Off-diagonal esitmates of heat kernels for jump processes

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Contents

1 Stable-like estimates of the heat kernel

Let (M,d) be a locally compact separable metric space and μ be a Radon measure with full support on M. Let $(\mathcal{E},\mathcal{F})$ be a regular jump type Dirichlet form with a symmetric jump kernel $J(x,y)$, that is,

$$\mathcal{E}(f,f)=\int\int_{M\times M}(f(x)-f(y))^2 J(x,y)d\mu(x)d\mu(y).$$

Let $\{P_t\}$ denote the associated heat semigroup, that is, $P_t=e^{-t\mathcal{L}}$, where $\mathcal{L}$ is the generator of $(\mathcal{E},\mathcal{F})$, and $p_t(x,y)$ be the heat kernel, that is, the integral kernel of P_t, should it exist.

For example, if $M=\mathbb{R}^n$ and

$$J(x,y)=\frac{c}{|x-y|^{n+\beta}}$$

where $0<\beta<2$, then $\mathcal{L}=(-\Delta)^{\beta/2}$ (that generates a symmetric stable process of index β), and

$$p_t(x,y)\simeq\frac{1}{t^{n/\beta}}\left(1+\frac{|x-y|}{t^{1/\beta}}\right)^{-(n+\beta)}. \tag{1}$$

Let now (M, d, μ) be a general metric measure space. Denote by $B(x, r)$ open metric balls in M. We assume always that μ is α-regular for some $\alpha > 0$, that is, for all $x \in M$ and $r > 0$,

$$\mu(B(x, r)) \simeq r^{\alpha} \tag{V}$$

(although the main results are available also in the setting of a doubling measure).

By a result of AG and T.Kumagai (2008), if the heat kernel of a jump type Dirichlet form on $L^2(M, \mu)$ satisfies a self-similar estimate

$$p_t(x, y) \simeq t^{-\gamma} \Phi\left(\frac{d(x, y)}{t^{1/\beta}}\right)$$

for some $\beta, \gamma > 0$ and some function Φ, then it is necessarily the following estimate:

$$p_t(x, y) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{2}$$

We refer to (2) as a *stable-like* estimate of the heat kernel because of its similarity to (1).

What is known about actual validity of (2) on general metric spaces?

Z.-Q. Chen and T.Kumagai proved in *Stoch.Process.Appl.* **108** (2003) that if $\beta < 2$ then (2) is equivalent to the condition

$$J(x,y) \simeq d(x,y)^{-(\alpha+\beta)} \quad \text{for all } x,y \in M. \tag{J}$$

However, on most of fractal sets there exist regular Dirichlet forms with the jump kernel satisfying (J) with $\beta > 2$. In this case one needs one more condition: a *generalized capacity* condition denoted shortly by $(Gcap)$ that will be explained below.

Condition $(Gcap)$ is closely related to the *cutoff Sobolev inequality* introduced by M.Barlow and R.Bass in *Trans.AMS* **356** (2004), and to the *energy inequality* of S.Andres and M.Barlow in *J.Reine Angew.Math.* **699** (2015).

With help of this condition, the following result was proved in AG, E.Hu, J.Hu, *Adv.Math.* **330** (2018) and in a more general setting – in Z.-Q.Chen, T.Kumagai, J.Wang, *Adv.Math.* **374** (2020).

Theorem 1 *Under the standing assumption* (V) *we have, for any* $\beta > 0$,

$$(Gcap) + (J) \Leftrightarrow (2). \tag{3}$$

2 Question about upper bounds of the heat kernel

The main question to be discussed here is how to obtain the estimates of the form

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\gamma}, \tag{4}$$

for all $t > 0$ and almost all $x, y \in M$, with some $\gamma > 0$. Using the identity,

$$J(x,y) = \lim_{t\to 0} \frac{p_t(x,y)}{2t}, \tag{5}$$

it is easy to obtain the following:
- if $\gamma > \alpha + \beta$ then $J(x,y) = 0$ (which excludes this case);
- if $\gamma = \alpha + \beta$ then

$$J(x,y) \le \frac{C}{d(x,y)^{\alpha+\beta}}; \tag{$J_\le$}$$

- and if $\gamma < \alpha + \beta$ then no useful bound for J.

It is natural to expect the opposite implication: if ($J_\le$) holds then (4) holds with $\gamma = \alpha+\beta$. However, one needs for that additional conditions.

3 Faber-Krahn inequality

Definition. We say that the *Faber-Krahn inequality* (FK) of index β holds if, for any precompact open set $\Omega \subset M$,

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-\beta/\alpha}, \tag{FK}$$

where $\lambda_1(\Omega) = \inf \operatorname{spec}\left(\mathcal{L}^{\Omega}\right)$.

Or, equivalently, (FK) holds if

$$\inf_{\varphi \in \mathcal{F} \cap C_0(\Omega) \setminus \{0\}} \frac{\mathcal{E}(\varphi, \varphi)}{\|\varphi\|_{L^2}^2} \geq c\mu(\Omega)^{-\beta/\alpha}.$$

It is known that (FK) is equivalent to the *diagonal upper estimate* of the heat kernel

$$\boxed{p_t(x,y) \leq Ct^{-\alpha/\beta}}. \tag{DUE}$$

It is also known that

$$J(x,y) \geq \frac{c}{d(x,y)^{\alpha+\beta}} \quad \Rightarrow \quad (FK). \tag{6}$$

In some sense, (FK) can be regarded as an integral version of the pointwise lower bound of J.

The following result was proved by A.Bendikov, AG, E.Hu, J.Hu in *Ann. Scuola Norm. Sup. Pisa* **22** (2021) in the setting of *ultra-metric* spaces.

Recall that (M, d) is called an ultra-metric space if d satisfies the ultra-metric triangle inequality

$$d(x, y) \leq \max(d(x, z), d(y, z)) \quad \forall x, y, z \in M.$$

For example, the field $\mathbb{Q}_p$ of p-adic numbers is an ultra-metric space with respect to the p-adic norm. Also, $\mathbb{Q}_p^n$ is an ultra-metric space with respect to l^∞ product distance.

Theorem 2 *Let (M, d) be an ultra-metric space. Assume that the following three hypotheses are satisfied: (V), (FK) and the tail estimate of the jump kernel:*

$$\boxed{\int_{B^c(x,r)} J(x, y) d\mu(y) \leq \frac{C}{r^\beta}}, \tag{TJ}$$

for all $x \in M$ and $r > 0$. Then

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-\beta}.$$

4 Condition $(Gcap)$

Let us now state $(Gcap)$. Recall that the *capacity* associated with $(\mathcal{E}, \mathcal{F})$ is defined as follows: for any open set $U \subset M$ and a Borel set $A \subset U$ set

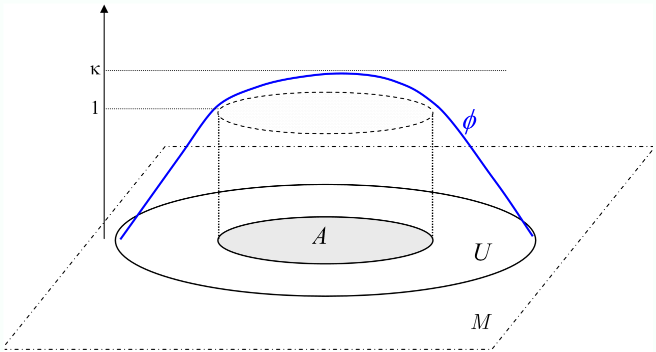
$$\operatorname{cap}(A, U) = \inf \left\{ \mathcal{E}\left(\phi, \phi\right) : \phi \in \mathcal{F},\ 0 \leq \phi \leq 1,\ \phi|_A = 1,\ \phi|_{U^c} = 0 \right\}.$$

Definition. For any bounded function $u \in \mathcal{F} + \text{const}$ and a real $\kappa \geq 1$, define the *generalized capacity* of the pair (A, U) by

$$\operatorname{cap}_u^{(\kappa)}(A, U) = \inf_{\phi} \mathcal{E}\left(u^2\phi, \phi\right)$$

where inf is taken over all $\phi \in \mathcal{F}$ such that

$$0 \leq \phi \leq \kappa, \qquad \phi|_A \geq 1, \qquad \phi|_{U^c} = 0.$$



For example, if $\kappa = 1$ and $u \equiv 1$ then

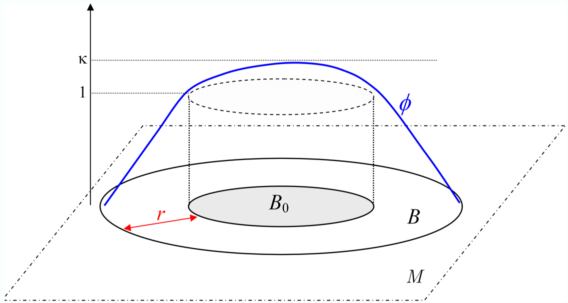
$$\operatorname{cap}_u^{(\kappa)}(A, U) = \operatorname{cap}(A, U).$$

Definition. We say that the *generalized capacity* condition ($Gcap$) of index β is satisfied if there exist two constants $\kappa \geq 1, C > 0$ such that, for any bounded function $u \in \mathcal{F}$+const and for all concentric balls $B_0 := B(x, R)$, $B := B(x, R + r)$ with $x \in M$ and $R, r > 0$,

$$\operatorname{cap}_u^{(\kappa)}(B_0, B) \leq \frac{C}{r^\beta} \int_B u^2 d\mu. \tag{$Gcap$}$$

Equivalently, this condition means that, for any pair of concentric balls B_0, B as above and for any bounded $u \in \mathcal{F} + \text{const}$,

there exists $\phi \in \mathcal{F}$ such that



$$0 \leq \phi \leq \kappa, \quad \phi|_{B_0} \geq 1, \quad \phi|_{B^c} = 0$$

and

$$\mathcal{E}\left(u^2\phi, \phi\right) \leq \frac{C}{r^\beta} \int_B u^2 d\mu. \tag{7}$$

Applying (7) with $u \equiv 1$ and replacing ϕ with $\phi \wedge 1$, we obtain the *capacity* condition:

$$\operatorname{cap}(B_0, B) \leq \frac{C}{r^\beta} \mu(B). \tag{cap}$$

Usually it is very difficult to verify ($Gcap$) (apart from some specific cases like ultra-metric spaces), and it is an open problem to develop methods for verification of ($Gcap$).

It would ideal if in all our results ($Gcap$) could be replaced by a much simpler condition (cap), but so far there is no technique for that.

Now we can state a result about the upper estimate (4) with $\gamma = \alpha + \beta$, that is, about

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{UE}$$

Denote by (C) the hypothesis that $(\mathcal{E}, \mathcal{F})$ is *conservative*, that is, $P_t 1 = 1$. Recall also the hypothesis

$$J(x,y) \le \frac{C}{d(x,y)^{\alpha+\beta}}. \tag{$J_\le$}$$

The following theorem can be extracted from the results of AG, J.Hu, K.-S.Lau K.-S. $Trans.AMS$ **366** (2014) and Z.-Q.Chen, T.Kumagai, J.Wang, $Mem.AMS$ **271** (2021).

Theorem 3 *Under the hypothesis* (V) *we have*

$$(FK) + (Gcap) + (J_\le) \Leftrightarrow (UE) + (C)\,.$$

Or, if we take (FK), $(Gcap)$ *and* (C) *as standing assumptions, then* $(J_\le) \Leftrightarrow (UE)$.

5 Strong generalized capacity condition and elliptic mean value inequality

In the definition of $(Gcap)$ in the inequality (7), that is, in

$$\mathcal{E}\left(u^2\phi,\phi\right)\leq\frac{C}{r^\beta}\int_B u^2 d\mu,$$

the cutoff function ϕ may depend on the weight u. Denote by $(Gcap')$ a stronger version of $(Gcap)$ when function ϕ depends only on the pair B_0, B of the balls and serves all functions u simultaneously.

The next theorem is the first one in a sequence of results leading to heat kernel upper bounds. We use again the following hypothesis about the tail of the jump kernel:

$$\boxed{\|J(x,\cdot)\|_{L^1(B^c(x,r))}\leq\frac{C}{r^\beta}},\quad \forall x\in M,\ r>0. \tag{TJ}$$

Theorem 4 (AG, E.Hu, J.Hu *J. Pure Appl. Funct. Anal.* (2023)) *Under the hypothesis* (V), *we have the implication*

$$(FK)+(Gcap)+(TJ)\Rightarrow(Gcap')\,.$$

The condition $(Gcap')$ will be used below for obtaining the *parabolic mean value inequality* that in turn is needed for heat kernel upper bounds. The proof of Theorem 4 uses the *elliptic mean value inequality* (EMV).

Definition. We say that (EMV) holds if, for any function $u \in \mathcal{F} \cap L^\infty$ that is non-negative and subharmonic in a ball $B = B(x_0, R)$, and for any $\varepsilon > 0$,

$$\operatorname*{esup}_{\frac{1}{2}B} u \leq C_\varepsilon \left(\fint_B u^2 \right)^{1/2} + \varepsilon \left\| u_+ \right\|_{L^\infty\left(\left(\frac{1}{2}B\right)^c\right)}.$$

The proof of Theorem 4 goes through the following implications (under the standing assumptions (V), (FK), (TJ)):

$$(Gcap) \Rightarrow (EMV) + (cap) \Rightarrow (Gcap').$$

6 Parabolic mean value inequality

Fix some $\beta > 0$, $q \in [1, \infty]$ and consider the following hypothesis for the tail of J:

$$\boxed{\|J(x,\cdot)\|_{L^q(B^c(x,r))} \le \frac{C}{r^{\alpha/q'+\beta}}}, \tag{TJ_q}$$

for all $x \in M$ and $r > 0$, where $q' = \frac{q}{q-1}$ is the Hölder conjugate of q.

It is easy to see that (TJ_q) becomes stronger when q increases.

For example, if $q = 1$ then $q' = \infty$ and (TJ_q) becomes

$$\int_{B^c(x,r)} J(x,y)d\mu(y) \le \frac{C}{r^\beta}, \tag{TJ_1}$$

which coincides with (TJ).

If $q = 2$ then $q' = 2$ and (TJ_q) becomes

$$\left(\int_{B^c(x,r)} J^2(x,y)d\mu(y)\right)^{1/2} \le \frac{C}{r^{\alpha/2+\beta}}. \tag{TJ_2}$$

If $q=\infty$ then $q'=1$ and (TJ_q) becomes

$$\operatorname*{essup}_{y\in B^c(x,r)} J(x,y) \leq \frac{C}{r^{\alpha+\beta}}, \tag{TJ_∞}$$

which is equivalent to $(J_\leq)$.

Theorem 5 (AG, E.Hu, J.Hu, preprint 2022) *For any* $q\in[1,\infty]$, *we have*

$$(FK)+(Gcap')+(TJ_q)\Rightarrow(PMV_q),$$

where (PMV_q) *stands for the* Parabolic Mean Value *inequality that means the following.* Fix an arbitrary ball $B=B(x,R)$ in M and set $T=R^\beta$. Let u be a bounded non-negative function on $M\times(0,T]$ that is *subcaloric* in the cylinder $B\times(0,T]$:

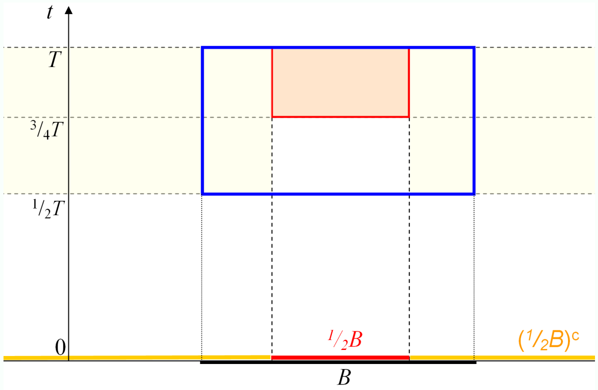
that is, for any $t\in(0,T]$,

$$u(\cdot,t)\in\mathcal{F}_+\cap L^\infty(M)$$

and u satisfies in $B\times(0,T]$

$$\partial_t u+\mathcal{L}u\leq 0$$

in a certain weak sense.



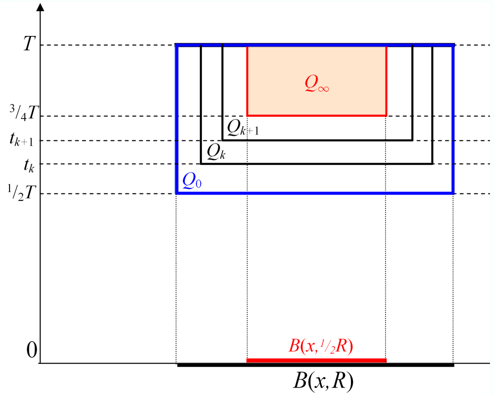
Then, for any $\varepsilon\in(0,1]$,

$$\sup_{t\in[\frac{3}{4}T,T]} \|u(\cdot,t)\|_{L^\infty(\frac{1}{2}B)} \leq C_\varepsilon \left(\fint_{B\times[\frac{1}{2}T,T]} u^2 \right)^{1/2} + \frac{\varepsilon}{R^{\alpha/q'}} \sup_{t\in[\frac{1}{2}T,T]} \|u(\cdot,t)\|_{L^{q'}((\frac{1}{2}B)^c)}. \quad (PMV_q)$$

For the proof, consider a shrinking sequence of cylinders $Q_k = B(x,r_k) \times [t_k, T]$, $k \geq 0$, and an increasing sequence $b_k > 0$. Set

$$a_k := \int_{Q_k} (u - b_k)_+^2 \, d\mu dt$$

so that a_k clearly decreases, and prove that



$$a_{k+1} \leq \frac{C}{(b_{k+1}-b_k)^{2\frac{\beta}{\alpha}}} \left(\frac{r_k}{r_k - r_{k+1}} \right)^C \left(\frac{1}{(r_k - r_{k+1})^\beta} + \frac{1}{t_{k+1} - t_k} + \frac{s_k}{b_{k+1} - b_k} \right)^{1+\frac{\beta}{\alpha}} a_k^{1+\frac{\beta}{\alpha}},$$

where

$$s_k = \sup_{t\in[t_k,T]} \operatorname*{essup}_{z\in B(x,\frac{r_k+r_{k+1}}{2})} \int_{B^c(x,r_k)} u(y,t) J(z,y) d\mu(y).$$

The proof of the relation between a_k and a_{k+1} uses essentially (FK) and $(Gcap')$.

Choose

$$r_k = (\tfrac{1}{2} + 2^{-k-1})R \quad \text{and} \quad t_k = (\tfrac{3}{4} - 2^{-\beta k-2})T,$$

so that

$$B \times [\tfrac{1}{2}T, T] = Q_0 \supset Q_k \supset Q_\infty = \tfrac{1}{2}B \times [\tfrac{3}{4}T, T].$$

Setting also $b_k = \left(1 - 2^{-k}\right) b$ for some $b > 0$, we obtain

$$a_{k+1} \le C 2^{Ck} \left(1 + \frac{R^\beta s_k}{b}\right)^{1+\frac{\beta}{\alpha}} \frac{a_k^{1+\frac{\beta}{\alpha}}}{(R^{\alpha+\beta} b^2)^{\frac{\beta}{\alpha}}}. \tag{8}$$

Iterating (8), we show that, for a large enough b,

$$\lim_{k\to\infty} a_k = 0,$$

which implies that

$$u \le b \ \text{ in } \ Q_\infty.$$

The choice of b depends on $\sup_k \frac{a_k}{R^{\alpha+\beta}} = \frac{a_o}{R^{\alpha+\beta}}$ and on an upper bound for $R^\beta s_k$. The value

$$\frac{a_0}{R^{\alpha+\beta}} \le \text{const} \fint_{B\times[\frac{1}{2}T,T]} u^2$$

yields the first term (PMV_q). Estimating s_k by means of the Hölder inequality and (TP_q) gives

$$\begin{aligned} R^{\beta} s_k &\leq R^{\beta} \sup_{t\in[\frac{1}{2}T,T]} \left\| u\left(\cdot,t\right)\right\|_{L^{q'}\left(\frac{1}{2}B\right)^c} \frac{C}{\left(r_k - r_{k+1}\right)^{\alpha/q'+\beta}} \\ &= \frac{C2^{Ck}}{R^{\alpha/q'}} \sup_{t\in[\frac{1}{2}T,T]} \left\| u\left(\cdot,t\right)\right\|_{L^{q'}\left(\frac{1}{2}B\right)^c} \end{aligned}$$

which yields the second term in (PMV_q).

7 Tail of the heat kernel

Consider the following hypotheses about the tail of the heat kernel $p_t(x,y)$:

$$\boxed{\|p_t(x,\cdot)\|_{L^q(B^c(x,r))} \le \frac{C}{t^{\alpha/(q'\beta)}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}} \simeq \frac{1}{t^{\alpha/(q'\beta)}} \wedge \frac{t}{r^{\alpha/q'+\beta}}, \tag{TP_q}$$

for all $x \in M$ and $r > 0$. By (5), we have

$$(TP_q) \Rightarrow (TJ_q). \tag{9}$$

The condition (TP_q) gets stronger when q increases. For $q=1$, we have

$$\int_{B^c(x,r)} p_t(x,y)d\mu(y) \le C\frac{t}{r^\beta}, \tag{TP_1}$$

for $q=2$, we have

$$\int_{B^c(x,r)} p_t^2(x,y)d\mu(y) \le \frac{C}{t^{\alpha/(2\beta)}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha/2+\beta)}, \tag{TP_2}$$

for $q=\infty$, (TP_∞) is equivalent (UE) as it becomes

$$\operatorname*{esssup}_{y\in B^c(x,r)} p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{TP_∞}$$

Consider also the following family of off-diagonal *upper estimates* of the heat kernel:

$$\boxed{p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}}, \tag{UE_q}$$

for all $t > 0$ and almost all $x, y \in M$. For example, for $q = \infty$ we have

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \tag{UE_∞}$$

which coincides with (UE) and (TP_∞).

For $q = 1$ we have a weaker estimate

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta}. \tag{UE_1}$$

Now we can state our main result.

Theorem 6 (AG, E.Hu, J.Hu, preprint, 2023) *Under the standing assumption* (V), *we have, for any* $q \in [2, \infty]$, *the following equivalence/implication:*

$$(FK) + (Gcap) + (TJ_q) \Leftrightarrow (TP_q) + (C) \Rightarrow (UE_q).$$

Or, considering (FK), $(Gcap)$, (C) as standing assumptions, we have

$$\boxed{(TJ_q) \Leftrightarrow (TP_q) \Rightarrow (UE_q)}.$$

The case $q = \infty$ coincides with the equivalence $(J_{\leq}) \Leftrightarrow (UE)$ of Theorem 3 while the case $q \in [2, \infty)$ is completely new. The range $q \in [1, 2)$ is not covered by Theorem 6.

However, if M is an ultra-metric space then a part of the above results is true for $q = 1$ because by Theorem 2 we have in this case

$$(FK) + (TJ_1) \Rightarrow (UE_1)\,.$$

In this setting there is an example of a jump kernel satisfying

$$J(x, y) \geq \frac{c}{d(x, y)^{\alpha+\beta}} \tag{$J_{\geq}$}$$

(which implies (FK)) and (TJ_1) where the upper bound (UE_1) is optimal in the sense that the exponent $-\beta$ cannot be replaced by $-(\beta + \varepsilon)$ for any $\varepsilon > 0$.

In the general case, replacing (FK) with a stronger condition $(J_{\geq})$ allows to obtain also a lower bound of the heat kernel.

Theorem 7 (AG, E.Hu, J.Hu, preprint, 2023) *Under the standing assumption* (V), *we have, for any* $q \in [2, \infty]$,

$$(J_{\geq}) + (Gcap) + (TJ_q) \Leftrightarrow (TP_q) + (LE) \Rightarrow (UE_q) + (LE),$$

where (LE) *stands for the lower estimate*

$$p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{LE}$$

That is, in this case the heat kernel satisfies the two-sides bounds:

$$\frac{c}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \leq p_t\,(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}.$$

8 Outline of the proof of Theorem 6

The most part of the proof is devoted to the implication

$$(FK) + (Gcap) + (TJ_q) \Rightarrow (TP_q)$$

Step 0. As it was already mentioned above,

$$(FK) \Rightarrow (DUE).$$

However, in our proof we do not use this implication because we work in a more general setting of doubling spaces where this result is unavailable. We use an alternative proof of (DUE) with help of the mean value inequality.

Step 1. By Theorem 4, we have

$$(FK) + (Gcap) + (TJ) \Rightarrow (Gcap'),$$

and, by Theorem 5,

$$(FK) + (Gcap') + (TJ_q) \Rightarrow (PMV_q).$$

Step 2. We prove that

$$(PMV_2) \Rightarrow (DUE)\,.$$

For that apply (PMV_2) with $u\,(\cdot,t) = P_t f$ where $f \in C_0(M)$ and $f \geq 0$, and observe that the both terms in the right hand side of (PMV_q) are bounded by $\frac{C}{R^{a/2}}\,\|f\|_{L^2}$ which yields

$$\|P_T f\|_\infty \leq \frac{C}{T^{\alpha/(2\beta)}}\,\|f\|_2\,,$$

which then implies (DUE). Consequently, we obtain that, for any $q \in [2,\infty]$,

$$(FK) + (Gcap) + (TJ_q) \Rightarrow (DUE)\,.$$

It follows from (DUE) that

$$\|p_t\,(x,\cdot)\|_{L^q(M)} \leq \frac{C}{t^{\alpha/(q'\beta)}}.$$

Hence, in order to prove (TP_q), it remains to prove

$$\boxed{\|p_t\,(x,\cdot)\|_{L^q(B^c(x,r))} \leq \frac{Ct}{r^{\alpha/q'+\beta}}} \tag{10}$$

assuming that $r^\beta \geq t$, which is done in the rest of the proof.

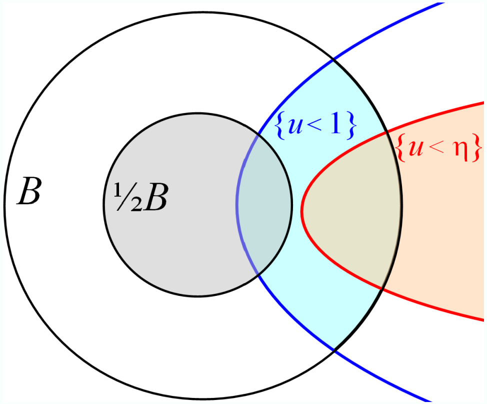
Step 3. We deduce from (PMV_1) a so called "Lemma of growth":

there exist some $\varepsilon, \eta \in (0,1)$ such that, for any ball $B \subset M$ and for any $u \in \mathcal{F}$ that is non-negative and bounded in M and superharmonic in B, if

$$\frac{\mu(B \cap \{u < 1\})}{\mu(B)} \le \varepsilon,$$

then

$$\operatorname*{essinf}_{\frac{1}{2}B} u \ge \eta.$$



For that observe that $v = \frac{1}{u+a}$ is subharmonic for any $a > 0$. For subharmonic functions, we obtain from (PMV_1) the following multiplicative form of the mean value inequality (by choosing ε):

$$\|v\|_{L^\infty(\frac{1}{2}B)} \le C A^\theta \max(A, T)^{1-\theta}, \tag{11}$$

where

$$A = \left(\fint_B v^2 d\mu \right)^{1/2}, \qquad T = \|v\|_{L^\infty((\frac{1}{2}B)^c)},$$

and $\theta = \theta(\alpha, \beta) \in (0,1)$.

Let us estimate A as follows:

$$A^2 = \frac{1}{\mu(B)} \left(\int_{B \cap u<1\}} + \int_{B \cap \{u \geq 1\}} \right) \frac{d\mu}{(u+a)^2}$$
$$\leq \frac{\mu(B \cap \{u<1\})}{\mu(B)} \frac{1}{a^2} + \frac{1}{(1+a)^2} \leq \frac{\varepsilon}{a^2} + \frac{1}{(1+a)^2} = \frac{2}{(1+a)^2},$$

for $a = \frac{1}{\varepsilon^{-1/2}-1}$. Estimating also trivially

$$\max(A, T) \leq \frac{1}{a},$$

we obtain from (11)

$$\operatorname*{essup}_{\frac{1}{2}B} \frac{1}{u+a} \leq C \left(\frac{2}{(1+a)^2} \right)^{\theta/2} \left(\frac{1}{a} \right)^{1-\theta} = \frac{C}{(1+a)^{\theta} a^{1-\theta}},$$

whence

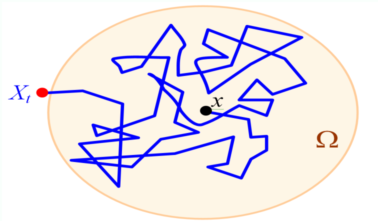
$$\operatorname*{essinf}_{\frac{1}{2}B} u \geq C^{-1} (1+a)^{\theta} a^{1-\theta} - a = a \left(C^{-1} \left(\tfrac{1}{a} + 1 \right)^{\theta} - 1 \right) =: \eta,$$

where $\eta > 0$ if a is small enough, that is, when ε is small enough.

Step 4. For any open set $\Omega \subset M$ and any $x \in \Omega$ set

$$E^{\Omega}(x) = \int_0^{\infty} P_t^{\Omega}\mathbf{1}(x)dt = \int_0^{\infty}\int_{\Omega} p_t^{\Omega}(x,y)d\mu(y)dt.$$

It has the probabilistic meaning of the *mean exit time* from Ω of the jump process X_t, associated with $(\mathcal{E},\mathcal{F})$, that starts at x: $E^{\Omega}(x) = \mathbb{E}_x(\tau^{\Omega})$, where τ^{Ω} is the first exit time from Ω.



In this step we prove that, under (FK), for any ball B of radius r,

$$\operatorname*{essup}_{B} E^{B} \le Cr^{\beta}. \tag{12}$$

Step 5. We prove the opposite inequality: the Lemma of growth and (cap) imply that

$$\operatorname*{essinf}_{\frac{1}{4}B} E^{B} \ge cr^{\beta}. \tag{13}$$

It is known that (12) and (13) imply (C).

Step 6. Using the upper and lower estimates of E^B, we deduce the *survival* inequality: there exist $\varepsilon > 0$ such that, for any ball B of radius r and for any $t > 0$,

$$P_t^B \mathbf{1}_B \geq \varepsilon - \frac{Ct}{r^\beta} \text{ in } \tfrac{1}{4}B. \tag{S}$$

In probabilistic terms,

$$P_t^B \mathbf{1}_B(x) = \mathbb{P}_x(\tau_B > t)$$

that is the probability of survival of the process in B up to time t assuming the killing condition in B^c.

Step 7. For any $\rho > 0$ consider a *truncated* Dirichlet form

$$\mathcal{E}^{(\rho)}(f,f) := \iint_{\{d(x,y)<\rho\}} (f(x) - f(y))^2 J(x,y) d(x) d\mu(y).$$

Denote by Q_t the heat semigroup of $\left(\mathcal{E}^{(\rho)}, \mathcal{F}\right)$ and by $q_t(x,y)$ its heat kernel. We prove that, under all the above hypotheses, the heat kernel of $\left(\mathcal{E}^{(\rho)}, \mathcal{F}\right)$ exists and satisfies the following diagonal upper bound

$$q_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right). \tag{14}$$

Step 8. We deduce from (S) a similar condition for the truncated semigroup Q_t:

$$Q_t^B \mathbf{1}_B \geq \varepsilon - Ct\left(r^{-\beta} + \rho^{-\beta}\right) \quad \text{in } \tfrac{1}{4}B$$

where $B = B(x, r)$. A certain iteration procedure allows to self-improve this estimate and to obtain that, for any $k \in \mathbb{N}$, if $r \geq 8k\rho$ then

$$Q_t^B \mathbf{1}_B \geq 1 - C(k)\left(\frac{t}{\rho^\beta}\right)^k,$$

which implies that

$$\int_{B^c(x,r)} q_t(x,y)d\mu(y) \leq C(k)\left(\frac{t}{\rho^\beta}\right)^k.$$

Combining this with (14), we obtain that, in the case $q < \infty$,

$$\|q_t(x,\cdot)\|_{L^q(B^c)} \leq \|q_t(x,\cdot)\|_{L^\infty(B^c)}^{1/q'} \|q_t(x,\cdot)\|_{L^1(B^c)}^{1/q} \leq \frac{C(k)}{t^{\alpha/(q'\beta)}} \exp\left(\frac{Ct}{\rho^\beta}\right)\left(\frac{t}{\rho^\beta}\right)^{\frac{k}{q}}. \tag{15}$$

In the case $q = \infty$ we improve (14) in a different way and obtain that if $r \geq 4k\rho$ then

$$\|q_t(x,\cdot)\|_{L^\infty(B^c)} \leq \frac{C(k)}{t^{\alpha/\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right)\left(1 + \frac{\rho^\beta}{t}\right)^{\alpha/\beta}\left(\frac{t}{\rho^\beta}\right)^k. \tag{16}$$

Step 9. We prove that, under all the above conditions, including (TJ_q), we have, for any $t > 0$ and for any ball $B = B(x, r)$,

$$\|p_t(x,\cdot)\|_{L^q(B^c)} \le \|q_t(x,\cdot)\|_{L^q(B^c)} + \frac{Ct}{\rho^{\alpha/q'+\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right). \tag{17}$$

Step 10. In the case $q < \infty$, combining (15) and (17), we obtain that if $r \ge 8k\rho$ then

$$\|p_t(x,\cdot)\|_{L^q(B^c)} \le \frac{C(k)}{t^{\alpha/(q'\beta)}} \exp\left(\frac{Ct}{\rho^\beta}\right)\left(\frac{t}{\rho^\beta}\right)^{k/q} + \frac{Ct}{\rho^{\alpha/q'+\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right).$$

Assuming that $r^\beta \ge t$ and setting $\rho = r/(8k)$, we obtain

$$\begin{aligned}\|p_t(x,\cdot)\|_{L^q(B^c)} &\le \frac{C(k)}{t^{\alpha/(q'\beta)}}\left(\frac{t}{r^\beta}\right)^{k/q} + \frac{C(k)t}{r^{\alpha/q'+\beta}} \\ &\le C\frac{t}{r^{\alpha/q'+\beta}},\end{aligned}$$

provided k is chosen so that

$$\left(\frac{t}{r^\beta}\right)^{k/q} \le \left(\frac{t}{r^\beta}\right)^{\frac{\alpha}{q'\beta}+1},$$

that is,

$$\frac{k}{q} \geq \frac{\alpha}{q'\beta} + 1.$$

This finishes the proof of (TP_q) if $q < \infty$.

In the case $q = \infty$ we obtain from (16) and (17), assuming that $r^\beta \geq t$ and setting $\rho = r/(4k)$ that

$$\begin{aligned}\|p_t(x,\cdot)\|_{L^q(B^c)} &\leq \frac{C(k)}{t^{\alpha/\beta}} \left(\frac{t}{r^\beta}\right)^{k-\frac{\alpha}{\beta}} + \frac{C(k)t}{r^{\alpha+\beta}} \\ &\leq C\frac{t}{r^{\alpha+\beta}},\end{aligned}$$

provided k is chosen so that

$$\left(\frac{t}{r^\beta}\right)^{k-\frac{\alpha}{\beta}} \leq \left(\frac{t}{r^\beta}\right)^{\frac{\alpha}{\beta}+1}$$

that is,

$$k \geq 2\frac{\alpha}{\beta} + 1.$$

Step 11. We prove now consequences of (TP_q). Let us first prove that if $q \in [2, \infty]$ then

$$\boxed{(TP_q) \Rightarrow (UE_q)}.$$

Setting $r = \frac{1}{2} d(x, y)$, we obtain by the semigroup property

$$\begin{aligned} p_{2t}(x, y) &= \int_M p_t(x, z)\, p_t(z, y)\, d\mu(z) \\ &\leq \left(\int_{B^c(x,r)} + \int_{B^c(y,r)} \right) p_t(x, z)\, p_t(z, y)\, d\mu(z). \end{aligned}$$

It suffices to estimate the first integral. By the Hölder inequality, we have

$$\int_{B^c(x,r)} p_t(x, z)\, p_t(z, y)\, d\mu(z) \leq \|p_t(x, \cdot)\|_{L^q(B^c(x,r))} \|p_t(\cdot, y)\|_{L^{q'}(M)}.$$

Since $q \geq 2$ and, hence, $q' \leq q$, we have not only (TP_q) but also $(TP_{q'})$. Hence,

$$\|p_t(x, \cdot)\|_{L^q(B^c(x,r))} \leq \frac{C}{t^{\alpha/(q'\beta)}} \left(1 + \frac{r}{t^{1/\beta}}\right)^{-(\alpha/q' + \beta)}$$

and

$$\|p_t(\cdot, y)\|_{L^{q'}(M)} \leq \frac{C}{t^{\alpha/(q\beta)}}.$$

Since $\frac{\alpha}{q'\beta}+\frac{\alpha}{q\beta}=\frac{\alpha}{\beta}$, we obtain

$$\int_{B^c(x,r)} p_t(x,z)\,p_t(z,y)\,d\mu(z) \leq \frac{C}{t^{\alpha/\beta}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}.$$

Estimating in the same manner the second integral, we obtain

$$p_{2t}(x,y) \leq \frac{C}{t^{\alpha/\beta}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)},$$

that is, (UE_q).

Step 12. Since $(UE_q) \Rightarrow (DUE) \Rightarrow (FK)$, we obtain that

$$\boxed{(TP_q) \Rightarrow (FK)}.$$

The implication

$$\boxed{(TP_q) \Rightarrow (TJ_q)}$$

was already mentioned in (9).

Step 13. Finally, the implication

$$\boxed{(TP_q) + (C) \Rightarrow (Gcap)}$$

is proved as follows. By (TP_q) we have also (TP_1), that is,

$$\int_{B^c(x,r)} p_t(x,y)d\mu(y) \le C\left(1 + \frac{r}{t^{1/\beta}}\right)^{-\beta} \le \frac{Ct}{r^\beta}.$$

This and (C) imply that

$$P_t^{B(x,r)}\mathbf{1}\,(x) \ge \varepsilon - \frac{Ct}{r^\beta}$$

that is, (S), and it is known that $(S) \Rightarrow (Gcap)$.