Upper bounds of heat kernels for jump processes

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Based on a joint work with Eryan Hu and Jiaxin Hu

References

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Non-local heat kernels in $\mathbb{R}^n$

Consider in $\mathbb{R}^n$ a fractional Laplace operator $\mathcal{L} = (-\Delta)^{\beta/2}$ that is a non-negative definite self-adjoint operator in $L^2(\mathbb{R}^n)$. If $\beta \in (0,2)$ then this operator is the generator of the symmetric stable Levy process of index β.

Denote by $p_t(x,y)$ the *heat kernel* of $\mathcal{L}$ that is the fundamental solution of the associated heat equation $\partial_t u = -\mathcal{L}u$ and, at the same time, the transition density of the Levy process. The heat kernel of $\mathcal{L}$ admits the estimate

$$p_t(x,y) \simeq \frac{1}{t^{n/\beta}} \left(1 + \frac{|x-y|}{t^{1/\beta}}\right)^{-(n+\beta)}, \tag{1}$$

where $A \simeq B$ means that $c_1 B \leq A \leq c_2 B$ for some positive constants c_1, c_2.

The operator $\mathcal{L} = (-\Delta)^{\beta/2}$ is the generator of the non-local Dirichlet form $(\mathcal{E}, \mathcal{F})$ given by

$$\mathcal{E}(f,f) = C_n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^2}{|x-y|^{n+\beta}} dxdy,$$

with the domain $\mathcal{F} = B^{\beta/2}_{2,2}(\mathbb{R}^n)$.

Jump type Dirichlet forms on metric measure spaces

Let (M, d) be a locally compact separable metric space and μ be a Radon measure with full support on M. Let $(\mathcal{E}, \mathcal{F})$ be a *regular jump type Dirichlet form*, where $\mathcal{F}$ is a dense subspace of $L^2(M, \mu)$ and $\mathcal{E}$ is a bilinear form on $\mathcal{F}$ given by

$$\mathcal{E}(f, f) = \int_{M \times M} \int (f(x) - f(y))^2 J(x, y) d\mu(x) d\mu(y),$$

Here $J(x, y)$ is a *jump kernel*, that is, a non-negative measurable symmetric function on $M \times M$.

Let $\mathcal{L}$ be the (non-negative definite) generator of $(\mathcal{E}, \mathcal{F})$ and $P_t = e^{-t\mathcal{L}}$, $t \geq 0$, be the heat semigroup of $(\mathcal{E}, \mathcal{F})$.

If, for any $t > 0$, the operator P_t is an integral operator with the integral kernel $p_t(x, y)$,

$$P_t f(x) = \int_M p_t(x, y) f(y) d\mu(y) \quad \text{for all } f \in L^2(M, \mu),$$

then $p_t(x, y)$ is referred to as *the heat kernel* of $(\mathcal{E}, \mathcal{F})$.

Major problem: *obtaining estimates of* $p_t(x, y)$ depending on the geometry of the underlying space and on J.

Two-sides estimates of the heat kernel

Denote by $B(x,r)$ open metric balls in M. In this talk we always assume that μ is α-regular for some $\alpha > 0$, that is, for all $x \in M$ and $r > 0$,

$$\mu\left(B(x,r)\right) \simeq r^{\alpha} \tag{V}$$

(although the main results are available also in the setting of a doubling measure).

By a result of AG and T.Kumagai (2008), if the heat kernel of a jump type Dirichlet form satisfies a self-similar estimate

$$p_t(x,y) \simeq \frac{1}{t^{\gamma}} \Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right)$$

for some $\beta, \gamma > 0$ and decreasing function Φ, then it is necessarily the following estimate:

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{2}$$

We refer to (2) as a *stable-like* estimate of the heat kernel because it matches (1) with $\alpha = n$. The number β is called the *index* of the corresponding Dirichlet form.

If (2) holds then, using the identity $J(x,y) = \lim_{t\to 0} \frac{1}{2t} p_t(x,y)$, we obtain that

$$J(x,y) \simeq d(x,y)^{-(\alpha+\beta)}. \tag{J}$$

Z.-Q. Chen and T.Kumagai (2003) proved that if $\beta < 2$ then, in fact, $(J) \Leftrightarrow (2)$.

To obtain the implication $(J) \Rightarrow (2)$ in the case $\beta > 2$ in general, one has to assume one more hypothesis: a *generalized capacity* condition (*Gcap*) that will be explained below. It was proved by AG, E.Hu, J.Hu (2018) and Chen, Kumagai, Wang (2020) that, for any $\beta > 0$,

$$(Gcap) + (J) \Leftrightarrow (2). \tag{3}$$

Our goal here is obtaining the upper estimates of the form

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\gamma}, \tag{4}$$

with some $\beta, \gamma > 0$, using certain information about the jump kernel J. Here necessarily $\gamma \le \alpha + \beta$ because otherwise (4) implies $J \equiv 0$. If $\gamma = \alpha + \beta$ then the necessary condition for (4) is

$$J(x,y) \le \frac{C}{d(x,y)^{\alpha+\beta}}. \tag{$J_\le$}$$

Tail conditions

The case $\gamma < \alpha+\beta$ will be of main interest. In order to obtain (4) with such γ, we will use a *tall estimate* of the jump kernel. Consider for any $q \in [1,\infty]$ the following hypothesis:

$$\boxed{\|J(x,\cdot)\|_{L^q(B^c(x,r))} \le \frac{C}{r^{\alpha/q'+\beta}}}, \quad \forall x \in M,\ r>0, \tag{TJ_q}$$

where $q' = \frac{q}{q-1}$. It is easy to verify that (TJ_q) becomes stronger when q increases.

For example, if $q=1$ then $q'=\infty$ so that (TJ_q) becomes

$$\int_{B^c(x,r)} J(x,y)d\mu(y) \le \frac{C}{r^\beta}, \tag{TJ_1}$$

If $q=2$ then $q'=2$ and (TJ_q) becomes

$$\left(\int_{B^c(x,r)} J^2(x,y)d\mu(y)\right)^{1/2} \le \frac{C}{r^{\alpha/2+\beta}}. \tag{TJ_2}$$

If $q=\infty$ then $q'=1$ and (TJ_q) is equivalent to $(J_{\le})$:

$$\operatorname*{esssup}_{y\in B^c(x,r)} J(x,y) \le \frac{C}{r^{\alpha+\beta}}. \tag{TJ_∞}$$

Consider also the following condition about the *tail estimate* of the heat kernel:

$$\boxed{\left\|p_t\left(x,\cdot\right)\right\|_{L^q(B^c(x,r))} \leq \frac{C}{t^{\alpha/(q'\beta)}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}} \simeq \frac{1}{t^{\alpha/(q'\beta)}} \wedge \frac{t}{r^{\alpha/q'+\beta}}, \tag{TP_q}$$

for all $x \in M$ and $t, r > 0$. Condition (TP_q) gets stronger when q increases. If $q = 1$ then $q' = \infty$ and (TP_q) is equivalent to

$$\int_{B^c(x,r)} p_t(x,y)d\mu(y) \leq C\frac{t}{r^\beta}, \tag{TP_1}$$

if $q = 2$ then $q' = 2$ and (TP_q) becomes

$$\left(\int_{B^c(x,r)} p_t^2(x,y)d\mu(y)\right)^{1/2} \leq \frac{C}{t^{\alpha/(2\beta)}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha/2+\beta)}, \tag{TP_2}$$

if $q = \infty$ then $q' = 1$ and (TP_q) is becomes

$$\operatorname*{esssup}_{y\in B^c(x,r)} p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \tag{TP_∞}$$

which is equivalent to (4) with $\gamma = \alpha + \beta$.

Lemma 1 *If $q \in [2, \infty]$ then (TP_q) implies the following pointwise upper estimate*

$$\boxed{p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha/q' + \beta)}}, \qquad (UE_q)$$

for all $t > 0$ and μ-almost all $x, y \in M$ (that is the estimate (4) with $\gamma = \alpha/q' + \beta$).

Clearly, (UE_q) gets stronger when q increases. For example, if $q = 1$ then (UE_q) becomes

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta}, \qquad (UE_1)$$

and if $q = \infty$ then (UE_q) becomes

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \qquad (UE_\infty)$$

which is equivalent to (TP_∞) and to (4) with $\gamma = \alpha + \beta$.

Using $J(x,y) = \lim_{t \to 0} \frac{1}{2t} p_t(x,y)$, one obtains easily the implication $(TP_q) \Rightarrow (TJ_q)$. Our main result states that, under some additional hypotheses, also the converse implication holds, that is, (*additional hypotheses*)+$(TJ_q) \Rightarrow (TP_q)$.

Additional hypothesis

We introduce here two hypotheses for any regular Dirichlet form $(\mathcal{E}, \mathcal{F})$.

Definition. We say that the *Faber-Krahn inequality* (FK) of index β is satisfied for $(\mathcal{E}, \mathcal{F})$ if, for any precompact open set $\Omega \subset M$,

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-\beta/\alpha}, \tag{FK}$$

where $\lambda_1(\Omega) = \inf \operatorname{spec}(\mathcal{L}^\Omega)$ and $\mathcal{L}^\Omega$ is the generator of the restricted form $(\mathcal{E}, \mathcal{F}(\Omega))$.

Equivalently, (FK) holds if, for any $\varphi \in \mathcal{F} \cap C_0(\Omega)$

$$\mathcal{E}(\varphi, \varphi) \geq c\mu(\Omega)^{-\beta/\alpha} \|\varphi\|_{L^2}^2.$$

It is known that (FK) is equivalent to the *diagonal upper estimate* of the heat kernel

$$p_t(x, y) \leq Ct^{-\alpha/\beta}. \tag{DUE}$$

It is also known that

$$J(x, y) \geq \frac{c}{d(x, y)^{\alpha+\beta}} \quad \Rightarrow \ (FK). \tag{5}$$

Hence, (FK) can be regarded as a weaker version of the lower bound of J.

Recall that the *capacity* associated with $(\mathcal{E}, \mathcal{F})$ is defined as follows: for any open set $U \subset M$ and a Borel set $A \subset U$ set

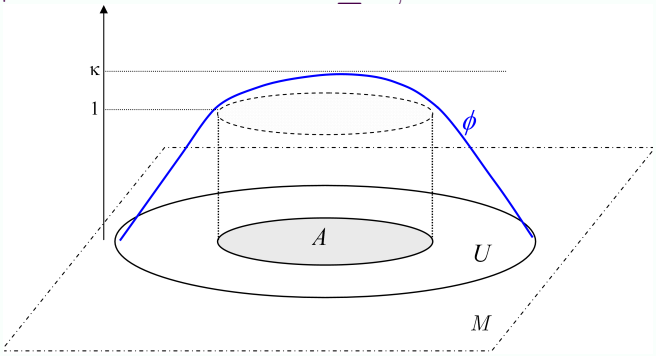
$$\mathrm{cap}(A, U) = \inf \{\mathcal{E}(\phi, \phi) : \phi \in \mathcal{F},\ 0 \le \phi \le 1,\ \phi|_A = 1,\ \phi|_{U^c} = 0\}\,.$$

Definition. For any bounded function $u \in \mathcal{F} + \mathrm{const}$ and a real $\kappa \ge 1$, define the *generalized capacity* of the pair (A, U) by

$$\mathrm{cap}_u^{(\kappa)}(A, U) = \inf_{\phi} \mathcal{E}(u^2\phi, \phi),$$

where inf is taken over all $\phi \in \mathcal{F}$ such that

$$0 \le \phi \le \kappa, \qquad \phi|_A \ge 1, \qquad \phi|_{U^c} = 0.$$



For example, if $\kappa = 1$ and $u \equiv 1$ then $\mathrm{cap}_u^{(\kappa)}(A, U) = \mathrm{cap}(A, U)$.

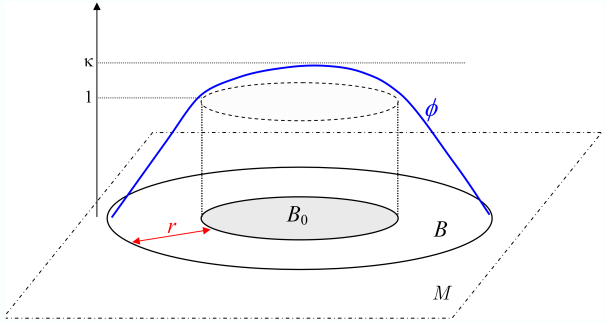
Definition. We say that the *generalized capacity condition* $(Gcap)$ of index β is satisfied for $(\mathcal{E}, \mathcal{F})$ if there exist $\kappa \ge 1, C > 0$ such that, for any bounded function $u \in \mathcal{F} + \mathrm{const}$ and for all concentric balls $B_0 := B(x, R)$, $B := B(x, R + r)$ with $x \in M$ and $R, r > 0$,

$$\mathrm{cap}_u^{(\kappa)}(B_0, B) \le \frac{C}{r^\beta} \int_B u^2 d\mu. \qquad (Gcap)$$

Equivalently, this condition means that, for any pair of concentric balls B_0, B as above and for any bounded $u \in \mathcal{F} + \text{const}$,
there exists $\phi \in \mathcal{F}$ such that

$$0 \le \phi \le \kappa, \quad \phi|_{B_0} \ge 1, \quad \phi|_{B^c} = 0$$

and the following inequality is true:



$$\mathcal{E}(u^2\phi, \phi) \le \frac{C}{r^\beta} \int_B u^2 d\mu. \tag{6}$$

Setting $u \equiv 1$ in (6) and replacing ϕ with $\phi \wedge 1$, we obtain the *capacity* condition:

$$\text{cap}(B_0, B) \le \frac{C}{r^\beta} \mu(B). \tag{cap}$$

Usually it is very difficult to verify (*Gcap*) (apart from some specific cases), and it is an open problem to develop methods for verification of (*Gcap*). In contrast to that, the capacity condition (*cap*) can be proved in many examples of interest.

Conjecture. *If in all our results (Gcap) can be replaced by (cap).*

Main result

Denote by (C) the hypothesis that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is conservative, that is, $P_t 1 \equiv 1$ for all $t > 0$.

Theorem 2 *Let (V) be satisfied. Then, for any $q \in [1, \infty]$,*

$$(FK) + (Gcap) + (TJ_q) \Leftrightarrow (TP_q) + (UE_q) + (C). \tag{7}$$

Recall for comparison that if $(\mathcal{E}, \mathcal{F})$ is a strongly local Dirichlet form then

$$(FK) + (Gcap) \Leftrightarrow \left\{ p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \exp\left(-c \left(\frac{d^\beta(x,y)}{t} \right)^{\frac{1}{\beta-1}} \right) \right\} + (C),$$

by S.Andres and M.Barlow (2015) and AG, J.Hu, K.-S. Lau (2015).

In the present case of jump-type Dirichlet form, we add one more condition (TJ_q) about the tail of J, and obtain both tail and pointwise estimates of the heat kernel.

If $q \geq 2$ then, by Lemma 1, $(TP_q) \Rightarrow (UE_q)$ so that Theorem 2 can be restated as follows:

Theorem 3 *If $q \in [2, \infty]$ then*

$$(FK) + (Gcap) + (TJ_q) \Leftrightarrow (TP_q) + (C)\,. \tag{8}$$

In fact, we have proved Theorem 3 in a more general setting when measure μ is doubling, and the scaling function r^β is replaced by a general scaling function $W(x, r)$.

Recall some previously known related results. In the case $q = \infty$ we obtain from (8)

$$(FK) + (Gcap) + (J_{\leq}) \Leftrightarrow (UE_\infty) + (C)\,,$$

which can also be extracted from the results of AG, J.Hu, K.-S.Lau (2014) and Z.-Q.Chen, T.Kumagai, J.Wang (2021).

Let M be an ultra-metric space, that is, d satisfies the ultra-metric triangle inequality

$$d(x, y) \leq \max(d(x, z), d(y, z)) \quad \forall x, y, z \in M.$$

It is known that in ultrametric spaces $(TJ_1) \Rightarrow (Gcap)$, and we obtain from (7) that

$$(FK) + (TJ_1) \Rightarrow (UE_1) \quad \text{that is, } p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-\beta},$$

which was previously proved by A.Bendikov, AG, E.Hu, J.Hu (2021).

Approach to the proof of Theorem 2

Strong generalized capacity condition and elliptic mean value inequality

In the definition of (*Gcap*), in the inequality (6), that is, in

$$\mathcal{E}\left(u^2\phi,\phi\right) \leq \frac{C}{r^\beta}\int_B u^2 d\mu,$$

the cutoff function ϕ may depend on the weight u. Denote by (*Gcap′*) a stronger version of (*Gcap*) when function ϕ depends only on the pair B_0, B of the balls and serves all functions u simultaneously.

The next theorem is the first one in a sequence of results leading to heat kernel upper bounds. We use again the following hypothesis about the tail of the jump kernel:

$$\boxed{\|J(x,\cdot)\|_{L^1(B^c(x,r))} \leq \frac{C}{r^\beta}}, \quad \forall x \in M,\, r>0. \tag{TJ}$$

Theorem 4 *Under the hypothesis* (*V*)*, we have the implication*

$$(FK) + (Gcap) + (TJ) \Rightarrow (Gcap')\,.$$

The condition $(Gcap')$ will be used below for obtaining the *parabolic mean value inequality* that in turn is needed for heat kernel upper bounds. The proof of Theorem 4 uses the *elliptic mean value inequality* (EMV).

Definition. We say that (EMV) holds if, for any function $u \in \mathcal{F} \cap L^\infty$ that is non-negative and subharmonic in a ball $B = B(x_0, R)$, and for any $\varepsilon > 0$,

$$\operatorname*{esup}_{\frac{1}{2}B} u \le C_\varepsilon \left(\fint_B u^2 \right)^{1/2} + \varepsilon \, \|u_+\|_{L^\infty\left(\left(\frac{1}{2}B\right)^c\right)} .$$

The proof of Theorem 4 goes through the following implications (under the standing assumptions (V), (FK), (TJ)):

$$(Gcap) \Rightarrow (EMV) + (cap) \Rightarrow (Gcap') .$$

Parabolic mean value inequality

Theorem 5 *For any $q \in [1, \infty]$, we have*

$$(FK) + (Gcap') + (TJ_q) \Rightarrow (PMV_q),$$

where (PMV_q) stands for the Parabolic Mean Value *inequality that means the following.* Fix an arbitrary ball $B = B(x, R)$ in M and set $T = R^\beta$. Let u be a bounded non-negative function on $M \times (0, T]$ that is *subcaloric* in the cylinder $B \times (0, T]$:

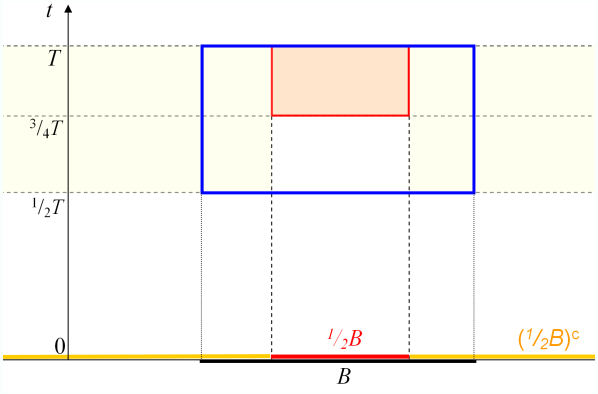
that is, for any $t \in (0, T]$,

$$u(\cdot, t) \in \mathcal{F}_+ \cap L^\infty(M)$$

and u satisfies in $B \times (0, T]$

$$\partial_t u + \mathcal{L} u \le 0$$

in a certain weak sense.



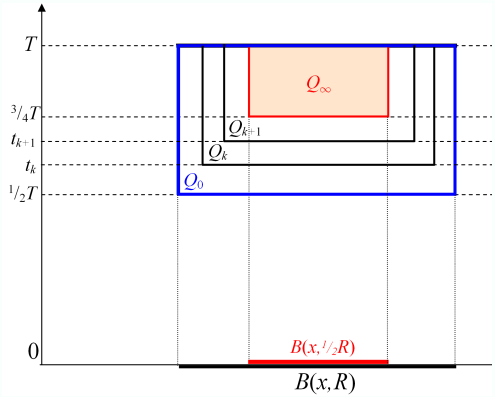
Then, for any $\varepsilon \in (0, 1]$,

$$\sup_{t \in [\frac{3}{4}T, T]} \|u(\cdot, t)\|_{L^\infty(\frac{1}{2}B)} \le C_\varepsilon \left(\fint_{B \times [\frac{1}{2}T, T]} u^2 \right)^{1/2} + \frac{\varepsilon}{R^{\alpha/q'}} \sup_{t \in [\frac{1}{2}T, T]} \|u(\cdot, t)\|_{L^{q'}((\frac{1}{2}B)^c)}. \quad (PMV_q)$$

For the proof, consider a shrinking sequence of cylinders $Q_k = B(x, r_k) \times [t_k, T]\,, \quad k \geq 0,$ and an increasing sequence $b_k > 0$. Set

$$a_k := \int_{Q_k} (u - b_k)_+^2 \, d\mu dt$$

so that a_k clearly decreases, and prove that



$$a_{k+1} \leq \frac{C}{(b_{k+1} - b_k)^{2\frac{\beta}{\alpha}}} \left(\frac{r_k}{r_k - r_{k+1}} \right)^C \left(\frac{1}{(r_k - r_{k+1})^\beta} + \frac{1}{t_{k+1} - t_k} + \frac{s_k}{b_{k+1} - b_k} \right)^{1+\frac{\beta}{\alpha}} a_k^{1+\frac{\beta}{\alpha}},$$

where

$$s_k = \sup_{t \in [t_k, T]} \operatorname*{essup}_{z \in B(x, \frac{r_k + r_{k+1}}{2})} \int_{B^c(x, r_k)} u(y,t) J(z,y) d\mu(y).$$

The proof of the relation between a_k and a_{k+1} uses essentially (FK) and $(Gcap')$.

Choose

$$r_k = (\tfrac{1}{2} + 2^{-k-1})R \quad \text{and} \quad t_k = (\tfrac{3}{4} - 2^{-\beta k - 2})T,$$

so that

$$B \times [\tfrac{1}{2}T, T] = Q_0 \supset Q_k \supset Q_\infty = \tfrac{1}{2}B \times [\tfrac{3}{4}T, T].$$

Setting also $b_k = \left(1 - 2^{-k}\right) b$ for some $b > 0$, we obtain

$$a_{k+1} \leq C2^{Ck} \left(1 + \frac{R^\beta s_k}{b}\right)^{1+\frac{\beta}{\alpha}} \frac{a_k^{1+\frac{\beta}{\alpha}}}{\left(R^{\alpha+\beta} b^2\right)^{\frac{\beta}{\alpha}}}. \tag{9}$$

Iterating (9), we show that if b is large enough then $\lim_{k\to\infty} a_k = 0$,which implies that $u \leq b$ in Q_∞.The choice of b depends on $\sup_k \frac{a_k}{R^{\alpha+\beta}} = \frac{a_o}{R^{\alpha+\beta}}$ and on an upper bound for $R^\beta s_k$. The value

$$\frac{a_0}{R^{\alpha+\beta}} \leq \text{const} \fint_{B\times[\frac{1}{2}T,T]} u^2$$

yields the first term (PMV_q). Estimating s_k by means of the Hölder inequality and (TP_q) gives

$$\begin{aligned} R^\beta s_k &\leq R^\beta \sup_{t\in[\frac{1}{2}T,T]} \|u(\cdot,t)\|_{L^{q'}(\frac{1}{2}B)^c)} \frac{C}{(r_k - r_{k+1})^{\alpha/q'+\beta}} \\ &= \frac{C2^{Ck}}{R^{\alpha/q'}} \sup_{t\in[\frac{1}{2}T,T]} \|u(\cdot,t)\|_{L^{q'}(\frac{1}{2}B)^c)} \end{aligned}$$

which yields the second term in (PMV_q).

Outline of the proof of Theorem 2

Most of the proof is devoted to the implication

$$(FK) + (Gcap) + (TJ_q) \Rightarrow (TP_q)$$

Step 0. As it was already mentioned above,

$$(FK) \Rightarrow (DUE).$$

However, this implication does not work in a more general setting of doubling spaces, where we use an alternative proof of (DUE) with help of the mean value inequality of Theorem 5.

Step 1. By Theorem 4, we have

$$(FK) + (Gcap) + (TJ) \Rightarrow (Gcap')\,,$$

and, by Theorem 5,

$$(FK) + (Gcap') + (TJ_q) \Rightarrow (PMV_q).$$

Step 2. We prove that

$$(PMV_2) \Rightarrow (DUE)\,.$$

For that apply (PMV_2) with $u(\cdot,t) = P_t f$ where $f \in C_0(M)$ and $f \geq 0$, and observe that the both terms in the right hand side of (PMV_q) are bounded by $\frac{C}{R^{\alpha/2}} \|f\|_{L^2}$ which yields

$$\|P_T f\|_\infty \leq \frac{C}{T^{\alpha/(2\beta)}} \|f\|_2\,,$$

which then implies (DUE). Consequently, we obtain that, for any $q \in [2,\infty]$,

$$(FK) + (Gcap) + (TJ_q) \Rightarrow (DUE)\,.$$

It follows from (DUE) that

$$\|p_t(x,\cdot)\|_{L^q(M)} \leq \frac{C}{t^{\alpha/(q'\beta)}}.$$

Hence, in order to prove (TP_q), it remains to prove

$$\boxed{\|p_t(x,\cdot)\|_{L^q(B^c(x,r))} \leq \frac{Ct}{r^{\alpha/q'+\beta}}} \tag{10}$$

assuming that $r^\beta \geq t$, which is done in the rest of the proof.

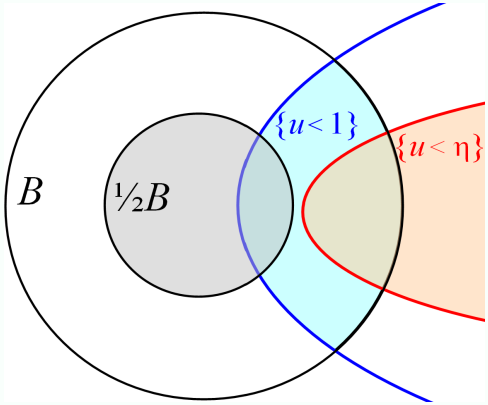
Step 3. We deduce from (PMV_1) a so called "Lemma of growth":

there exist some $\varepsilon, \eta \in (0,1)$ such that, for any ball $B \subset M$ and for any $u \in \mathcal{F}$ that is non-negative and bounded in M and superharmonic in B, if

$$\frac{\mu(B \cap \{u < 1\})}{\mu(B)} \leq \varepsilon,$$

then

$$\operatorname*{essinf}_{\frac{1}{2}B} u \geq \eta.$$



For that observe that $v = \frac{1}{u+a}$ is subharmonic for any $a > 0$. For subharmonic functions, we obtain from (PMV_1) the following multiplicative form of the mean value inequality (by choosing ε):

$$\|v\|_{L^\infty(\frac{1}{2}B)} \leq C A^\theta \max(A, T)^{1-\theta}, \tag{11}$$

where

$$A = \left(\fint_B v^2 d\mu \right)^{1/2}, \qquad T = \|v\|_{L^\infty((\frac{1}{2}B)^c)},$$

and $\theta = \theta(\alpha, \beta) \in (0,1)$.

Let us estimate A as follows:

$$A^2 = \frac{1}{\mu(B)} \left(\int_{B \cap u<1\}} + \int_{B \cap \{u \geq 1\}} \right) \frac{d\mu}{(u+a)^2}$$
$$\leq \frac{\mu(B \cap \{u<1\})}{\mu(B)} \frac{1}{a^2} + \frac{1}{(1+a)^2} \leq \frac{\varepsilon}{a^2} + \frac{1}{(1+a)^2} = \frac{2}{(1+a)^2},$$

for $a = \frac{1}{\varepsilon^{-1/2}-1}$. Estimating also trivially

$$\max(A, T) \leq \frac{1}{a},$$

we obtain from (11)

$$\operatorname*{essup}_{\frac{1}{2}B} \frac{1}{u+a} \leq C \left(\frac{2}{(1+a)^2} \right)^{\theta/2} \left(\frac{1}{a} \right)^{1-\theta} = \frac{C}{(1+a)^{\theta} a^{1-\theta}},$$

whence

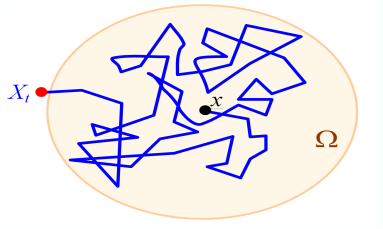
$$\operatorname*{essinf}_{\frac{1}{2}B} u \geq C^{-1} (1+a)^{\theta} a^{1-\theta} - a = a \left(C^{-1} \left(\tfrac{1}{a} + 1 \right)^{\theta} - 1 \right) =: \eta,$$

where $\eta > 0$ if a is small enough, that is, when ε is small enough.

Step 4. For any open set $\Omega \subset M$ and any $x \in \Omega$ set

$$E^{\Omega}(x) = \int_0^{\infty} P_t^{\Omega}\mathbf{1}(x)dt = \int_0^{\infty}\int_{\Omega} p_t^{\Omega}(x,y)d\mu(y)dt.$$

It has the probabilistic meaning of the *mean exit time* from Ω of the jump process X_t, associated with $(\mathcal{E},\mathcal{F})$, that starts at x: $E^{\Omega}(x) = \mathbb{E}_x(\tau^{\Omega})$, where τ^{Ω} is the first exit time from Ω.



In this step we prove that, under (FK), for any ball B of radius r,

$$\operatorname*{essup}_B E^B \le Cr^{\beta}. \tag{12}$$

Step 5. We prove the opposite inequality: the Lemma of growth and (cap) imply that

$$\operatorname*{essinf}_{\frac{1}{4}B} E^B \ge cr^{\beta}. \tag{13}$$

It is known that (12) and (13) imply (C).

Step 6. Using the upper and lower estimates of E^B, we deduce the *survival* inequality: there exist $\varepsilon > 0$ such that, for any ball B of radius r and for any $t > 0$,

$$P_t^B \mathbf{1}_B \geq \varepsilon - \frac{Ct}{r^\beta} \text{ in } \tfrac{1}{4}B. \tag{S}$$

In probabilistic terms,

$$P_t^B \mathbf{1}_B(x) = \mathbb{P}_x\left(\tau_B > t\right)$$

that is the probability of survival of the process in B up to time t assuming the killing condition in B^c.

Step 7. For any $\rho > 0$ consider a *truncated* Dirichlet form

$$\mathcal{E}^{(\rho)}(f,f) := \iint_{\{d(x,y)<\rho\}} (f(x)-f(y))^2 J(x,y) d(x) d\mu(y).$$

Denote by Q_t the heat semigroup of $\left(\mathcal{E}^{(\rho)}, \mathcal{F}\right)$ and by $q_t(x,y)$ its heat kernel. We prove that, under all the above hypotheses, the heat kernel of $\left(\mathcal{E}^{(\rho)}, \mathcal{F}\right)$ exists and satisfies the following diagonal upper bound

$$q_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right). \tag{14}$$

Step 8. We deduce from (S) a similar condition for the truncated semigroup Q_t:

$$Q_t^B \mathbf{1}_B \geq \varepsilon - Ct\left(r^{-\beta} + \rho^{-\beta}\right) \quad \text{in } \tfrac{1}{4}B$$

where $B = B(x,r)$. A certain iteration procedure allows to self-improve this estimate and to obtain that, for any $k \in \mathbb{N}$, if $r \geq 8k\rho$ then

$$Q_t^B \mathbf{1}_B \geq 1 - C(k)\left(\frac{t}{\rho^\beta}\right)^k,$$

which implies that

$$\int_{B^c(x,r)} q_t(x,y)d\mu(y) \leq C(k)\left(\frac{t}{\rho^\beta}\right)^k.$$

Combining this with (14), we obtain that, in the case $q < \infty$,

$$\|q_t(x,\cdot)\|_{L^q(B^c)} \leq \|q_t(x,\cdot)\|_{L^\infty(B^c)}^{1/q'} \|q_t(x,\cdot)\|_{L^1(B^c)}^{1/q} \leq \frac{C(k)}{t^{\alpha/(q'\beta)}} \exp\left(\frac{Ct}{\rho^\beta}\right)\left(\frac{t}{\rho^\beta}\right)^{\frac{k}{q}}. \tag{15}$$

In the case $q = \infty$ we improve (14) in a different way and obtain that if $r \geq 4k\rho$ then

$$\|q_t(x,\cdot)\|_{L^\infty(B^c)} \leq \frac{C(k)}{t^{\alpha/\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right)\left(1 + \frac{\rho^\beta}{t}\right)^{\alpha/\beta}\left(\frac{t}{\rho^\beta}\right)^k. \tag{16}$$

Step 9. We prove that, under all the above conditions, including (TJ_q), we have, for any $t > 0$ and for any ball $B = B(x, r)$,

$$\|p_t(x,\cdot)\|_{L^q(B^c)} \le \|q_t(x,\cdot)\|_{L^q(B^c)} + \frac{Ct}{\rho^{\alpha/q'+\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right). \tag{17}$$

Step 10. In the case $q < \infty$, combining (15) and (17), we obtain that if $r \ge 8k\rho$ then

$$\|p_t(x,\cdot)\|_{L^q(B^c)} \le \frac{C(k)}{t^{\alpha/(q'\beta)}} \exp\left(\frac{Ct}{\rho^\beta}\right) \left(\frac{t}{\rho^\beta}\right)^{k/q} + \frac{Ct}{\rho^{\alpha/q'+\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right).$$

Assuming that $r^\beta \ge t$ and setting $\rho = r/(8k)$, we obtain

$$\begin{aligned}\|p_t(x,\cdot)\|_{L^q(B^c)} &\le \frac{C(k)}{t^{\alpha/(q'\beta)}} \left(\frac{t}{r^\beta}\right)^{k/q} + \frac{C(k)t}{r^{\alpha/q'+\beta}} \\ &\le C\frac{t}{r^{\alpha/q'+\beta}},\end{aligned}$$

provided k is chosen so that

$$\left(\frac{t}{r^\beta}\right)^{k/q} \le \left(\frac{t}{r^\beta}\right)^{\frac{\alpha}{q'\beta}+1},$$

that is,

$$\frac{k}{q} \geq \frac{\alpha}{q'\beta} + 1.$$

This finishes the proof of (TP_q) if $q < \infty$.

In the case $q = \infty$ we obtain from (16) and (17), assuming that $r^\beta \geq t$ and setting $\rho = r/(4k)$ that

$$\begin{aligned} \|p_t(x,\cdot)\|_{L^q(B^c)} &\leq \frac{C(k)}{t^{\alpha/\beta}} \left(\frac{t}{r^\beta}\right)^{k-\frac{\alpha}{\beta}} + \frac{C(k)t}{r^{\alpha+\beta}} \\ &\leq C\frac{t}{r^{\alpha+\beta}}, \end{aligned}$$

provided k is chosen so that

$$\left(\frac{t}{r^\beta}\right)^{k-\frac{\alpha}{\beta}} \leq \left(\frac{t}{r^\beta}\right)^{\frac{\alpha}{\beta}+1}$$

that is,

$$k \geq 2\frac{\alpha}{\beta} + 1.$$

Step 11. We prove now consequences of (TP_q). Let us first prove Lemma 1, that is, if $q \in [2, \infty]$ then

$$\boxed{(TP_q) \Rightarrow (UE_q)}.$$

Setting $r = \frac{1}{2} d(x, y)$, we obtain by the semigroup property

$$\begin{aligned} p_{2t}(x, y) &= \int_M p_t(x, z)\, p_t(z, y)\, d\mu(z) \\ &\leq \left(\int_{B^c(x,r)} + \int_{B^c(y,r)} \right) p_t(x, z)\, p_t(z, y)\, d\mu(z). \end{aligned}$$

It suffices to estimate the first integral. By the Hölder inequality, we have

$$\int_{B^c(x,r)} p_t(x, z)\, p_t(z, y)\, d\mu(z) \leq \|p_t(x, \cdot)\|_{L^q(B^c(x,r))}\, \|p_t(\cdot, y)\|_{L^{q'}(M)}.$$

Since $q \geq 2$ and, hence, $q' \leq q$, we have not only (TP_q) but also $(TP_{q'})$. Hence,

$$\|p_t(x, \cdot)\|_{L^q(B^c(x,r))} \leq \frac{C}{t^{\alpha/(q'\beta)}} \left(1 + \frac{r}{t^{1/\beta}}\right)^{-(\alpha/q' + \beta)}$$

and

$$\|p_t(\cdot, y)\|_{L^{q'}(M)} \leq \frac{C}{t^{\alpha/(q\beta)}}.$$

Since $\frac{\alpha}{q'\beta} + \frac{\alpha}{q\beta} = \frac{\alpha}{\beta}$, we obtain

$$\int_{B^c(x,r)} p_t(x,z)\, p_t(z,y)\, d\mu(z) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}.$$

Estimating in the same manner the second integral, we obtain

$$p_{2t}(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)},$$

that is, (UE_q).

Step 12. Since $(UE_q) \Rightarrow (DUE) \Rightarrow (FK)$, we obtain that

$$\boxed{(TP_q) \Rightarrow (FK)}.$$

The implication

$$\boxed{(TP_q) \Rightarrow (TJ_q)}$$

was already mentioned in (**??**).

Step 13. Finally, the implication

$$\boxed{(TP_q) + (C) \Rightarrow (Gcap)}$$

is proved as follows. By (TP_q) we have also (TP_1), that is,

$$\int_{B^c(x,r)} p_t(x,y)d\mu(y) \leq C\left(1 + \frac{r}{t^{1/\beta}}\right)^{-\beta} \leq \frac{Ct}{r^\beta}.$$

This and (C) imply that

$$P_t^{B(x,r)}\mathbf{1}(x) \geq \varepsilon - \frac{Ct}{r^\beta}$$

that is, (S), and it is known that $(S) \Rightarrow (Gcap)$.