertices without arrow, that is, $Y = \{\bullet, \bullet\}$. Then the join $X * Y$ is called a *suspension* of X. Since $\widetilde{\beta}_0(Y) = 1$ and $\widetilde{\beta}_q(Y) = 0$ for all $q \geq 1$, we obtain from (4.7) that

$$\widetilde{\beta}_r(X * Y) = \widetilde{\beta}_{r-1}(X).$$

Let us define a *digraph n-sphere* S^n as follows: $S^0 = \{\bullet, \bullet\}$ and S^{n+1} is the suspension of S^n for all $n \geq 0$. It follows by induction that the only positive reduced Betti number of S^n is $\widetilde{\beta}_n(S^n) = 1$.

For example, S^1 is a *diamond* and S^2 is an *octahedron.*

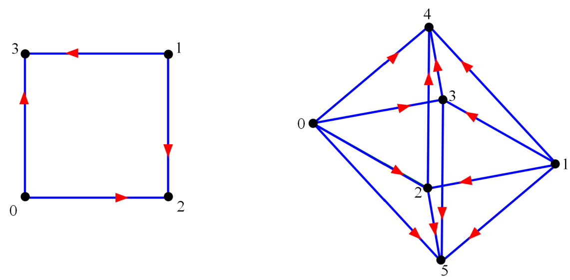

Figure 6: Digraph spheres: S^1 is a diamond and S^2 is an octahedron.

Using (4.7) one can show that $H_1(S^1)$ is generated by $e_{02} - e_{12} + e_{13} - e_{03}$ and $H_2(S^2)$ is generated by

$$e_{024} - e_{025} - e_{034} + e_{035} - e_{124} + e_{125} + e_{134} - e_{135}.$$

5 Homotopy of digraphs

In this section we use again the chain complex (2.3).

5.1 Digraphs morphisms

We write $a \overset{\rightarrow}{=} b$ if either $a \to b$ or $a = b$.

Definition 5.1 A *morphism* from a digraph $G = (V, E)$ to a digraph $G' = (V', E')$ is a map $f : V \to V'$ such that

$$\text{if } a \overset{\rightarrow}{=} b \text{ on } G \quad \text{then} \quad f(a) \overset{\rightarrow}{=} f(b) \;\; \text{on } G'. \tag{5.1}$$

That is, if $a \to b$ in G then either $f(a) \to f(b)$ or $f(a) = f(b)$ in G'. We will refer to such morphisms also as *digraphs maps* and denote them shortly by $f : G \to G'$.

Given a map $f : V \to V'$, define for any $p \geq 0$ the *induced map*

$$f_* : \Lambda_p(V) \to \Lambda_p(V')$$

by the rule

$$f_*\left(e_{i_0 \ldots i_p}\right) = e_{f(i_0) \ldots f(i_p)}, \tag{5.2}$$

extended by $\mathbb{K}$-linearity to all elements of $\Lambda_p(V)$. It is obvious that

$$f_*(\mathcal{R}_p(V)) \subset \mathcal{R}_p(V') \quad \text{and} \quad f_*(\mathcal{A}_p(G)) \subset \mathcal{A}_p(G').$$

It follows from (2.1) and (5.2) that $\partial f_* = f_* \partial$, which implies that also

$$f_*(\Omega_p(G)) \subset \Omega_p(G'). \tag{5.3}$$

Hence, the map $f_* : \Omega_p(G) \to \Omega_p(G')$ is a morphism of the chain complexes $\Omega_*(G) \to \Omega_*(G')$. In induces a homomorphism of homology groups $H_*(G) \to H_*(G')$ that will also be denoted by f_*.

The set of all digraphs with digraphs maps form a *category of digraphs*. Of course, it is desirable to state the results in the language of the category theory, but it is not always possible. For example, the Cartesian product from Section 3 is not a product in the category of digraphs as understood in category theory.

5.2 Homotopy

For any $n \geq 1$ define a *linear digraph* I_n as any digraph with vertices $\{0, 1, \dots, n\}$ such that if $|i-j| = 1$ then either $i \to j$ or $j \to i$, and if $|i-j| \neq 1$ then there is no arrow between i and j.

Here is an example of a linear digraph I_3: $\overset{0}{\bullet} \to \overset{1}{\bullet} \leftarrow \overset{2}{\bullet} \to \overset{3}{\bullet}$.

Definition 5.2 Let X and Y be digraphs. Two digraph maps $f, g \colon X \to Y$ are called *homotopic* if there exists a linear digraph I_n with some $n \geq 1$ and a digraph map

$$\Phi \colon X \Box I_n \to Y$$

such that

$$\Phi|_{X \times 0} = f \quad \text{and} \quad \Phi|_{X \times n} = g. \tag{5.4}$$

In this case we write $f \simeq g$.

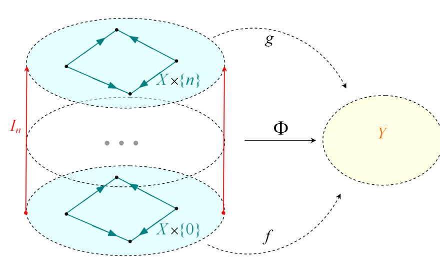

Figure 7: A homotopy Φ

Definition 5.3 Two digraphs X and Y are called *homotopy equivalent* if there exist digraph maps

$$f : X \to Y, \quad g : Y \to X \tag{5.5}$$

such that

$$f \circ g \simeq \mathrm{id}_Y, \qquad g \circ f \simeq \mathrm{id}_X. \tag{5.6}$$

In this case we write $X \simeq Y$.

Clearly, the relation $\simeq$ is an equivalence relation. If $X \simeq \{\bullet\}$ then X is called *contractible*.

5.3 Invariance of homology groups under homotopy

Now we can present the main result about connections between homotopy and homology on digraphs.

Theorem 5.4 *Let X and Y be two digraphs.*

(a) *Let $f, g \colon X \to Y$ be two digraph maps. If $f \simeq g$ then the induced maps of homology groups*

$$f_* : H_p(X) \to H_p(Y) \quad \textit{and} \quad g_* : H_p(X) \to H_p(Y)$$

are identical, that is, $f_* = g_*$.

(b) *If the digraphs X and Y are homotopy equivalent then they are also homologically equivalent, that is, $H_*(X) \cong H_*(Y)$.*

Example 5.5 The n-torus T^n is not contractible because its homology groups are non-trivial as was shown in Section 3.3.

A digraph Y is called an *induced subgraph* of a digraph X if the set of vertices of Y is a subset of that of X and the arrows of Y are all those arrows of X whose adjacent vertices belong to Y.

Definition 5.6 Let X be a digraph and Y be its induced subgraph. A *retraction* of X onto Y is a digraph map $r : X \to Y$ such that $r|_Y = \mathrm{id}_Y$.

Theorem 5.7 *Let $r : X \to Y$ be a retraction of a digraph X onto an induced subgraph Y. Assume that*

$$\textit{either } x \,\overset{\rightarrow}{=}\, r(x) \quad \textit{for all } x \in X \quad \textit{or} \quad r(x) \,\overset{\rightarrow}{=}\, x \quad \textit{for all } x \in X. \tag{5.7}$$

Then $X \simeq Y$ and, consequently, $H_*(X) \cong H_*(Y)$.

A retraction that satisfies (5.7) is called a *deformation retraction.*

Example 5.8 An obvious projection of the n-cube I^n onto I^{n-1} is a deformation retraction, which implies that all the cubes I^n are homotopy equivalent and, hence, are contractible. Consequently, all the cubes I^n are homologically trivial, which was also a consequence of Theorem 3.6. Similarly, an obvious projection of the complete digraph K_n onto K_{n-1} is a deformation retraction, which implies that all the simplices K_n are contractible. Consequently, I^n and K_n are homotopy equivalent.

Example 5.9 Consider the following two digraphs:

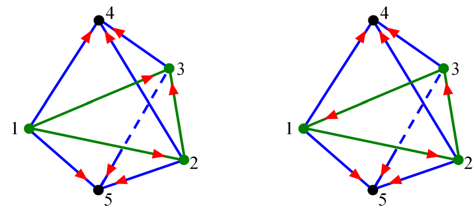

Figure 8:

The digraph at the left panel is contractible as there is a sequence of two deformation retractions r_1 and r_2 reducing it to $\{\bullet\}$:

$$r_1(4) = r_1(5) = 3, \quad r_2(1) = r_2(2) = 3.$$

Consequently, the digraph at the left panel is homologically trivial. The digraph at the right panel differs only by one arrow $3 \to 1$, but it is not contractible because $H_2 \neq \{0\}$. In fact, for this digraph H_2 is generated by a cycle

$$e_{124} + e_{234} + e_{314} - e_{125} - e_{235} - e_{315}.$$

Example 5.10 Let a be a vertex in a digraph G and let $b_0, b_1, ..., b_n$ be all the neighboring vertices of a in G. Assume that the following condition is satisfied:

$$\forall i = 1, ..., n \quad a \to b_i \Rightarrow b_0 \to b_i \text{ and } a \leftarrow b_i \Rightarrow b_0 \leftarrow b_i. \tag{5.8}$$

Denote by H the digraph that is obtained from G by removing the vertex a with all the adjacent arrows. Clearly, the map $r : G \to H$ given by $r(a) = b_0$ and $r|_H = \mathrm{id}$, is a deformation retraction, whence $G \simeq H$.

For example, consider the digraph G on the following diagram:

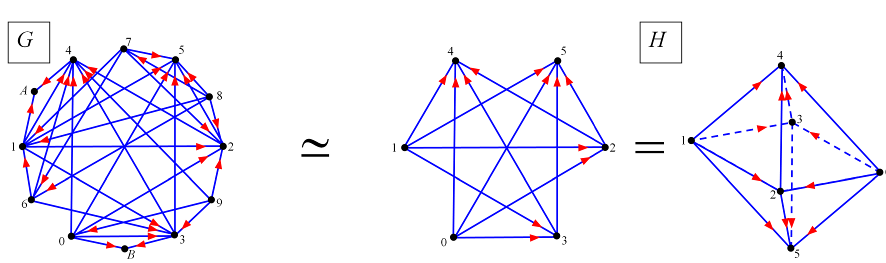

Figure 9:

Removing successively the vertices $A, B, 8, 9, 6, 7$ each time satisfying (5.8), we obtain an induced digraph H with the vertex set $\{0, 1, 2, 3, 4, 5\}$ that is homotopy (and homology) equivalent to G. As it is shown on the same diagram, the digraph H is identical to the octahedron.

6 Hodge Laplacian

6.1 Definition and basic properties

In this section, we use the chain complex (2.3) over the field $\mathbb{K} = \mathbb{R}$. Let fix an arbitrary inner product $\langle \cdot, \cdot \rangle$ in each of the spaces $\mathcal{R}_p$ so that we have an inner product also in all Ω_p. However, in all examples we use the *natural inner* product given by

$$\left\langle e_{i_0 \dots i_p}, e_{j_0 \dots j_p} \right\rangle = \delta_{i_0 \dots i_p}^{j_0 \dots j_p},$$

that is, all elementary regular p-paths form an orthonormal basis in $\mathcal{R}_p$.

For the operator $\partial : \Omega_p \to \Omega_{p-1}$ $(p \geq 0)$, consider the adjoint operator $\partial^* : \Omega_{p-1} \to \Omega_p$ so that

$$\langle \partial u, v \rangle = \langle u, \partial^* v \rangle \quad \text{for all } u \in \Omega_p \text{ and } v \in \Omega_{p-1}.$$

Definition 6.1 For any $p \geq 0$, define the *Hodge-Laplace operator* $\Delta_p : \Omega_p \to \Omega_p$ by

$$\Delta_p u = \partial^* \partial u + \partial \partial^* u. \tag{6.1}$$

Then Δ_p is a self-adjoint and non-negative definite operator in Ω_p.

A path $u \in \Omega_p$ is called *harmonic* if $\Delta_p u = 0$. It is easy to prove that a path $u \in \Omega_p$ is harmonic if and only if $\partial u = 0$ and $\partial^* u = 0$. Denote by $\mathcal{H}_p$ the space of all harmonic p-path in Ω_p.

Theorem 6.2 (Hodge decomposition) *The space Ω_p is an orthogonal sum:*

$$\Omega_p = \partial \Omega_{p+1} \bigoplus \partial^* \Omega_{p-1} \bigoplus \mathcal{H}_p. \tag{6.2}$$

Consequently, we obtain a natural linear isomorphism $H_p \cong \mathcal{H}_p$ between the homology group H_p and the space $\mathcal{H}_p$. In particular, $\dim \mathcal{H}_p = \beta_p$; that is, the multiplicity of 0 as an eigenvalue of Δ_p is equal to the Betti number β_p.

Theorem 6.3 *Let a digraph G contains T triangles, D double arrows, and let the maximal number of linearly independent squares be S. Then*

$$\operatorname{trace} \Delta_1 = 2E + 3T + 2S + 4D. \tag{6.3}$$

Note for comparison that $\operatorname{trace} \Delta_0 = 2E$. It may be interesting to find a similar formula for $\operatorname{trace} \Delta_p$ for $p > 1$.

For any finite-dimensional self-adjoint operator A denote by $\operatorname{spec} A$ the spectrum of A, that is, an unordered sequence of eigenvalues counted with the multiplicities. This sequence can be denoted either by $\{\lambda_i\}$ or by $\{(\alpha_i)_{m_i}\}$ where m_i is the multiplicity of α_i. Observe that the problem of determination of the spectrum of A amounts to computation of the dimensions of eigenspaces $\{Ax = \lambda x\}$ for all $\lambda \in \mathbb{R}$, that is, to the multiplicities of all real λ.

We will use the operation *disjoint union* of unordered sequences: $\{\lambda_i\} \sqcup \{\mu_j\} = \{\lambda_i, \mu_j\}$. If the same value λ occurs in the both sequences, then its multiplicities add up in the disjoint union.

6.2 Hodge spectra of the products

The following theorems contain the results of computation of spectra of the Hodge Laplacian Δ_p on n-cube and n-torus.

Theorem 6.4 *For all $1 \leq p \leq n$ we have*

$$\operatorname{spec}\Delta_p(I^n) = \left\{ \left(\frac{2k}{p}\right)_{\binom{n}{k}\binom{k-1}{p-1}} \right\}_{k=p}^{n} \sqcup \left\{ \left(\frac{2k}{p+1}\right)_{\binom{n}{k}\binom{k-1}{p}} \right\}_{k=p+1}^{n}. \tag{6.4}$$

In particular,

$$\lambda_{\max}\left(\Delta_p(I^n)\right) = \left(\frac{2n}{p}\right)_{\binom{n-1}{p-1}} \quad \text{and} \quad \lambda_{\min}\left(\Delta_p(I^n)\right) = 2_{\binom{n+1}{p+1}}.$$

Theorem 6.5 *For all $1 \leq p \leq n$ we have*

$$\operatorname{spec}\Delta_p(T^n) = \left\{ \left(\frac{3k}{p}\right)_{2^k\binom{n}{k}\binom{n-1}{p-1}} \right\}_{k=0}^{n} \sqcup \left\{ \left(\frac{3k}{p+1}\right)_{2^k\binom{n}{k}\binom{n-1}{p}} \right\}_{k=0}^{n}. \tag{6.5}$$

In particular,

$$\lambda_{\max}\left(\Delta_p(T^n)\right) = \left(\frac{3n}{p}\right)_{2^n\binom{n-1}{p-1}} \quad \text{and} \quad \lambda_{\min}\left(\Delta_p(T^n)\right) = 0_{\binom{n}{p}}.$$

6.3 Hodge spectra on the joins

Let D_m denote the digraph that consists of $m \geq 1$ disjoint vertices and no arrows, that is,

$$D_m = \{ \underbrace{\bullet, ..., \bullet}_{n \text{ vertices}} \}.$$

Consider for any $n \geq 1$ the digraph

$$D_m^n = \underbrace{D_m * ... * D_m}_{n \text{ times}}.$$

Theorem 6.6 *We have, for all $n, m \geq 1$ and $r \geq 2$,*

$$\operatorname{spec}\Delta_{r-1}(D_m^n) = \left\{ ((n-k)m)_{(m-1)^k\binom{r}{k}\binom{n}{r}} \right\}_{k=0}^{r}. \tag{6.6}$$

More explicitly, (6.6) means the following: if $n < r$ then $\operatorname{spec}\Delta_{r-1}(D_m^n) = \emptyset$, while for $n \geq r$ the spectrum of $\Delta_{r-1}(D_m^n)$ consists of the following $r+1$ eigenvalues

$$(n-r)m,\ (n-r+1)m,\ (n-r+2)m, ..., (n-1)m,\ nm, \tag{6.7}$$

with the multiplicities

$$(m-1)^r\binom{n}{r},\quad (m-1)^{r-1}r\binom{n}{r},\ (m-1)^{r-2}\binom{r}{2}\binom{n}{r}, ..., (m-1)r\binom{n}{r},\ \binom{n}{r}. \tag{6.8}$$

Example 6.7 Let $m = 1$, that is, $D_1 = \{\bullet\}$. Clearly, D_1^n coincides with a complete digraph K_n. In this case all the multiplicities in (6.8) are 0 except for the last one $\binom{n}{r}$. Hence, $\operatorname{spec}\Delta_{r-1}(K_n)$ consists of a single eigenvalue n with the multiplicity $\binom{n}{r}$.

Example 6.8 Let $m = 2$, that is, $D_2 = \{\bullet, \bullet\}$. Then D_2^n coincides with a digraph sphere S^{n-1}. In this case (6.6) becomes

$$\operatorname{spec}\Delta_{r-1}(S^{n-1}) = \left\{(2(n-k))_{\binom{r}{k}\binom{n}{r}}\right\}_{k=0}^{r}.$$

Example 6.9 Let $m = 3$ and $n = 2$. Then we have $D_3^2 = K_{3,3}$ that is a complete bipartite digraph.

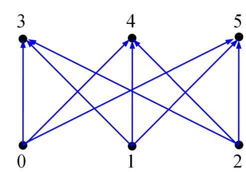

Figure 10: Digraph $K_{3,3}$

Then (6.6) yields for $r = 2$ that

$$\operatorname{spec}\Delta_1(K_{3,3}) = \left\{(3(2-k))_{\binom{2}{k}\binom{2}{2}2^k}\right\}_{k=0}^{2} = \{0_4,\, 3_4,\, 6\}.$$

7 Torsion

Let us fix a positive integer N and denote by Ω the following truncated version of the chain complex (2.3) over $\mathbb{R}$:

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \cdots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \cdots \overset{\partial}{\leftarrow} \Omega_N \leftarrow 0. \tag{7.1}$$

7.1 Reidemeister torsion

Denote $B_p = \partial\Omega_{p+1}$, $Z_p = \ker \partial|_{\Omega_p}$ and $H_p = Z_p/B_p$. In any $p = 0, ..., N$, choose a basis ω_p in Ω_p and a basis h_p in H_p. For each element of h_p choose its representative in Z_p and denote the resulting independent set by $\widetilde{h}_p$.

Let b_p be *any* basis in B_p. For each element $w \in b_{p-1}$ choose one element $v \in \partial^{-1}w \subset \Omega_p$ so that $\partial v = w$. Let $\widetilde{b}_p$ be the collection of chosen elements v so that

$$b_{p-1} = \partial\widetilde{b}_p. \tag{7.2}$$

Note that always $\widetilde{b}_0 = \emptyset$. Since b_{p-1} is linearly independent, the set $\widetilde{b}_p$ is also linearly independent. Clearly, the union $(b_p, \widetilde{h}_p)$ is a basis in Z_p. Since the subspaces Z_p and $\operatorname{span}(\widetilde{b}_p)$ of Ω_p have a trivial intersection $\{0\}$, by the rank-nullity theorem we conclude that the direct sum of these subspaces is Ω_p. Hence, the union $(b_p, \widetilde{h}_p, \widetilde{b}_p)$ of the these sequences is a basis in Ω_p.

If U and W are two bases in an n-dimensional linear space, then denote by (U/W) the transformation matrix from W to U and set

$$[U/W] = |\det (U/W)| .$$

In the case $n = 0$ set $[U/W] = 1$.

Denote ω the collection $\{\omega_p\}$ of the bases in Ω_p and similarly let $h = \{h_p\}$ be the collection of the bases in H_p.

Definition 7.1 The *R-torsion* $\tau(\Omega, \omega, h)$ of the chain complex Ω with the preferred bases ω and h is a positive real number defined by

$$\log \tau(\Omega, \omega, h) = \sum_{p=0}^{N} (-1)^p \log[b_p, \widetilde{h}_p, \widetilde{b}_p \,/\, \omega_p]. \tag{7.3}$$

One can prove that the value of $\tau\,(\Omega, \omega, h)$ does not depend on the choice of the bases b_p, the representatives in $\widetilde{b}_p$ and the representatives in $\widetilde{h}_p$ (which justifies the notation $\tau(\Omega, \omega, h)$).

Now let us fix an inner product $\langle\cdot, \cdot\rangle$ in each Ω_p. Then we have the induced inner product in the subspaces B_p, Z_p and $\mathcal{H}_p$. Using the natural isomorphism $H_p \cong \mathcal{H}_p$ we transfer the inner product to H_p. Hence, in this case we choose orthonormal bases ω_p in Ω_p and h_p in H_p. In fact, we can identify h_p with an orthonormal basis in $\mathcal{H}_p$ and set $\widetilde{h}_p = h_p$. One can show that the torsion $\tau\,(\Omega, \omega, h)$ does not depend on the choice of orthonormal bases ω and h.

Hence, with this choice of ω and h, we define the R-torsion of Ω by

$$\tau(\Omega) = \tau\,(\Omega, \omega, h) .$$

7.2 Analytic torsion

Denote by $\{\lambda_i\}_{i=1}^{\dim \Omega_p}$ the sequence of all the eigenvalues of Δ_p. The zeta function $\zeta_p\,(s)$ of Δ_p is defined by

$$\zeta_p(s) = \sum_{\lambda_i > 0} \frac{1}{\lambda_i^s}.$$

Definition 7.2 The *analytic torsion* $T(\Omega)$ of the chain complex Ω is defined by

$$\log T(\Omega) = \frac{1}{2} \sum_{p=0}^{N} (-1)^p \, p \, \zeta_p'(0). \tag{7.4}$$

The next theorem is the main results of this section.

Theorem 7.3 *We have* $\tau(\Omega) = T\,(\Omega)$.

Example 7.4 One can show that, for a complete digraph K_n,

$$T(K_n) = \sqrt{n},$$

and for the n-cube I^n

$$T(I^n) = 2^{n/2} \prod_{p=2}^{n} (p!)^{\frac{1}{2}(-1)^p 2^{n-p}\binom{n}{p}}.$$

In particular, $T(K_3) = \sqrt{3}$ and $T(I^2) = 2\sqrt{2}$. Note that all K_n and I^n are homologically and even homotopically equivalent (see Section 5.3) while their torsions are different.

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