# Tails of heat kernels for jump processes

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*Dedicated to the memory of Ka-Sing Lau*

Based on a joint work with Eryan Hu and Jiaxin Hu

# Heat kernels in $\mathbb{R}^n$

The heat kernel of the Laplace operator $\Delta = \sum_{i=1}^n \partial_{x_i x_i}$ in $\mathbb{R}^n$ is the following function of $x, y \in \mathbb{R}^n$ and $t > 0$:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

that is called the *Gauss-Weierstrass function*. This function is a fundamental solution of the heat equation $\partial_t u = \Delta u$ as well as the transition density of Brownian motion in $\mathbb{R}^n$.

For any $\beta \in (0,2)$ consider the operator $\mathcal{L} = (-\Delta)^{\beta/2}$ and the associated heat equation $\partial_t u = -\mathcal{L}u$. Its fundamental solution $p_t(x,y)$ is the transition density of the symmetric stable Levy process of index $\beta$. In the case $\beta = 1$, it coincides with the Cauchy distribution

$$p_t(x,y) = \frac{c_n t}{\left(t^2 + |x-y|^2\right)^{\frac{n+1}{2}}} = \frac{c_n}{t^n}\left(1 + \frac{|x-y|^2}{t^2}\right)^{-\frac{n+1}{2}}.$$

For a general $\beta$, the heat kernel of $\mathcal{L}$ admits the estimate

$$p_t(x,y) \simeq \frac{1}{t^{n/\beta}}\left(1 + \frac{|x-y|}{t^{1/\beta}}\right)^{-(n+\beta)}, \tag{1}$$

where $\simeq$ means that the ratio of the both sides is bounded from above and below by positive constants.

The operator $\mathcal{L} = (-\Delta)^{\beta/2}$ is a *non-local* operator given by

$$\mathcal{L}f(x) = p.v. \int_{\mathbb{R}^n} C_n \frac{f(x) - f(y)}{|x-y|^{n+\beta}} dy,$$

where the function

$$J(x,y) = \frac{C_n}{|x-y|^{n+\beta}}$$

is called the *jump kernel* of $\mathcal{L}$ and of the corresponding Levy process.

The Laplace operator $\Delta$ is associated with the Dirichlet integral by the Green formula

$$-(\Delta f, f)_{L^2} = \int_{\mathbb{R}^n} |\nabla f|^2 \, dx.$$

Similarly, the operator $\mathcal{L} = (-\Delta)^{\beta/2}$ is associated with the following quadratic form

$$(\mathcal{L}f, f)_{L^2} = \iint_{\mathbb{R}^n \times \mathbb{R}^n} (f(x) - f(y))^2 \, J(x,y) dx dy$$

that is referred to as the *Dirichlet form* of $\mathcal{L}$.

## Dirichlet forms and heat kernels

Let $(M, d)$ be a locally compact separable metric space and $\mu$ be a Radon measure with full support on $M$. Let $(\mathcal{E}, \mathcal{F})$ be a *regular jump type Dirichlet form*, that is, $\mathcal{F}$ is a dense subspace of $L^2(M, \mu)$ and $\mathcal{E}$ is a quadratic form on $\mathcal{F}$ given by

$$\mathcal{E}(f, f) = \iint_{M \times M} (f(x) - f(y))^2 J(x, y) d\mu(x) d\mu(y),$$

where $J(x, y)$ is a non-negative symmetric function that is called the jump kernel.

Let $\mathcal{L}$ be the generator of $(\mathcal{E}, \mathcal{F})$ that is a non-negative definite self-adjoint operator in $L^2(M, \mu)$ satisfying the identity $(\mathcal{L}f, f)_{L^2} = \mathcal{E}(f, f)$. For any $t \geq 0$, set $P_t = e^{-t\mathcal{L}}$ so that $\{P_t\}_{t \geq 0}$ is the *heat semigroup* of $(\mathcal{E}, \mathcal{F})$. For any $f \in L^2(M, \mu)$, the function $u(x, t) = P_t f(x)$ solves in some sense the heat equation $\partial_t u = -\mathcal{L}u$.

If, for any $t > 0$, the operator $P_t$ is an integral operator with the integral kernel $p_t(x, y)$,

$$P_t f(x) = \int_M p_t(x, y) f(y) d\mu(y) \text{ for all } f \in L^2(M, \mu),$$

then $p_t(x, y)$ and is referred to as the heat kernel of $(\mathcal{E}, \mathcal{F})$ or $\mathcal{L}$.

Main problem: *obtaining upper and lower bounds of* $p_t(x, y)$.

Denote by $B(x,r)$ open metric balls in $M$. We assume always that $\mu$ is $\alpha$-regular in the sense of Ahlfors for some $\alpha > 0$, that is, for all $x \in M$ and $r > 0$,

$$\mu\left(B(x,r)\right) \simeq r^{\alpha} \tag{$V$}$$

(although the main results are available also in the setting of a doubling measure).

By a result of AG and T.Kumagai (2008), if the heat kernel of a jump type Dirichlet form satisfies a self-similar estimate

$$p_t(x,y) \simeq \frac{1}{t^{\gamma}}\Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right)$$

for some $\beta, \gamma > 0$ and decreasing function $\Phi$, then it is necessarily the following estimate:

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}}\left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{2}$$

We refer to (2) as a *stable-like* estimate of the heat kernel because of its similarity to (1). The number $\beta$ is called the index of the corresponding process.

If (2) holds then, using the identity $J(x,y) = \lim_{t\to 0} \frac{1}{2t} p_t(x,y)$, we obtain that

$$J(x,y) \simeq d(x,y)^{-(\alpha+\beta)}. \tag{$J$}$$

Z.-Q. Chen and T.Kumagai (2003) proved that if $\beta < 2$ then, in fact, $(J)\Leftrightarrow(2)$.

However, on many fractal sets there exist regular Dirichlet forms with the jump kernel satisfying (2) with $\beta > 2$. Indeed, by works of M.Barlow, J.Kigami, et al, on large families of p.c.f. fractals and Sierpinski carpets, there are diffusion processes with heat kernels satisfying the following *sub-Gaussian* estimate:

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta^*}} \exp\left(-c\left(\frac{d^{\beta^*}(x,y)}{t}\right)^{\frac{1}{\beta^*-1}}\right),$$

where $\beta^* > 2$ is the *walk dimension* of the process. Using a subordination techniques, one obtains a jump process satisfying (2) for any index $\beta < \beta^*$ . Clearly, $\beta$ can be $> 2$.

To handle the case $\beta > 2$ in general, one needs one more condition: a *generalized capacity* condition $(Gcap)$ that will be explained below. This condition is closely related to the *cutoff Sobolev inequality* introduced by M.Barlow and R.Bass (2004), and to the *energy inequality* of S.Andres and M.Barlow (2015).

**Theorem 1** (AG, E.Hu, J.Hu (2018), Chen, Kumagai, Wang (2020)) *For any* $\beta > 0$,

$$(Gcap) + (J) \Leftrightarrow (2). \tag{3}$$

# Upper bounds of the heat kernel

The main question to be discussed here is how to obtain the estimates of the form

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\gamma}, \tag{4}$$

with some $\beta, \gamma > 0$. Since $J(x,y) = \lim_{t\to 0} \frac{1}{2t} p_t(x,y)$, it is easy to obtain the following:

- if $\gamma > \alpha + \beta$ then $J(x,y) = 0$ (which excludes this case);
- if $\gamma = \alpha + \beta$ then

$$J(x,y) \leq \frac{C}{d(x,y)^{\alpha+\beta}}; \tag{$J_\leq$}$$

- and if $\gamma < \alpha + \beta$ then there is no useful bound for $J$.

It is natural to expect the opposite implication: ($J_\leq$)$\Rightarrow$(4) where $\gamma = \alpha + \beta$. However, for that one needs additional conditions.

**Definition.** We say that the *Faber-Krahn inequality* $(FK)$ of index $\beta$ holds if, for any precompact open set $\Omega \subset M$,

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-\beta/\alpha}, \tag{$FK$}$$

where $\lambda_1(\Omega) = \inf \operatorname{spec}(\mathcal{L}^\Omega)$ and $\mathcal{L}^\Omega$ is the generator of the restricted form $(\mathcal{E}, \mathcal{F}(\Omega))$.

Equivalently, $(FK)$ holds if, for any $\varphi \in \mathcal{F} \cap C_0(\Omega)$

$$\mathcal{E}(\varphi, \varphi) \geq c\mu(\Omega)^{-\beta/\alpha} \|\varphi\|_{L^2}^2.$$

It is known that $(FK)$ is equivalent to the *diagonal upper estimate* of the heat kernel

$$\boxed{p_t(x,y) \leq Ct^{-\alpha/\beta}}. \tag{$DUE$}$$

It is also known that

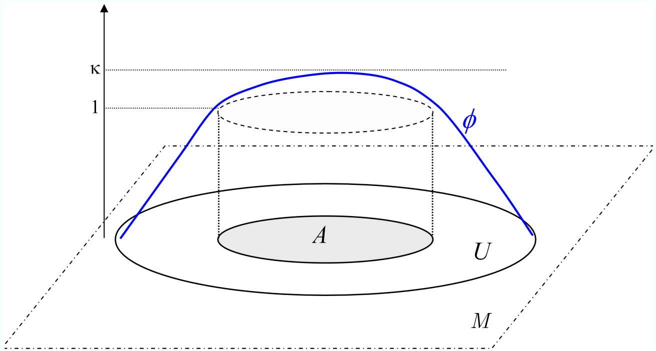
$$J(x,y) \geq \frac{c}{d(x,y)^{\alpha+\beta}} \quad \Rightarrow \ (FK). \tag{5}$$

In some sense, $(FK)$ can be regarded as an integral version of the pointwise lower bound of $J$.

Recall that the *capacity* associated with $(\mathcal{E}, \mathcal{F})$ is defined as follows: for any open set $U \subset M$ and a Borel set $A \subset U$ set

$$\operatorname{cap}(A, U) = \inf\left\{\mathcal{E}(\phi, \phi) : \phi \in \mathcal{F},\ 0 \leq \phi \leq 1,\ \phi|_A = 1,\ \phi|_{U^c} = 0\right\}.$$

**Definition.** For any bounded function $u \in \mathcal{F} + \text{const}$ and a real $\kappa \geq 1$, define the *generalized capacity* of the pair $(A, U)$ by




$$\operatorname{cap}_u^{(\kappa)}(A, U) = \inf_{\phi} \mathcal{E}(u^2\phi, \phi),$$

where inf is taken over all $\phi \in \mathcal{F}$ such that

$$0 \leq \phi \leq \kappa, \quad \phi|_A \geq 1, \quad \phi|_{U^c} = 0.$$

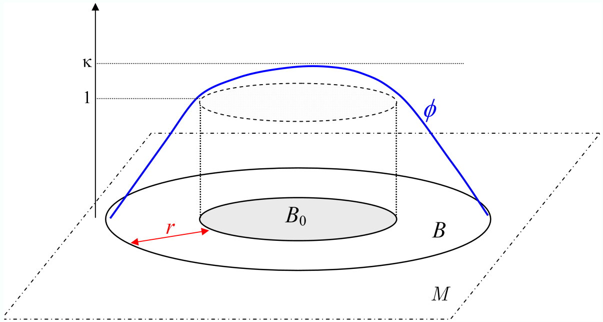
For example, if $\kappa = 1$ and $u \equiv 1$ then

$$\operatorname{cap}_u^{(\kappa)}(A, U) = \operatorname{cap}(A, U).$$

**Definition.** We say that the *generalized capacity condition* $(Gcap)$ of index $\beta$ is satisfied if there exist two constants $\kappa \geq 1, C > 0$ such that, for any bounded function $u \in \mathcal{F} + \text{const}$ and for all concentric balls $B_0 := B(x, R)$, $B := B(x, R + r)$ with $x \in M$ and $R, r > 0$,

$$\operatorname{cap}_u^{(\kappa)}(B_0, B) \leq \frac{C}{r^\beta} \int_B u^2 d\mu. \qquad (Gcap)$$

Equivalently, this condition means that, for any pair of concentric balls $B_0, B$ as above and for any bounded $u \in \mathcal{F} + \text{const}$, there exists $\phi \in \mathcal{F}$ such that




$$0 \le \phi \le \kappa, \quad \phi|_{B_0} \ge 1, \quad \phi|_{B^c} = 0$$

and

$$\mathcal{E}(u^2\phi, \phi) \le \frac{C}{r^\beta} \int_B u^2 d\mu. \tag{6}$$

Applying (6) with $u \equiv 1$ and replacing $\phi$ with $\phi \wedge 1$, we obtain the *capacity* condition:

$$\operatorname{cap}(B_0, B) \le \frac{C}{r^\beta} \mu(B). \tag{$cap$}$$

Usually it is very difficult to verify ($Gcap$) (apart from some specific cases), and it is an open problem to develop methods for verification of ($Gcap$).

It would ideal if in all our results ($Gcap$) could be replaced by a much simpler condition ($cap$), but so far there is no technique for that.

Now we can state a result about the upper estimate (4) with $\gamma = \alpha + \beta$, that is,

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{$UE$}$$

Denote by $(C)$ the hypothesis that $(\mathcal{E}, \mathcal{F})$ is *conservative*, that is, $P_t 1 = 1$. Recall also the hypothesis

$$J(x,y) \leq \frac{C}{d(x,y)^{\alpha+\beta}}. \tag{$J_\leq$}$$

The following theorem can be extracted from the results of AG, J.Hu, K.-S.Lau (2014) and Z.-Q.Chen, T.Kumagai, J.Wang (2021).

**Theorem 2** *Under the hypothesis $(V)$ we have*

$$(Gcap) + (FK) + (J_\leq) \Leftrightarrow (UE) + (C).$$

Our next goal is to replace the assumption $(J_\leq)$ containing pointwise upper bound of the jump kernel $J$ by a weaker (and more robust) assumption involving *tail estimates* of $J$.

## $L^1$-tails of jump kernels

We discuss now the situations when the pointwise upper bound of $J$ is replaced the integrals of $J$ outside balls as follows: for all $x \in M$ and $r > 0$

$$\boxed{\int_{B^c(x,r)} J(x,y)d\mu(y) \leq \frac{C}{r^\beta}}. \tag{$TJ$}$$

The following result is proved in the setting of *ultra-metric* spaces. Recall that $(M, d)$ is called an ultra-metric space if $d$ satisfies the ultra-metric triangle inequality

$$d(x,y) \leq \max\left(d(x,z), d(y,z)\right) \quad \forall x, y, z \in M.$$

For example, the field $\mathbb{Q}_p$ of $p$-adic numbers is an ultra-metric space with respect to the $p$-adic norm. Also, $\mathbb{Q}_p^n$ is an ultra-metric space with respect to $l^\infty$ product distance.

It is interesting that on ultra-metric spaces $(TJ) \Rightarrow (Gcap)$.

**Theorem 3** (A.Bendikov, AG, E.Hu, J.Hu, 2021) *Let $(M, d)$ be an ultra-metric space satisfying $(V)$. Then*

$$(FK) + (TJ) \;\Rightarrow\; p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}}\left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta}.$$

## $L^q$-tail of jump kernels

Fix $\beta > 0$, $q \in [1, \infty]$ and consider the following hypothesis: for all $x \in M$ and $r > 0$

$$\boxed{\|J(x, \cdot)\|_{L^q(B^c(x,r))} \leq \frac{C}{r^{\alpha/q'+\beta}}}, \tag{$TJ_q$}$$

where $q' = \frac{q}{q-1}$. It is easy to see that ($TJ_q$) becomes stronger when $q$ increases.

For example, if $q = 1$ then $q' = \infty$ so that ($TJ_q$) is equivalent to ($TJ$):

$$\int_{B^c(x,r)} J(x, y) d\mu(y) \leq \frac{C}{r^{\beta}}, \tag{$TJ_1$}$$

If $q = 2$ then $q' = 2$ and ($TJ_q$) is equivalent to

$$\left( \int_{B^c(x,r)} J^2(x, y) d\mu(y) \right)^{1/2} \leq \frac{C}{r^{\alpha/2+\beta}}. \tag{$TJ_2$}$$

If $q = \infty$ then $q' = 1$ and ($TJ_q$) is equivalent to ($J_\leq$):

$$\operatorname*{essup}_{y \in B^c(x,r)} J(x, y) \leq \frac{C}{r^{\alpha+\beta}}. \tag{$TJ_\infty$}$$

## Tails of the heat kernel

Consider the following hypotheses about the tail of the heat kernel $p_t(x,y)$:

$$\boxed{\left\| p_t\left(x,\cdot\right)\right\|_{L^q(B^c(x,r))} \le \frac{C}{t^{\alpha/(q'\beta)}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}} \simeq \frac{1}{t^{\alpha/(q'\beta)}} \wedge \frac{t}{r^{\alpha/q'+\beta}}, \tag{$TP_q$}$$

for all $x \in M$ and $r > 0$. Since $J(x,y) = \lim_{t\to 0} \frac{1}{2t} p_t(x,y)$, we have the implication

$$(TP_q) \Rightarrow (TJ_q). \tag{7}$$

Condition $(TP_q)$ gets stronger when $q$ increases. If $q = 1$ then $(TP_q)$ is equivalent to

$$\int_{B^c(x,r)} p_t(x,y)d\mu(y) \le C\frac{t}{r^\beta}, \tag{$TP_1$}$$

if $q = 2$ then $(TP_q)$ becomes

$$\left(\int_{B^c(x,r)} p_t^2(x,y)d\mu(y)\right)^{1/2} \le \frac{C}{t^{\alpha/(2\beta)}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha/2+\beta)}, \tag{$TP_2$}$$

if $q = \infty$ then $(TP_q)$ is equivalent $(UE)$ as it becomes

$$\operatorname*{esssup}_{y\in B^c(x,r)} p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{$TP_\infty$}$$

## Main result

Consider also the following family of off-diagonal *upper estimates* of the heat kernel:

$$\boxed{p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}}, \tag{$UE_q$}$$

for all $t>0$ and almost all $x,y\in M$. For example, if $q=\infty$ then ($UE_q$) becomes

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \tag{$UE_\infty$}$$

which coincides with ($UE$) and ($TP_\infty$). If $q=1$ then ($UE_q$) becomes

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta}, \tag{$UE_1$}$$

as was used in Theorem 3. Now we can state our main result.

**Theorem 4** (AG, E.Hu, J.Hu, in preparation) *If* ($V$) *is satisfied then, for any* $q\in[2,\infty]$,

$$(Gcap)+(FK)+(TJ_q) \Leftrightarrow (TP_q)+(C) \Rightarrow (UE_q). \tag{8}$$

The case $q = \infty$ coincides with Theorem 2 while the case $q \in [2, \infty)$ is completely new. In fact, we prove (8) in a much more general setting when measure $\mu$ is doubling, and the scaling function $r^\beta$ is replaced by a general scaling function $W(x, r)$.

In the present setting of $\alpha$-regular measure and the scaling function $r^\beta$, a similar equivalence can be proved for the entire range $q \in [1, \infty]$:

$$(FK) + (Gcap) + (TJ_q) \Leftrightarrow (TP_q) + (UE_q) + (C)\,. \tag{9}$$

Recall that if $M$ is an ultra-metric space then, by Theorem 3, we have

$$(FK) + (TJ_1) \Rightarrow (UE_1)\,,$$

which also follows from (9) with $q = 1$ because in the ultra-metric space $(TJ_1)\Rightarrow(Gcap)$.

In the next theorem we replace $(FK)$ with a stronger condition

$$J(x, y) \geq \frac{c}{d(x, y)^{\alpha+\beta}}, \tag{$J_\geq$}$$

and look for the lower estimate of the heat kernel

$$p_t(x, y) \geq \frac{c}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{$LE$}$$

**Theorem 5** (AG, E.Hu, J.Hu, 2023) *If* $(V)$ *is satisfied then, for any* $q \in [2, \infty]$,

$$(Gcap) + (J_{\geq}) + (TJ_q) \Leftrightarrow (TP_q) + (LE) \Rightarrow (UE_q) + (LE).$$

That is, in this case the heat kernel satisfies the two-sides bounds:

$$\frac{c}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \leq p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}. \tag{10}$$

In the present setting the implication

$$(Gcap) + (J_{\geq}) + (TJ_q) \Rightarrow (10)$$

remains true for all $q \in [1, \infty]$.

The lower bound in (10) is always sharp in the sense that the exponent $a + \beta$ cannot be reduced (otherwise it would imply $J \equiv \infty$). In the setting of ultra-metric spaces, there exists an example of a jump kernel that satisfies

$$(Gcap) + (J_{\geq}) + (TJ_1)$$

(the case $q = 1$) so that the corresponding heat kernel admits the estimates

$$\frac{c}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \leq p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta}.$$

The upper bound here is also sharp, in the sense that the exponent $\beta$ cannot be enlarged.

# Approach to the proof of Theorem 4

**Strong generalized capacity condition and elliptic mean value inequality**

In the definition of $(Gcap)$, in the inequality (6), that is, in

$$\mathcal{E}\left(u^2\phi,\phi\right)\leq\frac{C}{r^\beta}\int_B u^2 d\mu,$$

the cutoff function $\phi$ may depend on the weight $u$. Denote by $(Gcap')$ a stronger version of $(Gcap)$ when function $\phi$ depends only on the pair $B_0, B$ of the balls and serves all functions $u$ simultaneously.

The next theorem is the first one in a sequence of results leading to heat kernel upper bounds. We use again the following hypothesis about the tail of the jump kernel:

$$\boxed{\|J(x,\cdot)\|_{L^1(B^c(x,r))}\leq\frac{C}{r^\beta}},\quad\forall x\in M,\ r>0. \tag{$TJ$}$$

**Theorem 6** (AG, E.Hu, J.Hu, 2023) *Under the hypothesis $(V)$, we have the implication*

$$(FK)+(Gcap)+(TJ)\Rightarrow\left(Gcap'\right).$$

The condition $(Gcap')$ will be used below for obtaining the *parabolic mean value inequality* that in turn is needed for heat kernel upper bounds. The proof of Theorem 6 uses the *elliptic mean value inequality* $(EMV)$.

**Definition.** We say that $(EMV)$ holds if, for any function $u \in \mathcal{F} \cap L^\infty$ that is non-negative and subharmonic in a ball $B = B(x_0, R)$, and for any $\varepsilon > 0$,

$$\operatorname*{esup}_{\frac{1}{2}B} u \leq C_\varepsilon \left( \fint_B u^2 \right)^{1/2} + \varepsilon \left\| u_+ \right\|_{L^\infty\left(\left(\frac{1}{2}B\right)^c\right)}.$$

The proof of Theorem 6 goes through the following implications (under the standing assumptions $(V)$, $(FK)$, $(TJ)$):

$$(Gcap) \Rightarrow (EMV) + (cap) \Rightarrow (Gcap').$$

## Parabolic mean value inequality

**Theorem 7** (AG, E.Hu, J.Hu, 2023) *For any* $q \in [1,\infty]$, *we have*

$$(FK) + (Gcap') + (TJ_q) \Rightarrow (PMV_q),$$

*where* $(PMV_q)$ *stands for the* Parabolic Mean Value *inequality that means the following.* Fix an arbitrary ball $B = B(x,R)$ in $M$ and set $T = R^\beta$. Let $u$ be a bounded non-negative function on $M \times (0,T]$ that is *subcaloric* in the cylinder $B \times (0,T]$:

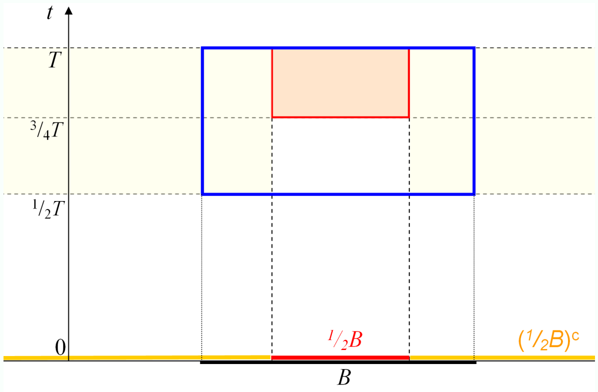
that is, for any $t \in (0,T]$,

$$u(\cdot,t) \in \mathcal{F}_+ \cap L^\infty(M)$$

and $u$ satisfies in $B \times (0,T]$

$$\partial_t u + \mathcal{L}u \leq 0$$

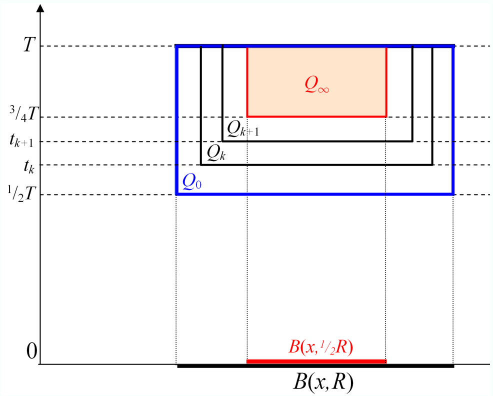
in a certain weak sense.




Then, for any $\varepsilon \in (0,1]$,

$$\sup_{t\in[\frac{3}{4}T,T]} \|u(\cdot,t)\|_{L^\infty(\frac{1}{2}B)} \leq C_\varepsilon \left( ⨍_{B\times[\frac{1}{2}T,T]} u^2 \right)^{1/2} + \frac{\varepsilon}{R^{\alpha/q'}} \sup_{t\in[\frac{1}{2}T,T]} \|u(\cdot,t)\|_{L^{q'}((\frac{1}{2}B)^c)}. \quad (PMV_q)$$

For the proof, consider a shrinking sequence of cylinders $Q_k = B(x, r_k) \times [t_k, T]$, $k \geq 0$, and an increasing sequence $b_k > 0$. Set




$$a_k := \int_{Q_k} (u - b_k)_+^2 \, d\mu dt$$

so that $a_k$ clearly decreases, and prove that

$$a_{k+1} \leq \frac{C}{(b_{k+1} - b_k)^{2\frac{\beta}{\alpha}}} \left( \frac{r_k}{r_k - r_{k+1}} \right)^C \left( \frac{1}{(r_k - r_{k+1})^\beta} + \frac{1}{t_{k+1} - t_k} + \frac{s_k}{b_{k+1} - b_k} \right)^{1+\frac{\beta}{\alpha}} a_k^{1+\frac{\beta}{\alpha}},$$

where

$$s_k = \sup_{t \in [t_k, T]} \operatorname*{essup}_{z \in B(x, \frac{r_k + r_{k+1}}{2})} \int_{B^c(x, r_k)} u(y,t) J(z,y) d\mu(y).$$

The proof of the relation between $a_k$ and $a_{k+1}$ uses essentially $(FK)$ and $(Gcap')$.

Choose

$$r_k = (\tfrac{1}{2} + 2^{-k-1})R \quad \text{and} \quad t_k = (\tfrac{3}{4} - 2^{-\beta k - 2})T,$$

so that

$$B \times [\tfrac{1}{2}T, T] = Q_0 \supset Q_k \supset Q_\infty = \tfrac{1}{2}B \times [\tfrac{3}{4}T, T].$$

Setting also $b_k = \left(1 - 2^{-k}\right) b$ for some $b > 0$, we obtain

$$a_{k+1} \leq C 2^{Ck} \left(1 + \frac{R^\beta s_k}{b}\right)^{1+\frac{\beta}{\alpha}} \frac{a_k^{1+\frac{\beta}{\alpha}}}{\left(R^{\alpha+\beta} b^2\right)^{\frac{\beta}{\alpha}}}. \tag{11}$$

Iterating (11), we show that if $b$ is large enough then $\lim_{k\to\infty} a_k = 0$,which implies that $u \leq b$ in $Q_\infty$.The choice of $b$ depends on $\sup_k \frac{a_k}{R^{\alpha+\beta}} = \frac{a_o}{R^{\alpha+\beta}}$ and on an upper bound for $R^\beta s_k$. The value

$$\frac{a_0}{R^{\alpha+\beta}} \leq \text{const} \fint_{B\times[\frac{1}{2}T,T]} u^2$$

yields the first term $(PMV_q)$. Estimating $s_k$ by means of the Hölder inequality and $(TP_q)$ gives

$$\begin{aligned} R^\beta s_k &\leq R^\beta \sup_{t\in[\frac{1}{2}T,T]} \|u(\cdot,t)\|_{L^{q'}(\frac{1}{2}B)^c)} \frac{C}{\left(r_k - r_{k+1}\right)^{\alpha/q'+\beta}} \\ &= \frac{C 2^{Ck}}{R^{\alpha/q'}} \sup_{t\in[\frac{1}{2}T,T]} \|u(\cdot,t)\|_{L^{q'}(\frac{1}{2}B)^c)} \end{aligned}$$

which yields the second term in $(PMV_q)$.

**Outline of the proof of Theorem 4**

Most of the proof is devoted to the implication

$$(FK) + (Gcap) + (TJ_q) \Rightarrow (TP_q)$$

**Step 0.** As it was already mentioned above,

$$(FK) \Rightarrow (DUE).$$

However, in our proof we do not use this implication because we work in a more general setting of doubling spaces where this result is unavailable. We use an alternative proof of $(DUE)$ with help of the mean value inequality of Theorem 7.

**Step 1.** By Theorem 6, we have

$$(FK) + (Gcap) + (TJ) \Rightarrow (Gcap'),$$

and, by Theorem 7,

$$(FK) + (Gcap') + (TJ_q) \Rightarrow (PMV_q).$$

**Step 2**. We prove that

$$(PMV_2) \Rightarrow (DUE)\,.$$

For that apply $(PMV_2)$ with $u(\cdot,t) = P_t f$ where $f \in C_0(M)$ and $f \geq 0$, and observe that the both terms in the right hand side of $(PMV_q)$ are bounded by $\frac{C}{R^{a/2}} \|f\|_{L^2}$ which yields

$$\|P_T f\|_\infty \leq \frac{C}{T^{\alpha/(2\beta)}} \|f\|_2\,,$$

which then implies $(DUE)$. Consequently, we obtain that, for any $q \in [2,\infty]$,

$$(FK) + (Gcap) + (TJ_q) \Rightarrow (DUE)\,.$$

It follows from $(DUE)$ that

$$\|p_t(x,\cdot)\|_{L^q(M)} \leq \frac{C}{t^{\alpha/(q'\beta)}}.$$

Hence, in order to prove $(TP_q)$, it remains to prove

$$\boxed{\|p_t(x,\cdot)\|_{L^q(B^c(x,r))} \leq \frac{Ct}{r^{\alpha/q'+\beta}}} \tag{12}$$

assuming that $r^\beta \geq t$, which is done in the rest of the proof.

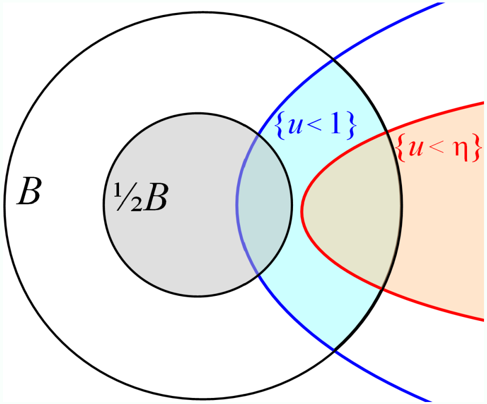
**Step 3**. We deduce from $(PMV_1)$ a so called "Lemma of growth":

there exist some $\varepsilon, \eta \in (0,1)$ such that, for any ball $B \subset M$ and for any $u \in \mathcal{F}$ that is non-negative and bounded in $M$ and superharmonic in $B$, if

$$\frac{\mu(B \cap \{u < 1\})}{\mu(B)} \le \varepsilon,$$

then

$$\operatorname*{essinf}_{\frac{1}{2}B} u \ge \eta.$$




For that observe that $v = \frac{1}{u+a}$ is subharmonic for any $a > 0$. For subharmonic functions, we obtain from $(PMV_1)$ the following multiplicative form of the mean value inequality (by choosing $\varepsilon$):

$$\|v\|_{L^\infty(\frac{1}{2}B)} \le CA^\theta \max(A,T)^{1-\theta}, \tag{13}$$

where

$$A = \left(\fint_B v^2 d\mu\right)^{1/2}, \qquad T = \|v\|_{L^\infty((\frac{1}{2}B)^c)},$$

and $\theta = \theta(\alpha, \beta) \in (0,1)$.

Let us estimate $A$ as follows:

$$A^2 = \frac{1}{\mu(B)} \left( \int_{B\cap u<1} + \int_{B\cap\{u\geq 1\}} \right) \frac{d\mu}{(u+a)^2}$$
$$\leq \frac{\mu(B\cap\{u<1\})}{\mu(B)} \frac{1}{a^2} + \frac{1}{(1+a)^2} \leq \frac{\varepsilon}{a^2} + \frac{1}{(1+a)^2} = \frac{2}{(1+a)^2},$$

for $a = \frac{1}{\varepsilon^{-1/2}-1}$. Estimating also trivially

$$\max(A,T) \leq \frac{1}{a},$$

we obtain from (13)

$$\operatorname*{essup}_{\frac{1}{2}B} \frac{1}{u+a} \leq C \left( \frac{2}{(1+a)^2} \right)^{\theta/2} \left( \frac{1}{a} \right)^{1-\theta} = \frac{C}{(1+a)^{\theta} a^{1-\theta}},$$

whence

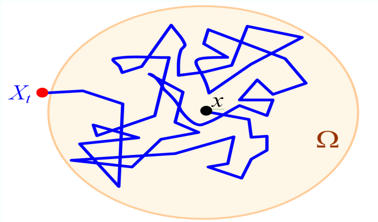
$$\operatorname*{essinf}_{\frac{1}{2}B} u \geq C^{-1} (1+a)^{\theta} a^{1-\theta} - a = a \left( C^{-1} \left( \tfrac{1}{a} + 1 \right)^{\theta} - 1 \right) =: \eta,$$

where $\eta > 0$ if $a$ is small enough, that is, when $\varepsilon$ is small enough.

**Step 4.** For any open set $\Omega \subset M$ and any $x \in \Omega$ set

$$E^{\Omega}(x) = \int_0^{\infty} P_t^{\Omega}\mathbf{1}(x)dt = \int_0^{\infty}\int_{\Omega} p_t^{\Omega}(x,y)d\mu(y)dt.$$

It has the probabilistic meaning of the *mean exit time* from $\Omega$ of the jump process $X_t$, associated with $(\mathcal{E},\mathcal{F})$, that starts at $x$: $E^{\Omega}(x) = \mathbb{E}_x(\tau^{\Omega})$, where $\tau^{\Omega}$ is the first exit time from $\Omega$.




In this step we prove that, under $(FK)$, for any ball $B$ of radius $r$,

$$\underset{B}{\operatorname{essup}}\, E^B \le Cr^{\beta}. \tag{14}$$

**Step 5**. We prove the opposite inequality: the Lemma of growth and $(cap)$ imply that

$$\underset{\frac{1}{4}B}{\operatorname{essinf}}\, E^B \ge cr^{\beta}. \tag{15}$$

It is known that (14) and (15) imply $(C)$ .

**Step 6**. Using the upper and lower estimates of $E^B$, we deduce the *survival* inequality: there exist $\varepsilon > 0$ such that, for any ball $B$ of radius $r$ and for any $t > 0$,

$$P_t^B \mathbf{1}_B \geq \varepsilon - \frac{Ct}{r^\beta} \quad \text{in } \tfrac{1}{4}B. \tag{$S$}$$

In probabilistic terms,

$$P_t^B \mathbf{1}_B(x) = \mathbb{P}_x \left(\tau_B > t\right)$$

that is the probability of survival of the process in $B$ up to time $t$ assuming the killing condition in $B^c$.

**Step 7**. For any $\rho > 0$ consider a *truncated* Dirichlet form

$$\mathcal{E}^{(\rho)}(f,f) := \iint_{\{d(x,y)<\rho\}} (f(x) - f(y))^2 J(x,y) d(x) d\mu(y).$$

Denote by $Q_t$ the heat semigroup of $\left(\mathcal{E}^{(\rho)}, \mathcal{F}\right)$ and by $q_t(x,y)$ its heat kernel. We prove that, under all the above hypotheses, the heat kernel of $\left(\mathcal{E}^{(\rho)}, \mathcal{F}\right)$ exists and satisfies the following diagonal upper bound

$$q_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right). \tag{16}$$

**Step 8**. We deduce from $(S)$ a similar condition for the truncated semigroup $Q_t$:

$$Q_t^B \mathbf{1}_B \geq \varepsilon - Ct\left(r^{-\beta} + \rho^{-\beta}\right) \quad \text{in } \tfrac{1}{4}B$$

where $B = B(x,r)$. A certain iteration procedure allows to self-improve this estimate and to obtain that, for any $k \in \mathbb{N}$, if $r \geq 8k\rho$ then

$$Q_t^B \mathbf{1}_B \geq 1 - C(k)\left(\frac{t}{\rho^\beta}\right)^k,$$

which implies that

$$\int_{B^c(x,r)} q_t(x,y)d\mu(y) \leq C(k)\left(\frac{t}{\rho^\beta}\right)^k.$$

Combining this with (16), we obtain that, in the case $q < \infty$,

$$\|q_t(x,\cdot)\|_{L^q(B^c)} \leq \|q_t(x,\cdot)\|_{L^\infty(B^c)}^{1/q'} \|q_t(x,\cdot)\|_{L^1(B^c)}^{1/q} \leq \frac{C(k)}{t^{\alpha/(q'\beta)}} \exp\left(\frac{Ct}{\rho^\beta}\right)\left(\frac{t}{\rho^\beta}\right)^{\frac{k}{q}}. \tag{17}$$

In the case $q = \infty$ we improve (16) in a different way and obtain that if $r \geq 4k\rho$ then

$$\|q_t(x,\cdot)\|_{L^\infty(B^c)} \leq \frac{C(k)}{t^{\alpha/\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right)\left(1 + \frac{\rho^\beta}{t}\right)^{\alpha/\beta}\left(\frac{t}{\rho^\beta}\right)^k. \tag{18}$$

**Step 9**. We prove that, under all the above conditions, including $(TJ_q)$, we have, for any $t>0$ and for any ball $B=B(x,r)$,

$$\|p_t(x,\cdot)\|_{L^q(B^c)} \le \|q_t(x,\cdot)\|_{L^q(B^c)} + \frac{Ct}{\rho^{\alpha/q'+\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right). \tag{19}$$

**Step 10**. In the case $q<\infty$, combining (17) and (19), we obtain that if $r \ge 8k\rho$ then

$$\|p_t(x,\cdot)\|_{L^q(B^c)} \le \frac{C(k)}{t^{\alpha/(q'\beta)}} \exp\left(\frac{Ct}{\rho^\beta}\right)\left(\frac{t}{\rho^\beta}\right)^{k/q} + \frac{Ct}{\rho^{\alpha/q'+\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right).$$

Assuming that $r^\beta \ge t$ and setting $\rho = r/(8k)$, we obtain

$$\begin{aligned}\|p_t(x,\cdot)\|_{L^q(B^c)} &\le \frac{C(k)}{t^{\alpha/(q'\beta)}}\left(\frac{t}{r^\beta}\right)^{k/q} + \frac{C(k)t}{r^{\alpha/q'+\beta}} \\ &\le C\frac{t}{r^{\alpha/q'+\beta}},\end{aligned}$$

provided $k$ is chosen so that

$$\left(\frac{t}{r^\beta}\right)^{k/q} \le \left(\frac{t}{r^\beta}\right)^{\frac{\alpha}{q'\beta}+1},$$

that is,

$$\frac{k}{q} \geq \frac{\alpha}{q'\beta} + 1.$$

This finishes the proof of $(TP_q)$ if $q < \infty$.

In the case $q = \infty$ we obtain from (18) and (19), assuming that $r^\beta \geq t$ and setting $\rho = r/(4k)$ that

$$\begin{aligned} \|p_t(x,\cdot)\|_{L^q(B^c)} &\leq \frac{C(k)}{t^{\alpha/\beta}} \left(\frac{t}{r^\beta}\right)^{k-\frac{\alpha}{\beta}} + \frac{C(k)t}{r^{\alpha+\beta}} \\ &\leq C\frac{t}{r^{\alpha+\beta}}, \end{aligned}$$

provided $k$ is chosen so that

$$\left(\frac{t}{r^\beta}\right)^{k-\frac{\alpha}{\beta}} \leq \left(\frac{t}{r^\beta}\right)^{\frac{\alpha}{\beta}+1}$$

that is,

$$k \geq 2\frac{\alpha}{\beta} + 1.$$

**Step 11**. We prove now consequences of $(TP_q)$. Let us first prove that if $q \in [2, \infty]$ then

$$\boxed{(TP_q) \Rightarrow (UE_q)}.$$

Setting $r = \frac{1}{2} d(x, y)$, we obtain by the semigroup property

$$\begin{aligned} p_{2t}(x, y) &= \int_M p_t(x, z) \, p_t(z, y) \, d\mu(z) \\ &\leq \left( \int_{B^c(x,r)} + \int_{B^c(y,r)} \right) p_t(x, z) \, p_t(z, y) \, d\mu(z) \, . \end{aligned}$$

It suffices to estimate the first integral. By the Hölder inequality, we have

$$\int_{B^c(x,r)} p_t(x, z) \, p_t(z, y) \, d\mu(z) \leq \| p_t(x, \cdot) \|_{L^q(B^c(x,r))} \| p_t(\cdot, y) \|_{L^{q'}(M)} \, .$$

Since $q \geq 2$ and, hence, $q' \leq q$, we have not only $(TP_q)$ but also $(TP_{q'})$. Hence,

$$\| p_t(x, \cdot) \|_{L^q(B^c(x,r))} \leq \frac{C}{t^{\alpha/(q'\beta)}} \left( 1 + \frac{r}{t^{1/\beta}} \right)^{-(\alpha/q' + \beta)}$$

and

$$\| p_t(\cdot, y) \|_{L^{q'}(M)} \leq \frac{C}{t^{\alpha/(q\beta)}}.$$

Since $\frac{\alpha}{q'\beta}+\frac{\alpha}{q\beta}=\frac{\alpha}{\beta}$, we obtain

$$\int_{B^c(x,r)} p_t(x,z)\, p_t(z,y)\, d\mu(z) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}.$$

Estimating in the same manner the second integral, we obtain

$$p_{2t}(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)},$$

that is, $(UE_q)$.

**Step 12**. Since $(UE_q) \Rightarrow (DUE) \Rightarrow (FK)$, we obtain that

$$\boxed{(TP_q) \Rightarrow (FK)}.$$

The implication

$$\boxed{(TP_q) \Rightarrow (TJ_q)}$$

was already mentioned in (7).

**Step 13**. Finally, the implication

$$\boxed{(TP_q) + (C) \Rightarrow (Gcap)}$$

is proved as follows. By $(TP_q)$ we have also $(TP_1)$, that is,

$$\int_{B^c(x,r)} p_t(x,y)d\mu(y) \le C\left(1 + \frac{r}{t^{1/\beta}}\right)^{-\beta} \le \frac{Ct}{r^\beta}.$$

This and $(C)$ imply that

$$P_t^{B(x,r)}\mathbf{1}(x) \ge \varepsilon - \frac{Ct}{r^\beta}$$

that is, $(S)$, and it is known that $(S) \Rightarrow (Gcap)$.