Isotropic Markov semigroups on ultra-metric spaces*

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Dedicated to the memories of
V.S. Vladimirov (1923–2012) and M.H. Taibleson (1929–2004)

Abstract

Let (X, d) be a separable ultra-metric space with compact balls. Given a reference measure μ on X and a distance distribution function σ on $[0, \infty)$, we construct a symmetric Markov semigroup $\{P^t\}_{t\geq 0}$ acting in $L^2(X, \mu)$. Let $\{\mathcal{X}_t\}$ be the corresponding Markov process. We obtain upper and lower bounds of its transition density and its Green function, give a transience criterion, estimate its moments and describe the Markov generator $\mathcal{L}$ and its spectrum which is pure point. In the particular case when $X = \mathbb{Q}_p^n$, where $\mathbb{Q}_p$ is the field of p-adic numbers, our construction recovers the Taibleson Laplacian (spectral multiplier), and we can also apply our theory to the study of the Vladimirov Laplacian. Even in this well established setting, several of our results are new. We also elaborate the relation between the Markov process $\{\mathcal{X}_t\}$ and Kigami's process on the boundary of a tree, which is induced by a random walk on the tree. In conclusion, we provide examples illustrating the interplay between the fractional derivatives and random walks.

Contents

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1 Introduction

In the past three decades there has been an increasing interest in various constructions of Markov chains on ultra-metric (totally disconnected) spaces, such as the Cantor set or the field of p-adic numbers. In this paper we introduce and study a class of symmetric Markov semigroups and their generators on ultra-metric spaces. Our construction is very transparent, and it leads to a number of new results as well as to a better understanding of previously known results.

Let (X,d) be a metric space. The metric d is called an *ultra-metric* if it satisfies the ultra-metric inequality

$$d(x,y) \leq \max\{d(x,z), d(z,y)\}, \tag{1.1}$$

that is obviously stronger than the usual triangle inequality. In this case (X,d) is called an ultra-metric space.

We will always assume in addition that the ultra-metric space (X,d) in question is separable, and that every closed ball

$$B_r(x) = \{y \in X : d(x,y) \leq r\} \tag{1.2}$$

is compact. The latter implies that (X,d) is complete.

The ultra-metric property (1.1) implies that the balls in an ultra-metric space (X,d) look very differently from familiar Euclidean balls. In particular, any two ultra-metric balls of the same radius are either disjoint or identical. Consequently, the collection of all distinct balls of the same radius r forms a partition of X.

One of the best known examples of an ultra-metric space is the field $\mathbb{Q}_p$ of p-adic numbers endowed with the p-adic norm $\|x\|_p$ and the p-adic ultra-metric $d(x,y) = \|x-y\|_p$. Moreover, for any integer $n \geq 1$, the p-adic n-space $\mathbb{Q}_p^n = \mathbb{Q}_p \times ... \times \mathbb{Q}_p$ is also an ultra-metric space with the ultra-metric $d_n(x, y)$ defined as

$$d_n(x, y) = \max\{d(x_1, y_1), ..., d(x_n, y_n)\}.$$

If the group of isometries of an ultra-metric space (X, d) acts transitively on X, then (X, d) is in fact a locally compact Abelian group, which in particular is the case for $\mathbb{Q}_p^n$.

In literature one distinguishes the following two subclasses of ultra-metric spaces:

(i) (X, d) is discrete and infinite.

(ii) (X, d) is perfect (that is, X contains no isolated point).

Various constructions of Markov processes in the setting (ii), when X in addition is a locally compact Abelian group, have been developed by Evans [22], Haran [29], [30], Ismagilov [33], Kochubei [38], [39], Albeverio and Karwowski [1], [2], Albeverio and Zhao [3], Del Muto and Figá-Talamanca [42], [43], Rodrigues-Vega and Zuniga-Galindo [66], [50]. They studied X-valued infinitely divisible random variables and processes by using tools of Fourier analysis; for general references, see Hewitt and Ross [31], Taibleson [55] and Kochubei [39]. Note that Taibleson's spectral multipliers on $\mathbb{Q}_p^n$ are early forerunners of the Laplacians that we are considering here.

Pearson and Bellissard [45] and Kigami [36], [37] considered random walks on the Cantor set, resp. the Cantor set minus one point. In [36], [37], a main focus is on the interplay between random walks on trees and jump process on their boundaries. In this context, we also mention Aldous and Evans [4] and Chen, Fukushima and Ying [15]. We shall come back to Kigami's work in the last three sections of this paper.

An entirely different approach was developed by Vladimirov, Volovich and Zelenov [57], [59]. They were concerned with p-adic analysis (Bruhat distributions, Fourier transform etc.) related to the concept of p-adic Quantum Mechanics, and introduced a class of pseudo-differential operators on $\mathbb{Q}_p$ and on $\mathbb{Q}_p^n$. In particular, they studied the p-adic Laplacian defined on $\mathbb{Q}_p^3$ and the corresponding p-adic Schrödinger equation. In particular, they explicitly computed (as series expansions) certain heat kernels as well as the Green function of the p-adic Laplacian. In connection with the theory of pseudo-differential operators on general totally disconnected groups we mention here the pioneering work of Saloff-Coste [51].

Discrete ultra-metric spaces (X, d) (as in (i)) were treated by Bendikov, Grigor'yan and Pittet [7], the direct forerunner of the present work. Among the examples of such spaces we mention the class of locally finite groups: a countable group G is locally finite if any of its finite subsets generates a finite subgroup. Every locally finite group G is the union of an increasing sequence of finite subgroups $\{G_n\}$. An ultra-metric d in G can be defined as follows: $d(x, y)$ is the minimal value of n such that x and y belong to a common coset of G_n.

Since locally finite groups are not finitely generated, the basic notions of geometric group theory such as the word metric, volume growth, isoperimetric inequalities, etc. (cf. e.g. [16], [28], [52], [46], [47], [48], , [56], [61]), do not apply in this setting. The notion of an ultra-metric can be used instead of the word metric in this setting (see [5], [7], [6]).

Selecting a set of generators for each subgroup G_n of a locally finite group G, one defines thereby a random walk, that is, a Markov kernel on G_n. Taking a convex combination of the Markov kernels across all G_n, one obtains a Markov kernel on G that determines a random walk on G. Such random walks have been studied by Darling and Erdös [17], Kesten and Spitzer [35], Flatto and Pitt [26], Fereig and Molchanov [25], Kasymdzhanova [34], Cartwright [13], Lawler [40], Brofferio and Woess [11], see also Bendikov and Saloff-Coste [9]. In particular, [40] has a remarkable general criterion of recurrence of such random walks. Further results on Markov processes on ultra-metric spaces can be found in [18], [19], [23], [24], [41], [49].

Many of the results in the above-mentioned literature are covered by our approach via ultra-metrics. We develop tools to analyse a natural class of Markov processes on ultra-metric spaces without assuming any group structure. In particular, the nature of our argument allows us to bring into consideration an arbitrary Radon measure μ on X (instead of the Haar measure in the case of groups), that is used as a speed measure for a Markov process.

So, given an ultra-metric space (X, d), fix a Radon measure μ on X with full support and define the family $\{\mathrm{Q}_r\}_{r>0}$ of averaging operators acting on non-negative or bounded Borel functions $f : X \to \mathbb{R}$ by

$$\mathrm{Q}_r f(x) = \frac{1}{\mu\left(B_r(x)\right)} \int_{B_r(x)} f \, d\mu. \tag{1.3}$$

Note that $0 < \mu\left(B_r\left(x\right)\right) < \infty$ for all $x \in X$ and $r > 0$. The operator Q_r has the kernel

$$K_r(x, y) = \frac{1}{\mu\left(B_r(x)\right)} \mathbf{1}_{B_r(x)}(y). \tag{1.4}$$

It is symmetric in x, y because $B_r(x) = B_r(y)$ for any $y \in B_r(x)$. Clearly, Q_r is a Markov operator on the space $\mathcal{B}_b\left(X\right)$ of bounded Borel functions on X, that is, $\mathrm{Q}_r f \geq 0$ if $f \geq 0$ and $\mathrm{Q}_r 1 = 1$. Hence, Q_r extends to a bounded self-adjoint operator in $L^2\left(X, \mu\right)$.

Let us choose a function σ that satisfies the following assumptions:

$$\begin{aligned} &\sigma : [0, \infty] \to [0, 1] \ \textit{is a strictly monotone increasing} \\ &\textit{left-continuous function, such that } \sigma\left(0+\right) = 0 \textit{ and } \sigma\left(\infty\right) = 1. \end{aligned} \tag{1.5}$$

Then the operator

$$Pf = \int_0^\infty \mathrm{Q}_r f \, d\sigma(r) \tag{1.6}$$

is also a Markov operator in $\mathcal{B}_b\left(X\right)$ as well as a bounded self-adjoint operator in $L^2\left(X, \mu\right)$.

The operator P determines a discrete time Markov chain $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ on X with the following transition rule: a random point $\mathcal{X}_{n+1}$ is μ-uniformly distributed in $B_r(\mathcal{X}_n)$ where the radius r is chosen at random according to the probability distribution σ. For that reason we refer to σ as the *distance distribution function.*

Note that the operator P is determined by the triple (d, μ, σ). We refer to P as an *isotropic Markov* operator associated with (d, μ, σ). The isotropic Markov operator P has some unique features arising from the ultra-metric property. First of all, let us mention the following simple identity:

$$\mathrm{Q}_r \, \mathrm{Q}_s = \mathrm{Q}_s \, \mathrm{Q}_r = \mathrm{Q}_{\max\{r,s\}}. \tag{1.7}$$

Indeed, for any ball B of radius r, *any* point $x \in B$ is a center of B. Since the value $\mathrm{Q}_r f\left(x\right)$ is the average of f in B, we see that $\mathrm{Q}_r f\left(x\right)$ does not depend on $x \in B$; that is, $\mathrm{Q}_r f = \mathrm{const}$ on B. Now, if $s \leq r$ then the application of Q_s to $\mathrm{Q}_r f$ does not change this constant, whence we obtain $\mathrm{Q}_s \mathrm{Q}_r f = \mathrm{Q}_r f$. On the other hand, if $s > r$ then any ball of radius s is the disjoint union of finitely many balls of radius r. Since the integrals of f and $\mathrm{Q}_r f$ over any such ball are the same, we obtain $\mathrm{Q}_s \mathrm{Q}_r f = \mathrm{Q}_s f$.

Since by (1.7) $\mathrm{Q}_r^2 = \mathrm{Q}_r$, we obtain that Q_r is an *orthoprojector*[1] in L^2. In particular, $\operatorname{spec} \mathrm{Q}_r \subset [0, 1]$.

It follows from (1.6) that the spectral projectors in the spectral decomposition of P are the averaging operators Q_r, up to a change of variables (cf. (2.6). The fact that the spectral projectors are themselves Markov operators brings up a new insight, new technical possibilities, and a new type of results, that have no analogue in other commonly used settings.

[1]Let us mention for comparison, that the analogous averaging operator in $\mathbb{R}^n$ is also bounded and self-adjoint, but it has a non-empty negative part of the spectrum. In particular, it is not an orthoprojector.

In particular, the Markov operator P is non-negative definite, which allows us to define the powers P^t for all $t \geq 0$. Then $\left\{P^t\right\}_{t\geq 0}$ is a symmetric strongly continuous Markov semigroup. It follows from (1.6) that P^t admits for $t > 0$ the following representation:

$$P^t f(x) = \int_0^\infty \mathrm{Q}_r f(x)\, d\sigma^t(r)\,. \tag{1.8}$$

Alternatively, one can *define* P^t by (1.8) and then use formula (1.7) to derive that $P^s P^t = P^{s+t}$.

The semigroup $\left\{P^t\right\}_{t\geq 0}$ determines a continuous time Markov process $\{\mathcal{X}_t\}_{t\geq 0}$. Since the n-step transition operator of the discrete time Markov chain $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ is P^n, we see that the discrete time Markov chain coincides with the restriction of the continuous time Markov process $\{\mathcal{X}_t\}$ to integer values of t. This allows us to concentrate on the study of the continuous time process $\{\mathcal{X}_t\}_{t\geq 0}$ only.

We refer to the Markov semigroup $\left\{P^t\right\}_{t\geq 0}$ defined by (1.3)-(1.8) as an *isotropic semigroup*, and to the jump process $\{\mathcal{X}_t\}_{t\geq 0}$ as an *isotropic process,* associated with the triple (d, μ, σ).

Let us briefly describe the content of the present paper that is devoted to the study of isotropic semigroups.

In Section 2 we construct the isotropic semigroup as above and provide explicit formulas for its heat kernel $p\,(t, x, y)$ (=the transition density of the process $\{\mathcal{X}_t\}$). As indicated above, our approach is based upon the observation that the building blocks of the operator P, namely, the averaging operators Q_r of (1.3), are orthogonal projectors in $L^2(X, \mu)$, which enables us to engage at an early stage the methods of spectral theory and functional calculus.

We establish some basic properties of the heat kernel, for example, its continuity away from the diagonal, and prove upper and lower bounds in terms of t and $d\,(x, y)$.

For example, in $\mathbb{Q}_p$ with the p-adic ultra-metric $\|x-y\|_p$ and the Haar measure μ, the most natural choice of the distance distribution function is

$$\sigma\,(r) = \exp\left(-\left(\frac{p}{r}\right)^\alpha\right), \quad \alpha > 0. \tag{1.9}$$

Then the associated heat kernel admits the estimate

$$p_t\,(x, y) \simeq \frac{t}{(t^{1/\alpha} + \|x-y\|_p)^{1+\alpha}} \tag{1.10}$$

for all $t > 0$ and $x, y \in \mathbb{Q}_p$. Note that the estimate (1.10) is similar to the heat kernel bound for a symmetric α-stable process in $\mathbb{R}$.

We also obtain explicit expression for the Green function of the isotropic semigroup and provide a transience criterion in terms of the volume growth. Unlike the previously known transience criteria (cf. [40]), ours does not assume any group structure.

In Section 3 we are concerned with the spectral properties of the *isotropic Laplacian* $\mathcal{L}$ that is the (positive definite) generator of the isotropic semigroup, that is, $P^t = e^{-t\mathcal{L}}$. We provide a full description of the spectrum of $\mathcal{L}$, in particular, we show that the spectrum is pure point. We list explicitly all the eigenfunctions of $\mathcal{L}$ and we prove that the spectra of the extensions of $\mathcal{L}$ in the spaces L^p, $1 \leq p < \infty$, do not depend on p.

A striking property of the isotropic Laplacian $\mathcal{L}$ is that, for any increasing bijection ψ : $[0, \infty) \to [0, \infty)$, the operator $\psi\,(\mathcal{L})$ is also an isotropic Laplacian (for another distance distribution function). In particular, $\mathcal{L}^\alpha$ is an isotropic Laplacian for any $\alpha > 0$. Recall for comparison that, for a general symmetric Markov generator $\mathcal{L}$, the operator $\mathcal{L}^\alpha$ generates a Markov semigroup only for $0 < \alpha \leq 1$.

In Section 4 we obtain two sided estimates of moments of the isotropic process $\{\mathcal{X}_t\}$.

In the case when X is a locally compact group, our results apply with an arbitrary Radon measure μ instead of the Haar measure. Some of the aforementioned questions are particularly sensitive to the choice of the measure μ, for example, the heat kernel and Green function estimates. On the other hand, the spectrum of the Laplacian and the moment bounds do not depend on μ. These quantities depend strongly on the choice of the ultra-metric d, whereas the eigenfunctions depend both on d and μ.

In Section 5 we compare our isotropic Laplacian with other previously known "differential" operators in $\mathbb{Q}_p$ and $\mathbb{Q}_p^n$. The notion of fractional derivative $\mathfrak{D}^\alpha$ on functions on $\mathbb{Q}_p$ was introduced by Vladimirov [57] by means of Fourier transform in $\mathbb{Q}_p$. The operator $\mathfrak{D}^\alpha$ coincides with the operator of Taibleson [55], introduced in a quite different context of Riesz multipliers on $\mathbb{Q}_p^n$. We show that $\mathfrak{D}^\alpha$ coincides with our isotropic Laplacian $\mathcal{L}_\alpha$ associated with the distance distribution function (1.9). In particular, this implies that the heat kernel of $\mathfrak{D}^\alpha$ satisfies the estimate (1.10). Note that previously only an upper bound for the heat kernel of $\mathfrak{D}^\alpha$ was known (cf. Kochubei [39, Ch.4.1, Lemma 4.1]). We also give a simple proof for a previously known explicit formula for the Green function of $\mathfrak{D}^\alpha$.

Using functional calculus of the operator $\mathfrak{D}^1$, we give a full description of the class of all rotation invariant Markov generators on $\mathbb{Q}_p$. This class includes but is not restricted to the isotropic Laplacians. As a consequence, we obtain that the class of all rotation invariant Markov processes in $\mathbb{Q}_p$ coincides with the class of Markov processes constructed by Albeverio and Karwowski [2] by use of much more involved technical tools.

Next we consider "partial differential" operators on $\mathbb{Q}_p^n$. The p-adic Laplacian of Vladimirov on $\mathbb{Q}_p^n$ is defined as a direct sum of the operators $\mathfrak{D}^\alpha$ acting separately on each coordinate. Although this operator is not an isotropic Laplacian, it can be studied within our setting, which gives simple direct proofs of many results of [59], without using Fourier Analysis and the theory of Bruhat distributions.

Another multidimensional generalization of $\mathfrak{D}^\alpha$ is the Taibleson operator $\mathfrak{T}^\alpha$ in $\mathbb{Q}_p^n$ that is defined by means of Fourier transform in $\mathbb{Q}_p^n$. We show that the operator $\mathfrak{T}^\alpha$ is an isotropic Laplacian, which allows to obtain detailed analytic results.

In Section 6 we use the fact that every locally compact ultra-metric space arises as the boundary of a locally finite tree. Using that we relate random walks[2] on the tree with isotropic jump processes on its boundary. Kigami in [36] starts with a transient nearest neighbour random walk on a tree and constructs a naturally associated jump process on the boundary of the tree: given the Dirichlet form of the random walk on the tree, the boundary process is induced by the Dirichlet form that reproduces the energy of a harmonic function on the tree via its boundary values. This is analogous to the well-known Douglas integral [21] on the unit disk. Using this approach, [36] undertakes a detailed analysis of the process on the boundary.

Restricting attention at first to the compact case, we answer in Section 7 the obvious question how the approach of Kigami and that of the present paper are related. The relation is basically one-to-one: every boundary process induced by a random walk is an isotropic process in our sense. Conversely, we show that, up to a unique linear time change, every isotropic process on the boundary of a tree arises from a uniquely determined random walk on the tree as in [36]. In addition, we explain how the boundary process on a tree transforms into an isotropic process on the non-compact ultra-metric space given by a punctured boundary of the tree. This should be compared with [37].

Finally, in Section 8 we construct explicitly the random walks on the trees, which correspond to fractional derivatives on the (compact) group $\mathbb{Z}_p$ of p-adic integers and on the whole of $\mathbb{Q}_p$.

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[2]Discrete time random walks of nearest neighbour type on a tree are very well understood – see the book by Woess [63, Ch. 9]

2 Isotropic semigroup and the heat kernel

Throughout this paper, (X, d) is an ultra-metric space which is separable, and such that all d-balls $B_r(x)$ are compact.

2.1 Averaging operator

Recall that for any $r > 0$,

$$\mathrm{Q}_r f(x) = \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f \, d\mu$$

is an orthoprojector in $L^2 \equiv L^2(X, \mu)$ (cf. (1.3)), and the image of Q_r is the subspace $\mathcal{V}_r$ of L^2 that consists of all functions taking constant values on each ball radius r.

Clearly, the family $\{\mathcal{V}_r\}_{r>0}$ is monotone decreasing with respect to set inclusion. It follows that there exists the limit

$$\mathrm{Q}_\infty := s\text{-}\lim_{r\to\infty} \mathrm{Q}_r$$

in the strong operator topology, which is an orthoprojector onto $\mathcal{V}_\infty = \bigcap_{r>0} \mathcal{V}_r$. It follows that $\mathcal{V}_\infty$ consists of constant functions. If $\mu(X) = \infty$ then $\mathcal{V}_\infty = \{0\}$ and $\mathrm{Q}_\infty = 0$, while in the case $\mu(X) < \infty$ we have $\dim \mathcal{V}_\infty = 1$ and

$$\mathrm{Q}_\infty f = \frac{1}{\mu(X)} \int_X f d\mu. \tag{2.1}$$

Set also $\mathrm{Q}_0 := \mathrm{id}$.

Lemma 2.1 *The family $\{\mathrm{Q}_r\}_{r\in[0,\infty)}$ of orthoprojectors is strongly right continuous in r.*

Proof. Let us first show that $r \mapsto \mathrm{Q}_r$ is strongly continuous at $r = 0$, that is,

$$s\text{-}\lim_{s\to 0+} \mathrm{Q}_s = \mathrm{id}. \tag{2.2}$$

Let f be a continuous function on X with compact support. Then, for any $x \in X$,

$$\mathrm{Q}_s f(x) \to f(x) \text{ as } s \to 0.$$

Since the family $\{\mathrm{Q}_s f\}_{s\in(0,1)}$ is uniformly bounded by $\sup |f|$ and is uniformly compactly supported, it follows by the dominated convergence theorem that

$$\|\mathrm{Q}_s f - f\|_{L^2} \to 0 \text{ as } s \to 0. \tag{2.3}$$

Since the space of continuous functions with compact support is dense in L^2, by a standard approximation argument (2.3) extends to all $f \in L^2$, whence (2.2) follows.

Next, let us prove that $r \mapsto \mathrm{Q}_r$ is strongly right continuous at any $r > 0$, that is,

$$s\text{-}\lim_{s\to r+} \mathrm{Q}_s = \mathrm{Q}_r. \tag{2.4}$$

It suffices to show that, for any continuous function f with compact support,

$$\|\mathrm{Q}_s f - \mathrm{Q}_r f\|_{L^2} \to 0 \text{ as } s \to r+. \tag{2.5}$$

Indeed, for any $x \in X$, the function $r \mapsto \mathrm{Q}_r f(x)$ is right continuous by (1.3) as the balls are closed, whence (2.5) follows by the dominated convergence theorem. ■

For any $\lambda \in \mathbb{R}$ set

$$E_\lambda = \begin{cases} \mathrm{Q}_{1/\lambda}, & \lambda > 0, \\ 0, & \lambda \le 0. \end{cases} \tag{2.6}$$

Note that $E_{0+} = \mathrm{Q}_\infty$. It follows from the above properties of Q_r that the family $\{E_\lambda\}$ of ortho-projectors in L^2 is a left-continuous spectral resolution. Consequently, for any Borel function $\varphi : [0,\infty) \to \mathbb{R}$, the integral

$$\int_{[0,\infty)} \varphi(\lambda)\, dE_\lambda$$

determines a self-adjoint non-negative definite operator, which is bounded if and only if φ is bounded.

2.2 Basic properties of isotropic semigroup

Consider now the operator P defined by (1.6) with a function σ as in (1.5). Observe that the integral in (1.6) converges in the strong operator topology since, for any $f \in L^2$,

$$\int_0^\infty \|Q_r f\|_{L^2}\, d\sigma(r) < \infty.$$

On the other hand, for any $f \in \mathcal{B}_b(X)$, the integral (1.6) converges pointwise. Moreover, in this case the function Pf is continuous, because the function $x \mapsto \int_\varepsilon^\infty \mathrm{Q}_r f(x)\, d\sigma(r)$ is for any $\varepsilon > 0$ locally constant and, hence, continuous and it converges uniformly to $Pf(x)$ as $\varepsilon \to 0$.

As it was already observed, P is a self-adjoint operator in L^2 and $\operatorname{spec} P \subset [0,1]$. In particular, for any $t > 0$, the power P^t is well defined. Set also $P^0 := \mathrm{id}$. In the next statement we collect basic properties of P^t.

Theorem 2.2 *(a) The family $\{P^t\}_{t\ge0}$ is a strongly continuous symmetric Markov semigroup on $L^2(X,\mu)$.*

(b) For any $t > 0$, the operator P^t has the representation (1.8), that is,

$$P^t f = \int_{[0,\infty)} \mathrm{Q}_r f\, d\sigma^t(r).$$

(c) For any $t > 0$, the operator P^t admits an integral kernel $p(t,x,y)$, that is, for all $f \in \mathcal{B}_b$ and $f \in L^2$,

$$P^t f(x) = \int_X p(t,x,y) f(y) d\mu(y), \tag{2.7}$$

where $p(t,x,y)$ is given by

$$p(t,x,y) = \int_{[d(x,y),\infty)} \frac{d\sigma^t(r)}{\mu(B_r(x))}. \tag{2.8}$$

The function $p(t,x,y)$ is called the *heat kernel* of the semigroup $\{P^t\}$. It is clear from (2.8) that $p(t,x,y) < \infty$ for all $t > 0$ and $x \ne y$, whereas under certain conditions $p(t,x,x)$ can be equal to ∞.

For $f \in \mathcal{B}_b$ the identity (2.7) holds pointwise, that is, for all $x \in X$, whereas for $f \in L^2$ (2.7) is an identity of two L^2-functions, that it, it holds for μ-almost all x.

Proof. It follows from (1.6) by integration by parts that, for any $f \in L^2$,

$$Pf = \int_{[0,\infty)} \mathrm{Q}_r f\; d\sigma(r) = Q_\infty f - \int_{(0,\infty)} \sigma(r)\, d\mathrm{Q}_r f. \tag{2.9}$$

Changing $\lambda = 1/r$ and using (2.6), we obtain

$$Pf = (E_{0+}) f + \int_{(0,\infty)} \sigma(1/\lambda)\, dE_\lambda f = \int_{[0,\infty)} \sigma(1/\lambda)\, dE_\lambda f,$$

using the convention $\sigma(\infty) = 1$. Hence, we obtain the spectral decomposition of P in the following form:

$$P = \int_{[0,\infty)} \sigma(1/\lambda)\, dE_\lambda. \tag{2.10}$$

It follows that

$$P^t = \int_{[0,\infty)} \sigma^t(1/\lambda)\, dE_\lambda. \tag{2.11}$$

(a) The semigroup identity $P^tP^s = P^{t+s}$ is a straightforward consequence of (2.11) and the functional calculus. The strong continuity condition

$$s\text{-}\lim_{t\to 0+} P^t = \mathrm{id}$$

follows also from (2.11) because $\sigma(1/\lambda) > 0$ for $\lambda \in [0,\infty)$ and, hence, $\sigma^t(1/\lambda) \to 1$ as $t \to 0+$.

(b) Reversing the argument in the derivation of (2.11) from (2.9), we obtain that (2.11) implies

$$P^t f = \int_{[0,\infty)} \mathrm{Q}_r f \; d\sigma^t(r).$$

(c) It follows from (b), (1.3) and Fubini that, for any $f \in \mathcal{B}_b$,

$$\begin{aligned} P^t f(x) &= \int_{[0,\infty)} \left(\frac{1}{\mu(B_r(x))} \int_X \mathbf{1}_{B_r(x)}(y) f(y) d\mu(y) \right) d\sigma^t(r) \\ &= \int_X \left(\int_{[d(x,y),\infty)} \frac{1}{\mu(B_r(x))} d\sigma^t(r) \right) f(y) d\mu(y) \\ &= \int_X p(t,x,y) f(y)\, d\mu(y). \end{aligned}$$

For $f \in L^2$ it follows by approximation argument. ■

Remark 2.3 In the proof of Theorem 2.2 we have not used at full strength the fact that σ is *strictly* monotone increasing (cf. (1.5)). For that theorem, it suffices to assume that σ is monotone increasing and $\sigma(r) > 0$ for $r > 0$.

Remark 2.4 If one takes (1.8) as definition of the operator P^t, then one can prove the semigroup identity $P^tP^s = P^{t+s}$ by means of (1.7). Indeed, for any given $s, t > 0$ and $f \in L^2$, we have

$$\begin{aligned} P^s P^t f(x) &= \int_0^\infty d\sigma^s(r) \int_0^\infty d\sigma^t(r')\, \mathrm{Q}_r \mathrm{Q}_{r'} f(x) = \\ &= \int_0^\infty d\sigma^s(r) \int_0^\infty d\sigma^t(r')\, \mathrm{Q}_{\max\{r,r'\}} f(x). \end{aligned}$$

Let ξ_1 and ξ_2 be two independent random variables with distributions σ^s and σ^t, respectively. Then the distribution of the random variable $\xi = \max\{\xi_1, \xi_2\}$ is σ^{t+s}. It follows that

$$P^s P^t f(x) = \mathbb{E}\left(\mathrm{Q}_{\max\{\xi_1,\xi_2\}} f(x)\right) = \int_0^\infty \mathrm{Q}_r f(x)\, d\sigma^{t+s}(r) = P^{t+s} f(x).$$

Corollary 2.5 *For all $x, y \in X$ and $t > 0$, we have $p(t, x, y) > 0$,*

$$p(t, x, y) = p(t, y, x),$$

and

$$p(t,x,y) \leq \min\{p(t,x,x), p(t,y,y)\}. \tag{2.12}$$

Proof. The strict positivity of $p(t, x, y)$ follows from (2.8) and the strict monotonicity of σ.

In the integral in (2.8) we have $r \geq d(x, y)$ whence it follows that $B_r(x) = B_r(y)$ and $p(t, x, y) = p(t, y, x)$. Alternatively, the symmetry of the heat kernel follows also from the fact that P^t is self-adjoint.

By (2.8) we have

$$p(t,x,y) = \int_{[d(x,y),\infty)} \frac{d\sigma^t(r)}{\mu(B_r(x))} \leq \int_{[0,\infty)} \frac{d\sigma^t(r)}{\mu(B_r(x))} = p(t,x,x),$$

whence (2.12) follows. ■

Note that in general, heat kernels only satisfy the estimate

$$p(t, x, y) \leq \sqrt{p(t, x, x)\, p(t, y, y)}.$$

The estimate (2.12) is obviously stronger, which reflects a special feature of ultra-metricity.

Corollary 2.6 *For any $t > 0$, the function*

$$x, y \mapsto \begin{cases} \frac{1}{p(t,x,y)}, & x \neq y, \\ 0, & x = y, \end{cases} \tag{2.13}$$

is an ultra-metric.

Proof. Set

$$F(x, r) = \left(\int_{[r,+\infty)} \frac{d\sigma^t(s)}{\mu(B_s(x))} \right)^{-1} \quad \text{for } r > 0,$$

$F(x, 0) = 0$, and observe the following two properties of F:

(a) $r \mapsto F(x, r)$ is monotone increasing in r;

(b) $F(x, r) = F(y, r)$ wherever $r \geq d(x, y)$ as in this case $B_s(x) = B_s(y)$ for all $s \geq r$.

For any function F with these properties, $\rho(x, y) := F(x, d(x, y))$ is an ultra-metric, as the symmetry follows from (b), while the ultra-metric inequality (1.1) follows from (a) and (b): if $d(x, y) \leq d(x, z)$ then

$$\rho(x, y) = F(x, d(x, y)) \leq F(x, d(x, z)) = \rho(x, z),$$

and if $d(x, y) \leq d(y, z)$ then

$$\rho(x, y) = F(y, d(x, y)) \leq F(y, d(y, z)) = \rho(y, z).$$

■

2.3 Spectral distribution function

For the Markov semigroup $\{P^t\}$ associated with the triple (d, μ, σ), define the *intrinsic ultra-metric* d_* by

$$\frac{1}{d_*(x,y)} = \log \frac{1}{\sigma(d(x,y))}. \tag{2.14}$$

Since d_* is expressed as a strictly monotone increasing function of d, which vanishes at 0, it follows that d_* is an ultra-metric on X. Denote by $B^*_r(x)$ the metric balls of d_*.

Lemma 2.7 *For any $r \geq 0$ set*

$$s = \frac{1}{\log \frac{1}{\sigma(r)}}.$$

Then the following identity holds for all $x \in X$:

$$B^*_s(x) = B_r(x).$$

Consequently, the metrics d and d_* determine the same set of balls and the same topology.

Proof. We have

$$\begin{aligned} B^*_s(x) &= \{y \in X : d_*(x,y) \leq s\} \\ &= \{y \in X : \sigma(d(x,y)) \leq \sigma(r)\} \\ &= \{y \in X : d(x,y) \leq r\} \\ &= B_r(x), \end{aligned}$$

where we have used that σ is strictly monotone increasing. ■

Definition 2.8 For any $x \in X$ we define the *spectral distribution function* $N(x, \cdot) : [0, \infty) \to [0, \infty)$ as

$$N(x,\tau) = \frac{1}{\mu\left(B^*_{1/\tau}(x)\right)}. \tag{2.15}$$

(See Figures 1, 2 and 3).

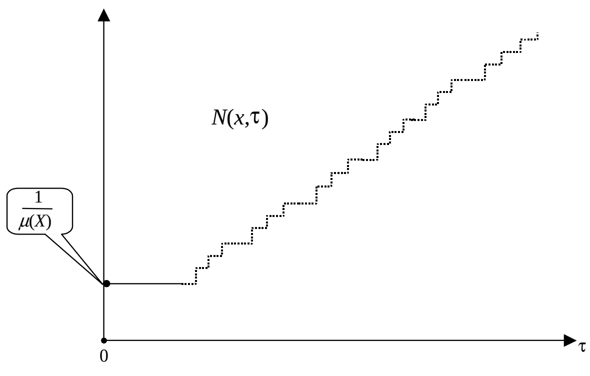

Figure 1: The graph of the function $\tau \mapsto N(x,\tau)$ in the case when $\mu(X) < \infty$

Let us define $\sigma_*(r)$ as the distribution function of "inverse exponential distribution", that is, set

$$\sigma_*(r) = \exp\left(-\frac{1}{r}\right), \quad r > 0. \tag{2.16}$$

As a distance distribution function, σ_* will play an important role in what follows.

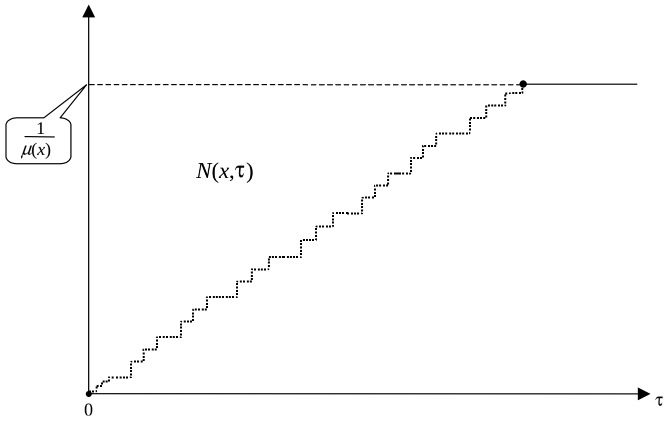

Figure 2: The graph of the function $\tau \mapsto N(x,\tau)$ in the case, when $\mu(x) > 0$

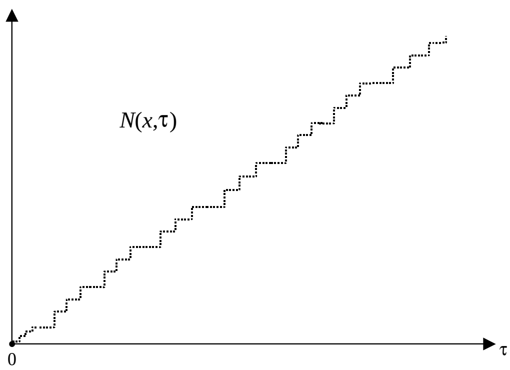

Figure 3: The graph of the function $\tau \mapsto N(x,\tau)$ in the case when $\mu(x) = 0$ and $\mu(X) = \infty$

Definition 2.9 An isotropic Markov operator P associated with a triple (d,μ,σ_*) will be referred to as a *standard* Markov operator, associated with (d,μ).

Theorem 2.10 *Let d_* and σ_* be defined by (2.14) and (2.16).*

(a) *The triples (d,μ,σ) and (d_*,μ,σ_*) induce the same isotropic Markov operators.*

(b) *The heat kernel $p(t,x,y)$ associated with the triple (d,μ,σ) satisfies for all $x,y\in X$ and $t>0$ the following identities:*

$$p(t,x,y)=\int_0^{t/d_*(x,y)} N\left(x,\frac{s}{t}\right)e^{-s}ds \tag{2.17}$$

and

$$p(t,x,y)=t\int_0^{1/d_*(x,y)} N(x,\tau)\exp(-\tau t)\,d\tau. \tag{2.18}$$

Consequently, $p(t,x,y)$ is a finite continuous function of t,x,y for all $t>0$ and $x\neq y$.

As it follows from (a), any isotropic Markov operator is at the same time the standard Markov operator, associated with (d_*,μ).

Proof. (a) It suffices to show that

$$p(t,x,y)=\int_{[d_*(x,y),\infty)}\frac{d\sigma_*^t(u)}{\mu(B_u^*(x))}, \tag{2.19}$$

where by Theorem 2.2 the right hand side represents the heat kernel associated with the triple (d_*,μ,σ_*). Consider the function

$$u(r)=\frac{1}{\log\frac{1}{\sigma(r)}},\quad r\in[0,\infty)$$

and observe that

1. $u(d(x,y))=d_*(x,y),\quad u(\infty)=\infty;$
2. $\sigma_*(u(r))=\exp\left(-\frac{1}{u(r)}\right)=\sigma(r);$
3. $B_{u(r)}^*(x)=B_r(x)$ by Lemma 2.7.

Making the change $u=u(r)$ in the integral in (2.19), we obtain

$$\int_{[d_*(x,y),\infty)}\frac{d\sigma_*^t(u)}{\mu(B_u^*(x))}=\int_{[d(x,y),\infty)}\frac{d\sigma^t(r)}{\mu(B_r(x))},$$

which together with (2.8) implies (2.19). Clearly, (2.20) follows from (2.17) as $d_*(x,x)=0$.

(b) The change $s=t/u$ in (2.19) yields

$$\begin{aligned}p(t,x,y) &= \int_{[d_*(x,y),\infty)}\frac{d\exp\left(-\frac{t}{u}\right)}{\mu(B_u^*(x))}\\ &= \int_{t/d_*(x,y)}^0\frac{de^{-s}}{\mu\left(B_{t/s}^*(x)\right)}\\ &= \int_0^{t/d_*(x,y)} N\left(x,\frac{s}{t}\right)e^{-s}ds\end{aligned}$$

which proves (2.17). Another change $s = t\tau$ transforms (2.17) to (2.18). ∎

In the case $x = y$ we obtain from (2.17) and (2.18)

$$p(t, x, x) = \int_0^\infty N\left(x, \frac{s}{t}\right) e^{-s} ds = t \int_0^\infty N(x, \tau) \exp(-\tau t)\, d\tau. \tag{2.20}$$

Depending on the function $N(x, \tau)$, the on-diagonal value $p(t, x, x)$ can be equal to ∞. For any $x \in X$ set

$$T(x) := \limsup_{\tau\to\infty} \frac{\log N(x, \tau)}{\tau}. \tag{2.21}$$

Corollary 2.11 *The function $t \mapsto p(t, x, x)$ is monotone decreasing and $p(t, x, x) < \infty$ for all $t > T(x)$.*

Proof. The monotonicity of $p(t, x, x)$ follows from the first identity in (2.20), while the second claim follows from the second identity in (2.20). Observe also that if $\lim_{\tau\to\infty} \frac{\log N(x,\tau)}{\tau}$ exists and hence is equal to $T(x)$ then $p(t, x, x) = \infty$ for $t < T(x)$. ∎

Proposition 2.12 *Assume that $T(x) < \infty$ for some $x \in X$.*

(a) *For all $y \in X$,*

$$\lim_{t\to\infty} p(t, x, y) = \frac{1}{\mu(X)},$$

where the convergence is locally uniform in $y \in X$.

(b) *For all $y \in X$,*

$$\lim_{t\to\infty} \frac{p(t, x, y)}{p(t, x, x)} = 1,$$

where the convergence is locally uniform in $y \in X$.

Proof. (a) As $t \to \infty$ we have

$$N\left(x, \frac{s}{t}\right) \to N(x, 0) = \frac{1}{\mu(X)}$$

and $t/d_*(x, y) \to \infty$. Hence, we obtain from (2.17)

$$\lim_{t\to\infty} p(t, x, y) = \int_0^\infty \frac{1}{\mu(X)} e^{-s} ds = \frac{1}{\mu(X)},$$

provided we justify that the integral and lim are interchangeable. The latter follows from the dominated convergence theorem, because the hypothesis $T(x) < \infty$ implies that, for some $A, a > 0$ and all $\tau > 0$,

$$N(x, \tau) \le A \exp(a\tau) \tag{2.22}$$

whence

$$N\left(x, \frac{s}{t}\right) e^{-s} \le A \exp\left(\left(\frac{a}{t} - 1\right) s\right) \le A \exp\left(-\frac{1}{2} s\right) \tag{2.23}$$

for $t > 2a$, so that the domination condition is satisfied.

(b) Set $r = d_*(x, y)$. It follows from (2.17) and (2.20) that

$$p(t, x, x) - p(t, x, y) = \int_{t/r}^\infty N\left(x, \frac{s}{t}\right) e^{-s} ds.$$

Assuming $t > 2a$ and applying (2.23), we obtain

$$p\left(t,x,x\right) - p\left(t,x,y\right) \leq A \int_{t/r}^{\infty} e^{-\frac{1}{2}s} ds = 2A \exp\left(-\frac{t}{2r}\right),$$

whereas

$$p(t,x,x) \geq \int_{\frac{t}{4r}}^{\infty} N\left(x, \frac{s}{t}\right) e^{-s} ds \geq N\left(x, \frac{1}{4r}\right) \exp\left(-\frac{t}{4r}\right).$$

It follows that

$$\frac{p(t,x,x) - p(t,x,y)}{p(t,x,x)} \leq \frac{2A \exp\left(-\frac{t}{4r}\right)}{N\left(x, \frac{1}{4r}\right)} \to 0 \quad \text{as } t \to \infty.$$

■

2.4 Estimates of the heat kernel

The purpose of this section is to provide some estimates of the isotropic heat kernel. Recall that by Theorem 2.10

$$p(t,x,y) = \int_0^{t/d_*(x,y)} N\left(x, \frac{s}{t}\right) e^{-s} ds. \tag{2.24}$$

Definition 2.13 A monotone increasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to satisfy the *doubling property* if there exists a constant $D > 0$ such that

$$\Phi(2s) \leq D\Phi(s) \quad \text{for all } s > 0.$$

It is known (Potter's theorem) that if Φ is doubling then

$$\Phi(s_2) \leq D \left(\frac{s_2}{s_1}\right)^{\gamma} \Phi(s_1) \quad \text{for all } 0 < s_1 < s_2\,, \text{ where } \gamma = \log_2 D. \tag{2.25}$$

Theorem 2.14 *Suppose that, for some $x \in X$, the function $\tau \mapsto N(x,\tau)$ is doubling. Then*

$$\frac{c\,t}{t + d_*(x,y)} N\bigg(x, \frac{1}{t + d_*(x,y)}\bigg) \leq p(t,x,y) \leq \frac{C\,t}{t + d_*(x,y)} N\bigg(x, \frac{1}{t + d_*(x,y)}\bigg) \tag{2.26}$$

for all $t > 0$, $y \in X$ and some constants $C, c > 0$ depending on the doubling constant.

In what follows we will use the relation $f \simeq g$ between two positive function f, g, which means that the ratio f/g is bounded from above and below by positive constants, for a specified range of the variables. In particular, we can write (2.26) shortly in the form

$$p\left(t,x,y\right) \simeq \frac{t}{t + d_*(x,y)} N\bigg(x, \frac{1}{t + d_*(x,y)}\bigg) \tag{2.27}$$

for a fixed x and all $y \in X, t > 0$.

Example 2.15 Assume that, for some $x \in X$ and $\alpha > 0$,

$$N\left(x,\tau\right) \simeq \tau^{\alpha} \quad \text{for all } \tau > 0.$$

Then by (2.27)

$$p(t,x,y) \simeq \frac{t}{\left(t + d_*(x,y)\right)^{1+\alpha}} \simeq \frac{t}{\left(t^2 + d_*(x,y)^2\right)^{\frac{1+\alpha}{2}}},$$

that is, $p\left(t,x,y\right)$ behaves like the Cauchy distribution in "α-dimensional" space.

Example 2.16 More generally, assume that, for some $\alpha, \beta \geq 0$,

$$N(x,\tau) \simeq \begin{cases} \tau^{\alpha}, & 0 < \tau \leq 1, \\ \tau^{\beta}, & \tau > 1. \end{cases} \tag{2.28}$$

Then we obtain by (2.27)

$$p(t,x,y) \simeq \begin{cases} \dfrac{t}{(t+d_*(x,y))^{1+\beta}}, & t + d_*(x,y) \leq 1, \\ \dfrac{t}{(t+d_*(x,y))^{1+\alpha}}, & t + d_*(x,y) > 1. \end{cases} \tag{2.29}$$

For example, let X be a discrete locally finite group, like $X = \bigoplus_{k=1}^{\infty} \mathbb{Z}(n_k)$, and μ be the Haar measure, normalized to $\mu(x) = 1$. With the discrete ultra-metric we obtain by (2.15) that $N(x,\tau) \simeq 1$ for large enough τ. Assuming additionally that

$$N(x,\tau) \simeq \tau^{\alpha} \text{ for small } \tau,$$

we see that (2.28) and, hence, (2.29) hold with $\beta = 0$ (cf. [13]).

Example 2.17 Assume that $\tau \mapsto N(x,\tau)$ is doubling and, for some $\alpha > 0$,

$$N(x,\tau) \simeq \left(\log \frac{1}{\tau}\right)^{-\alpha} \quad \text{for } \tau < \frac{1}{2}.$$

Then by (2.27)

$$p(t,x,y) \simeq \frac{t}{(t+d_*(x,y)) \log^{\alpha}(t+d_*(x,y))}$$

provided $t + d_*(x,y) > 2$.

Example 2.18 Assume that, for some $\alpha > 0$,

$$N(x,\tau) \simeq \exp\left(-\tau^{-\alpha}\right).$$

In this case Theorem 2.14 does not apply. An ad hoc method of estimating the integral in (2.24) yields in this case

$$p(t,x,y) \leq \frac{C_3 t}{t+d_*(x,y)} \exp\left(-c_3\left(t^{\frac{\alpha}{\alpha+1}} + d_*(x,y)^{\alpha}\right)\right)$$

and

$$p(t,x,y) \geq \frac{C_4 t}{t+d_*(x,y)} \exp\left(-c_4\left(t^{\frac{\alpha}{\alpha+1}} + d_*(x,y)^{\alpha}\right)\right),$$

for all $x, y \in X$, $t > 0$ and some positive constants C_3, C_4, c_3, c_4.

For the proof of Theorem 2.14 we need a sequence of lemmas.

Lemma 2.19 *For all $x, y \in X$ and $t > 0$ the following estimates hold.*

(a)

$$p(t,x,y) \leq \frac{t}{d_*(x,y)} N\left(x, \frac{1}{d_*(x,y)}\right). \tag{2.30}$$

(b)

$$p(t,x,y) \geq \frac{1}{2e} \begin{cases} \frac{t}{d_*(x,y)} N\left(x, \frac{1}{2d_*(x,y)}\right), & t \leq d_*(x,y), \\ N\left(x, \frac{1}{2t}\right), & t \geq d_*(x,y). \end{cases} \tag{2.31}$$

(c)

$$p(t,x,x) \geq \frac{1}{e} N\left(x, \frac{1}{t}\right). \tag{2.32}$$

Proof. (a) Inequality (2.30) follows from (2.24) using the monotonicity of $\tau \mapsto N(x,\tau)$ that yields

$$N\left(x, \frac{s}{t}\right) e^{-s} \leq N\left(x, \frac{1}{d_*(x,y)}\right).$$

(b) Set $a = \min\left(1, \frac{t}{d_*(x,y)}\right)$. It follows from (2.24) that

$$p(t,x,y) \geq \int_{a/2}^{a} N(x, \frac{s}{t}) e^{-s}\, ds \geq N\left(x, \frac{a}{2t}\right) \frac{a}{2e},$$

which is equivalent to (2.31).

(c) We have by (2.20)

$$p(t,x,x) \geq \int_1^\infty N(x, \frac{s}{t}) e^{-s} ds \geq N\left(x, \frac{1}{t}\right) \int_1^\infty e^{-s} ds,$$

whence (2.32) follows. ■

Lemma 2.20 *The following inequalities hold for all $x, y \in X$ and $t > 0$:*

$$p(t,x,y) \geq \frac{1}{2e} \frac{t}{t + d_*(x,y)} N\left(x, \frac{1}{2\left(t + d_*(x,y)\right)}\right), \tag{2.33}$$

and

$$p(t,x,y) \leq 2e \frac{t}{t + d_*(x,y)} p\left(\frac{t + d_*(x,y)}{2}, x, x\right). \tag{2.34}$$

Proof. The lower bound (2.33) follows immediately from (2.31). To prove (2.34), observe that by (2.30) and (2.32)

$$p(t,x,y) \leq e \frac{t}{d_*(x,y)} p\left(d_*(x,y), x, x\right),$$

which yields (2.34) in the case $t \leq d_*(x,y)$ as the function $p(\cdot, x, x)$ is monotone decreasing. In the case $t > d_*(x,y)$ (2.34) follows trivially from (2.12), that is, from

$$p(t,x,y) \leq p(t,x,x),$$

using again the monotonicity of $p(\cdot, x, x)$. ■

Lemma 2.21 *For any given $x \in X$, the following two properties are equivalent.*

(i) *For some constant C and all $t > 0$,*

$$p(t,x,x) \leq CN\left(x, \frac{1}{t}\right). \tag{2.35}$$

(*ii*) *The function* $\tau \mapsto N(x,\tau)$ *is doubling, that is, for some constant* D,

$$N(x,2\tau) \le DN(x,\tau.)$$

Proof. $(ii) \Rightarrow (i)$. The estimate (2.35) follows from (2.20) and (2.25) as follows:

$$\begin{aligned} p(t,x,x) &= N\left(x,\frac{1}{t}\right)\int_0^\infty \frac{N\left(x,\frac{s}{t}\right)}{N\left(x,\frac{1}{t}\right)}e^{-s}ds \\ &\le DN\left(x,\frac{1}{t}\right)\int_0^\infty \max\{1,s^\gamma\}e^{-s}ds \\ &= CN\left(x,\frac{1}{t}\right). \end{aligned}$$

$(i) \Rightarrow (ii)$. The upper bound (2.35) implies, for any $t>0$,

$$\begin{aligned} CN\left(x,\frac{1}{t}\right) &\ge p(t,x,x) \ge \int_2^\infty N(x,\frac{s}{t})e^{-s}ds \\ &\ge e^{-2}N\left(x,\frac{2}{t}\right), \end{aligned}$$

whence the doubling property of $N(x,\cdot)$ follows. ■

Proof of Theorem 2.14. The lower bound in (2.26) follows from (2.33), the upper bound follows from (2.34) and (2.35). ■

In conclusion of this section we provide practicable conditions for the validity of the doubling property of $N(x,\cdot)$.

Definition 2.22 A monotone increasing function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to satisfy the *reverse doubling property,* if there is a constant $\delta \in (0,1)$ such that for all $r>0$

$$\Psi(r) \ge 2\Psi(\delta r).$$

Proposition 2.23 *Fix some* $x \in X$. *The function* $\tau \mapsto N(x,\tau)$ *is doubling provided the following two conditions hold:*

(*i*) *The function* $\Psi(r) = -1/\log\sigma(r)$ *satisfies the reverse doubling property.*

(*ii*) *The volume function* $r \mapsto \mu(B_r(x))$ *satisfies the doubling property.*

Proof. We use the following short notation for the balls centered at x: $B_r = B_r(x)$ and $B_r^* = B_r^*(x)$. It follows from the Definition 2.8 of the spectral distribution function that $\tau \mapsto N(x,\tau)$ is doubling if and only if the function $s \mapsto \mu(B_s^*)$ is doubling. Set $\Phi = \Psi^{-1}$ and observe that the reverse doubling property for Ψ is equivalent to the doubling property for Φ. By Lemma 2.7 we have $B^*_{\Psi(r)} = B_r$ which implies that $B_s^* = B_{\Phi(s)}$. Using the hypotheses (ii) and (2.25) for the function $\mu(B_r)$, we obtain

$$\mu\left(B_{2s}^*\right) = \mu\left(B_{\Phi(2s)}\right) \le D\left(\frac{\Phi(2s)}{\Phi(s)}\right)^\gamma \mu\left(B_{\Phi(s)}\right) \le \text{const}\,\mu\left(B_s^*\right),$$

which was to be proved. ■

2.5 Heat kernels in $\mathbb{Q}_p$

Given a prime p, the p-adic norm on $\mathbb{Q}$ is defined as follows: if $x = p^n \frac{a}{b}$, where a, b are integers not divisible by p, then

$$\|x\|_p := p^{-n}.$$

If $x = 0$ then $\|x\|_p := 0$. The p-adic norm on $\mathbb{Q}$ satisfies the ultra-metric inequality. Indeed, if $y = p^m \frac{c}{d}$ and $m \le n$ then

$$x + y = p^m \left(\frac{p^{n-m}a}{b} + \frac{c}{d} \right)$$

whence

$$\|x + y\|_p \le p^{-m} = \max \left\{ \|x\|_p , \|y\|_p \right\}.$$

Hence, $\mathbb{Q}$ with the metric $d(x, y) = \|x - y\|_p$ is an ultra-metric space, and so is its completion $\mathbb{Q}_p$ – the field of p-adic numbers.

Every p-adic number x has a representation

$$x = \sum_{k=-N}^{\infty} a_k p^k = ...a_k...a_2 a_1 a_0 . a_{-1} a_{-2} ... a_{-N} \tag{2.36}$$

where $N \in \mathbb{N}$ and $a_k \in \{0, \dots, p-1\}$ are p-adic digits. The rational number $0.a_{-1}...a_{-N} = \sum_{k=-N}^{k=-1} a_k p^k$ is called the fractional part of x and the rest $\sum_{k=0}^{\infty} a_k p^k$ is the integer part of x.

For any $n \in \mathbb{Z}$, the d-ball $B_{p^{-n}}(x)$ consists of all numbers

$$y = \sum_{k=-N}^{\infty} b_k p^k = ...b_k...b_2 b_1 b_0 . b_{-1} b_{-2} ... b_{-N}$$

such that b_k are arbitrary for $k \ge n$ and $b_k = a_k$ for $k < n$. It follows that $B_{p^{-n}}(x)$ decomposes into a disjoint union of p balls of radii $p^{-(n+1)}$ depending on the choice of b_n.

For example, $B_1(0)$ coincides with the set $\mathbb{Z}_p$ of all p-adic integers, that is, any $y \in B_1(0)$ has the form

$$y = ...b_k...b_2 b_1 b_0$$

with arbitrary p-adic digits b_k. For any fixed $c = 0, 1, ..., p-1$, the additional restriction $b_0 = c$ determines a ball of radius $1/p$ centered at c, so that $B_1(0)$ is a disjoint union of p such balls, as on the following diagram, where every cell renders one of the balls $B_{1/p}(c)$:

$...b_k...b_2 b_1 0$	$...b_k...b_2 b_1 1$	...	$...b_k...b_2 b_1 (p-1)$

Let μ be the additive Haar measure on $\mathbb{Q}_p$ normalized so that $\mu(B_1(0)) = 1$. Since

$$B_r(x) = x + B_r(0)$$

and μ is translation invariant, we obtain that $\mu(B_r(x))$ does not depend on x. The above argument with the decomposition of the ball $B_{p^{-n}}(x)$ implies that

$$\mu\left(B_{p^{-n}}(x)\right) = p\mu\left(B_{p^{-(n+1)}}(x)\right),$$

whence it follows that

$$\mu\left(B_{p^{-n}}(x)\right) = p^{-n}. \tag{2.37}$$

For any $r > 0$, the ball $B_r(x)$ coincides with $B_{p^{-n}}(x)$, where $n \in \mathbb{Z}$ is such that $p^{-n} \le r < p^{-(n-1)}$, which implies that, for all $r > 0$,

$$r/p < \mu(B_r(x)) \le r. \tag{2.38}$$

Example 2.24 Let (X, d, μ) be $\mathbb{Q}_p$ with p-adic distance and the Haar measure μ. Consider the distance distribution function

$$\sigma(r) = \exp\left(-(b/r)^\alpha\right),$$

where $\alpha, b > 0$. Since

$$\Psi(r) := \frac{1}{\log \frac{1}{\sigma(r)}} = (r/b)^\alpha,$$

we obtain by (2.14)

$$d_*(x, y) = \Psi\left(d\left(x, y\right)\right) = \left(\frac{\|x - y\|_p}{b}\right)^\alpha. \tag{2.39}$$

By Lemma 2.7, we have

$$B_s^*\left(x\right) = B_{\Psi^{-1}(s)}\left(x\right),$$

which together with (2.38) yields

$$\mu\left(B_s^*(x)\right) \simeq s^{1/\alpha}. \tag{2.40}$$

Consequently, we obtain

$$N(x, \tau) \simeq \tau^{1/\alpha}.$$

Since this function is doubling, Theorem 2.14 (cf. also Example 2.15) yields the estimate

$$p(t, x, y) \simeq \frac{t}{\left(t + d_*(x, y)\right)^{1+1/\alpha}} \simeq \frac{t}{\left(t^{1/\alpha} + \|x - y\|_p\right)^{1+\alpha}}.$$

In particular, for all $t > 0$ and $x \in X$

$$p\left(t, x, x\right) \simeq t^{1/\alpha}.$$

Example 2.25 Let $X = \mathbb{Z}_p$, that is, X is the unit ball $B_1\left(0\right)$ in $\mathbb{Q}_p$, with the p-adic distance and the Haar measure μ. Consider the distance distribution function

$$\sigma\left(r\right) = \exp\left(1 - \exp r^{-\alpha}\right),$$

for some $\alpha > 0$. Since for $r \leq 1$

$$\Psi(r) := \frac{1}{\log \frac{1}{\sigma(r)}} = \frac{1}{\exp r^{-\alpha} - 1} \simeq \exp\left(-r^{-\alpha}\right),$$

we obtain that

$$d_*(x, y) = \Psi\left(d\left(x, y\right)\right) \simeq \exp\left(-\left\|x - y\right\|_p^{-\alpha}\right).$$

By Lemma 2.7 and (2.38), we have, for all $s \leq \frac{1}{2}$,

$$\mu\left(B_s^*\left(x\right)\right) = \mu\left(B_{\Psi^{-1}(s)}\left(x\right)\right) \simeq \Psi^{-1}\left(s\right) \simeq \frac{1}{\log^{1/\alpha} \frac{1}{s}},$$

whereas for $s > \frac{1}{2}$ we have $\mu\left(B_s^*\left(x\right)\right) \simeq 1$. Therefore, we obtain, for all $\tau > 0$,

$$N\left(x, \tau\right) = \frac{1}{\mu\left(B_{1/\tau}^*\left(x\right)\right)} \simeq \log^{1/\alpha}\left(2 + \tau\right).$$

Hence, the function $N\left(x, \tau\right)$ is doubling, and we obtain by (2.27) that

$$p(t, x, y) \simeq \frac{t}{t + \exp\left(-\left\|x - y\right\|_p^{-\alpha}\right)} \log^{1/\alpha}\left(2 + \frac{1}{t + \exp\left(-\left\|x - y\right\|_p^{-\alpha}\right)}\right).$$

Example 2.26 Let X be the subset of $\mathbb{Q}_p$ consisting of all p-adic fractions, that is, the numbers of the form $x = 0.a_{-1}....a_{-N}$. Then the p-adic distance d on X takes only integer values so that (X, d) is a discrete space. Let μ be the counting measure on X, that is, $\mu(x) = 1$ for any $x \in X$. Consider the following distance distribution function

$$\sigma(r) = \exp\left(-\frac{1}{\log^\alpha(2r)}\right) \quad \text{for } r \geq 1, \tag{2.41}$$

that is arbitrarily extended to $r < 1$ to be strictly monotone increasing and to have $\sigma(0) = 0$. Since

$$\Psi(r) := \frac{1}{\log \frac{1}{\sigma(r)}} = \log^\alpha(2r) \quad \text{for } r \geq 1,$$

we obtain, for $x \neq y$,

$$d_*(x, y) = \Psi(d(x, y)) = \log^\alpha\left(2\|x - y\|_p\right). \tag{2.42}$$

For $s \geq s_0 := \log^\alpha 2$, we have

$$\mu(B_s^*(x)) = \mu\left(B_{\Psi^{-1}(s)}(x)\right) \simeq \Psi^{-1}(s) = \frac{1}{2}\exp\left(s^{1/\alpha}\right), \tag{2.43}$$

whereas for $s < s_0$ we have $\mu(B_s^*(x)) \simeq \mu(x) = 1$. We see that (2.43) holds for all $s > 0$. It follows that, for all $\tau > 0$,

$$N(x, \tau) = \frac{1}{\mu\left(B_{1/\tau}^*(x)\right)} \simeq \exp\left(-\tau^{-1/\alpha}\right). \tag{2.44}$$

By Example 2.18, we obtain

$$p(t, x, y) \leq \frac{Ct}{t + \log_+^\alpha\left(2\|x - y\|_p\right)} \exp\left(-c\left(t^{\frac{1}{\alpha+1}} + \log_+\left(2\|x - y\|_p\right)\right)\right),$$

and a similar lower bound.

2.6 Green function and transience

Given an isotropic heat semigroup $\{P^t\}$, define the Green operator G on non-negative Borel functions f on X by

$$Gf(x) = \int_0^\infty P^t f(x)\, dt.$$

Of course, the value of $Gf(x)$ could be ∞. By Fubini's theorem, we obtain

$$Gf(x) = \int_X g(x, y) f(y)\, d\mu(y)$$

where

$$g(x, y) = \int_0^\infty p(t, x, y)\, dt.$$

Substituting the heat kernel from (2.18) and using again Fubini's theorem, we obtain

$$g(x, y) = \int_0^{1/d_*(x,y)} \frac{N(x, \tau)\, d\tau}{\tau^2} = \int_{d_*(x,y)}^\infty \frac{ds}{\mu(B_s^*(x))}, \tag{2.45}$$

where the second identity follows from (2.15). The function $g(x, y)$ is called the *Green function* of the semigroup $\{P^t\}$. Note that the Green function can be identically equal to ∞. For example, this is the case when $\mu(X) < \infty$ (cf. Figure 1) and the second integral (2.45) diverges at ∞.

Definition 2.27 The process $\{\mathcal{X}_t\}$ and the semigroup $\{P^t\}$ are called *transient* if Gf is a bounded function whenever f is bounded and has compact support, and *recurrent* otherwise.

Theorem 2.28 *The following statements are equivalent.*

(i) *The semigroup* $\{P^t\}$ *is transient.*

(ii) $g(x,y) < \infty$ *for some/all distinct* $x, y \in X$.

(iii) *For some/all* $x \in X$,

$$\int^{\infty} \frac{ds}{\mu\left(B_s^*\left(x\right)\right)} < \infty. \tag{2.46}$$

The inequality (2.46) is equivalent to

$$\int_0 \frac{N(x,\tau)\, d\tau}{\tau^2} < \infty\,. \tag{2.47}$$

Observe that, in the transient case, the function $x, y \mapsto \frac{1}{g(x,y)}$ determines an ultra-metric on X, which is proved similarly to Corollary 2.6.

Proof. The validity of the condition (2.46) is independent of the choice of x because for any two $x, x' \in X$ the balls $B_s^*\left(x\right)$ and $B_s^*\left(x'\right)$ are identical provided $s \geq d\left(x, x'\right)$. The finiteness of the second integral in (2.45) for $x \neq y$ is clearly equivalent to (2.46), whence the equivalence $(ii) \Leftrightarrow (iii)$ follows, with all combinations of some/all options.

The finiteness of Gf for any bounded function f with compact support clearly implies that $g\left(x,y\right) \not\equiv \infty$, that is, $(i) \Rightarrow (ii)$. So, it remains to prove $(iii) \Rightarrow (i)$. It suffices to show that Gf is bounded for $f = \mathbf{1}_A$ where A is a bounded Borel subset of X. Let R be the diameter of A with respect to the distance d^*. Then we have $A \subset B_R^*\left(x\right)$ for any $x \in A$ whence by (2.45)

$$\begin{aligned}
Gf\left(x\right) &= \int_A g\left(x,y\right) d\mu\left(y\right) \leq \int_{B_R^*(x)} g\left(x,y\right) d\mu\left(y\right) \\
&= \int_{B_R^*(x)} \int_0^{\infty} \mathbf{1}_{[d_*(x,y),\infty)}\left(s\right) \frac{ds}{\mu\left(B_s^*\left(x\right)\right)} d\mu\left(y\right) \\
&= \int_0^{\infty} \frac{1}{\mu\left(B_s^*\left(x\right)\right)} \left(\int_{B_R^*(x)} \mathbf{1}_{[0,s]}\left(d_*\left(x,y\right)\right) d\mu\left(y\right)\right) ds \\
&= \int_0^{\infty} \frac{1}{\mu\left(B_s^*\left(x\right)\right)} \mu\left(B_R^*\left(x\right) \cap B_s^*\left(x\right)\right) ds.
\end{aligned}$$

For $s \geq R$ the integrand is equal to $\frac{1}{\mu(B_s^*(x))}\mu\left(B_R^*\left(x\right)\right)$ so that the convergence at ∞ follows from (2.46). The convergence is clearly uniform in $x \in A$ because $\mu\left(B_R^*\left(x\right)\right)$ and $\mu\left(B_s^*\left(x\right)\right)$ are independent of $x \in A$ for $s \geq R$. For $s \leq R$ the integrand is equal to

$$\frac{1}{\mu\left(B_s^*\left(x\right)\right)} \mu\left(B_s^*\left(x\right)\right) = 1,$$

whence the uniform convergence at 0 follows. Hence, $\sup_A Gf\left(x\right) < \infty$. That $\sup_X Gf\left(x\right) < \infty$ follows from the decay of $g\left(x,y\right)$ in $d_*\left(x,y\right)$. ■

Let us note that if X is a locally finite group with the Haar measure μ, then the transience criterion (iii) of Theorem 2.28 coincides with the general sufficient condition of transience of [40].

Now let us provide some estimate of the Green function. Set

$$V(x,r) = \mu\left(B_r^*(x)\right). \tag{2.48}$$

Theorem 2.29 *Assume that there exist constants $1 < c < c' < c''$ such that for all $r > r_0 \geq 0$ and some $x \in X$*

$$c' \leq \frac{V(x, cr)}{V(x, r)} \leq c''. \tag{2.49}$$

Then the semigroup $\{P^t\}$ is transient and, for all $y \in X$ such that $r := d^*(x, y) > r_0$, we have

$$g(x, y) \simeq \frac{r}{V(x, r)}.$$

Note that the condition $\frac{V(x,cr)}{V(x,r)} \leq c''$ is equivalent to the doubling property of $r \mapsto V(x, r)$ (cf. Definition 2.13), whereas the condition $\frac{V(x,cr)}{V(x,r)} \geq c'$ with $c' > c$ is somewhat stronger than the reverse doubling property (cf. Definition 2.22). For example, (2.49) holds for $V(x, r) \simeq r^\alpha$ if and only if $\alpha > 1$.

Proof. Set for simplicity of notation $V(s) := V(x, s)$. For $r > r_0$ we have

$$g(x, y) = \int_r^\infty \frac{ds}{V(s)} = \sum_{k=0}^\infty \int_{c^k r}^{c^{k+1} r} \frac{ds}{V(s)} = \sum_{k=0}^\infty c^k \int_r^{cr} \frac{ds}{V(c^k s)}$$

Using the lower bound in (2.49), we obtain

$$\int_r^\infty \frac{ds}{V(s)} \leq \sum_{k=0}^\infty c^k \int_r^{cr} \frac{(c')^{-k} ds}{V(s)} \leq \sum_{k=0}^\infty \left(\frac{c}{c'}\right)^k \frac{cr}{V(r)} \leq \text{const}\, \frac{r}{V(r)},$$

where the series converges due to $c' > c$. Similarly, using the upper bound in (2.49), we obtain

$$\int_r^\infty \frac{ds}{V(s)} \geq \int_r^{cr} \frac{ds}{V(s)} \geq \frac{(c-1)r}{V(cr)} \geq \text{const}\, \frac{r}{V(r)},$$

which finishes the proof. ■

Example 2.30 Let (X, d, μ) and σ be as in Example 2.24, that is, $X = \mathbb{Q}_p$ is the field of p-adic numbers with ultra-metric $d(x, y) = \|x - y\|_p$ and $\sigma(r) = \exp(-(b/r)^\alpha)$. Then by (2.39) we have

$$d_*(x, y) = \text{const}\, \|x - y\|_p^\alpha$$

and by (2.40)

$$V(x, r) \simeq r^{1/\alpha}.$$

Therefore, by Theorem 2.28, the semigroup $\{P^t\}$ is transient if and only if $\alpha < 1$. Moreover, the condition (2.49) is fulfilled also if and only if $\alpha < 1$, and in this case we obtain by Theorem 2.29 that, for all x, y,

$$g(x, y) \simeq d_*(x, y)^{1 - \frac{1}{\alpha}} \simeq \|x - y\|_p^{\alpha - 1}.$$

Example 2.31 Let (X, d, μ) and σ be as in Example 2.26, that is, X is the set of fractional p-adic numbers and σ is given by (2.41). By (2.42) we have, for $x \neq y$,

$$d_*(x, y) = \log^\alpha \left(2 \|x - y\|_p\right)$$

and by (2.43)

$$V(x, r) \simeq \exp\left(r^{1/\alpha}\right).$$

By Theorem 2.28 we conclude that the semigroup $\{P^t\}$ is transient. Theorem 2.29 does not apply in this case, but a direct estimate of the integral in (2.45) yields, for $r := d^* (x, y)$,

$$g(x,y) = \int_r^\infty \frac{ds}{V(x,s)} \simeq \int_r^\infty \exp\left(-s^{1/\alpha}\right) ds \simeq r^{1-1/\alpha} \exp\left(-r^{1/\alpha}\right),$$

whence, for $x \neq y$,

$$g(x,y) \simeq \|x-y\|_p^{-1} \log^{\alpha-1}\left(2\|x-y\|_p\right).$$

3 Isotropic Laplacian and its spectrum

In this section we are concerned with the properties of the generator of the isotropic semigroup $\{P^t\}$. By definition, the generator $\mathcal{L}$ of a strongly continuous semigroup $\{P_t\}_{t\geq 0}$ in a Banach space is defined by

$$\mathcal{L}f = s\text{-}\lim_{t\to 0} \frac{f - P_t f}{t}$$

and the domain $\mathrm{dom}_{\mathcal{L}}$ consists of those f for which the above limit exists. Since the isotropic semigroup $\{P^t\}$ is symmetric and acts in a Hilbert space $L^2(X,\mu)$, the above definition is equivalent to the following: $\mathcal{L}$ is a self-adjoint (unbounded) operator in $L^2(X,\mu)$ such that

$$P^t = \exp(-t\mathcal{L}) \text{ for all } t > 0.$$

Obviously, this is equivalent to $P = \exp(-\mathcal{L})$, which leads to the identity

$$\mathcal{L} = \log \frac{1}{P},$$

where the right hand side is understood in the sense of functional calculus of self-adjoint operators. We refer to $\mathcal{L}$ as an *isotropic Laplace operator* associated with (d, μ, σ).

3.1 Subordination

Using the spectral decomposition (2.10) of P, we obtain that

$$\mathcal{L} = \int_{[0,+\infty)} \log \frac{1}{\sigma(1/\lambda)} dE_\lambda$$

where $\{E_\lambda\}$ is the spectral resolution defined by (2.6). Denote for simplicity

$$\varphi(\lambda) := \log \frac{1}{\sigma(1/\lambda)} \tag{3.1}$$

so that

$$\mathcal{L} = \int_{[0,+\infty)} \varphi(\lambda)\, dE_\lambda . \tag{3.2}$$

The domain $\mathrm{dom}_{\mathcal{L}}$ is then given by

$$\mathrm{dom}_{\mathcal{L}} = \left\{ f \in L^2 : \int_0^\infty \varphi(\lambda)^2 d(E_\lambda f, f) < \infty \right\}.$$

Observe that the function φ has the following properties that follow from the assumptions (1.5) about σ:

$$\begin{array}{l} \varphi : [0,\infty] \to [0,\infty] \text{ is a strictly monotone increasing} \\ \text{right-continuous function, such that } \varphi(0) = 0 \text{ and } \varphi(\infty-) = \infty. \end{array} \tag{3.3}$$

Conversely, any function φ satisfying (3.3) determines the function

$$\sigma(\lambda) = \exp(-\varphi(1/\lambda))$$

that satisfies (1.5). This observation leads us to the following interesting subordination property of isotropic Laplacians.

Theorem 3.1 *Let $\mathcal{L}$ be an isotropic Laplacian associated with (d, μ, σ). Let ψ be any function satisfying (3.3). Then $\psi(\mathcal{L})$ is also an isotropic Laplacian associated with $(d, \mu, \widetilde{\sigma})$ for some other distance distribution function $\widetilde{\sigma}$.*

Proof. It follows from (3.2) that

$$\psi(\mathcal{L}) = \int_{[0,+\infty)} \psi \circ \varphi(\lambda)\, dE_\lambda.$$

Since the composition $\psi \circ \varphi$ also satisfies (3.3), we obtain that $\psi(\mathcal{L})$ is an isotropic Laplacian. Moreover, using (3.1), we obtain the following formula for $\widetilde{\sigma}$:

$$\widetilde{\sigma}(r) = \exp\left(-\psi\left(\log\frac{1}{\sigma(r)}\right)\right).$$

■

Remark 3.2 Any a non-negative definite, self-adjoint operator $\mathcal{L}$ in L^2 generates a semigroup $\left\{e^{-t\mathcal{L}}\right\}_{t\geq 0}$. We refers to $\mathcal{L}$ as a Laplacian if the semigroup $\left\{e^{-t\mathcal{L}}\right\}$ is Markovian. In general, by Bochner's theorem, for any Laplacian $\mathcal{L}$, the operator $\psi(\mathcal{L})$ is again a Laplacian, provided ψ is a Bernstein function (see, for example, Schilling, Song and Vondraček [53]). It is known that $\psi(\lambda) = \lambda^\alpha$ is a Bernstein function if and only if $0 < \alpha \leq 1$. Thus, for a general Laplacian $\mathcal{L}$, the power $\mathcal{L}^\alpha$ is guaranteed a Laplacian only for $\alpha \leq 1$. For example, for the classical Laplace operator $\mathcal{L} = -\Delta$ in $\mathbb{R}^n$, the power $(-\Delta)^\alpha$ with $\alpha > 1$ is not a Laplacian. In a striking contrast to that, by Theorem 3.1, the powers $\mathcal{L}^\alpha$ of the isotropic Laplacian are again Laplacians for all $\alpha > 0$.

3.2 L^2-spectrum

Our next goal is to give an explicit expression for $\mathcal{L}f$ and to describe the spectrum of $\mathcal{L}$. Recall that by Theorem 2.10 the triples (d, μ, σ) and (d_*, μ, σ_*) induce the same Markov operator P and, hence, the same Laplace operator $\mathcal{L}$, where d_* is the intrinsic ultra-metric defined by (2.14) and

$$\sigma_*(r) = \exp\left(-\frac{1}{r}\right)$$

From now on we will use only the metric d_* and σ_*. Let the spectral resolution $\{E_\lambda\}$ be also defined using the metric d_*, which means that in the definition (2.6) of E_λ we now use the averaging operator Q_r with respect to the metric d_*. The function φ_* associated with σ_* by (3.1) has especially simple form: $\varphi_*(\lambda) = \lambda$. Therefore, we obtain from (3.2) the spectral decomposition of $\mathcal{L}$ in the classical form

$$\mathcal{L} = \int_{[0,+\infty)} \lambda dE_\lambda = \int_{(0,\infty)} \lambda dE_\lambda. \tag{3.4}$$

The change $s = \frac{1}{\lambda}$ gives

$$\mathcal{L} = -\int_{(0,\infty)} \frac{1}{s} d\mathrm{Q}_s.$$

For any $x \in X$, denote by $\Lambda(x)$ the set of values of $d_*(x,y)$ for all $y \in X$, $y \neq x$, that is,

$$\Lambda(x) = \{d(x,y) : y \in X \setminus \{x\}\}. \tag{3.5}$$

Lemma 3.3 *The set $\Lambda(x)$ has no accumulation point in $(0,\infty)$. Consequently, $\Lambda(x)$ is at most countable.*

Proof. Let $r \in (0,\infty)$ be an accumulation point of $\Lambda(x)$, that is, there is a sequence $\{r_k\}$ from $\Lambda(x) \setminus \{r\}$ such that $r_k \to r$ as $k \to \infty$. Then $r_k = d_*(x, y_k)$ for some $y_k \in X$. Since the sequence $\{y_k\}$ is bounded, by the compactness of all balls in X it has a convergent subsequence. Without loss of generality, we can then assume that $\{y_k\}$ converges, say to $y \in X$. Then we have $r = d(x,y)$. Since $r > 0$, we have for large enough k that $r_k > r/2$ and $d(y, y_k) < r/2$. Then we obtain by the ultra-metric inequality that

$$r_k \leq \max(r, d(y, y_k)) = r$$

and analogously

$$r \leq \max(r_k, d(y, y_k)) = r_k$$

whence $r_k = r$, which contradicts the assumptions. ■

Definition 3.4 For any ball B in X denote by $\rho(B)$ the minimal d_*-radius of B.

Note that $\rho(B)$ exists because all balls are defined as closed balls, and $\rho(B)$ coincides with the d_*-diameter of B.

Lemma 3.5 *If $\rho(B) > 0$ then $\rho(B) \in \Lambda(x)$ for any $x \in B$. Conversely, any number in $\Lambda(x)$ is equal to $\rho(B)$ for some ball B containing x.*

Proof. Set $r = \rho(B)$ so that $B = B_r^*(x)$. For any $y \in B$ we have $d_*(x,y) \leq r$, and we have to show that $d_*(x,y) = r$ for some y. Assume that $d_*(x,y) < r$ for all $y \in B$. Then the set $\{d_*(x,y) : y \in B \setminus \{x\}\}$ is a subset of $(0,r) \cap \Lambda(x)$. By Lemma 3.3, the latter set has a maximal element, say r'. Then $B \subset B_{r'}^*(x)$, which contradicts the minimality of radius r. Conversely, if $r \in \Lambda(x)$ then the ball $B = B_r(x)$ has $\rho(B) = r$ since there exists $y \in X$ with $d(x,y) = r$. ■

Definition 3.6 Let B, C be two balls in X such that $C \subset B$. We say that C is a *child* or *successor* of B (and B is a *parent* or *predecessor* of C) if $C \neq B$ and, for any ball A, such that $C \subset A \subset B$ we have $A = C$ or $A = B$. In other words, B is a minimal ball containing C as a proper subset. If C is a child of B then we write $C \prec B$.

Denote by $\mathcal{K}$ be the family of all balls C in X with positive radii. If $C = B_r^*(x)$ is a ball from $\mathcal{K}$ with $r > 0$ then for the minimal radius $\rho(C)$ we have two possibilities:

1. either $\rho(C) > 0$,
2. or $\rho(C) = 0$ and the center of C is an isolated point of X.

Lemma 3.7 *For any ball $C \in \mathcal{K}$ such that $C \neq X$ there is a unique parent ball B. For any ball B with $\rho(B) > 0$ the number $\deg(B)$ of its children satisfies $2 \leq \deg(B) < \infty$. Moreover, all the children of B are disjoint and their union is equal to B.*

Proof. Fix some $x \in C$. It follows from Lemma 3.3 and the definition of $\mathcal{K}$ that the set $(\rho(C), \infty) \cap \Lambda(x)$ has a minimum that we denote by r. Then the ball $B_r^*(x)$ is a parent of C. The uniqueness of the parent follows from definition.

If C_1 and C_2 are two distinct children of B then C_1 and C_2 are disjoint. Indeed, if they intersect then one of them contains the other, say $C_1 \subset C_2$. By definition of a parent/child, we must have then $C_2 = C_1$ or $C_2 = B$, whence $C_1 = C_2$ follows.

Let us show that for any $x \in B$ there is a ball C such that $x \in C \prec B$. Indeed, if the set $(0, \rho(B)) \cap \Lambda(x)$ is empty, then $C = B_0^*(x) = \{x\}$ is the child of B. If the set $(0, \rho(B)) \cap \Lambda(x)$ is non-empty then by Lemma 3.3 is has a maximum, say r. Then $C = B_r^*(x)$ is a child of B. Hence, the set of all children of B is a covering of B.

Each child C of B is an open set (being also a closed ball) because C coincides with an open ball of radius $\rho(B)$. Since B is compact, it follows that the set of its children is finite, that is, $\deg(B) < \infty$. Finally, $\deg(B)$ cannot be equal to 1 since then B would coincide with its only child. Hence, $\deg(B) \geq 2$. ■

For any $C \in \mathcal{K}$ define the function f_C on X as follows. If C is a proper subset of X then, denoting by B the parent of C, set

$$f_C = \frac{1}{\mu(C)} \mathbf{1}_C - \frac{1}{\mu(B)} \mathbf{1}_B \tag{3.6}$$

(note that always $\mu(C) > 0$). Set also $\lambda(C) := 1/\rho(B)$. If $C = X$ (which can only be the case when X is compact), then set $f_C \equiv 1$ and $\lambda(C) = 0$.

Theorem 3.8 *For any $C \in \mathcal{K}$ the function f_C is an eigenfunction of $\mathcal{L}$ with the eigenvalue $\lambda(C)$. The family $\{f_C : C \in \mathcal{K}\}$ is complete (its linear span is dense) in $L^2(X, \mu)$. Consequently, the operator $\mathcal{L}$ has a complete system of compactly supported eigenfunctions.*

Proof. Fix a ball $C \in \mathcal{K}$ or radius $r = \rho(C)$, and let B be the parent of radius $r' = \rho(B)$. Any ball of radius $s < r'$ either is disjoint with C or is contained in C, which implies that $\mathbf{1}_C$ is constant in any such ball. It follows that, for any $s < r'$, we have $\mathrm{Q}_s \mathbf{1}_C = \mathbf{1}_C$ and, similarly $\mathrm{Q}_s \mathbf{1}_B = \mathbf{1}_B$, whence

$$\mathrm{Q}_s f_C = f_C.$$

For $s \geq r'$ any ball of radius s either contains both balls C, B or is disjoint from B. Since the averages of the two functions $\frac{1}{\mu(C)} \mathbf{1}_C$ and $\frac{1}{\mu(B)} \mathbf{1}_B$ over any ball containing C and B are equal, we obtain that in this case $\mathrm{Q}_s f_C = 0$. It follows that

$$\mathcal{L} f_C = -\int_{(0,\infty)} \frac{1}{s} \mathrm{Q}_s f_C \, ds = \frac{1}{r'} f_C = \lambda(C) f_C,$$

which proves that f_C is an eigenfunction of $\mathcal{L}$ with the eigenvalue $\lambda(C)$. In the case of compact X we have $\mathrm{Q}_s f_X = f_X$ for all $s > 0$, whence $\mathcal{L} f_X = 0 = \lambda(X)$.

Let us show that the system $\{f_C : C \in \mathcal{K}\}$ is complete. We assume that some function $f \in L^2$ is orthogonal to all functions f_C and prove that $f \equiv \text{const}$. We have for any $r > 0$,

$$(\mathrm{Q}_r f, f_C)_{L^2} = (f, \mathrm{Q}_r f_C)_{L^2} = \text{const}\,(f, f_C)_{L^2} = 0,$$

where we have used the fact that any eigenfunction of $\mathcal{L}$ is also eigenfunction of Q_r with an eigenvalue that we denoted by const. Hence, $\mathrm{Q}_r f$ is also orthogonal to all f_C. We will prove below that $\mathrm{Q}_r f = 0$, which will imply by (2.2) that $f = 0$.

Since $\mathrm{Q}_r f$ is constant in any ball of radius r, by renaming $\mathrm{Q}_r f$ back to f we can assume from now on that f is constant in any ball of radius r. Fix some ball $C \in \mathcal{K}$ and its parent B. It follows from (3.6) that $(f, f_C)_{L^2} = 0$ is equivalent to

$$\frac{1}{\mu(C)} \int_C f \, d\mu = \frac{1}{\mu(B)} \int_B f \, d\mu,$$

that is, the average value of f over a ball is preserved when switching to its parent. Starting with two balls C_1 and C_2 of radii r, we can build a sequence of their predecessors which end up with the same (large enough) ball. This implies that the averages of f in C_1 and C_2 are the same. Since f is constant in C_1 and C_2, it follows that the values of these constants are the same. It follows that $f \equiv \text{const}$ on X. If $\mu(X) = \infty$ then we obtain $f \equiv 0$. If $\mu(X) < \infty$ then using the orthogonality of f to $f_X \equiv 1$ we obtain again that $f \equiv 0$. ■

For any ball B with $\rho(B) > 0$ define the subspace $\mathcal{H}_B$ of L^2 as follows:

$$\mathcal{H}_B = \operatorname{span}\{f_C : C \prec B\}. \tag{3.7}$$

By Theorem 3.8, all non-zero functions in $\mathcal{H}_B$ are the eigenfunctions of $\mathcal{L}$ with eigenvalue $\frac{1}{\rho(B)}$.

It follows from Lemma 3.7 that the functions $\{\mathbf{1}_C : C \prec B\}$ are linearly independent and

$$\sum_{C \prec B} \mathbf{1}_C = \mathbf{1}_B.$$

This entails

$$\sum_{C \prec B} \mu(C) f_C = 0 \tag{3.8}$$

and that this is the only dependence between functions f_C. Hence, we obtain that

$$\dim \mathcal{H}_B = \deg(B) - 1. \tag{3.9}$$

Clearly, the spaces $\mathcal{H}_B$ and $\mathcal{H}_{B'}$ are orthogonal provided the balls B, B' are disjoint.

Define the set

$$\Lambda := \{d_*(x,y) : x, y \in X,\ x \neq y\} = \bigcup_{x \in X} \Lambda(x). \tag{3.10}$$

Theorem 3.8 implies the following.

Corollary 3.9 *The spectrum* $\operatorname{spec}\mathcal{L}$ *of the Laplacian* $\mathcal{L}$ *is pure point and*

$$\operatorname{spec}\mathcal{L} = \overline{\left\{\frac{1}{r} : r \in \Lambda\right\}} \cup \{0\}.$$

The space $L^2(X,\mu)$ *decomposes into an orthogonal sum of finite-dimensional eigenspaces as follows: if* $\mu(X) = \infty$ *then*

$$L^2(X,\mu) = \bigoplus_{\rho(B)>0} \mathcal{H}_B,$$

and if $\mu(X) < \infty$ *then*

$$L^2(X,\mu) = \{\text{const}\} \oplus \bigoplus_{\rho(B)>0} \mathcal{H}_B\ .$$

Example 3.10 Let (X, d, μ) and be as in Example 2.24, that is, $X = \mathbb{Q}_p$, $d(x,y) = \|x-y\|_p$ is the p-adic distance and μ be the Haar measure. Set for some $\alpha > 0$

$$\sigma(r) = \exp\left(-\left(\frac{p}{r}\right)^{\alpha}\right),$$

so that by (2.39)

$$d_*(x,y) = \left(\frac{\|x-y\|_p}{p}\right)^{\alpha}.$$

Since the set of non-zero values of $\|x-y\|_p$ is $\left\{p^k\right\}_{k\in\mathbb{Z}}$, it follows that the set Λ of all non-zero values of $d_*(x,y)$ is

$$\Lambda=\left\{p^{\alpha k}:k\in\mathbb{Z}\right\} \ .$$

Hence,

$$\operatorname{spec}\mathcal{L}=\left\{p^{\alpha k}:k\in\mathbb{Z}\right\}\cup\{0\}\,.$$

Corollary 3.11 *Let (X,d) be a non-compact, proper ultra-metric space. Let $M\subset[0,\infty)$ be any closed set (unbounded, if X contains at least one non-isolated point) that accumulates at 0. Then the following is true.*

(a) There exists a proper ultra-metric d' on X that generates the same topology as d and the isotropic Laplacian $\mathcal{L}'$ of the triple (d',μ,σ_*) has the spectrum $\operatorname{spec}\mathcal{L}'=M$.

(b) Suppose in addition that there exists a partition of X into d-balls that consists of infinitely many non-singletons. Then the ultra-metric d' of part (a) can be chosen so that the collections of d-balls and d'-balls coincide.

Proof. The set

$$D=\left\{x\in(0,\infty):x^{-1}\in M\right\}\cup\{0\}$$

is a closed, unbounded subset of $[0\,,\,\infty)$ containing 0. The the statement (a) is equivalent to the existence of a proper ultra-metric d' on X that generates the same topology as d and such that the closure of the value set $\{d'(x,y)\}_{x,y\in X}$ of that metric coincides with D. This metric property is proved by Bendikov and Krupski [8, §2]. Then the isotropic Laplacian associated with the triple (d',μ,σ_*) has the required property by Corollary 3.9. The proof of (b) follows in the same way from [8, §2]. ■

3.3 The Dirichlet form and jump kernel

Let us construct a Dirichlet form $(\mathcal{E},\mathrm{dom}_{\mathcal{E}})$ associated with the isotropic semigroup $\left\{P^t\right\}$. It is well known that if $P^t1=1$, which is the case here, then

$$\mathcal{E}(f,f)=\lim_{t\to0}\frac{1}{2t}\int_X\int_X p_t(x,y)\left(f(x)-f(y)\right)^2\,d\mu(x)\,d\mu(y)$$

and

$$\mathrm{dom}_{\mathcal{E}}=\left\{f\in L^2:\mathcal{E}(f,f)<\infty\right\}$$

(see [27]). Using the identity (2.18), we obtain that

$$\frac{p(t,x,y)}{t}\nearrow\int_0^{1/d_*(x,y)}N(x,\tau)\,d\tau\quad\text{as }t\searrow0.$$

Setting

$$J(x,y):=\int_0^{1/d_*(x,y)}N(x,\tau)\,d\tau=\int_{d_*(x,y)}^{\infty}\frac{1}{V(x,s)}\frac{ds}{s^2},\tag{3.11}$$

we obtain by the monotone convergence theorem that, for all $f\in L^2$,

$$\mathcal{E}(f,f)=\frac{1}{2}\int_X\int_X\left(f(x)-f(y)\right)^2J(x,y)\,d\mu(x)\,d\mu(y)\,.$$

Note that $0<J(x,y)=J(y,x)<\infty$ for all $x\neq y$, while $J(x,x)=\infty$.

The polarization identity implies then, for all $f,g\in\mathrm{dom}_{\mathcal{E}}$ that

$$\mathcal{E}(f,g)=\frac{1}{2}\int_X\int_X\left(f(x)-f(y)\right)\left(g(x)-g(y)\right)J(x,y)\,d\mu(x)\,d\mu(y)\,.\tag{3.12}$$

The function J is called the *jump kernel* of the Dirichlet form $\mathcal{E}$. We show here that it can be used also to describe the generator $\mathcal{L}$ of $\{P^t\}$. Recall that by the theory of Dirichlet forms, the generator $\mathcal{L}$ has the following equivalent definition: it is the self-adjoint operator in L^2 with $\mathrm{dom}_{\mathcal{L}} \subset \mathrm{dom}_{\mathcal{E}}$ such that

$$(\mathcal{L}f, g) = \mathcal{E}(f, g)$$

for all $f \in \mathrm{dom}_{\mathcal{L}}$ and $g \in \mathrm{dom}_{\mathcal{E}}$.

Denote by $\mathcal{V}_r$ the image of the operator Q_r (defined with respect to d_*), that is, the space of all L^2-functions that are constant on each ball of radius r. Set also

$$\mathcal{V} := \bigcup_{r>0} \mathcal{V}_r$$

and observe that $\mathcal{V}$ is a linear subspace of L^2. Observe also that the space $\mathcal{V}_c$ of all locally constant functions with compact support is contained in $\mathcal{V}$.

Theorem 3.12 *The space $\mathcal{V}$ is dense in L^2, it is a subset of* $\mathrm{dom}_{\mathcal{L}}$ *and, for any $f \in \mathcal{V}$,*

$$\mathcal{L}f(x) = \int_X (f(x) - f(y)) J(x, y)\, d\mu(y). \tag{3.13}$$

Proof. That $\mathcal{V}$ is dense in L^2 follows from (2.2). In fact, $\mathcal{V}_c$ is also dense in L^2, which follows from the fact that all the eigenfunctions of $\mathcal{L}$ lie in $\mathcal{V}_c$.

By (2.6) and (3.4) we have $\mathrm{Q}_r = \mathbf{1}_{[0,1/r)}(\mathcal{L})$. Therefore, $\mathcal{L}\mathrm{Q}_r$ is a bounded operator, which implies that $\mathrm{dom}_{\mathcal{L}} \supset \mathcal{V}_r$ and, hence, $\mathrm{dom}_{\mathcal{L}} \supset \mathcal{V}$.

Fix a function $f \in \mathcal{V}_r$ with $r > 0$, set

$$u(x) = \int_X |f(x) - f(y)| J(x, y)\, d\mu(y).$$

We show that $u \in L^2$. Observe that $f(x) = f(y)$ whenever $d_*(x, y) \le r$. Hence, we can restrict the integration to the domain $\{d_*(x, y) > r\}$. We have by the Cauchy-Schwarz inequality

$$u^2(x) \le \left(\int_X |f(x) - f(y)|^2 J(x, y)\, d\mu(y)\right)\left(\int_{\{y:d_*(x,y)>r\}} J(x, y)\, d\mu(y)\right). \tag{3.14}$$

Let us show that

$$\int_{\{y:d_*(x,y)>r\}} J(x, y)\, d\mu(y) \le \frac{1}{r}.$$

Indeed, by (3.11) and Fubini's theorem, the latter integral is equal to

$$\begin{aligned}
\int_{\{y:d_*(x,y)>r\}} \int_{\{s:s\ge d_*(x,y)\}}^{\infty} \frac{1}{V(x, s)} \frac{ds}{s^2}\, d\mu(y) &= \int_r^\infty \frac{ds}{s^2 V(x, s)} \int_{\{y:r<d_*(x,y)\le s\}} d\mu(y) \\
&= \int_r^\infty \frac{V(x, s) - V(x, r)}{s^2 V(x, s)}\, ds \\
&\le \int_r^\infty \frac{ds}{s^2} = \frac{1}{r}.
\end{aligned}$$

It follows from (3.14) that

$$\int_X u^2\, d\mu \le \frac{1}{r}\,\mathcal{E}(f, f).$$

Since $f \in \mathrm{dom}_{\mathcal{L}} \subset \mathrm{dom}_{\mathcal{E}}$, we obtain that $u \in L^2$. In particular, $u(x) < \infty$ for almost all $x \in X$. Consequently, for almost all $x \in X$, the function

$$y \mapsto (f(x) - f(y))\, J(x, y)$$

is in L^1, and its integral

$$v(x) = \int_X (f(x) - f(y)) J(x,y)\, d\mu(y)$$

is an L^2 function. We need to verify that $\mathcal{L} f = v$. For that purpose it suffices to verify that, for any $g \in \mathrm{dom}_{\mathcal{E}}$,

$$(v,g)_{L^2} = \mathcal{E}(f,g).$$

Indeed, using Fubini's theorem, we obtain

$$\begin{aligned}
(v,g)_{L^2} &= \int_X \int_X (f(x) - f(y))\, g(x)\, J(x,y)\, d\mu(y)\, d\mu(x) \\
&= \int_X \int_X (f(y) - f(x))\, g(y)\, J(y,x)\, d\mu(x)\, d\mu(y) \\
&= \frac{1}{2} \int_X \int_X (f(x) - f(y)) (g(x) - g(y))\, J(x,y)\ d\mu(x)\ d\mu(y) \\
&= \mathcal{E}(f,g),
\end{aligned}$$

which was to be proved. ■

3.4 L^p-spectrum

It is known that any continuous symmetric Markov semigroup can be extended to all spaces L^p, $1 \le p < \infty$, as a continuous contraction semigroup. In particular, this is true for the semigroup $\{P^t\}$. We use the same notation for the extended semigroup, while we denote by $\mathcal{L}_p$ its infinitesimal generator and by $\mathrm{dom}_{\mathcal{L}_p}$ its domain in L^p.

Theorem 3.13 *For all $1 \le p < \infty$ we have*

$$\mathrm{spec}\, \mathcal{L}_p = \mathrm{spec}\, \mathcal{L}_2 .$$

Proof. Since by Theorem 3.8 all the eigenfunctions of $\mathcal{L}_2$ are compactly supported, they belong also to L^p, which implies that

$$\mathrm{spec}\, \mathcal{L}_2 \subset \mathrm{spec}\, \mathcal{L}_p.$$

To prove the opposite inclusion, we choose $\lambda_0 \notin \mathrm{spec}\, \mathcal{L}_2$ and show that $\lambda_0 \notin \mathrm{spec}\, \mathcal{L}_p$. For that purpose it suffices to show that the resolvent operator

$$R := (\mathcal{L}_2 - \lambda_0\, \mathrm{id})^{-1}$$

being a bounded operator in L^2, extends to a bounded operator in L^p. The latter amounts to showing that, for any functions $f \in L^2 \cap L^p$ and $g \in L^2 \cap L^q$, where $q = \frac{p}{p-1}$ is the Hölder conjugate of p, the following inequality holds:

$$|(Rf,g)_{L^2}| \le C \|f\|_{L^p} \|g\|_{L^q}$$

with a constant C that does not depend on f, g.

Let us restrict to the case $\lambda_0 > 0$ (the case when $\lambda_0 < 0$ is simpler). Choose $a, b > 0$ such that $a < \lambda_0 < b$ and $[a,b]$ is disjoint from $\mathrm{spec}\, \mathcal{L}_2$. Using the spectral decomposition (3.4), we obtain

$$R = \int_{\mathrm{spec}\, \mathcal{L}_2} \frac{dE_\lambda}{\lambda - \lambda_0} = \int_{[0,a)} \frac{dE_\lambda}{\lambda - \lambda_0} + \int_{[b,\infty)} \frac{dE_\lambda}{\lambda - \lambda_0},$$

whence

$$(Rf,g) = \int_{[0,a)} \frac{d\,(E_\lambda f, g)}{\lambda - \lambda_0} + \int_{[b,\infty)} \frac{d\,(E_\lambda f, g)}{\lambda - \lambda_0}.$$

Integration by parts gives

$$\begin{aligned}(Rf,g) &= \frac{(E_a f,g)}{a-\lambda_0}+\int_{[0,a)}\frac{(E_\lambda f,g)}{(\lambda-\lambda_0)^2}\,d\lambda \\ &\quad -\frac{(E_b f,g)}{b-\lambda_0}+\int_{[b,\infty)}\frac{(E_\lambda f,g)}{(\lambda-\lambda_0)^2}\,d\lambda.\end{aligned}$$

Since $E_\lambda = \mathrm{Q}_{1/\lambda}$ is a Markov operator, it standardly extends to a bounded operator in L^p with the norm bound 1, so that

$$|(E_\lambda f,g)| \le \|f\|_{L^p}\,\|g\|_{L^q}\,.$$

It follows that

$$|(Rf,g)| \le \|f\|_{L^p}\,\|g\|_{L^q}\left(\frac{1}{\lambda_0-a}+\frac{1}{b-\lambda_0}+\int_{[0,a)\cup[b,\infty)}\frac{d\lambda}{(\lambda-\lambda_0)^2}\right),$$

which finishes the proof since the quantity in the large parentheses is finite. ■

The last theorem of this section concerns a Liouville property. Note that the semigroup $\{P^t\}$ defined by (1.6) acts on the space $\mathcal{B}_b$ of bounded Borel functions as a contraction semigroup, but it is not continuous unless X is discrete. Define convergence of sequence in $\mathcal{B}_b$ as a bounded pointwise convergence, that is, a sequence $\{f_k\}\subset\mathcal{B}_b$ converges in $\mathcal{B}_b$ to a function f if all sequence $\{f_k\}$ is uniformly bounded and $f_k(x)\to f(x)$ as $k\to\infty$ for all $x\in X$. Define a weak infinitesimal generator $\mathcal{L}_\infty$ of the semigroup $\{P^t\}$ in $\mathcal{B}_b$ as follows: the domain $\mathrm{dom}_{\mathcal{L}_\infty}$ consists of functions $f\in\mathcal{B}_b$ such that the limit

$$\mathcal{L}_\infty f := \lim_{t\to 0}\frac{f-P^t f}{t}$$

exists in the sense of convergence in $\mathcal{B}_b$. This yields $\mathcal{L}_\infty f\in\mathcal{B}_b$ for any $f\in\mathrm{dom}_{\mathcal{L}_\infty}$.

Theorem 3.14 (Strong Liouville property) *Any Borel function $f : X\to[0\,,\infty)$ that satisfies $Pf=f$ must be constant.*

Consequently, 0 is an eigenvalue of $\mathcal{L}_\infty$ of multiplicity 1.

Proof. Since P and Q_r commute, we obtain from $f=Pf$ and

$$Pf=\int_0^\infty \mathrm{Q}_s f\,d\sigma_*(s), \tag{3.15}$$

that, for all $r\ge 0$,

$$\mathrm{Q}_r f = P\mathrm{Q}_r f=\int_{[0,\infty)}\mathrm{Q}_s\mathrm{Q}_r f\,d\sigma_*(s)\,.$$

Observing that

$$\mathrm{Q}_s\mathrm{Q}_r=\mathrm{Q}_{\max(r,s)},$$

we obtain

$$\mathrm{Q}_r f=\int_{[0,r)}\mathrm{Q}_r f\;d\sigma_*(s)+\int_{[r,\infty)}\mathrm{Q}_s f\,d\sigma_*(s)\,.$$

The first integral here is equal to $\sigma_*(r)\,\mathrm{Q}_r f$, which implies

$$(1-\sigma_*(r))\,\mathrm{Q}_r f=\int_{[r,\infty)}\mathrm{Q}_s f\,d\sigma_*(s)\,. \tag{3.16}$$

Fix some $x \in X$. By Lemma 3.3, the set $\Lambda(x)$ of all values $d_*(x,y)$ for $y \neq x$ has no accumulation point in $(0,+\infty)$. Choose r_0 as follows: if $\Lambda(x)$ does not accumulate to 0, then $r_0 = 0$, and if $\Lambda(x)$ accumulates at 0 then r_0 is any value from $\Lambda(x)$. In the both cases the set $\Lambda(x) \cap (r,\infty)$ consists of a (finite or infinite) sequence $r_1 < r_2 < ...$ that converges to ∞ in the case when it is infinite. Applying (3.16) to $r = r_k$ and $r = r_{k+1}$ instead of r, where $k \geq 0$, we obtain

$$\begin{aligned}(1-\sigma_*(r_k))\,\mathrm{Q}_{r_k} f(x) - (1-\sigma_*(r_{k+1}))\,\mathrm{Q}_{r_{k+1}} f(x) &= \int_{[r_k,r_{k+1})} \mathrm{Q}_s f(x)\, d\sigma_*(s) \\ &= \mathrm{Q}_{r_k} f(x)\left(\sigma_*(r_{k+1}) - \sigma_*(r_k)\right),\end{aligned}$$

whence it follows that

$$(1-\sigma_*(r_{k+1}))\,\mathrm{Q}_{r_k} f(x) = (1-\sigma_*(r_{k+1}))\,\mathrm{Q}_{r_{k+1}} f(x)$$

and, hence,

$$\mathrm{Q}_{r_k} f(x) = \mathrm{Q}_{r_{k+1}} f(x).$$

Consequently, we obtain that

$$\mathrm{Q}_{r_k} f(x) = \mathrm{Q}_{r_0} f(x) \quad \text{for all } k \geq 1.$$

Since r_0 can be chosen arbitrarily close to 0, we obtain that $\mathrm{Q}_r f(x)$ does not depend on r. For any two points $x, y \in X$, we have $\mathrm{Q}_r f(x) = \mathrm{Q}_r f(y)$ for $r \geq d_*(x,y)$. Therefore, the function $\mathrm{Q}_r f(x)$ is constant both in r and x. It follows from (3.15) that $f = Pf$ is also a constant.

For the second statement of the theorem, 0 is an eigenvalue of $\mathcal{L}_\infty$ because $\mathcal{L}_\infty 1 = 0$. Assume that $\mathcal{L}_\infty f = 0$ and prove that $f \equiv \mathrm{const}$, which will imply that the multiplicity of 0 is 1. By assumption we have $f \in \mathcal{B}_b$ and

$$\frac{f - P^t f}{t} \xrightarrow{\mathcal{B}_b} 0 \text{ as } t \to 0.$$

Since the family $\left\{\frac{f-P^t f}{t}\right\}_{t>0}$ is uniformly bounded, we obtain by the dominated convergence theorem that, for any $r \geq 0$,

$$\mathrm{Q}_r\left(\frac{f - P^t f}{t}\right) \xrightarrow{\mathcal{B}_b} 0 \text{ as } t \to 0,$$

which in turn implies that, for all $s \geq 0$,

$$\frac{P^s f - P^{s+t} f}{t} = P^s\left(\frac{f - P^t f}{t}\right) \xrightarrow{\mathcal{B}_b} 0 \text{ as } t \to 0.$$

It follows that, for any $x \in X$, the function $s \mapsto P^s f(x)$ has derivative 0 and, hence, is constant. It follows that $f = Pf$, and by the first statement of the theorem, we conclude that $f = \mathrm{const}$.
■

4 Moments of the isotropic Markov process

Let $\{\mathcal{X}_t\}$ be the Markov process associated with the isotropic semigroup $\{P^t\}$. For any $\gamma > 0$, the moment of order γ of the process is defined as

$$M_\gamma(x,t) = \mathbb{E}_x\left(d_*(x,\mathcal{X}_t)^\gamma\right),$$

where $\mathbb{E}_x$ is expectation with respect to the probability measure on the trajectory space of $\{\mathcal{X}_t\}$ with $\mathcal{X}_0 = x$. In terms of the heat kernel $p(t, x, y)$ the moment is given by

$$M_\gamma(x,t) = \int_X d_*(x,y)^\gamma p(t,x,y)\, d\mu(y). \tag{4.1}$$

The aim of this section is to estimate $M_\gamma(x,t)$ as a function of t and γ.

Let us start with two lemmas. We use the volume function (2.48), that is

$$V(x,r) = \mu\left(B_r^*(x)\right)$$

and its average moment function of order γ, that is

$$R_\gamma(x,\tau) = \frac{1}{V(x,\tau)} \int_{(0,\tau]} r^\gamma\, dV(x,r).$$

Lemma 4.1 *For all $x \in X$, $t > 0$ and $\gamma > 0$,*

$$M_\gamma(x,t) = t \int_0^\infty R_\gamma\left(x, \frac{1}{\tau}\right) e^{-\tau t}\, d\tau = \int_0^\infty R_\gamma\left(x, \frac{t}{s}\right) e^{-s}\, ds.$$

Proof. Using the equations (4.1) and (2.24), as well as the Definition 2.8 of the spectral distribution function in terms of the volume function, we obtain

$$\begin{aligned} M_\gamma(x,t) &= \int_X d_*(x,y)^\gamma\, p(t,x,y)\, d\mu(y) \\ &= \int_{(0\,,\,\infty)} r^\gamma \left(t \int_0^{1/r} N(x,\tau)\, e^{-\tau t}\, d\tau \right) dV(x,r) \\ &= \int_0^\infty \left(\int_{(0\,,\,1/\tau)} \frac{r^\gamma}{V(x,1/\tau)}\, dV(x,r) \right) t\, e^{-\tau t}\, d\tau = \int_0^\infty R_\gamma\left(x, \frac{1}{\tau}\right) t\, e^{-\tau t}\, d\tau. \end{aligned}$$

In the 3rd identity, we have used Fubini's theorem. ■

The volume function $r \mapsto V(x,r)$ non-decreasing and takes values from 0 to $\mu(X)$. In the compact case, $V(x,r) = \mu(X)$ for all $r \geq r^*_{\max} = r^*_{\max}(x)$, the largest value in $\Lambda(x)$ (see (3.5)). When x is isolated, $V(x,r) = \mu\{x\}$ for all $0 \leq r < r_0^* = r_0^*(x)$, the smallest positive value in $\Lambda(x)$.

Lemma 4.2 *For any given $x \in X$ and $\gamma > 0$, the following properties hold.*

(a) *The function $\tau \mapsto R_\gamma(x,\tau)$ is non-decreasing.*

If X is compact $R_\gamma(x,\tau) = R_\gamma\left(x, r^*_{\max}(x)\right)$ for all $\tau \geq r^*_{\max}(x)$.

If X is discrete and infinite, $R_\gamma(x,\tau) = R_\gamma\left(x, r_0^*(x)\right)$ for all $0 < \tau \leq r_0^*(x)$.

(b) *For all $\tau > 0$, we have*

$$R_\gamma(x,\tau) \leq \tau^\gamma$$

and, if the volume function $r \mapsto V(x,r)$ satisfies the reverse doubling property, then there exists a constant $c > 0$, such that

$$R_\gamma(x,\tau) \geq c\,\tau^\gamma \tag{4.2}$$

for all $\tau > 0$. In the non-discrete compact case, if the volume function just satisfies the reverse doubling property at zero, (4.2) holds for all $0 < \tau < r^*_{\max}(x)$. In the discrete infinite case, if the volume function just satisfies the reverse doubling property at infinity, (4.2) holds for all $\tau > r_0^*(x)$.

Proof. For the first part of (a), we integrate by parts:

$$R_\gamma(x,\tau) = \frac{1}{V(x,\tau)}\left(\tau^\gamma\, V(x,\tau) - \int_{(0\,,\,\tau]} V(x,s)\, ds^\gamma\right) = \int_{(0\,,\,\tau]} \left(1 - \frac{V(x,s)}{V(x,\tau)}\right) ds^\gamma,$$

whence $\tau \mapsto \mathcal{R}_\gamma\,(x,\tau)$ is non-decreasing.

The second part (a) is straightforward.

Regarding (b), the general upper bound on $R_\gamma(x,\tau)$ is obvious. If the volume function satisfies the reverse doubling property, then in the respective range,

$$\begin{aligned} R_\gamma(x,\tau) &\geq \frac{1}{V(x,\tau)}(\delta\tau)^\gamma\,(V(x,\tau) - V(x,\delta\tau)) \\ &= (\delta\tau)^\gamma\left(1 - \frac{V(x,\delta\tau)}{V(x,\tau)}\right) \geq \delta^\gamma(1-\kappa)\tau^\gamma = c\,\tau^\gamma \end{aligned}$$

for suitable constants $0 < \kappa, c < 1$. ■

Now, in order to estimate the moment function $t \mapsto M_\gamma(x,t)$, we need to estimate a Laplace-type integral as given by the formula of Lemma 4.1. We will treat such estimates in the two technical Propositions 4.6 and 4.7 at the end of this section. Before that, in the next three theorems, we anticipate the statements of the results regarding the moment function.

Theorem 4.3 *Assume that (X,d) is non-compact and has no isolated points. Then the following properties hold.*

(1) *For all $x \in X$, $t > 0$ and $0 < \gamma < 1$,*

$$M_\gamma(x,t) \leq \frac{t^\gamma}{1-\gamma}.$$

(2) *If for some $x \in X$, the volume function satisfies the reverse doubling property, then for any $0 < \gamma < 1$,*

$$M_\gamma(x,t) \geq \frac{c}{1-\gamma}t^\gamma,$$

for all $x, t > 0$ and some $c > 0$. Moreover,

$$M_\gamma\,(z,t) = \infty,$$

for all z, $t > 0$ and $\gamma \geq 1$.

Theorem 4.4 *Assume that (X,d) is discrete and infinite. Then the following properties hold.*

(a) *For all x, $t > 0$ and $0 < \gamma < 1$,*

$$M_\gamma(x,t) \leq \frac{C}{1-\gamma}\min\{t, t^\gamma\}$$

for some $C > 0$.

(b) *If for some (equivalently, all) $x \in X$ the volume function satisfies the reverse doubling property at infinity, then for any $0 < \gamma < 1$,*

$$M_\gamma(z,t) \geq \frac{c}{1-\gamma}\min\{t, t^\gamma\}$$

for all z , $t > 0$ and for some $c > 0$. Moreover,

$$M_\gamma(z,t) = \infty$$

for all z, $t > 0$ and all $\gamma \geq 1$.

Assume now that (X,d) is compact and let D be its d_*-diameter. By Lemmas 4.1 and 4.2, for all $x \in X$, $\gamma > 0$ and $t > 0$,

$$M_\gamma(x,t) \le R_\gamma(x,D) \le D^\gamma,$$

whence we study the behavior of the moment function $t \mapsto M_\gamma(x,t)$ at zero.

Theorem 4.5 *Assume that (X,d) is non-discrete and compact. Then the following properties hold.*

(1) *There exists a constant $C > 0$ such that*

$$M_\gamma(x,t) \le C \begin{cases} t & \text{if} \quad \gamma > 1, \\ t\left(\log \frac{1}{t} + 1\right) & \text{if} \quad \gamma = 1, \\ t^\gamma & \text{if} \quad \gamma < 1, \end{cases}$$

holds for all x and all $0 < t \le 1$.

(2) *If for some $x \in X$ the volume function satisfies the reverse doubling property at zero, then there exists a constant $c > 0$ such that*

$$M_\gamma(z,t) \ge c \begin{cases} t & \text{if} \quad \gamma > 1, \\ t\left(\log \frac{1}{t} + 1\right) & \text{if} \quad \gamma = 1, \\ t^\gamma & \text{if} \quad \gamma < 1 \end{cases}$$

holds for all z and all $0 < t \le 1$.

We now provide the technical details regarding the Laplace-type estimates that imply Theorems 4.3, 4.4 and 4.5. In the following two propositions, M and R will always be two non-negative, non-decreasing functions related by the Laplace-type integral

$$M(t) = \int_0^\infty R\left(\frac{t}{\tau}\right) e^{-\tau}\, d\tau\,.$$

Proposition 4.6 *Let $\gamma > 0$ be given.*

(1) *Assume that*

$$A s^\gamma \ge R(s)\,, \quad \text{or that respectively} \quad R(s) \ge B\, s^\gamma \tag{4.3}$$

for some $A > 0$ (resp. $B > 0$) and all $s > 0$. Then the inequality

$$\frac{A\, t^\gamma}{1-\gamma} \ge M(t)\,, \quad \text{respectively} \quad M(t) \ge \frac{B\, t^\gamma}{(1-\gamma)\, e}$$

holds for all $0 < \gamma < 1$ and all $t > 0$.

(2) *Assume that there is $t_0 > 0$ such that $R(s) = 0$ for all $0 < s < t_0$. Assume also that one of the respective inequalities of (4.3) holds for all $s > t_0$. Then*

$$M(t) \le \frac{c}{1-\gamma} \min\left\{\frac{t}{t_0}, \left(\frac{t}{t_0}\right)^\gamma\right\}, \quad \text{respectively} \quad M(t) \ge \frac{c'}{1-\gamma} \min\left\{\frac{t}{t_0}, \left(\frac{t}{t_0}\right)^\gamma\right\},$$

for all $0 < \gamma < 1$, all $t > 0$ and some constants $c, c' > 0$.

(3) *The assumption $\gamma \ge 1$ and the lower bound $R(s) \ge B\, s^\gamma$ imply that $M(t) = \infty$ for all $t > 0$.*

Proof. It is known that for $0 < \gamma < 1$ the Gamma-function satisfies

$$\frac{1}{(1-\gamma)\,e} < \Gamma(1-\gamma) < \frac{1}{1-\gamma},$$

whence by monotonicity of the Laplace-type integral the first claim follows.

To prove the second statement, we write

$$M(t) = \int_{\{t/s \geq t_0\}} R\left(\frac{t}{s}\right) e^{-s}\, ds.$$

First assume that $R(\tau) \leq A\, s^{\gamma}$ for all $0 < s < \infty$. Then we obtain

$$\begin{aligned} M(t) &\leq A \int_{\{t/s\geq t_0\}} \left(\frac{t}{s}\right)^{\gamma} e^{-s}\, ds = A\, t^{\gamma} \int_{\{s \leq t/t_0\}} s^{-\gamma} e^{-s}\, ds \\ &\leq A\, t^{\gamma} \int_0^{t/t_0} s^{-\gamma}\, ds = \left(\frac{t}{t_0}\right) \frac{A t_0^{-\gamma}}{1-\gamma}, \quad \text{and} \end{aligned}$$

$$M(t) \leq A\, t^{\gamma} \int_0^{\infty} s^{-\gamma} e^{-s}\, ds \leq \frac{A\, t^{\gamma}}{1-\gamma} = \left(\frac{t}{t_0}\right)^{\gamma} \frac{A\, t_0^{\gamma}}{1-\gamma}.$$

It follows that

$$M(t) \leq \frac{A\, \max\left\{t_0, t_0^{-1}\right\}}{1-\gamma} \min\left\{\frac{t}{t_0}, \left(\frac{t}{t_0}\right)^{\gamma}\right\}.$$

Second, assume that $R(s) \geq B\, s^{\gamma}$, for all $s \geq t_0$. Then for $t/t_0 \geq 1$

$$M(t) \geq B\, t^{\gamma} \int_0^{t/t_0} s^{-\gamma} e^{-s}\, ds \geq \frac{B\, t^{\gamma}}{e} \int_0^1 s^{-\gamma}\, ds = \frac{B\, t^{\gamma}}{(1-\gamma) e} = \frac{B\, t_0^{\gamma}}{(1-\gamma)\, e} \left(\frac{t}{t_0}\right)^{\gamma}.$$

When $t/t_0 \leq 1$ we get

$$M(t) \geq B\, t^{\gamma} \int_0^{t/t_0} s^{-\gamma} e^{-s}\, ds \geq \frac{B\, t^{\gamma}}{e} \int_0^{t/t_0} s^{-\gamma}\, ds = \frac{B\, t^{\gamma}}{(1-\gamma)\, e} \left(\frac{t}{t_0}\right)^{1-\gamma} = \frac{B\, t_0^{\gamma}}{(1-\gamma)\, e} \left(\frac{t}{t_0}\right).$$

It follows that

$$M(t) \geq \frac{B\, t_0^{\gamma}}{(1-\gamma)\, e} \min\left\{\frac{t}{t_0}, \left(\frac{t}{t_0}\right)^{\gamma}\right\} \geq \frac{B \min\{t_0, 1\}}{(1-\gamma)\, e} \min\left\{\frac{t}{t_0}, \left(\frac{t}{t_0}\right)^{\gamma}\right\}.$$

This proves the second claim. For the third claim observe that that if $R(s) \geq B\, s^{\gamma}$ for all $s \geq t_0$ and $\gamma \geq 1$,

$$M(t) \geq B\, t^{\gamma} \int_0^{t/t_0} s^{-\gamma} e^{-s}\, ds = \infty$$

for all $t > 0$. ■

Proposition 4.7 *Assume that there is $t_0 > 0$ such that $R(s) = R(t_0)$ for all $s \geq t_0$. Assume also that one of the respective inequalities in (4.3) holds for all $0 < s \leq t_0$. Then*

$$M(t) \leq \begin{cases} c_1 \frac{t}{t_0} & \text{if} \quad \gamma > 1, \\ c_2\, t \left(\log \frac{t_0}{t} + 1\right) & \text{if} \quad \gamma = 1, \\ c_3 \left(\frac{t}{t_0}\right)^{\gamma} & \text{if} \quad \gamma < 1, \end{cases}$$

$$\text{respectively,} \quad M(t) \geq \begin{cases} c_1' \frac{t}{t_0} & \text{if} \quad \gamma > 1, \\ c_2'\, t \left(\log \frac{t_0}{t} + 1\right) & \text{if} \quad \gamma = 1, \\ c_3' \left(\frac{t}{t_0}\right)^{\gamma} & \text{if} \quad \gamma < 1, \end{cases}$$

for all $0 < t \leq t_0$ and some positive constants $c_1, c_1', c_2, c_2', c_3, c_3'$.

Proof. Let $\gamma > 1$ and $0 < t < t_0$. According to our assumption

$$M(t) = \int_{\{t/s \le t_0\}} R\left(\frac{t}{s}\right) e^{-s}\, ds + R(t_0)\left(1 - e^{-t/t_0}\right).$$

Observe that for $0 < t < t_0$,

$$\frac{t}{2t_0} \le \left(1 - e^{-t/t_0}\right) \le \frac{t}{t_0}.$$

First, if $R(s) \le A\, s^\gamma$ for all $0 < s < t_0$, then

$$\begin{aligned} M(t) &\le A\, t^\gamma \int_{t/t_0}^\infty s^{-\gamma} e^{-s}\, ds + \frac{R(t_0)t}{t_0} \le A\, s^\gamma \int_{t/t_0}^\infty s^{-\gamma}\, ds + \frac{R(t_0)t}{t_0} \\ &\le \frac{A\, t^\gamma}{\gamma - 1}\left(\frac{t}{t_0}\right)^{1-\gamma} + \frac{R(t_0)t}{t_0} = \frac{t}{t_0}\left(R(t_0) + \frac{A\, t_0^\gamma}{\gamma - 1}\right). \end{aligned}$$

Second, if $R(s) \ge B\, s^\gamma$, for all $0 < s < t_0$, then

$$M(t) \ge \frac{R(t_0)}{2}\frac{t}{t_0}.$$

Assume that $0 < \gamma < 1$ and $0 < t < t_0$. Again first, if $R(s) \le A\, s^\gamma$ for all $0 < \tau < t_0$, then

$$\begin{aligned} M(t) &\le At^\gamma \int_{t/t_0}^\infty s^{-\gamma} e^{-s}\, ds + \frac{R(t_0)t}{t_0} \le A\, t^\gamma\, \Gamma(1-\gamma) + R(t_0)\frac{t}{t_0} \\ &\le \frac{A\, t^\gamma}{1-\gamma} + R(t_0)\frac{t}{t_0} = \frac{A\, t_0^\gamma}{1-\gamma}\left(\frac{t}{t_0}\right)^\gamma + R(t_0)\frac{t}{t_0} \\ &\le \left(\frac{t}{t_0}\right)^\gamma \left(\frac{A\, T^\gamma}{1-\gamma} + R(t_0)\right). \end{aligned}$$

Second, once more, when $R(s) \ge B\, s^\gamma$, for all $0 < s < T$, then

$$M(t) \ge B\, t^\gamma \int_{t/t_0}^\infty s^{-\gamma} e^{-s}\, ds \ge B\, t^\gamma \int_1^\infty s^{-\gamma} e^{-s}\, ds \ge \left(\frac{t}{t_0}\right)^\gamma \left(\frac{B\, \min\{t_0, 1\}}{e^2}\right).$$

Finally, assume that $\gamma = 1$ and $0 < t < t_0$. First, if $R(s) \le A\, s^\gamma$ for all $0 < \tau < t_0$, then

$$\begin{aligned} M(t) &\le At \int_{t/T}^\infty s^{-1} e^{-s} ds + \frac{R(T)t}{T} \\ &= At\left(\int_1^\infty s^{-1} e^{-s}\, ds + \int_{t/t_0}^1 s^{-1} e^{-s}\, ds\right) + \frac{R(t_0)t}{t_0} \\ &\le A\, t\left(\int_1^\infty \frac{ds}{s^2} + \int_{t/t_0}^1 \frac{ds}{s}\right) + \frac{R(t_0)t}{t_0} = \left(A + \frac{R(t_0)}{t_0}\right) t \left(\log\frac{t_0}{t} + 1\right). \end{aligned}$$

And at last, if $R(s) \ge B\, s^\gamma$ for all $0 < \tau < t_0$, then

$$\begin{aligned} M(t) &\ge B\, t \int_{t/t_0}^\infty s^{-1} e^{-s}\, ds + \frac{R(t_0)t}{2t_0} \\ &\ge \frac{B\, t}{e} \int_{t/t_0}^1 \frac{ds}{s} + \frac{R(t_0)t}{2t_0} = \frac{B\, t}{e} \log\frac{t_0}{t} + \frac{R(t_0)t}{2t_0} \\ &= \frac{B\, t}{e}\left(\log\frac{t_0}{t} + \frac{R(t_o)e}{2B\, t_0}\right) \ge \min\left\{\frac{R(t_0)}{2t_0}, \frac{B}{e}\right\} t \left(\log\frac{t_0}{t} + 1\right). \end{aligned}$$

The proof is finished. ■

Theorems 4.3, 4.4 and 4.5 follow.

5 Analysis in $\mathbb{Q}_p$ and $\mathbb{Q}_p^n$

5.1 The p-adic fractional derivative

Consider the field $\mathbb{Q}_p$ of p-adic numbers endowed with the p-adic norm $\|x\|_p$ and the p-adic ultra-metric $d_p(x,y) = \|x-y\|_p$. Let μ_p be the Haar measure on $\mathbb{Q}_p$, normalized such that $\mu_p(\mathbb{Z}_p) = 1$. Let $\mathcal{V}_c$ be the space of locally constant functions on $\mathbb{Q}_p$ with compact support which will be considered as test functions on $\mathbb{Q}_p$.

The notion of p-adic fractional derivative, closely related to the concept of p-adic Quantum Mechanics, was introduced in the papers by Vladimirov [57], Vladimirov and Volovich [58] and Vladimirov, Volovich and Zelenov [59]. In particular, a one-parameter family $\{\mathfrak{D}^\alpha\}_{\alpha>0}$ of operators, called operators of fractional derivative of order α, was introduced in [57].

Recall that the Fourier transform $\mathcal{F} : f \mapsto \widehat{f}$ of a function f on the self-dual locally compact Abelian group $\mathbb{Q}_p$ is defined by

$$\widehat{f}(\theta) = \int_{\mathbb{Q}_p} \langle x, \theta\rangle\, f(x) d\mu_p(x),$$

where $x, \theta \in \mathbb{Q}_p$,

$$\langle x, \theta\rangle = \exp\left(2\pi\sqrt{-1}\,\{x\theta\}\right),$$

and $\{x\theta\}$ is the fractional part of the p-adic number $x\theta$ (cf. (2.36)). It is known that $\mathcal{F}$ is a linear isomorphism of $\mathcal{V}_c$ onto itself.

Definition 5.1 The operator $(\mathfrak{D}^\alpha, \mathcal{V}_c)$, $\alpha > 0$, is defined via the Fourier transform on the locally compact Abelian group $\mathbb{Q}_p$ by

$$\widehat{\mathfrak{D}^\alpha f}(\xi) = \|\xi\|_p^\alpha\, \widehat{f}(\xi), \qquad \xi \in \mathbb{Q}_p.$$

It was shown by the above named authors that the operator $(\mathfrak{D}^\alpha, \mathcal{V}_c)$ can be written as a Riemann-Liouville type singular integral operator

$$\mathfrak{D}^\alpha f(x) = \frac{p^\alpha - 1}{1 - p^{-\alpha-1}} \int_{\mathbb{Q}_p} \frac{f(x) - f(y)}{\|x-y\|_p^{1+\alpha}}\, d\mu_p(y). \tag{5.1}$$

The aim of this section is in particular to show that the operator $(\mathfrak{D}^\alpha, \mathcal{V}_c)$ is in fact the restriction to $\mathcal{V}_c$ of an appropriate isotropic Laplacian. We use the following distance distribution function

$$\sigma_\alpha(r) = \exp\left(-\left(\frac{p}{r}\right)^\alpha\right).$$

Denote by $\{P_\alpha^t\}$ the isotropic semigroup associated with the triple $(d_p, \mu_p, \sigma_\alpha)$, and let $\mathcal{L}_\alpha$ be the corresponding Laplacian.

Theorem 5.2 *For any $\alpha > 0$, we have*

$$(\mathcal{L}_\alpha, \mathcal{V}_c) = (\mathfrak{D}^\alpha, \mathcal{V}_c). \tag{5.2}$$

Proof. By Theorem 3.12, we have, for any $f \in \mathcal{V}_c$,

$$\mathcal{L}_\alpha f(x) = \int_{\mathbb{Q}_p} (f(x) - f(y))\, J_\alpha(x,y)\, d\mu_p(y),$$

where

$$J_\alpha(x,y) = \int_{d_*(x,y)}^\infty \frac{s^{-2} ds}{\mu_p\left(B_s^*(x)\right)}.$$

As in Example (2.24), we have

$$d_*(x,y) = \left(\frac{\|x-y\|_p}{p}\right)^{\alpha}, \tag{5.3}$$

whence

$$B_s^*(x) = B_{ps^{1/\alpha}}(x).$$

The change $r = ps^{1/\alpha}$ yields

$$J_\alpha(x,y) = p^\alpha \int_{\|x-y\|_p}^{\infty} \frac{\alpha r^{-\alpha-1}\,dr}{\mu_p\left(B_r(x)\right)}.$$

Since the value set of the metric $\|x-y\|_p$ is $\{p^n\}_{k\in\mathbb{Z}}$, we obtain from (2.37) that

$$\mu_p\left(B_r\left(x\right)\right) = p^n \quad \text{if} \quad p^n \le r < p^{n+1}, \tag{5.4}$$

which implies, for $\|x-y\|_p = p^k$, that

$$\begin{aligned}
\int_{p^k}^{\infty} \frac{\alpha\, r^{-\alpha-1}\,dr}{\mu_p\left(B_r(x)\right)} &= \sum_{n\ge k} \int_{p^n}^{p^{n+1}} \frac{\alpha\, r^{-\alpha-1}\,dr}{\mu_p\left(B_r(x)\right)} \\
&= \sum_{n\ge k} \int_{p^n}^{p^{n+1}} \frac{-dr^{-\alpha}}{p^n} = \sum_{n\ge k} \frac{1}{p^n}\left(\frac{1}{p^{n\alpha}} - \frac{1}{p^{(n+1)\alpha}}\right) \\
&= \left(1-\frac{1}{p^\alpha}\right)\sum_{n\ge k}\frac{1}{p^{n(\alpha+1)}} = \left(1-\frac{1}{p^\alpha}\right)\frac{p^{-k(\alpha+1)}}{1-p^{-(\alpha+1)}} \\
&= \frac{1-p^{-\alpha}}{1-p^{-(\alpha+1)}}\left(\frac{1}{p^k}\right)^{\alpha+1} = \frac{1-p^{-\alpha}}{1-p^{-(\alpha+1)}}\left(\frac{1}{\|x-y\|_p}\right)^{\alpha+1}.
\end{aligned}$$

Hence, we obtain the identity

$$J_\alpha(x,y) = \frac{p^\alpha - 1}{1-p^{-\alpha-1}}\,\frac{1}{\|x-y\|_p^{\alpha+1}}, \tag{5.5}$$

which in view of (5.1) finishes the proof. ■

The heat kernel for the semigroup $\left\{P_\alpha^t\right\}$ was estimated in Example 2.24. We restate this estimate here as a theorem.

Theorem 5.3 *The semigroup $\{P_\alpha^t\}$ admits a continuous transition density $p_\alpha(t,x,y)$ with respect to Haar measure μ_p, which satisfies for all $t>0$ and $x,y\in\mathbb{Q}_p$ the estimate*

$$p_\alpha(t,x,y) \simeq \frac{t}{\left(t^{1/\alpha} + \|x-y\|_p\right)^{1+\alpha}}. \tag{5.6}$$

The upper bound in (5.6) was also obtained by a different method by Kochubei [39, Ch.4.1, Lemma 4.1].

Theorem 5.4 *The semigroup $\{P_\alpha^t\}$ is transient if and only if $\alpha<1$. In the transient case, its Green function g_α is given explicitly by*

$$g_\alpha(x,y) = \frac{1-p^{-\alpha}}{1-p^{\alpha-1}}\,\|x-y\|_p^{\alpha-1}. \tag{5.7}$$

The formula (5.7) for a fundamental solution of $\mathfrak{D}_\alpha$ acting in the space $\mathcal{V}'_c$ of Bruhat distribution, was obtained by Vladimirov [57, Thm 1, p.51] and Kochubei [39, Ch.2.2].

Proof. That $\alpha < 1$ is equivalent to transience was shown in Example 2.30. Assuming $\alpha < 1$, we obtain by (2.45)

$$g_\alpha(x,y) = \int_{d_*(x,y)}^{\infty} \frac{ds}{\mu_p(B_s^*(x))} = \frac{1}{p^\alpha} \int_{\|x-y\|_p}^{\infty} \frac{\alpha r^{\alpha-1}\, dr}{\mu_p\left(B_r(x)\right)}.$$

Setting $\|x-y\|_p = p^k$ and using (5.4), we obtain

$$\begin{aligned} g_\alpha(x,y) &= \frac{1}{p^\alpha} \sum_{n\geq k} \int_{p^n}^{p^{n+1}} \frac{dr^\alpha}{p^n} = \frac{1}{p^\alpha} \sum_{n\geq k} \frac{1}{p^n}\left(p^{(n+1)\alpha} - p^{n\alpha}\right) \\ &= \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} p^{(\alpha-1)k}, \end{aligned}$$

which finishes the proof. ■

Denote by $\mathcal{L}_{\alpha,q}$ the generator of the semigroup $\{P_\alpha^t\}$ acting in $L^q(\mu_p)$, $1 \leq q < \infty$. Applying Corollary 3.9 and Theorem 3.13, we obtain the following.

Theorem 5.5 *For any $\alpha > 0$ and $1 \leq q < \infty$, we have*

$$\operatorname{spec} \mathcal{L}_{\alpha,q} = \{p^{\alpha k} : k \in \mathbb{Z}\} \cup \{0\}.$$

Each $\lambda_k = p^{\alpha k}$ is an eigenvalue with infinite multiplicity.

Proof. We only need to show that the multiplicity of λ_k is infinite. In the general setting of Theorem 3.8 and Corollary 3.9, some eigenvalues may well have finite multiplicity and some not. Indeed, each ball B with the minimal positive d_*-radius ρ generates a finite dimensional eigenspace $\mathcal{H}_B$ that consists of eigenfunctions with the eigenvalue $\frac{1}{\rho}$. It follows that the eigenvalue $\frac{1}{\rho}$ has finite multiplicity if and only if there is only a finite number of distinct balls of d_*-radius ρ.

In the present setting in $\mathbb{Q}_p$ there are infinitely many disjoint balls of the same radius ρ, as they all can be obtained by translations of one such ball. Thus, all the eigenvalues have infinite multiplicity. ■

Let $\{X_t\}$ be the Markov process on $\mathbb{Q}_p$ driven by the Markov semigroup $\{P_\alpha^t\}_{t>0}$. The semigroup is translation invariant, whence the process has independent and stationary increments. For any given $\gamma > 0$ and $t > 0$, consider the moment of order γ of $\mathcal{X}_t$ defined in terms of the p-adic distance $d_p(x,y)$:

$$\mathcal{M}_\gamma(t) = \mathbb{E}(\|\mathcal{X}_t\|_p^\gamma),$$

where $\mathbb{E}$ is expectation with respect to the probability measure on the trajectory space of the process starting at 0. Applying Theorem 4.3 and using the relation (5.3) between d_* and $\|\cdot\|_p$, we obtain the following estimates.

Theorem 5.6 *The moment $\mathcal{M}_\gamma(t)$ is finite if and only if $\gamma < \alpha$. In that case, there exists a constant $\kappa = \kappa(\alpha) > 0$ such that*

$$\frac{\kappa\, t^{\gamma/\alpha}}{\alpha-\gamma} \leq \mathcal{M}_\gamma(t) \leq \frac{\alpha\, t^{\gamma/\alpha}}{\alpha-\gamma}.$$

5.2 Rotation invariant Markov semigroups

Let $\{P_t\}_{t\geq 0}$ be a symmetric, translation invariant Markov semigroup on the additive Abelian group $\mathbb{Q}_p$. This semigroup acts in $C_0(\mathbb{Q}_p)$, the Banach space of continuous functions vanishing at ∞. It follows that there exists a weakly continuous convolution semigroup $\{p_t\}_{t>0}$ of symmetric probability measures on $\mathbb{Q}_p$ such that

$$P_t f(x) = p_t * f(x). \tag{5.8}$$

As the probability measures p_t are symmetric, the following identity holds, which is basic in the theory of infinite divisible distributions:

$$\widehat{p_t}(\zeta) = \exp\left(-t\,\Psi(\zeta)\right),$$

where $\Psi : \mathbb{Q}_p \mapsto \mathbb{R}_+$ is a negative definite symmetric function on $\mathbb{Q}_p$. By the Lévy-Khinchin formula,

$$\Psi(\zeta) = \int_{\mathbb{Q}_p\setminus\{0\}} \left(1 - \mathrm{Re}\langle x,\zeta\rangle\right)\, d\mathfrak{J}(x),$$

where $\mathfrak{J}$ is a symmetric Radon measure on $\mathbb{Q}_p \setminus \{0\}$ – the Levy measure associated with the negative definite function Ψ (see for the details the book of Berg and Forst [10]).

Definition 5.7 For any $a \in \mathbb{Q}_p$ with $\|a\|_p = 1$ define the rotation operator $\theta_a : \mathbb{Q}_p \to \mathbb{Q}_p$ by $\theta_a(x) = ax$. We say that the Markov semigroup $\{P_t\}$ as above is *rotation invariant* if

$$\theta_a(p_t) = p_t \ \text{ for all } a \in \mathbb{Q}_p \text{ with } \|a\|_p = 1, \tag{5.9}$$

Let $\mathcal{L}$ be the (positive definite) generator of P_t, that is, $P_t = \exp(-t\mathcal{L})$. It is easy to see that (5.9) is equivalent to $\theta_a \circ \mathcal{L} = \mathcal{L} \circ \theta_a$. In this case we also say that $\mathcal{L}$ is rotation invariant. By construction, any isotropic Markov semigroup $\left\{P^t\right\}$ defined on the ultra-metric measure space $(\mathbb{Q}_p, d_p, \mu_p)$ is rotation invariant. As we will see the class of all isotropic Markov semigroups is indeed a proper subset of the class of rotation invariant Markov semigroups.

Assume that the semigroup $\{P_t\}$ is rotation invariant. Then, for all a such that $\|a\|_p = 1$ we have

$$\Psi(a\zeta) = \Psi(\zeta) \quad \text{and} \quad \theta_a(\mathfrak{J}) = \mathfrak{J}. \tag{5.10}$$

Since the Haar measure μ_p of each sphere is strictly positive, (5.9) and (5.10) imply that the measures p_t and $\mathfrak{J}$ are absolutely continuous with respect to μ_p and have densities $p_t(x)$ and $J(x)$ which depend only on $\|x\|_p$. The same is true for the function Ψ, so that

$$J(x) = \mathfrak{j}(\|x\|_p) \quad \text{and} \quad \Psi(\zeta) = \psi(\|\zeta\|_p).$$

All the above shows that, for the generator $\mathcal{L}$ of $\{P_t\}$, we have $\mathcal{V}_c \subset \mathrm{dom}_{\mathcal{L}}$ and

$$\mathcal{L}u = \psi(\mathfrak{D})u, \quad u \in \mathcal{V}_c, \tag{5.11}$$

where $\mathfrak{D} = \mathfrak{D}^1$ is the operator of fractional derivative of order $\alpha = 1$, which we identify with the isotropic Laplacian $\mathcal{L}_1$ by Theorem 5.2.

It follows from (5.11) and (5.2) that the eigenfunctions of the operator $(\mathcal{L}, \mathcal{V}_c)$ in L^2 has a complete system of eigenfunctions $\{f_C : C \in \mathcal{K}\}$ as described in Theorem 3.8. Associated with each ball B of radius p^m, there is the $(p-1)$-dimensional eigenspace $\mathcal{H}_B$ spanned by the functions f_C, where C runs through all balls that are children of B, and the corresponding eigenvalue is

$$\lambda(m) = \psi(p^{-m+1}).$$

Let $\{a(m)\}_{m\in\mathbb{Z}}$ be a sequence of real numbers satisfying

$$a(m) \geq a(m+1), \quad a(+\infty) = 0 \quad \text{and} \quad 0 < a(-\infty) = W \leq +\infty. \tag{5.12}$$

Define the sequence $\{\lambda(m)\}_{m\in\mathbb{Z}}$ by

$$\lambda(m) = a(m) - (p-1)^{-1}\{a(m+1) - a(m)\}. \tag{5.13}$$

Theorem 5.8 *A sequence $\{\lambda(m)\}_{m\in\mathbb{Z}}$ of reals represents the spectrum* $\operatorname{spec}\mathcal{L}$ *of a rotation invariant Laplacian $\mathcal{L}$ on $\mathbb{Q}_p$ if and only if it is given by* (5.13) *with a sequence $a(m)$ that satisfies* (5.12).

Proof. Consider a rotation invariant Laplacian $\mathcal{L} = \psi(\mathfrak{D})$. Let us compute the negative definite function $\Psi(\zeta) = \psi(\|\zeta\|_p)$ associated with $\mathcal{L}$. We have

$$\begin{aligned} \psi(\|\zeta\|_p) &= \int_{\mathbb{Q}_p\setminus\{0\}} (1 - \operatorname{Re}\langle x,\zeta\rangle)\, \mathfrak{j}(\|x\|_p)\, d\mu_p(x) \\ &= \sum_{k\in\mathbb{Z}} \mathfrak{j}(p^k) \int_{\{x:\|x\|_p = p^k\}} (1 - \operatorname{Re}\langle x,\zeta\rangle)\, d\mu_p(x). \end{aligned}$$

According to Vladimirov [57, Example 4],

$$\int_{\{x:\|x\|_p=p^k\}} \langle x,\zeta\rangle\, d\mu_p(x) = \begin{cases} p^k - p^{k-1} & \text{if} \quad \|\zeta\|_p \leq p^{-k}, \\ -p^{k-1} & \text{if} \quad \|\zeta\|_p = p^{-k+1}, \\ 0 & \text{if} \quad \|\zeta\|_p \geq p^{-k+2}. \end{cases}$$

In particular, we have

$$\int_{\{x:\|x\|_p=p^k\}} d\mu_p(x) = p^k - p^{k-1}.$$

Let $\|\zeta\|_p = p^{-m+1}$, then the above computations yield

$$\psi(p^{-m+1}) = \mathfrak{j}(p^m)\, p^m + \left(1 - p^{-1}\right) \sum_{k\geq m+1} \mathfrak{j}(p^k)\, p^k. \tag{5.14}$$

Define the non-increasing sequence $\{a(m)\}_{m\in\mathbb{Z}}$ by

$$a(m) = \left(1-p^{-1}\right) \sum_{k\geq m} \mathfrak{j}(p^k)\, p^k = \left(1-p^{-1}\right) \int_{\{x:\|x\|_p\geq p^m\}} \mathfrak{j}(\|x\|_p)\, d\mu_p(x). \tag{5.15}$$

By (5.15), the equation (5.14) will get the following form

$$\begin{aligned} \psi(p^{-m+1}) &= \frac{p}{p-1}\,(a(m) - a(m+1)) + a(m+1) \\ &= a(m) - (p-1)^{-1}\,(a(m+1) - a(m))\}. \end{aligned} \tag{5.16}$$

Let $\lambda(m)$ be the eigenvalue of the Laplacian $(\psi(\mathfrak{D}), \mathcal{V}_c)$ corresponding to the ball B of radius p^m. Then $\lambda(m) = \psi(p^{-m+1})$ and the identity (5.16) gives the desired result, namely, the equation (5.13).

Conversely, given a sequence $\{a(m)\}$ as in (5.12), we define the sequence $\{\lambda(m)\}$ by (5.13) and set

$$\begin{aligned} \Psi(\xi) &= \psi(\|\xi\|_p), \quad \text{where} \quad \psi(p^m) = \lambda(-m+1), \quad \text{and} \\ J(x) &= \mathfrak{j}(\|x\|_p), \quad \text{where} \quad \mathfrak{j}(p^m) = (a(m) - a(m+1))/\left(p^m - p^{m-1}\right). \end{aligned} \tag{5.17}$$

It is straightforward to show that

$$\Psi(\zeta) = \int_{\mathbb{Q}_p\setminus\{0\}} (1 - \mathrm{Re}\langle x,\zeta\rangle)\; J(x)\, d\mu_p(x)\,,$$

whence Ψ is a negative definite function. It follows that the function $\exp(-t\,\Psi)$ is positive definite, whence it is the Fourier transform of a probability measure p_t. Clearly, $\{p_t\}_{t>0}$ is a weakly continuous convolution semigroup of probability measures. By construction, each measure p_t is rotation invariant. Finally, we define the translation invariant Markov semigroup by $P_t f = f * p_t$. ■

Corollary 5.9 *In the above notation the following statements are equivalent.*

(1) *The sequence $\lambda(m)$ is non-increasing.*

(2) *The sequence $\psi(p^m)$ is non-decreasing.*

(3) *The sequence $\mathfrak{j}(p^m)$ is non-increasing.*

In particular, if the sequence $a(m)$ is convex, then each of the equivalent properties (1)−(3) *holds.*

Proof. The equivalence (1) $\Longleftrightarrow$ (2) follows from the relation $\lambda(m) = \psi(p^{-m+1})$. To prove that (1) $\Longleftrightarrow$ (3), we apply (5.17) and obtain

$$\lambda(m) - \lambda(m+1) = (p^m - p^{m-1})\left(\mathfrak{j}(p^m) - \mathfrak{j}(p^{m+1})\right).$$

The equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follows. Finally, (5.13) and the convexity of $a\,(m)$ yield (1). ■

Next, we consider strict monotonicity.

Corollary 5.10 *The following statements are equivalent*

(i) *The sequence $\lambda(m)$ is strictly decreasing, and $\lambda(-\infty) = +\infty$.*

(ii) *The sequence $\psi(p^m)$ is strictly increasing, and $\psi(+\infty) = +\infty$.*

(iii) *The sequence $\mathfrak{j}(p^m)$ is strictly decreasing, and $\int \mathfrak{j}(\|x\|_p)\, d\mu_p(x) = +\infty$.*

(iv) *The associated rotation invariant Markov semigroup $\{P_t\}$ is isotropic.*

In particular, if the sequence $a(m)$ is strictly convex and $a(-\infty) = +\infty$, then each of the equivalent properties (i)–(iv) holds.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follows by the same arguments as in the proof of Corollary 5.9. The convexity of $a\,(m)$ together with $a\,(-\infty) = +\infty$ imply (i) following the same argument. We are left to show that (iv) $\Longleftrightarrow$ (ii).

Assume that $\{P_t\}$ is a isotropic Markov semigroup as constructed in (1.3) – (1.8). The semigroup admits a continuous transition density $p(t,x,y) = p_t(x-y)$ with respect to the Haar measure μ_p; the function p_t is given by

$$p_t(y) = \int_0^\infty q_s(y)\, d\sigma^t(s), \quad \text{where} \quad q_s(y) = \frac{1}{\mu_p\left(B_s(0)\right)}\,\mathbf{1}_{B_s(0)}(y)\,. \tag{5.18}$$

To find the Fourier transform $\widehat{p_t}(\xi)$, we argue as follows. The ball $B_s(0)$, $p^k \le s < p^{k+1}$, is the compact subgroup $p^{-k}\mathbb{Z}_p$ of $\mathbb{Q}_p$, whence the measure $\omega_s = q_s\,\mu_p$ coincides with the normed Haar measure of that compact subgroup. Since for any locally compact Abelian group, the

Fourier transform of the normed Haar measure of any compact subgroup is the indicator of its annihilator group and, in our particular case, the annihilator of the group $p^{-k}\mathbb{Z}_p$ is the group $p^k\mathbb{Z}_p$, we obtain

$$\widehat{\omega_s}(\xi) = \mathbf{1}_{p^k\mathbb{Z}_p}(\xi) = \mathbf{1}_{[0,p^{-k}]}(\|\xi\|_p), \quad \text{where} \;\; p^k \leq s < p^{k+1}.$$

It follows that when $\|\xi\|_p = p^{-l}$,

$$\widehat{p_t}(\xi) = \sum_{k:k\leq l} \left(\sigma^t(p^{k+1}) - \sigma^t(p^k)^t\right) = \sigma^t(p^{l+1}) = \exp\left(-t\,\psi(\|\xi\|_p)\right),$$

whence

$$\psi(p^{-l}) = \log \frac{1}{\sigma(p^{l+1})}.$$

According to (1.5), the sequence $\sigma(p^l)$ is assumed to be strictly increasing and to tend to zero as $l \to -\infty$. Thus, $\psi(p^m)$ is as claimed in (ii).

Conversely, if a strictly increasing sequence $\psi(p^m)$ as in (ii) is given, we define the strictly increasing sequence

$$\sigma(p^m) = \exp\left(-\psi(p^{-m+1})\right).$$

Let $\sigma : [0\,,\,\infty) \to [0\,,\,1)$ be any increasing bijection which takes the values $\sigma(p^m)$ at the points p^m. We define the function $p_t(y)$ by the equation (5.18). As $\sigma(+\infty) = 1$, this is a probability density with respect to μ_p. It is straightforward that $\{p_t\}_{t>0}$ gives rise to a weakly continuous convolution semigroup of probability measures on $\mathbb{Q}_p$. Moreover, each p_t is rotation invariant by construction. Thus, the semigroup $P_tf = f * p_t$ is isotropic. ■

Remark 5.11 In [2], Albeverio and Karwowski started with a sequence $\{a(m)\}_{m\in\mathbb{Z}}$ as in (5.12) and used the classical approach of backward and forward Kolmogorov equations to construct a Markov semigroup $\{P_t\}$ on the ultra-metric measure space $(\mathbb{Q}_p, d_p, \mu_p)$. In particular, they showed in [2, Theorem 3.2 9] that the Laplacian $\mathcal{L}$ of that semigroup has a pure point spectrum $\{\lambda(m)\}$ as in (5.13), and the $\lambda(m)$-eigenspace is spanned by the functions f_B, where B runs over all balls of radius p^{m-1}. Our Theorem 5.8 shows that in fact the class of Markov semigroups constructed in [2] coincides with the class of rotation invariant Markov semigroups.

5.3 Product spaces

Let $\{(X_i, d_i)\}_{i=1}^n$ be a finite sequence of ultra-metric spaces; we assume that all (X_i, d_i) are separable and that all balls are compact. Let (X, d) be their Cartesian product: $X = X_1 \times ... \times X_n$ and, for $x = (x_i) \in X$ and $y = (y_i) \in Y$, we set

$$d(x, y) = \max\{d_i(x_i, y_i) : i = 1, 2, ..., n\}.$$

Thus (X, d) is a separable ultra-metric space, all balls in (X, d) are compact, and, moreover, each d-ball $B_r(a)$ in X is a product of d_i-balls $B_r^i(a_i)$ in X_i of the same radius.

Given a Radon measure μ_i on each (X_i, d_i) we define $\mu = \bigotimes \mu_i$ on (X, d). Let $\mathcal{V}_c$ be the set of all compactly supported locally constant functions on (X, d).

Consider the ultra-metric measure space (X, d, μ). According to the previous sections, there exists a rich class of isotropic Markov semigroups and corresponding Laplacians on (X, d, μ) as constructed in (1.3) – (1.8). Thanks to the product structure of (X, d, μ) one can define in a natural way a non-trivial and interesting class of Markov semigroups and Laplacians which are *not* isotropic. Namely, choosing on each (X_i, d_i, μ_i) an isotropic Markov semigroup $\{P_i^t\}$, we define a Markov semigroup $\{P_t\}$ on (X, d, μ) as the tensor product of the $\{P_i^t\}$,

$$P_t = \bigotimes_{i=1}^{n} P_i^t.$$

The semigroup $\{P_t\}$ has the following heat kernel:

$$p(t, x, y) = \prod_{i=1}^{n} p_i(t, x_i, y_i),$$

where p_i is the heat kernel of $\left\{P_i^t\right\}$.

The generator $\mathcal{L}$ of P_t can be described as follows: $\mathcal{V}_c \subset \mathrm{dom}_{\mathcal{L}}$ and for any $f \in \mathcal{V}_c$ we have

$$\mathcal{L}f(x) = \sum_{i=1}^{n} \mathcal{L}_i f(x) \tag{5.19}$$

where $x = (x_1, ..., x_n)$ and $\mathcal{L}_i$ acts on x_i. It follows that

$$\mathcal{L}f(x) = \int_X (f(x) - f(y))\ J(x, dy)$$

where

$$J(x, dy) = \sum_{i=1}^{n} J_i(x_i, y_i)\, d\mu_i(y_i)\,,$$

and $J_i(x_i, y_i)$ is the jump kernel of $\mathcal{L}_i$.

In particular, we see that for each $x \in X$ the measures $J(x, dy)$ and $\mu(dy)$ are not necessarily mutually absolutely continuous (in the case when at least one of X_l is perfect, $J(x, dy)$ is singular with respect to μ), which implies that the semigroup $\{P_t\}$ is not necessarily an isotropic Markov semigroup.

In this paper we do not intend to develop a general theory on product spaces. Our aim is to study in detail two specific examples related to p-adic analysis.

In the first example we consider the Vladimirov Laplacian that matches well the above general construction. In the second example we consider the Taibleson Laplacian defined in terms of the multidimensional Riesz kernels, see Taibleson [55] and Rodriguez-Vega and Zuniga-Galindo [50]. We show that the Taibleson Laplacian is isotropic. This will allow us to improve the heat kernel bounds from [50] and to obtain some new results (transience/recurrence, independence on $1 \le p < \infty$ of the L^p-spectrum, precise bounds of the moments of the corresponding Markov process etc.)

Consider the linear space $\mathbb{Q}_p^n = \mathbb{Q}_p \times ... \times \mathbb{Q}_p$ over the field $\mathbb{Q}_p$ and define in $\mathbb{Q}_p^n$ a norm

$$\|z\|_p = \max\left\{\|z_i\|_p : i = 1, 2, ..., n\right\}. \tag{5.20}$$

It clearly satisfies the ultra-metric triangle inequality (1.1) and is homogeneous in the following sense:

$$\|az\|_p = \|a\|_p \|z\|_p\,, \quad \text{for all} \quad a \in \mathbb{Q}_p,\ z \in \mathbb{Q}_p^n.$$

Set

$$d_p(x, y) = \|x - y\|_p$$

so that $(\mathbb{Q}_p^n, d_p)$ is an ultra-metric space.

Let $\mu_p = \bigotimes \mu_{p,i}$ be the additive Haar measure on the Abelian group $\mathbb{Q}_p^n$. As before, let $\mathcal{V}_c$ be the set of all compactly supported locally constant functions on the ultra-metric space $(\mathbb{Q}_p^n, d_p)$. Recall that $\mathcal{V}_c$ is a dense subset in $L^2 = L^2(\mathbb{Q}_p^n, \mu_p)$.

5.3.1 The Vladimirov Laplacian

For any given n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ with entries $\alpha_i > 0$ we define the ultra-metric

$$d_{p,\alpha}(x,y) = \max\left\{ \|x_i - y_i\|_p^{\alpha_i} : i = 1, 2, ..., n \right\}.$$

In particular, the ultra-metric $d_p(x,y)$ defined above corresponds to the case $\alpha = (1, ..., 1)$. The identity map

$$\left(\mathbb{Q}_p^n, d_{p,\alpha}\right) \to \left(\mathbb{Q}_p^n, d_p\right)$$

is a homeomorphism, but not bi-Lipschitz, unless $\alpha_i = 1$ for all i. This fact plays an essential role in the study of the class of Laplacians introduced next as a special instance of (5.19).

Definition 5.12 Let $\alpha = (\alpha_1, \ldots, \alpha_n)$. For any function $f \in \mathcal{V}_c$ we define the operator

$$\mathfrak{V}^\alpha f(x) = \sum_{i=1}^n \mathfrak{D}_{x_i}^{\alpha_i} f(x),$$

where $x = (x_1, x_2, ..., x_n)$ and $\mathfrak{D}_{x_i}^{\alpha_i}$ is the p-adic fractional derivative of order α_i acting on x_i.

The operator $\mathfrak{V}^\alpha$ on $\mathbb{Q}_p^3$ with $\alpha = (2, 2, 2)$, was introduced by Vladimirov [57] as an analogue of the classical Laplace operator in $\mathbb{R}^3$. This operator, which we denote briefly by $\mathfrak{V}^2$, is translation invariant and homogeneous, that is,

$$\mathfrak{V}^2 \tau_y(f) = \tau_y(\mathfrak{V}^2 f), \quad \text{where} \quad \tau_y f(x) = f(x+y).$$

and

$$\mathfrak{V}^2 \theta_a(f) = \|a\|_p^2 \, \theta_a(\mathfrak{V}^2 f), \quad \text{where} \quad \theta_a f(x) = f(ax_1, ax_2, ax_3).$$

It follows that the Green function $g(x,y)$ of the operator $\mathfrak{V}^2$ on $\mathbb{Q}_p^3$ is also translation invariant and homogeneous:

$$g(x,y) = g(x-z, y-z) \quad \text{and} \quad g(ax, ay) = g(x,y) / \|a\|_p \ , \ a \in \mathbb{Q}_p \, .$$

In particular, setting $\mathfrak{E}(x) = g(x, 0)$, we obtain for all non-zero $a \in \mathbb{Q}_p$ the identity

$$\mathfrak{E}(a,a,a) = \frac{\mathfrak{E}(1,1,1)}{\|a\|_p}.$$

This identity was observed in [57]. It gives an idea of how the Green function of the operator $\mathfrak{V}^2$ (in Vladimirov's terminology, the fundamental solution of the equation $\mathfrak{V}^2 \, \mathfrak{E} = \delta$) behaves at infinity/at zero. Below, in Proposition 5.15, we will prove that, for all non-zero $a = (a_1, a_2, a_3) \in \mathbb{Q}_p^3$,

$$\mathfrak{E}(a_1, a_2, a_3) \simeq \frac{1}{\|a\|_p}. \tag{5.21}$$

In fact, we shall prove similar estimate for more general operators $\mathfrak{V}^\alpha$ without the homogeneity property. We start by listing some properties of the operator $(\mathfrak{V}^\alpha, \mathcal{V}_c)$ from Definition 5.12 which follow directly from the corresponding properties of the "one-dimensional Laplacians" $\mathfrak{D}^{\alpha_i}$.

1. $(\mathfrak{V}^\alpha, \mathcal{V}_c)$ is a non-negative definite symmetric operator.

2. $(\mathfrak{V}^\alpha, \mathcal{V}_c)$ admits a complete system of compactly supported eigenfunctions. In particular, the operator $(\mathfrak{V}^\alpha, \mathcal{V}_c)$ is essentially self-adjoint.

3. The semigroup $\exp(-t\,\mathfrak{V}^{\alpha})$ is symmetric and Markovian. It admits the heat kernel $p_{\alpha}(t,x,y)$ which has the following form

$$p_{\alpha}(t,x,y)=\prod_{i=1}^{n}p_{\alpha_i}(t,x_i,y_i).$$

4. The semigroup $\exp(-t\,\mathfrak{V}^{\alpha})$ is transient if and only if $A:=\sum_{i=1}^{n}\frac{1}{\alpha_i}>1$.

5. For all $f\in\mathcal{V}_c$

$$\mathfrak{V}^{\alpha}f(x)=\int_{\mathbb{Q}_p^n}\left(f(x)-f(y)\right)\,J_{\alpha}(x,dy)$$

where

$$J_{\alpha}(x,dy)=\sum_{i_1}^{n}J_{\alpha_i}(x_i-y_i)\,d\mu_{p,i}(y_i)$$

and

$$J_{\alpha_i}(x_i-y_i)=\frac{p^{\alpha_i}-1}{1-p^{-\alpha_i-1}}\frac{1}{\|x_i-y_i\|_p^{1+\alpha_i}}\,.$$

In particular, the semigroup $\exp(-t\,\mathfrak{V}^{\alpha})$ is *not* an isotropic Markov semigroup, if $n>1$.

Observe that thanks to the group structure of $\mathbb{Q}_p^n$, the functions $(x,y)\mapsto p_{\alpha}(t,x,y)$ and $(x,y)\mapsto g_{\alpha}(x,y)$ are translation invariant. Hence, setting

$$p_{\alpha}(t,z)=p_{\alpha}(t,z,0)\quad\text{and}\quad g_{\alpha}(z)=g_{\alpha}(z,0)\,,$$

we obtain

$$p_{\alpha}(t,x,y)=p_{\alpha}(t,x-y)\quad\text{and}\quad g_{\alpha}(x,y)=g_{\alpha}(x-y).$$

Proposition 5.13 *Set*

$$A=\sum_{i=1}^{n}\frac{1}{\alpha_i}.$$

Then the heat kernel satisfies the following estimate

$$p_{\alpha}(t,z)\simeq t^{-A}\prod_{i=1}^{n}\min\left\{1,\frac{t^{1+1/\alpha_i}}{\|z_i\|_p^{1+\alpha_i}}\right\}\tag{5.22}$$

uniformly for all $t>0$ and $z\in\mathbb{Q}_p^n$. In particular, for all $t>\|z\|_{p,\alpha}$,

$$p_{\alpha}(t,z)\simeq t^{-A}\tag{5.23}$$

Proof. By Theorem 5.3 we have

$$p_{\alpha_i}(t,z_i)\simeq\frac{t}{\left(t^{1/\alpha_i}+\|z_i\|_p\right)^{1+\alpha_i}}\simeq\frac{1}{t^{1/\alpha_i}}\min\left\{1,\frac{t^{1+1/\alpha_i}}{\|z_i\|_p^{1+\alpha_i}}\right\},$$

whence the claim follows. ■

Proposition 5.14 *The semigroup* $\exp(-t\,\mathfrak{V}^{\alpha})$ *is transient if and only if* $A>1$. *If* $A>1$ *then, for all* $z\in\mathbb{Q}_p^n$ *and some* $C_1>0$,

$$g_{\alpha}(z)\geq C_1\left(\frac{1}{\|z\|_{p,\alpha}}\right)^{A-1}.$$

For any $\kappa>0$, *we define the set*

$$\Omega(\kappa)=\left\{x\in\mathbb{Q}_p^n:\max_i\left\{\|x_i\|_p^{\alpha_i}\right\}\leq\kappa\min_i\left\{\|x_i\|_p^{\alpha_i}\right\}\right\}.$$

Then, for all $z\in\Omega(\kappa)$ *and some constant* $C_2>0$ *which depends on* κ,

$$g_{\alpha}(z)\leq C_2\left(\frac{1}{\|z\|_{p,\alpha}}\right)^{A-1}.$$

Proof. The transience criterion $A>1$ follows from $p_a(t,x,x)\simeq t^{-A}$. To prove the lower bound, we use (5.23) and write

$$g_{\alpha}(z)=\int_0^{\infty}p_{\alpha}(t,z)\,dt\geq\int_{\|z\|_{p,\alpha}}^{\infty}p_{\alpha}(t,z)\,dt\geq C_1\int_{\|z\|_{p,\alpha}}^{\infty}t^{-A}\,dt=c_1\left(\frac{1}{\|z\|_{p,\alpha}}\right)^{A-1}.$$

On the other hand we have

$$g_{\alpha}(z)=\left(\int_0^{\|z\|_{p,\alpha}}+\int_{\|z\|_{p,\alpha}}^{\infty}\right)p_{\alpha}(t,z)\,dt=:I+II.$$

To estimate the second term II, we use again (5.23):

$$II\simeq\int_{\|z\|_{p,\alpha}}^{\infty}t^{-A}\,dt\simeq\left(\frac{1}{\|z\|_{p,\alpha}}\right)^{A-1}.$$

To estimate the first term we use (5.22):

$$\begin{aligned}I&\leq c\int_0^{\|z\|_{p,\alpha}}t^{-A}\prod_{i=1}^{n}\frac{t^{1+1/\alpha_i}}{\|z_i\|_p^{1+\alpha_i}}\,dt\\&=c\int_0^{\|z\|_{p,\alpha}}\prod_{i=1}^{n}\frac{1}{\|z_i\|_p^{1+\alpha_i}}t^{n}\,dt=c'\prod_{i=1}^{n}\frac{1}{\|z_i\|_p^{1+\alpha_i}}\|z\|_{p,\alpha}^{n+1}.\end{aligned}$$

When $z\in\Omega(\kappa)$, we obtain

$$\begin{aligned}I&\leq c''\prod_{i=1}^{n}\frac{1}{\|z_i\|_p^{1+\alpha_i}}\left(\min\left\{\|z_i\|_p^{\alpha_i}\right\}\right)^{n+1}\leq c''\min\left\{\|z_i\|_p^{\alpha_i}\right\}\prod_{i=1}^{n}\frac{1}{\|z_i\|_p}\\&=c''\min\left\{\|z_i\|_p^{\alpha_i}\right\}\prod_{i=1}^{n}\frac{1}{(\|z_i\|_p^{\alpha_i})^{1/\alpha_i}}.\end{aligned}$$

Next,

$$\prod_{i=1}^{n}\frac{1}{(\|z_i\|_p^{\alpha_i})^{1/\alpha_i}}\leq\prod_{i=1}^{n}\frac{1}{\left(\min\left\{\|z_j\|_p^{\alpha_j}\right\}\right)^{1/\alpha_i}}=\left(\frac{1}{\min\left\{\|z_j\|_p^{\alpha_j}\right\}}\right)^{A},$$

whence

$$I \le c'' \left(\frac{1}{\min\left\{ \|z_j\|_p^{\alpha_j} \right\}} \right)^{A-1}.$$

Again using the fact that $z \in \Omega(\kappa)$, we write

$$\left(\frac{1}{\min\left\{ \|z_j\|_p^{\alpha_j} \right\}} \right)^{A-1} \le \left(\frac{\kappa}{\max\left\{ \|z_j\|_p^{\alpha_j} \right\}} \right)^{A-1} = c(\kappa) \left(\frac{1}{\|z\|_{p,\alpha}} \right)^{A-1}.$$

The obtained upper bounds on the integrals I and II imply the desired upper bound for $g_\alpha(z)$. ■

Proposition 5.15 *Let $\alpha = (\alpha_1, \dots, \alpha_n) = (\beta, \dots, \beta)$ be an n-tuple having all entries equal to β. Assume that $(n-1)/2 < \beta < n$. Then the semigroup $\exp(-t\,\mathfrak{V}^\alpha)$ is transient and the Green function $g_\alpha(z)$ satisfies the estimates*

$$g_\alpha(z) \simeq \left(\frac{1}{\|z\|_{p,\alpha}} \right)^{A-1}, \tag{5.24}$$

for all $z \in \mathbb{Q}_p^n$ and some $c_1, c_2 > 0$.

Since $A = \frac{n}{\beta}$ and $\|z\|_{p,\alpha} = \|z\|_p^\beta$, the estimate (5.24) is equivalent to

$$g_\alpha(z) \simeq \left(\frac{1}{\|z\|_p} \right)^{n-\beta}. \tag{5.25}$$

Proof. Transience follows from Proposition 5.14 because $A = n/\beta > 1$. The same Proposition yields the desired lower bound of the Green function. To prove the upper bound, we observe that the Laplacian $\mathfrak{V}^\alpha$ is homogeneous, that is

$$\mathfrak{V}^\alpha \circ \theta_a = \|a\|_p^\beta \cdot \theta_a \circ \mathfrak{V}^\alpha,$$

for all $a \in \mathbb{Q}_p$. This implies that also the Green function $g_\alpha(z)$ is homogeneous, that is

$$g_\alpha(az) = \|a\|_p^{n-\beta}\, g_\alpha(z),$$

for all $a \in \mathbb{Q}_p$ and $z \in \mathbb{Q}_p^n$.

Without loss of generality assume that $\|z\|_{p,\alpha} = \|z_1\|_p^\beta > 0$. Then

$$\begin{aligned} g_\alpha(z) &= g_\alpha\left(z_1(1, z_2/z_1, ..., z_n/z_1)\right) = \|z_1\|_p^{n-\beta}\, g_\alpha(1, z_2/z_1, ..., z_n/z_1) \\ &= \left(\frac{1}{\|z\|_{p,\alpha}} \right)^{A-1} g_\alpha(1, z_2/z_1, ..., z_n/z_1) \\ &\le \left(\frac{1}{\|z\|_{p,\alpha}} \right)^{A-1} \sup\left\{ g_\alpha(1, x_2, ..., x_n) : x_i \in \mathbb{Z}_p \right\}. \end{aligned}$$

Next we apply our assumption $\beta > (n-1)/2$ and obtain from (5.22)

$$\begin{aligned} g_\alpha(1, x_2, ..., x_n) &= \int_0^\infty p_\alpha(t, (1, x_2, ..., x_n)\, dt \\ &= \left(\int_0^1 + \int_1^\infty \right) p_\alpha(t, (1, x_2, ..., x_n)\, dt \\ &\le c \int_0^1 t^{-\frac{n}{\beta}}\, t^{1+\frac{1}{\beta}}\, dt + c' \int_1^\infty t^{-\frac{n}{\beta}}\, dt = c_2 < \infty, \end{aligned}$$

which implies the desired upper bound. ■

5.3.2 The Taibleson Laplacian

The Fourier transform $\mathcal{F} : f \mapsto \widehat{f}$ of a function f on the locally compact Abelian group $\mathbb{Q}_p^n$ is defined by

$$\widehat{f}(\theta) = \int_{Q_p^n} \langle x, \theta \rangle \, f(x) d\mu_p^n(x),$$

where $x = (x_1, ..., x_n) \in \mathbb{Q}_p^n$, $\theta = (\theta_1, ..., \theta_n) \in (\mathbb{Q}_p^n)^* = \mathbb{Q}_p^n$,

$$\langle x, \theta \rangle = \prod_{k=1}^{n} \langle x_k, \theta_k \rangle \, ,$$

and $d\mu_p^n(x) = d\mu_p(x_1)...d\mu_p(x_n)$ is the Haar measure on $\mathbb{Q}_p^n$. It is known that $\mathcal{F}$ is a linear isomorphism from $\mathcal{V}_c$ onto itself, which justifies the following Definition (compare with Definition 5.1).

Definition 5.16 *The Taibleson operator $\mathfrak{T}^\alpha$ for $\alpha > 0$ is defined on functions $f \in \mathcal{V}_c$ by*

$$\widehat{\mathfrak{T}^\alpha f}(\zeta) = \|\zeta\|_p^\alpha \, \widehat{f}(\zeta), \quad \zeta \in \mathbb{Q}_p^n.$$

It follows that $(\mathfrak{T}^\alpha, \mathcal{V}_c)$ is an essentially self-adjoint and non-negative definite operator in L^2. This operator was introduced by Taibleson [55], and the associated semigroup $\exp(-t\,\mathfrak{T}^\alpha)$ was studied by Rodriguez-Vega and Zuniga-Galindo [50]. In particular, it was shown that

$$\mathfrak{T}^\alpha f(x) = \frac{p^\alpha - 1}{1 - p^{-\alpha - n}} \int_{\mathbb{Q}_p^n} \frac{f(x) - f(y)}{\|x - y\|_p^{\alpha + n}} \, d\mu_p^n(y). \tag{5.26}$$

The equation (5.26) implies that the operator $(-\mathfrak{T}^\alpha, \mathcal{V}_c)$ satisfies the max-principle, whence its semigroup is Markovian. Our aim is to show that $\exp(-t\,\mathfrak{T}^\alpha)$ is an isotropic Markov semigroup on the ultra-metric measure space $(\mathbb{Q}_p^n, d_p, \mu_p^n)$.

Our first observation is that the spectrum of the symmetric operator $(\mathfrak{T}^\alpha, \mathcal{V}_c)$ coincides with the range of the function $\zeta \mapsto \|\zeta\|_p^\alpha$,

$$\operatorname{spec} \mathfrak{T}^\alpha = \{p^{k\alpha} : k \in \mathbb{Z}\} \cup \{0\}.$$

The eigenspace $\mathcal{H}(\lambda)$ of the operator $(\mathfrak{T}^\alpha, \mathcal{V}_c)$ corresponding to the eigenvalue $\lambda = p^{k\alpha}$, is spanned by the function

$$f_k = \frac{1}{\mu_p^n(p^k \mathbb{Z}_p^n)} \mathbf{1}_{p^k \mathbb{Z}_p^n} - \frac{1}{\mu_p^n(p^{k-1} \mathbb{Z}_p^n)} \mathbf{1}_{p^{k-1} \mathbb{Z}_p^n}$$

and all its shifts $f_k(\cdot + a)$ with $a \in \mathbb{Q}_p^n / p^k \mathbb{Z}_p^n$. Indeed, computing the Fourier transform of the function f_k,

$$\widehat{f_k}(\zeta) = \mathbf{1}_{\{\|\zeta\|_p \le p^k\}} - \mathbf{1}_{\{\|\zeta\|_p \le p^{k-1}\}} = \mathbf{1}_{\{\|\zeta\|_p = p^k\}},$$

we obtain

$$\widehat{\mathfrak{T}^\alpha f_k}(\zeta) = \|\zeta\|_p^\alpha \, \widehat{f_k}(\zeta) = p^{k\alpha} \widehat{f_k}(\zeta).$$

All the above shows that the operator $\mathfrak{T}^\alpha$ coincides with an isotropic Laplacian $\mathcal{L}_\alpha$ on $(\mathbb{Q}_p^n, d_p, \mu_p^n)$ associated with the distance distribution function

$$\sigma_\alpha(r) = \exp\left(-\left(\frac{p}{r}\right)^\alpha\right),$$

and the semigroup $\exp(-t\,\mathfrak{T}^\alpha)$ coincides with the isotropic semigroup $\{P_\alpha^t\}$.

Observe that the associated intrinsic ultra-metric is

$$d_{p*}(x,y)=\left(\frac{\|x-y\|_p}{p}\right)^{\alpha}.$$

The spectral distribution function $N_\alpha(x,\tau)=N_\alpha(\tau)$ is the non-decreasing, left-continuous staircase function which has jumps at the points $\tau_k=p^{k\alpha}$, $k\in\mathbb{Z}$, and takes values $N_\alpha(\tau_k)=p^{(k-1)n}$ at these points. It follows that

$$N_\alpha(\tau)\simeq\tau^{n/\alpha}.$$

In particular, $\tau\mapsto N_\alpha(\tau)$ is a doubling function, and Theorem 2.14 implies the following result.

Theorem 5.17 *The semigroup* $\exp(-t\,\mathfrak{T}^\alpha)$ *on* $\mathbb{Q}_p^n$ *admits a continuous heat kernel* $p_\alpha(t,x,y)$ *that satisfies the estimate*

$$p_\alpha(t,x,y)\simeq\frac{t}{\left(t^{1/\alpha}+\|x-y\|_p\right)^{n+\alpha}}\,, \tag{5.27}$$

In particular, the semigroup $\exp(-t\,\mathfrak{T}^\alpha)$ *is transient if and only if* $\alpha<n$. *In the transient case, the Green function (=Taibleson's Riesz kernel) satisfies the identity*

$$g_\alpha(x,y)=\frac{1-p^{-\alpha}}{1-p^{\alpha-n}}\,\frac{1}{\|x-y\|_p^{n-\alpha}}.$$

Note that the upper bound in (5.27) was proved in [50].

Definition 5.7 of a rotation invariant Laplacian on $\mathbb{Q}_p$ can be carried over to $\mathbb{Q}_p^n$. The Taibleson operator $\mathfrak{T}^\alpha$ is an example of a rotation invariant Laplacian. Theorem 5.8, Corollary 5.9 and Corollary 5.10 and their proofs remain valid also for $\mathbb{Q}_p^n$. Here we provide a short proof of a slightly weaker result that is of significance for us. Set $\mathfrak{T}=\mathfrak{T}^1$.

Theorem 5.18 *The equation* $(\mathcal{L},\mathcal{V}_c)=(\psi(\mathfrak{T}),\mathcal{V}_c)$, *where* ψ *is an arbitrary increasing bijection* $[0\,,\,\infty)\to[0\,,\,\infty)$, *gives a complete description of the class of isotropic Laplacians on the ultra-metric measure space* $(\mathbb{Q}_p^n,d_p,\mu_p^n)$.

Proof. Let $\psi:[0,\infty)\mapsto[0,\infty)$ be an increasing bijection. By Theorem 3.1, the operator $(\psi(\mathfrak{T}),\mathcal{V}_c)$ is an isotropic Laplacian.

Conversely, let $(\mathcal{L},\mathcal{V}_c)$ be an isotropic Laplacian on $(\mathbb{Q}_p^n,d_p,\mu_p^n)$. Let d_{p*} be the intrinsic distance associated with $\mathcal{L}$. By construction, d_{p*} is an increasing function of d_p, see (2.14). Since the range of d_p is the set $\{p^k:k\in\mathbb{Z}\}\cup\{0\}$, one can choose an increasing bijection $\varphi:[0\,,\,\infty)\to[0\,,\,\infty)$ such that $d_{p*}=\phi(d_p)$. Let $\lambda(B)$ and $\tau(B)$ be the eigenvalues of $(\mathcal{L},\mathcal{V}_c)$ and $(\mathfrak{T},\mathcal{V}_c)$, respectively, corresponding to the ball $B\subset\mathbb{Q}_p^n$. Since the intrinsic distance associated with $\mathfrak{T}$ is $p^{-1}d_p$, we get

$$\begin{aligned}\lambda(B)&=\frac{1}{\mathrm{diam}_{p*}(B)}=\frac{1}{\varphi\left(\mathrm{diam}_p(B)\right)}\\&=\frac{1}{\varphi\left(p/\tau(B)\right)}=:\psi\left(\tau(B)\right),\end{aligned}$$

where $\psi(s)=1/\phi(p/s)$, an increasing bijection of $[0\,,\,\infty)$ onto itself.

Since both $(\mathcal{L},\mathcal{V}_c)$ and $(\psi(\mathfrak{T}),\mathcal{V}_c)$ are isotropic Laplacians defined on the ultra-metric measure space $(\mathbb{Q}_p^n,d_p,\mu_p^n)$ whose sets of eigenvalues coincide, we get

$$(\mathcal{L},\mathcal{V}_c)=(\psi(\mathfrak{T}),\mathcal{V}_c),$$

or equivalently, in terms of the Fourier transform,

$$\widehat{\mathcal{L}f}(\zeta)=\psi(\|\zeta\|_p)\,\widehat{f}(\zeta),$$

for all $f\in\mathcal{V}_c$ and $\zeta\in\mathbb{Q}_p^n$, which finishes the proof. ■

6 Random walks on a tree and jump processes on its boundary

6.1 Rooted trees and their boundaries

A tree is a connected graph T without cycles (closed paths of length ≥ 3). We tacitly identify T with its vertex set, which is assumed to be infinite. We write $u \sim v$ if $u, v \in T$ are neighbours. For any pair of vertices $u, v \in T$, there is a unique shortest path, called *geodesic segment*

$$\pi(u, v) = [u = v_0\,, v_1\,, \ldots, v_k = v]$$

such that $v_{i-1} \sim v_i$ and all v_i are disctinct. If $u = v$ then this is the *empty* or *trivial* path. The number k is the *length* of the path (the graph distance between u and v). In T we choose and fix a *root vertex* o. We write $|v|$ for the length of $\pi(o, v)$. The choice of the root induces a partial order on T, where $u \leq v$ when $u \in \pi(o, v)$. Every $v \in T \setminus \{o\}$ has a unique *predecessor* v^- with respect to o, which is the unique neighbour of v on $\pi(o, v)$. Thus, the set of all (unoriented) edges of T is

$$E(T) = \{[v^-, v] : v \in T\,,\ v \neq o\}\,.$$

For $u \in T$, the elements of the set

$$\{v \in T : v^- = u\}$$

are the *successors* of u, and its cardinality $\deg^+(u)$ is the *forward degree* of u.

In this and the next section, we assume that

$$2 \leq \deg^+(u) < \infty \quad \text{for every} \ \ u \in T\,. \tag{6.1}$$

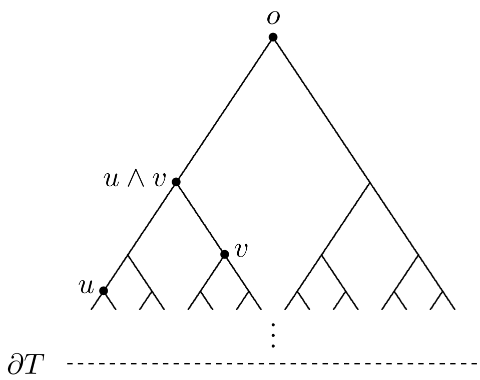

Figure 4

A *(geodesic) ray* in T is a one-sided infinite path $\pi = [v_0\,, v_1\,, v_2\,, \ldots]$ such that $v_{n-1} \sim v_n$ and all v_n are disctinct. Two rays are *equivalent* if their symmetric difference (as sets of vertices) is finite. An *end* of T is an equivalence class of rays. We shall typically use letters x, y, z to denote ends (and letters u, v, w for vertices). The set of all ends of T is denoted ∂T. This is the *boundary* at infinity of the tree. For any $u \in T$ and $x \in \partial T$, there is a unique ray $\pi(u, x)$ which is a representative of the end x and starts at u. We write

$$\widehat{T} = T \cup \partial T.$$

For $u \in T$, the *branch of T rooted at u* is the subtree T_u that we identify with its set of vertices

$$T_u = \{v \in T : u \leq v\}\,, \tag{6.2}$$

so that $T_o = T$. We write ∂T_u for the set of all ends of T which have a representative path contained in T_u, and $\widehat{T}_u = T_u \cup \partial T_u\,.$

For $w, z \in \widehat{T}$, we define their *confluent* $w \wedge z = w \wedge_o z$ with respect to the root o by the relation

$$\pi(o, w \wedge z) = \pi(o, w) \cap \pi(o, z)\,.$$

It is the last common element on the geodesics $\pi(o, w)$ and $\pi(o, z)$, and it is a vertex of T unless $w = z \in \partial T$. See Figure 4.

One of the most common ways to define an ultra-metric on $\widehat{T}$ is

$$d_e(z, w) = \begin{cases} 0\,, & \text{if } z = w\,, \\ e^{-|z \wedge w|}\,, & \text{if } z \neq w\,. \end{cases} \tag{6.3}$$

Then $\widehat{T}$ is compact, and T is open and dense. We are mostly interested in the compact ultra-metric space ∂T. In the metric d_e of (6.3), each d_e-ball with centre $x \in \partial T$ is of the form ∂T_u for some $u \in \pi(o, x)$. Indeed

$$\partial T_u = B_{e^{-|u|}}(x) \quad \text{for every } u \in \pi(o, x)\,, \quad \text{and} \quad \Lambda_{d_e}(x) = \{e^{-|u|} : u \in \pi(o, x)\}\,.$$

Conversely, we now start with a compact ultra-metric space (X, d) that does not possess isolated points, and construct a tree T as follows: The vertex set of T is the collection

$$\mathcal{B} = \{B_r(x) : x \in X\,,\ r > 0\}$$

of all closed balls in (X, d), already encountered in §3. Here, we may assume (if we wish) that $r \in \Lambda_d(x)$.

We now consider any ball $v = B \in \mathcal{B}$ as a vertex of a tree T. We choose our root vertex as $o = X$, which belongs to $\mathcal{B}$ by compactness. Neighborhood is given by the predecessor relation of balls, as given by Definition 3.6. That is, if $v = B$ then $u = B'$ is the predecessor vertex v^- of v in the tree T. By compactness, each x has only finitely many successors, and since there are no isolated points in X, every vertex has at least 2 successors, so that (6.1) holds.

This defines the tree structure. For any $x \in X$, the collection of all balls $B_r(x)$, $r \in \Lambda_d(x)$, ordered decreasingly, forms the set of vertices of a ray in T that starts at o. Via a straightforward exercise, the mapping that associates to x the end of T represented by that ray is a homeomorphism from X onto ∂T. Thus, we can identify X and ∂T as ultra-metric spaces.

In this identification, if originally a vertex u was interpreted as a ball $B_r(x)$, $r \in \Lambda_d(x)$, then the set ∂T_u of ends of the branch T_u just coincides with the ball $B_r(x)$. That is, we are identifying each vertex u of T with the set ∂T_u.

If we start with an arbitrary locally finite tree and take its space of ends as the ultra-metric space X, then the above construction does not recover vertices with forward degree 1, so that in general we do not get back the tree we started with. However, via the above construction, the correspondence between compact ultra-metric spaces without isolated points (perfect ultra-metric spaces) and locally finite rooted trees with forward degrees ≥ 2 is bijective (cf. [32]).

It is well known that any ultra-metric space X which is both compact and perfect is homeomorphic to the ternary Cantor set $C \subset [0, 1]$. When X is not compact but still perfect we have a homeomorphism $X \simeq C \setminus \{p\}$, where $p \in C$ is any fixed point.

For the rest of this and the next section, we shall abandon the notation X for compact and perfect ultra-metric space.

We consider X as the boundary ∂T of a locally finite, rooted tree with forward degrees ≥ 2.

At the end, we shall comment on how one can handle the presence of vertices with forward degree 1, as well as the non-compact case.

There are many ways to equip ∂T with an ultra-metric that has the same topology and the same compact-open balls ∂T_x , $x \in T$, possibly with different radii than in the standard metric (6.3). The following is a kind of ultra-metric analogue of a length element.

Definition 6.1 Let T be a locally finite, rooted tree with $\deg^+(x) \geq 2$ for all x. An *ultra-metric element* is a function $\phi : T \to (0\,,\infty)$ with

$$\text{(i)} \quad \phi(v^-) > \phi(v) \quad \text{for every } v \in T \setminus \{o\}\,,$$
$$\text{(ii)} \quad \lim \phi(v_n) = 0 \quad \text{along every geodesic ray } \pi = [v_0\,, v_1\,, v_2\,, \ldots]\,.$$

It induces the ultra-metric d_ϕ on ∂T given by

$$d_\phi(x,y) = \begin{cases} 0\,, & \text{if } x = y\,, \\ \phi(x \wedge y)\,, & \text{if } x \neq y\,. \end{cases}$$

The balls in this ultra-metric are again the sets

$$\partial T_u = B_{\phi(u)}(x)\,, \quad x \in \partial T_u\,.$$

Note that condition (ii) in the definition is needed for having that each end of T is non-isolated in the metric d_ϕ. The metric d_e of (6.3) is of course induced by $\phi(x) = e^{-|x|}$.

Lemma 6.2 *For a tree as in Definition 6.1, every ultra-metric on ∂T whose closed balls are the sets ∂T_u, $u \in T$, is induced by an ultra-metric element on T.*

Proof. Given an ultra-metric d as stated, we set $\phi(v) = \operatorname{diam}(\partial T_v)$, the diameter with respect to the metric d. Since $\deg^+(v^-) \geq 2$ for any $v \in T \setminus \{o\}$, the ball ∂T_{v^-} is the disjoint union of at least two balls ∂T_u with $u^- = v^-$. Therefore we must have $\operatorname{diam}(\partial T_v) < \operatorname{diam}(\partial T_{v^-})$, and property (i) holds. Since no end is isolated, ϕ satisfies (ii). It is now straightforward that $d_\phi = d$. ■

In view of the correspondence between the ultra-metric and the ultra-metric element, in the sequel we shall replace in the notation the subscript d referring to the metric $d = d_\phi$ by the subscript ϕ referring to the ultra-metric element. We note that

$$\operatorname{diam}_\phi(\partial T) = \phi(o)\,, \quad \Lambda_\phi(x) = \{\phi(u) : u \in \pi(o,x)\} \quad \text{and} \quad \Lambda_\phi = \{\phi(v) : v \in T\}. \tag{6.4}$$

We also note that, for any $x \in \partial T$ and $v \in \pi(o,x)$, the balls of the metric d_ϕ satisfy the identity

$$B_r(x) = B_r^\phi(x) = \begin{cases} \partial T_v & \text{for } \phi(v) \leq r < \phi(v^-)\,, \text{ if } v \neq o \\ \partial T & \text{for } r \geq \phi(o)\,, \text{ if } v = o\,. \end{cases} \tag{6.5}$$

6.2 Isotropic jump processes on the boundary of a tree

In view of the explanations given above, we can consider the isotropic jump processes of (1.3)–(1.8) on the space $X = \partial T$. Since this space is compact, we may assume that the reference measure μ is a probability measure on ∂T. Given a measure μ on ∂T, a distance distribution function σ with properties (1.5), and an ultra-metric element ϕ on T, we obtain the (d_ϕ, μ, σ)-process on ∂T, that will be referred to as the (ϕ, μ, σ)-process on ∂T. We can write the semigroup and its transition probabilities in detail as follows. For $x \in \partial T$ and $\pi(0,x) = [o = v_0\,, v_1\,, v_2\,, \ldots]$, using (6.5),

$$P^t f(x) = \sum_{n=0}^\infty c_n^t \, \mathrm{Q}_{\phi(v_n)} f(x)\,,$$
$$\text{where} \quad c_0^t = 1 - \sigma^t\left(\phi(v_0)\right) \quad \text{and} \quad c_n^t = \sigma^t\left(\phi(v_{n-1})\right) - \sigma^t\left(\phi(v_n)\right) \quad \text{for } n \geq 1\,.$$

Thus, for arbitrary $u \in T$ and $x \in \partial T$ as above

$$\mathbb{P}[X_t \in \partial T_u \mid X_0 = x] = \sum_{n=0}^\infty c_n^t \, \frac{\mu(\partial T_{v_n} \cap \partial T_u)}{\mu(\partial T_{v_n})}. \tag{6.6}$$

The standard (d,μ)-process in the sense of Definition 2.9 in the case of metric $d = d_\phi$ will be referred to as the standard (ϕ,μ)-process.

6.3 Nearest neighbour random walks on a tree

On trees there is a class of well-studied stochastic processes, namely *random walks.* Our aim is to analyze how random walks on a tree are related with isotropic jump processes on the boundary of the tree. A good part of the material outlined next is taken from the book of Woess [63]. An older, recommended reference is the seminal paper of Cartier [12].

A *nearest neighbour random walk* on the locally finite, infinite tree T is induced by its stochastic transition matrix $\mathcal{P} = (p(u,v))_{u,v\in T}$ with the property that $p(u,v) > 0$ if and only if $u \sim v$. The resulting discrete-time Markov chain (random walk) is written $(Z_n)_{n\geq 0}$. Its n-step transition probabilities

$$p^{(n)}(u,v) = \mathbb{P}_u[Z_n = v], \quad u,v \in T,$$

are the elements of the n^{th} power of the matrix $\mathcal{P}$. The notation $\mathbb{P}_u$ refers to the probability measure on the space of trajectories starting at u. We assume that the random walk is *transient,* i.e., with probability 1 it visits any finite set only finitely often. Thus, $0 < G(u,v) < \infty$ for all $u,v \in T$, where

$$G(u,v) = \sum_{n=0}^{\infty} p^{(n)}(u,v)$$

is the *Green kernel* of the random walk. In addition, we shall also make crucial use of the quantities

$$F(u,v) = \mathbb{P}_u[Z_n = v \text{ for some } n \geq 0] \quad \text{and} \quad U(v,v) = \mathbb{P}_v[Z_n = v \text{ for some } n \geq 1]\,.$$

We shall need several identities relating them and start with a few of them, valid for all $u,v \in T$.

$$G(u,v) = F(u,v)G(v,v) \tag{6.7}$$

$$G(v,v) = \frac{1}{1-U(v,v)} \tag{6.8}$$

$$U(v,v) = \sum_u p(v,u)F(u,v) \tag{6.9}$$

$$F(u,v) = F(u,w)F(w,v) \quad \text{whenever} \quad w \in \pi(u,v) \tag{6.10}$$

The first three identities hold for arbitrary denumerable Markov chains, while (6.10) is true specifically for trees. The identities show that the quantities G, U, F are completely determined just by the values of $F(u,v)$ for $u \sim v$. More identities from [63, Chapter 9] will be cited and used later on.

By transience, the random walk Z_n must converge to a random end (see e.g. [12] or [63, Theorem 9.18]).

Lemma 6.3 *There is a ∂T-valued random variable Z_∞ such that for every starting point $u \in T$,*

$$\mathbb{P}_u[Z_n \to Z_\infty \textit{ in the topology of } \widehat{T}] = 1.$$

In brief, the argument is as follows: by transience, random walk trajectories must accumulate at ∂T almost surely. If such a trajectory had two distinct accumulation points, say x and y, then by the nearest neighbour property, the trajectory would visit the vertex $x \wedge_u y$ infinitely often, which can occur only with probability 0.

For any $u \in T$ consider the *limit distribution* ν_u that is a Borel measure on ∂T defined for any Borel set $B \subset \partial T$ by

$$\nu_u(B) = \mathbb{P}_u[Z_\infty \in B]\,.$$

The sets ∂T_u , $u \in T$ (plus the empty set), form a semi-algebra that generates the Borel σ-algebra of ∂T. Thus, each measure ν_u is determined by the values on those sets. There is an explicit formula (cf. [12] or [63, Proposition 9.23]) that holds for $v \neq o$:

$$\nu_u(\partial T_v) = \begin{cases} F(u,v)\dfrac{1-F(v,v^-)}{1-F(v^-,v)F(v,v^-)}\,, & \text{if } u \in \{v\} \cup (T \setminus T_v)\,, \\ 1 - F(u,v)\dfrac{F(v,v^-)-F(v^-,v)F(v,v^-)}{1-F(v^-,\nu)F(v,v^-)}\,, & \text{if } u \in T_v\,. \end{cases} \tag{6.11}$$

A *harmonic function* is a function $h : T \to \mathbb{R}$ with $\mathcal{P}h = h$, where

$$Ph(u) = \sum_v p(u,v)h(v)\,.$$

For any Borel set $B \subset \partial T$, the function $u \mapsto \nu_u(B)$ is a bounded harmonic function. It is possible to prove that all ν_u are comparable in the following sense: $p^{(k)}(u,v)\,\nu_u \leq \nu_v$, where k is the length of $\pi(u,v)$. Thus, for any function $\varphi \in L^1(\partial T, \nu_o)$, the function h_φ defined by

$$h_\varphi(u) = \int_{\partial T} \varphi\, d\nu_u$$

is finite and harmonic on T. It is often called the *Poisson transform* of φ.

Next we define a measure m on T as follows: $\mathsf{m}(o) = 1$, and for $v \in T \setminus \{o\}$ with $\pi(o,v) = [o = v_0\,, v_1\,, \dots, v_k = v]$,

$$\mathsf{m}(v) = \frac{p(v_0,v_1)p(v_1,v_2)\cdots p(v_{k-1},v_k)}{p(v_1,v_0)p(v_2,v_1)\cdots p(v_k,v_{k-1})}\,. \tag{6.12}$$

Then, for all $u, v \in T$,

$$\mathsf{m}(u)p(u,v) = \mathsf{m}(v)p(v,u)\,, \quad \text{and consequently} \quad \mathsf{m}(u)G(u,v) = \mathsf{m}(v)G(v,u)\,. \tag{6.13}$$

Hence, the random walk is *reversible.* This will allow us to use the *electrical network* interpretation of $(T, \mathcal{P}, \mathsf{m})$: see e.g. Yamasaki [65], Soardi [54], or – with notation as used here – [63, Chapter 4]. Each edge $e = [v^-, v] \in E(T)$ is thought of as an electric conductor with *conductance*

$$a(v^-, v) = \mathsf{m}(v)p(v, v^-).$$

The Dirichlet form $\mathcal{E}_T = \mathcal{E}_{T,\mathcal{P}}$ for functions $f, g : T \to \mathbb{R}$ is defined by

$$\mathcal{E}_T(f,g) = \sum_{[v^-,v]\in E(T)} \big(f(v) - f(v^-)\big)\,\big(g(v) - g(v^-)\big)\; a(v^-,v)\,. \tag{6.14}$$

The domain of this Dirichlet form is the following space:

$$\mathcal{D}(T) = \mathcal{D}(T,\mathcal{P}) = \{f : T \to \mathbb{R} \mid \mathcal{E}_T(f,f) < \infty\}. \tag{6.15}$$

6.4 Harmonic functions of finite energy and their boundary values

We are interested in the subspace

$$\mathcal{HD}(T) = \mathcal{HD}(T,\mathcal{P}) = \{h \in \mathcal{D}(T,\mathcal{P}) : \mathcal{P}h = h\}$$

of harmonic functions with *finite energy.* The terminology comes from the interpretation of such a function as the potential of an electric flow (current), and then $\mathcal{E}_T(h,h)$ is the energy of that flow.

Every function in $\mathcal{HD}(T,\mathcal{P})$ is the Poisson transform of some function $\varphi \in L^2(\partial T, \nu_o)$. This is valid not only for trees, but for general finite range reversible Markov chains, and follows from the following facts.

(1) Every function in $\mathcal{HD}$ is the difference of two non-negative functions in $\mathcal{HD}$.

(2) Every non-negative function in $\mathcal{HD}$ can be approximated by a monotone increasing sequence of non-negative bounded functions in $\mathcal{HD}$.

(3) Every bounded harmonic function is the Poisson transform of a bounded function on the boundary.

The boundary ∂T is the (active part of) the Martin boundary, with ν_u being the limit distribution on ∂T of the Markov chain, starting at u. The facts (1) and (2) are contained in [65] and [54], while (3) is part of general Martin boundary theory, see e.g. [63, Theorem 7.61].

Thus, we introduce a form $\mathcal{E}_{\mathcal{HD}}$ on ∂T by setting

$$\begin{aligned} \mathcal{D}(\partial T, \mathcal{P}) &= \{\varphi \in L^1(\partial T, \nu_o) : \mathcal{E}_T(h_\varphi, h_\varphi) < \infty\}\,, \\ \mathcal{E}_{\mathcal{HD}}(\varphi, \psi) &= \mathcal{E}_T(h_\varphi, h_\psi) \quad \text{for } \varphi, \psi \in \mathcal{D}(\partial T, \mathcal{P}). \end{aligned} \tag{6.16}$$

6.5 Jump processes on the boundary of a tree

Kigami [36] elaborates an expression for the form $\mathcal{E}_{\mathcal{HD}}(\varphi, \psi)$ of (6.16) by considerable effort, shows its regularity properties and then studies the jump process on ∂T induced by this Dirichlet form. We call this the *boundary process* associated with the random walk on T.

Let us show that there is a rather simple expression for $\mathcal{E}_{\mathcal{HD}}$. Define the *Naïm kernel* on $\partial T \times \partial T$ by

$$\Theta_o(x, y) = \begin{cases} \dfrac{\mathsf{m}(o)}{G(o,o)F(o, x\wedge y)F(x \wedge y, o)}\,, & \text{if } x \neq y\,, \\ +\infty\,, & \text{if } x = y\,. \end{cases} \tag{6.17}$$

In our case $\mathsf{m}(o) = 1$, but we do not use this to ensure the applicability of (6.17) in more general cases (think of a different choice of the base point, or a different normalization of the measure m).

Theorem 6.4 *For any transient nearest neighbour random walk on the tree T with root o, and all functions φ, ψ in $\mathcal{D}(\partial T, \mathcal{P})$,*

$$\mathcal{E}_{\mathcal{HD}}(\varphi, \psi) = \frac{1}{2}\int_{\partial T}\int_{\partial T} (\varphi(x) - \varphi(y))\,(\psi(x) - \psi(y))\,\Theta_o(x, y)\,d\nu_o(x)\,d\nu_o(y)\,.$$

A proof of Theorem 6.4 is given in [20] in a setting of potential theory on Green spaces, which are locally Euclidean. The definition of the Naïm kernel in [44] refers to the same type of setting. However, the trees, even when seen as metric graphs, are not locally Euclidean. In this sense, so far the definition of the Naïm kernel and a proof of Theorem 6.4 in a suitable for us setting have not been well accessible in the literature. In a forthcoming paper, Georgakopoulos and Kaimanovich will provide those "missing links" in full generality.

We give here a direct and simple proof of Theorem 6.4 for the specific case of trees. We start with the following observation.

Lemma 6.5 *The measure $\Theta_o(x, y)\,d\nu_o(x)\,d\nu_o(y)$ on $\partial T \times \partial T$ is invariant with respect to changing the base point (root) o.*

Proof. We want to replace the base point o with some other $u \in T$. We may assume that $u \sim o$. Indeed, then we may step by step replace the current base point by one of its neighbours to obtain the result for arbitrary u.

Recall that the confluent that appears in the definition (6.17)) of Θ_o depends on the root o, while for Θ_x it becomes the one with respect to x as the new root. It is a well-known fact that

$$\frac{d\nu_u}{d\nu_o}(x) = K(u,x) := \frac{G(u, u \wedge_o x)}{G(o, u \wedge_o x)}\,,$$

where K is called the *Martin kernel.* Thus, we have to show that for all $x, y \in \partial T$ $(x \neq y)$

$$\frac{\mathsf{m}(o)}{G(o,o)F(o, x \wedge_o y)F(x \wedge_o y, o)} = \frac{\mathsf{m}(u)K(u,x)K(u,y)}{G(u,u)F(u, x \wedge_u y)F(x \wedge_u y, u)}\,.$$

Consider four cases.

Case 1. $x, y \in \partial T_u$. Then $x \wedge_o y = x \wedge_u y =: v \in T_u$, and $u \wedge_o x = u \wedge_o y = u$. Thus, using (6.7), (6.10) and the fact that by (6.13)

$$\mathsf{m}(u)/G(o,u) = \mathsf{m}(o)/G(u,o),$$

we obtain

$$\begin{aligned}
\frac{\mathsf{m}(u)K(u,x)K(u,y)}{G(u,u)F(u, x \wedge_u y)F(x \wedge_u y, u)} &= \frac{\mathsf{m}(u)}{G(u,u)F(u,v)F(v,u)}\left(\frac{G(u,u)}{G(o,u)}\right)^2 \\
&= \frac{\mathsf{m}(o)G(u,u)}{F(u,v)F(v,u)G(o,u)G(u,o)} \\
&= \frac{\mathsf{m}(o)}{F(u,v)F(v,u)F(o,u)F(u,o)G(o,o)} \\
&= \frac{\mathsf{m}(o)}{F(o,v)F(v,o)G(o,o)}\,,
\end{aligned}$$

as required.

Case 2. $x, y \in \partial T \setminus \partial T_u$. Then

$$x \wedge_o y = x \wedge_u y =: w \in T \setminus T_u, \quad \text{and} \quad u \wedge_o x = u \wedge_o y = o.$$

Case 3. $x \in \partial T_u$, $y \in \partial T \setminus \partial T_u$. Then

$$x \wedge_o y = o, \quad x \wedge_u y = u, \quad u \wedge_o x = u \quad \text{and} \quad u \wedge_o y = o.$$

Case 4. $x \in \partial T \setminus \partial T_u$, $y \in \partial T_u$. This is similar to Case 3, exchanging the roles of x and y.

In all cases 2–4, the proof is done similarly to Case 1. ■

For the proof of Theorem 6.4, we need a few more facts related with the network setting; compare e.g. with [63, §4.D].

The space $\mathcal{D}(T)$ of (6.15) is a Hilbert space when equipped with the inner product

$$(f, g) = \mathcal{E}_T(f, g) + f(o)g(o)\,.$$

The subspace $\mathcal{D}_0(T)$ is defined as the closure of the space of finitely supported functions in $\mathcal{D}(T)$. It is a proper subspace if and only if the random walk is transient, and then the function $G_v(u) = G(u,v)$ is in $\mathcal{D}_0(T)$ for any $v \in T$ [65], [54]. We need the formula

$$\mathcal{E}_T(f, G_v) = \mathsf{m}(v)f(v) \quad \text{for every } f \in \mathcal{D}_0(T)\,. \tag{6.18}$$

Given a branch T_w of T (for $w \in T \setminus \{o\}$), we can consider it as a subnetwork equipped with the same conductances $a(u,v)$ for $[u,v] \in E(T_w)$. The associated measure on T_w is

$$\mathsf{m}_{T_w}(u) = \sum_{v \in T_z : v \sim u} a(u,v) = \begin{cases} \mathsf{m}(u) & \text{if } u \in T_w \setminus \{w\}\,, \\ \mathsf{m}(w) - a(w,w^-) & \text{if } u = w\,. \end{cases}$$

The resulting random walk on T_w has transition probabilities

$$p_{T_w}(u,v) = \frac{a(v,w)}{\mathsf{m}_{T_w}(u)} = \begin{cases} p(u,v) & \text{if } u \in T_w \setminus \{w\}\,,\ v \sim u\,, \\[2ex] \dfrac{p(w,v)}{1 - p(w,w^-)} & \text{if } u = w\,,\ v \sim u\,. \end{cases}$$

We have $F_{T_w}(u,u^-) = F(u,u^-)$ and thus also $F_{T_w}(u,w) = F(u,w)$ for every $u \in T_w \setminus \{w\}$, because before its first visit to w, the random walk on T_w obeys the same transition probabilities as the original random walk on T. It is then easy to see [63, p. 241] that the random walk on T_w is transient if and only if for the original random walk, $F(w,w^-) < 1$, which in turn holds if and only if $\nu_o(\partial T_w) > 0$. (In other parts of this and the preceding two sections, this is always assumed, but for the proof of Theorem 6.4, we just assume the random walk on the whole of T to be transient.) Conversely, if $F(w,w^-) = 1$ then $F(u,w) = 1$ for all $u \in T_w$.

Below, we shall need the following formula for the limit distributions.

Lemma 6.6 *For $u \in T \setminus \{o\}$,*

$$\nu_u(\partial T_u) = 1 - p(u,u^-)\big(G(u,u) - G(u^-,u)\big)\,.$$

Proof. By (7.7),

$$G(u,u)p(u,u^-) = \frac{F(u,u^-)}{1 - F(u,u^-)F(u^-,u)}$$

Thus,

$$p(u,u^-)\big(G(u,u) - G(u^-,u)\big) = \big(1 - F(u^-,u)\big)\,G(u,u)p(u,u^-) = 1 - \nu_u(\partial T_u)$$

after a short computation using (6.11) ■

Proof of Theorem 6.4. We first prove the Doob-Naïm formula (shortly, D-N-formula) for the case when $\varphi = \mathbf{1}_{\partial T_v}$ and $\psi = \mathbf{1}_{\partial T_w}$ for two proper branches T_v and T_w of T. They are either disjoint, or one of them contains the other.

Case 1. $T_w \subset T_v$. (The case $T_v \subset T_w$ is analogous by symmetry.)

This means that $w \in T_v$. For $x, y \in \partial T$ we have

$$(\varphi(x) - \varphi(y))\,(\psi(x) - \psi(y)) = 1$$

if $x \in \partial T_w$ and $y \in \partial T \setminus \partial T_v$ or conversely, and

$$(\varphi(x) - \varphi(y))\,(\psi(x) - \psi(y)) = 0$$

otherwise. By Lemma 6.5, we may choose v as the base point. Thus, the right hand side of the identity is

$$\int_{\partial T \setminus \partial T_v} \int_{\partial T_w} \Theta_v(x,y)\, d\nu_v(x)\, d\nu_v(y) = \frac{\mathsf{m}(v)}{G(v,v)}\, \nu_v(\partial T \setminus \partial T_v)\nu_v(\partial T_w)\,,$$

since $x \wedge_v y = v$ and $F(v,v) = 1$.

Let us now turn to the left hand side of the D-N-formula. The Poisson transforms of φ and ψ are

$$h_\varphi(u) = \nu_u(\partial T_v) \quad \text{and} \quad h_\psi(u) = \nu_u(\partial T_w).$$

By (6.11),

$$\begin{aligned} h_\varphi(u) &= F(u,v)\nu_v(\partial T_v)\,, \quad u \in \{v\} \cup (T \setminus T_v) \\ 1 - h_\varphi(u) &= F(u,v)\nu_v(\partial T \setminus \partial T_v)\,, \quad u \in T_v\,. \end{aligned}$$

We set $F_v(u) = F(u,v)$ and write

$$h_\varphi(u) - h_\varphi(u^-) = \big(1 - h_\varphi(u^-)\big) - (1 - h_\varphi(u))$$

whenever this is convenient, and analogously for h_ψ. Then we get

$$\begin{aligned} &\mathcal{E}_T(h_\varphi\,, h_\psi) \\ &= \sum_{[u,u^-]\in E(T)\setminus E(T_v)} a(u,u^-)\big(F(u,v) - F(u^-,v)\big)\,\nu_v(\partial T_v)\,\big(F(u,v) - F(u^-,v)\big)\,\nu_v(\partial T_w) \\ &\quad - \sum_{[u,u^-]\in E(T_v)\setminus E(T_w)} a(u,u^-)\big(F(u,v) - F(u^-,v)\big)\,\nu_v(\partial T \setminus \partial T_v)\,\big(F(u,v) - F(u^-,v)\big)\,\nu_v(\partial T_w) \\ &\quad + \sum_{[u,u^-]\in E(T_w)} a(u,u^-)\big(F(u,v) - F(u^-,v)\big)\,\nu_v(\partial T \setminus \partial T_v)\,\big(F(u,v) - F(u^-,v)\big)\,\nu_v(\partial T \setminus \partial T_w) \\ &= \mathcal{E}_T(F_v\,, F_w)\nu_v(\partial T_v)\nu_v(\partial T_w) - \mathcal{E}_{T_v}(F_v\,, F_w)\nu_v(\partial T_w) + \mathcal{E}_{T_w}(F_v\,, F_w)\nu_v(\partial T \setminus \partial T_v)\,, \end{aligned}$$

where of course $\mathcal{E}_{T_v}$ is the Dirichlet form of the random walk on the branch T_v, as discussed above, and analogously for $\mathcal{E}_{T_w}$. Now $F_v = G_v/G(v,v)$ by (6.7), whence (6.18) yields

$$\mathcal{E}_T(F_v, F_w) = \frac{\mathcal{E}_T(G_v, F_w)}{G(v,v)} = \frac{\mathsf{m}(v)F(v,w)}{G(v,v)}\,. \tag{6.19}$$

Recall that for the random walk on T_v, we have $F_{T_v}(u,v) = F(u,v)$ for every $u \in T_v$. Also,

$$\mathsf{m}_{T_v}(v) = \mathsf{m}(v) - a(v,v^-) = \mathsf{m}(v)\,\big(1 - p(v,v^-)\big)\,.$$

We apply (6.19) to that random walk and obtain

$$\mathcal{E}_{T_v}(F_v, F_w) = \frac{\mathsf{m}(v)\,(1 - p(v,v^-))\,F(v,w)}{G_{T_v}(v,v)}\,.$$

Applying (6.8), (6.9) and

$$p_{T_v}(v,u) = \frac{p(v,u)}{1 - p(v,v^-)}$$

for $u \in T_v$, we obtain

$$\begin{aligned} \frac{1 - p(v,v^-)}{G_{T_v}(v,v)} &= 1 - p(v,v^-) - \big(1 - p(v,v^-)\big)\,U_{T_v}(v,v) \\ &= 1 - p(v,v^-) - \sum_{u:u^-=v} p(v,u)F(u,v) \\ &= 1 - p(v,v^-) - \big(U(v,v) - p(v,v^-)F(v^-,v)\big) \\ &= \frac{1}{G(v,v)} - p(v,v^-)\,\big(1 - F(v^-,v)\big) = \frac{\nu_v(\partial T_v)}{G(v,v)}, \end{aligned}$$

where in the last step we have used Lemma 6.6. It follows that

$$\mathcal{E}_{T_v}(F_v, F_w) = \frac{\mathsf{m}(v)F(v,w)}{G(v,v)}\,\nu_v(\partial T_v).$$

In the same way, exchanging roles between T_w and T_v and using reversibility (6.13),

$$\mathcal{E}_{T_w}(F_v, F_w) = \frac{\mathsf{m}(w)F(w,v)}{G(w,w)}\,\nu_w(\partial T_w) = \frac{\mathsf{m}(v)F(v,w)}{G(v,v)}\,\nu_w(\partial T_w) = \frac{\mathsf{m}(v)}{G(v,v)}\,\nu_v(\partial T_w)$$

Putting things together, we get that

$$\mathcal{E}_T(h_\varphi\,, h_\psi) = \mathcal{E}_{T_w}(F_v\,, F_w)\nu_v(\partial T \setminus \partial T_v) = \frac{\mathsf{m}(v)}{G(v,v)}\,\nu_v(\partial T_w)\nu_v(\partial T \setminus \partial T_v),$$

as proposed.

Case 2. $T_w \cap T_v = \emptyset$.

In view of Lemma 6.5, both sides of the D-N-formula are independent of the root o. Thus we may declare our root to be one of the neighbours of v that is not on $\pi(v,w)$. Also, let $\bar{v}$ be the neighbour of v on $\pi(w,v)$. Then, with our chosen new root, the complement of the "old" T_v is $T_{\bar{v}}$, which contains T_w (The latter remains the same with respect to the new root).

Thus, we can apply the result of case 1 to $T_{\bar{v}}$ and T_w. This means that we have to replace the functions φ and h_φ with $1-\varphi$ and $1-h_\varphi$, respectively, which just means that we change the sign on both sides of the identity. We are re-conducted to Case 1 without further computations.

We deduce from what we have done so far, and from linearity of the Poisson transform as well of bilinearity of the forms on both sides of the D-N-formula , that it holds for linear combinations of indicator functions of sets ∂T_v. Those indicator functions are dense in the space $C(\partial T)$ with respect to the max-norm. Thus, the D-N-formula holds for all continuous functions on ∂T. The extension to all of $\mathcal{D}(\partial T, \mathcal{P})$ is by standard approximation. ∎

7 The duality of random walks on trees and isotropic processes on their boundaries

When looking at our isotropic processes and at the boundary process of Kigami [36], it is natural to ask the following two questions.

Question I. Given a transient random walk on T associated with the Dirichlet form $\mathcal{E}_T$ of (6.14), does the boundary process on ∂T induced by the form $\mathcal{E}_{\mathcal{HD}}$ of (6.16) coincide with an isotropic process (1.8) on ∂T with transition probabilities (6.6), induced by the measure $\mu = \nu_o$ on ∂T, some ultra-metric element ϕ on T and a suitable distance distribution function σ on $[0\,, \infty)$?

Question II. Conversely, given data μ, ϕ and σ, is there a random walk on T with limit distribution $\nu_o = \mu$ such that the isotropic process induced by μ, ϕ and σ is the boundary process with Dirichlet form $\mathcal{E}_{\mathcal{HD}}$?

Before answering both questions, we need to specify the assumptions more precisely. When starting with (ϕ, μ, σ), we always assume as before that μ is supported by the whole of ∂T.

Thus, on the side of the random walk, we also want that $\text{supp}(\nu_o) = \partial T$. This is equivalent with the requirement that $\nu_o(\partial T_v) > 0$ for every $v \in T$. By (6.11) this is in turn equivalent with

$$F(v, v^-) < 1 \quad \text{for every } v \in T \setminus \{o\}. \tag{7.1}$$

Indeed, we shall see that we need a bit more, namely that

$$\lim_{v\to\infty} G(v,o) = 0\,, \tag{7.2}$$

that is, for every $\varepsilon > 0$ there is a finite set $A \subset T$ such that $G(v,o) < \varepsilon$ for all $v \in T \setminus A$. This condition is necessary and sufficient for solvability of the *Dirichlet problem:* for any $\varphi \in C(\partial X)$, its Poisson transform h_φ provides the unique continuous extension of φ to $\widehat{T}$ which is harmonic in T. See e.g. [63, Corollary 9.44].

We shall restrict attention to random walks with properties (7.1) and (7.2) on a rooted tree with forward degrees ≥ 2.

7.1 Answer to Question I

We start with a random walk that fulfills the above requirements. We know from §1 that each (μ, ϕ, σ)-process arises as the standard process of Definition 2.9 with respect to the intrinsic metric (cf. Theorem 2.10): given ϕ and σ, the intrinsic metric is induced by the ultra-metric element

$$\phi_*(u) = -1\big/ \log \sigma\left(\phi(u)\right)\,. \tag{7.3}$$

Thus, we can eliminate σ from our considerations by just looking for an ultra-metric element ϕ such that the boundary process is the standard process on ∂T associated with (ϕ, ν_0).

Since the processes are determined by the Dirichlet forms, we infer from Theorems 3.12 and 6.4 that we are looking for ϕ such that $J(x,y) = \Theta_o(x,y)$ for all $x, y \in \partial T$ with $x \neq y$, where $J(x,y)$ is given by (3.11). Rewriting $J(x,y)$ in terms of ϕ, ν_o and the tree structure, this becomes

$$\frac{1}{\phi(o)} + \int_{1/\phi(o)}^{1/\phi(x\wedge y)} \frac{dt}{\nu_o\left(B^\phi_{1/t}(x)\right)} = \frac{\mathsf{m}(o)}{G(o,o)F(o,x\wedge y)F(x\wedge y,o)}. \tag{7.4}$$

In our case, $\mathsf{m}(o) = 1$, but we keep track of what happens when one changes the root or the normalisation of m. First of all, since $\deg^+(o) \geq 2$, there are $x, y \in \partial T$ such that $x \wedge y = o$. We insert these two boundary points in (7.4). Since $F(o,o) = 1$, we see that we must have

$$\phi(o) = G(o,o)/\mathsf{m}(o)\,.$$

Now take $v \in T \setminus \{o\}$. Since forward degrees are ≥ 2, there are $x, y, y' \in \partial T$ such that $x \wedge y = v$ and $x \wedge y' = v^-$. We write (7.4) first for (x, y') and then for (x, y) and then take the difference, leading to the equation

$$\int_{1/\phi(v^-)}^{1/\phi(v)} \frac{dt}{\nu_o\left(B^\phi_{1/t}(x)\right)} = \frac{\mathsf{m}(o)}{G(o,o)F(o,v)F(v,o)} - \frac{\mathsf{m}(o)}{G(o,o)F(o,v^-)F(v^-,o)}. \tag{7.5}$$

By (6.5), within the range of the last integral we must have $B^\phi_{1/t}(v) = \partial T_v$, whence that integral reduces to

$$\left(\frac{1}{\phi(v)} - \frac{1}{\phi(v^-)}\right)\frac{1}{\nu_o(\partial T_v)}.$$

We multiply equation (7.5) by $\nu_o(\partial T_v)$ and simplify the resulting right hand side

$$\left(\frac{\mathsf{m}(o)}{G(o,o)F(o,v)F(v,o)} - \frac{\mathsf{m}(o)}{G(o,o)F(o,v^-)F(v^-,o)}\right)\nu_o(\partial T_v)$$

by use of the identities (6.7) – (6.10)) and the first of the two formulas of (6.11) (for ν_o). We obtain that the ultra-metric element that we are looking for should satisfy

$$\frac{1}{\phi(v)} - \frac{1}{\phi(v^-)} = \frac{\mathsf{m}(o)}{G(v,o)} - \frac{\mathsf{m}(o)}{G(v^-,o)} \quad \text{for every } v \in T \setminus \{o\}\,. \tag{7.6}$$

This determines $1/\phi(v)$ recursively, and with $\mathsf{m}(o) = 1$, we obtain

$$\phi(v) = G(v, o) .$$

Since by (6.7) and (6.10)

$$G(v, o) = F(v, v^-)G(v^- o),$$

the assumptions (7.1) and (7.2) yield that ϕ is an ultra-metric element. Reversing the last computations, we see that with this choice of ϕ, we have indeed that $J(x, y) = \Theta_o(x, y)$ for all $x, y \in \partial T$ with $x \neq y$. Hence, we have proved the following result.

Theorem 7.1 *Let T be a locally finite, rooted tree with forward degrees ≥ 2. Consider a transient nearest neighbour random walk on T that satisfies (6.7) and (6.10). Then the boundary process on ∂T induced by the Dirichlet form (6.16) coincides with the standard process associated with ultra-metric element $\phi = G(\cdot, o)$ and the limit distribution ν_o of the random walk.*

Let $\mathcal{L}$ be the Laplacian associated with the boundary process of Theorem 7.1. $\mathcal{L}$ acts on locally constant functions f by

$$\mathcal{L}f(x) = \int_{\partial T} (f(x) - f(y))\, \Theta_o(x, y)\, d\nu_o(y).$$

In view of the identification of balls in ∂T with vertices of T, the eigenfunctions of (3.6) now become

$$f_v = \frac{\mathbf{1}_{\partial T_v}}{\nu_o(\partial T_v)} - \frac{\mathbf{1}_{\partial T_{v^-}}}{\nu_o(\partial T_{v^-})},\ v \in T \backslash \{o\}.$$

In addition, we set $f_o = \mathbf{1}$ and note that it is an eigenfunction of $\mathcal{L}$ with eigenvalue 0. Applying Theorem 3.8 we obtain the following.

Corollary 7.2 *For $v \in T \setminus \{o\}$, we have $\mathcal{L}f_v = G(v^-, o)^{-1} f_v$ and the set of eigenfunctions $\{f_v\}_{v \in T}$ is complete. In particular, we have*

$$\operatorname{spec} \mathcal{L} = \overline{\{G(v, o)^{-1} : v \in T\}} \cup \{0\}.$$

Remark 7.3 For any two vertices v and w in $T \setminus \{o\}$ such that $v^- = w^- = u$ the functions f_v and f_w are eigenfunctions of $\mathcal{L}$ corresponding to the eigenvalue $\lambda = 1/G(u, o)$. Hence the eigenspace $\mathcal{H}(u)$ corresponding to the vertex u is spanned by functions $\{f_v : v^- = u\}$. Since the rank of the system $\{f_v : v^- = u\}$ is $\deg^+(u) - 1$, where $\deg^+(u) \geq 2$ is the forward degree of the vertex u, we obtain

$$\dim \mathcal{H}(u) = \deg^+(u) - 1$$

(cf. (3.9)).

Remark 7.4 Given the random walk on T and the associated boundary process on ∂T, we might want to realize the latter as the (ν_o, ϕ, σ)-process for an ultra-metric element ϕ different from $G(\cdot, o)$. This means that we have to look for a suitable distance distribution σ on $[0\,, \infty)$, different from the inverse exponential distribution (2.16). In view of (7.3), we are looking for σ such that for our given generic ϕ,

$$\sigma\left(\phi(v)\right) = e^{-1/G(v,o)}.$$

For this it is necessary that $\phi(u) = \phi(v)$ whenever $G(u, o) = G(v, o)$: we need ϕ to be constant on equipotential sets. In that case, the distribution function $\sigma(r)$ is determined by the above equation for r in the value set Λ_ϕ of the ultra-metric d_ϕ. Then that function can be extended to $[0, +\infty)$.

7.2 Answer to Question II

Answering Question II means that we start with ϕ and μ and then look for a random walk with limit distribution $\nu_o = \mu$ such that the standard (ϕ,μ)-process is the boundary process associated with the random walk. We know from Theorem 7.1 that in this case, we should have $\phi(v) = G(v,o)$, whence in particular, $\phi(o) > 1$. Thus we cannot expect that every ϕ is suitable. The most natural choice is to replace ϕ by $C \cdot \phi$ for some constant $C > 0$. For the standard processes associated with ϕ and $C \cdot \phi$, respectively, this just gives rise of a linear time change: if the old process is $\{X_t\}_{t>0}$, then the new one is $\{X_{t/C}\}_{t>0}$.

Theorem 7.5 *Let T be a locally finite, rooted tree with forward degrees ≥ 2. Consider an ultra-metric element ϕ on T and a fully supported probability measure μ on ∂T. Then there are a unique constant $C > 0$ and a unique transient nearest neighbour random walk on T that satisfies (6.7) and (6.10) with the following properties:*

1. *$\mu = \nu_o$ is the limit distribution of the random walk.*
2. *The associated boundary process coincides with the standard process on ∂T induced by the ultra-metric element $C \cdot \phi$ and the given measure μ.*

For the proof, we shall need three more formulas. The first two are taken from [63, Lemma 9.35], while the third is immediate from (6.11) and (6.10)

$$G(u,u)\,p(u,v) = \frac{F(u,v)}{1 - F(u,v)F(v,u)} \quad \text{if } u \sim v\,, \quad \text{and} \tag{7.7}$$

$$G(u,u) = 1 + \sum_{v:v\sim u} \frac{F(u,v)F(v,u)}{1 - F(u,v)F(v,u)} \tag{7.8}$$

$$F(v^-,v) = \frac{\nu_o(\partial T_v)/F(o,v^-)}{1 - F(v,v^-) + F(v,v^-)\;\nu_o(\partial T_v)/F(o,v^-)}. \tag{7.9}$$

Proof of Theorem 7.5. We proceed as follows: we start with ϕ and μ and replace ϕ by a new ultra-metric element $C \cdot \phi$, with C to be determined, and μ being the candidate for the limit distribution of the random walk that we are looking for.

Using the various formulas at our disposal, we first construct in the only possible way the quantities $F(u,v)$, $u,v,\in T$, in particular when $u \sim v$. In turn, they lead to the Green kernel $G(u,v)$. So far, these will be only "would-be" quantities whose feasibility will have to be verified. Until that verification, we shall denote them $\widetilde{F}(u,v)$ and $\widetilde{G}(u,v)$. Via (7.7), they will lead to definitions of transition probabilities $p(u,v)$. Stochasticity of the resulting transition matrix $\mathcal{P}$ will also have to be verified.

Only then, we will use a potential theoretic argument to show that $\widetilde{G}(u,v)$ really is the Green kernel associated with $\mathcal{P}$, so that the question mark that is implicit in the "$\,\widetilde{\ }\,$" symbol can be removed.

First of all, in view of Theorem 7.1, we must have

$$C \cdot \phi(v) = \widetilde{G}(v,o)\,,$$

whence by (6.7) and (6.10)

$$\widetilde{F}(v,v^-) = \phi(v)/\phi(v^-) \quad \text{for } v \in T \setminus \{o\}\,, \tag{7.10}$$

and more generally

$$\widetilde{F}(v,u) = \phi(v)/\phi(u) \quad \text{when } u \leq v\,.$$

We note immediately that $0 < \widetilde{F}(v,u) < 1$ when $u < v$, and that $\widetilde{F}(u,u) = 1$.

Next, we use (7.9) to construct recursively $\widetilde{F}(v^-,v)$ and $\widetilde{F}(o,v)$. We start with $\widetilde{F}(o,o) = 1$. If $v \neq o$ and $\widetilde{F}(o,v^-)$ is already given, with

$$\mu(\partial T_{v^-}) \leq \widetilde{F}(o,v^-) \leq 1$$

(the lower bound is required by (6.11)), then we have to set

$$\widetilde{F}(v^-,v) = \frac{\mu(\partial T_v)/\widetilde{F}(o,v^-)}{1 - \widetilde{F}(v,v^-) + \widetilde{F}(v,v^-)\ \mu(\partial T_v)/\widetilde{F}(o,v^-)} \tag{7.11}$$

and

$$\widetilde{F}(o,v) = \widetilde{F}(o,v^-)\widetilde{F}(v^-,v)\,.$$

Since

$$\widetilde{F}(o,v^-) \geq \mu(\partial T_{v^-}) \geq \mu(\partial T_v),$$

we see that

$$0 < \widetilde{F}(v^-,v) \leq 1.$$

We set – as imposed by (6.10) –

$$\widetilde{F}(o,v) = \widetilde{F}(o,v^-)\widetilde{F}(v^-,v).$$

Formula (7.11) transforms into

$$\mu(\partial T_v) = \widetilde{F}(o,v^-)\widetilde{F}(v^-,v)\frac{1-\widetilde{F}(v,v^-)}{1-\widetilde{F}(v,v^-)\widetilde{F}(v^-,v)} \leq \widetilde{F}(o,v) \leq 1\,, \tag{7.12}$$

as needed for our recursive construction. At this point, we have all values of $\widetilde{F}(u,v)$, initially for $u \sim v$, and consequently for all u,v by taking products along geodesic paths.

We now can compute the constant C: (7.8), combined with (7.10) and (7.12) for $u \sim o$ forces

$$\begin{aligned} C\phi(o) &= \widetilde{G}(o,o) = 1 + \sum_{u:u\sim o} \frac{\widetilde{F}(o,u)\widetilde{F}(u,o)}{1-\widetilde{F}(o,u)\widetilde{F}(u,o)} \\ &= 1 + \sum_{u:u\sim o} \frac{\widetilde{F}(u,o)}{1-\widetilde{F}(u,o)}\mu(\partial T_u) \\ &= 1 + \sum_{u:u\sim o} \frac{\phi(u)/\phi(o)}{1-\phi(u)/\phi(o)}\mu(\partial T_u) \end{aligned}$$

Therefore, we have

$$C = \frac{1}{\phi(o)} + \sum_{u:u\sim o} \frac{\phi(u)/\phi(o)}{\phi(o)-\phi(u)}\mu(\partial T_u)\,. \tag{7.13}$$

We now construct $\widetilde{G}(u,u)$ via (7.8):

$$\widetilde{G}(u,u) = 1 + \sum_{v:v\sim u} \frac{\widetilde{F}(u,v)\widetilde{F}(v,u)}{1-\widetilde{F}(u,v)\widetilde{F}(v,u)}. \tag{7.14}$$

For $u = o$, we know that this is compatible with our choice of C. At last, our only choice for the Green kernel is

$$\widetilde{G}(u,v) = \widetilde{F}(u,v)\widetilde{G}(v,v)\,, \quad u,v \in T\,.$$

Now we finally arrive at the only way how to define the transition probabilities, via (7.7):

$$p(u,v) = \frac{1}{\widetilde{G}(u,u)} \frac{\widetilde{F}(u,v)}{1-\widetilde{F}(u,v)\widetilde{F}(v,u)}. \tag{7.15}$$

Claim 1. $\mathcal{P}$ is stochastic.

Proof of Claim 1. Combining (7.15) with (7.14), we deduce that we have to verify that for every $u \in T$,

$$\sum_{v:v\sim u} \frac{\widetilde{F}(u,v)\left(1-\widetilde{F}(v,u)\right)}{1-\widetilde{F}(u,v)\widetilde{F}(v,u)} = 1\,. \tag{7.16}$$

If $u = o$, then by (7.12) this is just

$$\sum_{v:v\sim o} \mu(\partial T_v) = 1.$$

If $u \neq o$ then, again by (7.12), the left hand side of (7.16) is

$$\begin{aligned}
&\sum_{v:v^-=u} \frac{\widetilde{F}(u,v)\left(1-\widetilde{F}(v,u)\right)}{1-\widetilde{F}(u,v)\widetilde{F}(v,u)} + \frac{\widetilde{F}(u,u^-)\left(1-\widetilde{F}(u^-,u)\right)}{1-\widetilde{F}(u,u^-)\widetilde{F}(u^-,u)} \\
&\quad = \sum_{v:v^-=u} \frac{\mu(\partial T_v)}{\widetilde{F}(o,u)} + 1 - \frac{1-\widetilde{F}(u^-,u)}{1-\widetilde{F}(u,u^-)\widetilde{F}(u^-,u)} = 1.
\end{aligned}$$

This proves Claim 1.

Claim 2. For any $u_0 \in T$, the function $\tilde{g}_{u_0}(u) = \widetilde{G}(u,u_0)$ satisfies $\mathcal{P}\tilde{g}_{u_0} = \tilde{g}_{u_0} - \mathbf{1}_{u_0}$.

Proof of Claim 2. First, we combine (7.14) with (7.15) to get

$$P\tilde{g}_{u_0}(u_0) = \sum_{v:v\sim u_0} p(u_0,v)\widetilde{F}(v,u_0)\widetilde{G}(u_0,u_0) = \sum_{v:v\sim u_0} \frac{\widetilde{F}(u_0,v)\widetilde{F}(v,u_0)}{1-\widetilde{F}(u_0,v)\widetilde{F}(v,u_0)} = \tilde{g}_{u_0}(u_0) - 1\,,$$

and Claim 2 is true at $u = u_0$. Second, for $u \neq u_0$, let w be the neighbour of u on $\pi(u,u_0)$. Then

$$\begin{aligned}
P\tilde{g}_{u_0}(u) &= \sum_{v:v\sim u, v\neq w} p(u,v)\widetilde{F}(v,u)\widetilde{G}(u,u_0) + p(u,w)\widetilde{G}(w,u_0) \\
&= \underbrace{\sum_{v:v\sim u} \frac{\widetilde{F}(u,v)\widetilde{F}(v,u)}{1-\widetilde{F}(u,v)\widetilde{F}(v,u)}}_{\widetilde{G}(u,u)-1} \frac{\widetilde{G}(u,u_0)}{\widetilde{G}(u,u)} - p(u,w)\widetilde{F}(w,u)\widetilde{G}(u,u_0) + p(u,w)\widetilde{G}(w,u_0) \\
&= G(u,u_0)\left(1 - \frac{1}{\widetilde{G}(u,u)} - p(u,w)\widetilde{F}(w,u) + p(u,w)\frac{1}{\widetilde{F}(u,w)}\right) = \tilde{g}_{u_0}(u)
\end{aligned}$$

since

$$p(u,w)/\widetilde{F}(u,w) - p(u,w)\widetilde{F}(w,u) = 1/\widetilde{G}(u,u)$$

by (7.15). This completes the proof of Claim 2.

Now we can conclude: the function $\tilde{g}_{u_0}$ is non-constant, positive and superharmonic. Therefore the random walk with transition matrix $\mathcal{P}$ given by (7.15) is transient and does posses a Green function $G(u,v)$. Furthermore, by the Riesz decomposition theorem, we have

$$\tilde{g}_{u_0} = Gf + h\,,$$

where h is a non-negative harmonic function and the charge f of the potential

$$Gf(u) = \sum_v G(u,v)f(v)$$

is $f = \tilde{g}_{u_0} - P\tilde{g}_{u_0} = \mathbf{1}_{u_0}$. That is,

$$\widetilde{G}(u,u_0) = G(u,u_0) + h(x) \quad \text{for all } u \in T.$$

Now let $x \in \partial T$ and $v = u_0 \wedge x$. If $u \in T_v$ then by our construction

$$\widetilde{G}(u,u_0) = \widetilde{G}(u,o)\frac{\widetilde{G}(v,u_0)}{\phi(v)}\phi(u) \to 0 \quad \text{as } u \to x\,.$$

Therefore $\widetilde{G}(\cdot,u_0)$ vanishes at infinity, and the same must hold for h. By the Maximum Principle, $h \equiv 0$.

We conclude that $\widetilde{G}(u,v) = G(u,v)$ for all $u,v \in T$. But then, by our construction, also $\widetilde{F}(u,v) = F(u,v)$. Comparing (7.12) with (6.11), we see that $\mu = \nu_o$. This completes the proof. ■

7.3 The non-compact case

The approach of the present work is not restricted to compact spaces. In case of a non-compact, locally compact ultra-metric space without isolated points, one constructs the tree in the same way: the vertex set corresponds to the collection of all closed balls, and neighbourhood in the resulting tree is defined as above: if a vertex v corresponds to a ball B, then the predecessor v^- is the vertex corresponding to the ball B' (see Definition 3.6), and there is the edge $[v^-,v]$. Now *every* vertex has a predecessor (while in the compact case, the root vertex has none), and the tree has its root at infinity, i.e., the ultra-metric space becomes $\partial^*T = \partial T \setminus \{\varpi\}$, where ϖ is a fixed reference end of T. See Figure 5 below.

We now start with this situation: given a tree T and a reference end $\varpi \in \partial T$, the predecessor $v^- = v^-_\varpi$ of a vertex v with respect to ϖ is the neighbour of v on the geodesic $\pi(v,\varpi)$. Given two elements $w,z \in \widehat{T} \setminus \{\varpi\}$, their confluent $w \curlywedge z$ with respect to ϖ is again defined as the last common element on the geodesics $\pi(\varpi,w)$ and $\pi(\varpi,z)$, a vertex, unless $v = w \in \partial^*T$ (Figure 5). Again, it is natural to assume that each vertex has at least two forward neighbours.

In this situation, for the Definition 6.1 of an ultra-metric element $\phi : T \to (0\,,\infty)$, we need besides monotonicity $[\phi(v) < \phi(v^-)]$ that ϕ tends to ∞ along $\pi(o,\varpi)$, while it has to tend to 0 along any geodesic going to ∂^*T. The associated ultra-metric on ∂^*T is then given in the same way as before:

$$d_\phi(x,y) = \begin{cases} 0\,, & \text{if } x = y\,,\\ \phi(x \curlywedge y)\,, & \text{if } x \neq y\,. \end{cases}$$

Let us note here that also when ϕ does not tend to ∞ along $\pi(x,\varpi)$, this does define an ultra-metric, but then (∂^*T, d_ϕ) will not be complete. Also, if the inequality $\phi(v) \le \phi(v^-)$ is not strict, one gets an ultra-metric, but then the above construction of the tree of closed balls does not recover the original tree from (∂^*T, d_ϕ). Finally, if ϕ does not tend to 0 along some geodesic $\pi(o,x)$, $x \in \partial^*T$, then x will be an isolated point in (∂^*T, d_ϕ). (The last two observations are also true in the compact case, for a tree with a root vertex.)

Returning to our setting, the reference measure μ of a (ϕ,μ,σ)-process may have infinite mass: a Radon measure supported on the whole of ∂^*T. Again, we know that it is sufficient to study the standard (ϕ,μ)-process. We give a brief outline of the duality of such processes with random walks on T. This should be compared with the final part of Kigami's second paper

[37] (whose preprint became available when the largest part of this work had been done, and in particular, the preliminary version [64], containing Sections 6 – 7 of the present work, had been circulated).

With respect to ϖ, the branch of T rooted at $u \in T$ is now

$$T_u = T_{\varpi,u} = \{v \in T : u \in \pi(v,\varpi)\}.$$

Then ∂T_u is a compact subset of $\partial^* T$, a ball with d_ϕ-diameter $\phi(u)$. Here, it will be good to write $T_{o,u}$ for the branch with respect to a root vertex $o \in T$, as defined in (6.2). We note that $T_{\varpi,u} = T_{o,u}$ iff $u \notin \pi(o,\varpi)$. In addition to the reference end ϖ, we choose such a root o and write o_n for its n-th predecessor, that is, the vertex on $\pi(o,\varpi)$ at graph distance n from o.

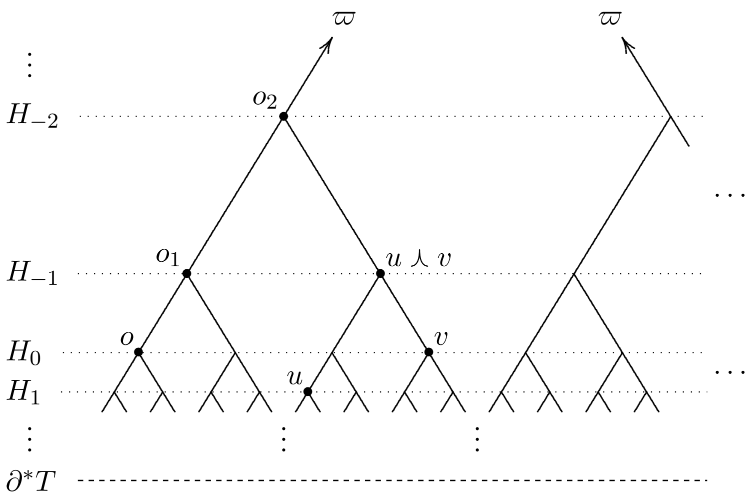

Figure 5

Now let $\mathcal{P} = (p(u,v))_{u,v\in T}$ be the transition matrix of a transient nearest neighbour random walk on T. We assume once more that (7.1) holds: $F(v,v^-) < 1$ for every $v \in T$, but now predecessors refer to ϖ. (Indeed, this implies (7.1) with respect to any choice of the root vertex.) We now consider the Dirichlet form $\mathcal{E}_{\mathcal{HD}}$ and look at the formula of Theorem 6.4. We would like to move o to ϖ in that formula. We know from Lemma 6.5 that the measures $\Theta_{o_n}(x,y)\, d\nu_{o_n}(x)\, d\nu_{o_n}(y)$ are the same for all n. However, the measures ν_{o_n} restricted to $\partial^* T$ will typically converge vaguely to 0. Thus, we normalise by defining

$$\mu_n = \frac{1}{\nu_{o_n}(\partial T_o)}\, \nu_{o_n} \quad \text{and} \quad J_n(x,y) = \Theta_{o_n}(x,y)\ \big(\nu_{o_n}(\partial T_o)\big)^2 \,.$$

For the following, recall that $T_u = T_{\varpi,u}$, and note that $u \curlywedge o = o_k$ for some $k \ge 0$.

Lemma 7.6 *Let $A \subset \partial^* T$ be compact, so that there is a vertex u such that $A \subset \partial T_u$. If $u \curlywedge o = o_k$ then for all $n \ge k$ and for all $x,y \in \partial T_{o_k}$,*

$$\mu_n(A) = \mu_k(A) =: \mu(A)\,, \quad \textit{and} \quad J_n(x,y) = J_k(x,y) =: J(x,y).$$

We have

$$J(x,y) = \mathsf{j}(x \curlywedge y) \quad \textit{with} \quad \mathsf{j}(v) = \frac{\vartheta^2}{K(v,\varpi)^2}\, \frac{G(v,v)}{\mathsf{m}(v)}\,,\ v \in T\,,$$

where

$$\vartheta = \frac{\mathsf{m}(o)\nu_o(\partial T_o)}{G(o,o)}\,, \quad \textit{and} \quad K(v,\varpi) = \frac{F(v, v \curlywedge o)}{F(o, v \curlywedge o)} = \frac{F(v, v \wedge_o \varpi)}{F(o, v \curlywedge_o \varpi)}$$

is the Martin kernel at ϖ.

Proof. Since ∂T_{o_k} contains both ∂T_o and A, we have for $n \geq k$

$$\mu_n(A) = \frac{\nu_{o_n}(A)}{\nu_{o_n}(\partial T_o)} = \frac{F(o_n, o_k)\nu_{o_k}(A)}{F(o_n, o_k)\nu_{o_k}(\partial T_o)} = \mu_k(A).$$

Analogously, Let $x, y \in \partial T_{o_k}$ and $x \curlywedge y = v$, an element of T_{o_k}. We use the identity $\mathsf{m}(v)G(v,w) = \mathsf{m}(w)G(w,v)$, which implies $\mathsf{m}(o_n)F(o_n,o) = \mathsf{m}(o_n)G(o_n,o)/G(o,o) = \mathsf{m}(o)G(o,o_n)/G(o,o)$, and compute for $n \geq k$

$$\begin{aligned} J_n(x,y) &= \frac{\nu_{o_n}(\partial T_o)^2\,\mathsf{m}(o_n)}{F(o_n,v)G(v,o_n)} = \nu_o(\partial T_o)^2\,\frac{\mathsf{m}(o_n)^2 F(o_n,o)^2}{\mathsf{m}(o_n)F(o_n,v)G(v,o_n)} \\ &= \frac{\nu_o(\partial T_o)^2 \mathsf{m}(o)^2}{G(o,o)^2}\,\frac{G(o,o_n)^2}{G(v,o_n)^2}\,\frac{G(v,v)}{\mathsf{m}(v)}\,, \end{aligned}$$

which yields the proposed formula, since $G(o,o_n) = F(o, o \curlywedge v)G(o \curlywedge v, o_n)$ and $G(v,o_n) = F(v, o \curlywedge v)G(o \curlywedge v, o_n)$. ■

Now its is not hard to deduce the following.

Theorem 7.7 *Let T and its reference end ϖ be as outlined above. Consider a nearest neighbour random walk on T that satisfies $F(v,v^-) < 1$ for every $v \in T$. Let μ and J be as in Lemma 7.6. Then for all compactly supported continuous functions φ, ψ on $\partial^* T$, the Dirichlet form (6.16) can be written as*

$$\begin{aligned} \mathcal{E}_{\mathcal{HD}}(\varphi,\psi) &= \mathcal{E}_J(\varphi,\psi) + \vartheta \cdot \nu_o(\{\varpi\}) \int_{\partial^* T} \varphi(x)\psi(x)\,d\mu(x)\,, \quad \textit{where} \\ \mathcal{E}_J(\varphi,\psi) &= \frac{1}{2}\int_{\partial^* T}\int_{\partial^* T} (\varphi(x)-\varphi(y))\,(\psi(x)-\psi(y))\; J(x,y)\,d\mu(x)\,d\mu(y)\,. \end{aligned}$$

When the random walk is regular, that is, $\nu_o(\{\varpi\}) = 0$, the form $\mathcal{E}_J = \mathcal{E}_{\mathcal{HD}}$ induces the standard (μ,ϕ)-process, where the ultra-metric element ϕ with respect to ϖ is given by

$$\phi(v) = \frac{1}{\vartheta}\,K(v,\varpi)\,,$$

and ϑ and the Martin kernel $K(v,\varpi)$ are as defined in Lemma 7.6.

In particular, the (μ,ϕ)-process is the boundary process with a time-change.

Proof. There is k such that the compact supports of φ and ψ are contained in ∂T_{o_k}. Let $n \geq k$. Using lemmas 6.5 and 7.6,

$$\begin{aligned} \mathcal{E}_{\mathcal{HD}}(\varphi,\psi) &= \frac{1}{2}\int_{\partial T}\int_{\partial T} (\varphi(x)-\varphi(y))\,(\varphi(x)-\varphi(y))\; J_n(x,y)\,d\mu_n(x)\,d\mu_n(y) \\ &= \frac{1}{2}\int_{\partial T_{o_n}}\int_{\partial T_{o_n}} (\varphi(x)-\varphi(y))\,(\varphi(x)-\varphi(y))\; J(x,y)\,d\mu(x)\,d\mu(y) \\ &\quad + \int_{\partial T_{o_n}} \varphi(x)\psi(x) \underbrace{\int_{\partial T \setminus \partial T_{o_n}} J_n(x,y)\,d\mu_n(y)}_{=: f_n(x)}\,d\mu(x) \end{aligned}$$

As $n \to \infty$,

$$\frac{1}{2}\int_{\partial T_{o_n}}\int_{\partial T_{o_n}} (\varphi(x)-\varphi(y))\,(\varphi(x)-\varphi(y))\; J(x,y)\,d\mu(x)\,d\mu(y) \to \mathcal{E}_J(\varphi,\psi)\,.$$

Let us look at the second term. We have

$$f_n(x) = \int_{\partial T \setminus \partial T_{o_n}} \Theta_{o_n}(x,y)\, \nu_{o_n}(\partial T_o)\, d\nu_{o_n}(y).$$

For $x \in \partial T_{o_n}$ and $y \in \partial T \setminus \partial T_{o_n}$, their confluent with respect to o_n is o_n itself. Therefore, using (6.17) and (6.11)

$$\begin{aligned} \Theta_{o_n}(x,y)\, \nu_{o_n}(\partial T_o) &= \frac{\mathsf{m}(o_n)}{G(o_n,o_n)} F(o_n,o)\, \nu_o(\partial T_o) = \frac{\mathsf{m}(o_n) G(o_n,o)}{G(o_n,o_n) G(o,o)} \nu_o(\partial T_o) \\ &= \frac{\mathsf{m}(o) G(o,o_n)}{G(o_n,o_n) G(o,o)} \nu_o(\partial T_o) = \frac{\mathsf{m}(o)}{G(o,o)} \nu_o(\partial T_o)\, F(o,o_n) = \vartheta\, F(o,o_n)\,. \end{aligned}$$

Now note that for $y \in \partial T \setminus \partial T_{o_n}$, we have $F(o,o_n)\, d\nu_{o_n}(y) = d\nu_o(y)$. Therefore

$$f_n(x) = \vartheta \int_{\partial T \setminus \partial T_{o_n}} F(o,o_n)\, d\nu_{o_n}(y) = \vartheta \cdot \nu_o(\partial T \setminus \partial T_{o_n}) \to \vartheta \cdot \nu_o(\{\varpi\})\,,$$

and as $n \to \infty$, we can use dominated convergence to get that

$$\begin{aligned} \int_{\partial T_{o_n}} \varphi(x)\psi(x) \int_{\partial T \setminus \partial T_{o_n}} J_n(x,y)\, d\mu_n(y)\, d\mu(x) &= \int_{\partial^* T} \varphi(x)\psi(x)\, f_n(x)\, d\mu(x) \\ &\to \vartheta \cdot \nu_o(\{\varpi\}) \int_{\partial^* T} \varphi(x)\psi(x)\, d\mu(x)\,, \end{aligned}$$

as proposed. To prove the formula for the associated ultra-metric element, we proceed as in the proof of Theorem 7.1, see (7.4) and the subsequent lines. We find that the ultra-metric element must satisfy

$$\frac{1}{\phi(v)} - \frac{1}{\phi(v^-)} = \big(\mathfrak{j}(v) - \mathfrak{j}(v^-)\big)\, \mu(\partial T_v)\,.$$

The right hand side of this equation can be computed: we have $v^- \curlywedge o = o_k$ for some $k \geq 0$, and combining the arguments after (7.4) with those of the proof of Lemma 7.6,

$$\begin{aligned} \big(\mathfrak{j}(v) - \mathfrak{j}(v^-)\big)\, \mu(\partial T_v) &= \left(\frac{\mathsf{m}(o_k)}{F(o_k,v) G(v,o_k)} - \frac{\mathsf{m}(o_k)}{F(o_k,v^-) G(v^-,o_k)} \right) \nu_{o_k}(\partial T_v)\, \nu_{o_k}(\partial T_o) \\ &= \left(\frac{\mathsf{m}(o_k)}{G(v,o_k)} - \frac{\mathsf{m}(o_k)}{G(v^-,o_k)} \right) F(o_k,o)\, \nu_o(\partial T_o) \\ &= \left(\frac{G(o,o_k)}{G(v,o_k)} - \frac{G(o,o_k)}{G(v^-,o_k)} \right) \frac{\mathsf{m}(o) \nu_o(\partial T_o)}{G(o,o)} = \frac{\vartheta}{K(v,\varpi)} - \frac{\vartheta}{K(v^-,\varpi)} \end{aligned}$$

We infer that $1/\phi(\cdot) - \vartheta/K(\cdot,\varpi)$ must be constant. By the regularity of the random walk, $K(v,\varpi) \to \infty$ as $v \to \varpi$. On the other hand, also $\phi(o_n)$ must tend to infinity. Thus, the constant is 0, and ϕ has the proposed form. ∎

Lemma 7.6 and Theorem 7.7 lead to clearer insight and simpler proofs concerning the material on random walks in [37, §10 – §11], in particular [37, Theorem 11.3]. Namely, our limit measure μ coincides with the ν_* of [37]. Note, that there are examples where $\mu(\partial^* T) = \infty$, as well as examples where $\mu(\partial^* T) < \infty$, even though the ultra-metric space is non-compact.

Remark 7.8 In Sections 6 – 7, we have always assumed that the ultra-metric space has no isolated points, which for the tree means that $\deg^+ \geq 2$. Theme of [7] is the opposite situation, where all points are isolated, i.e., the space is discrete. In that case the ultra-metric space is also the boundary of a tree, which does not consist of ends, but of terminal vertices, that is, vertices with only one neighbour.

From the point of view of the present section, the mixed situation works equally well. If we start with a locally compact ultra-metric space having both isolated and non-isolated points, we can construct the tree in the same way. The vertex set is the collection of all closed balls. The isolated points will then become *terminal vertices* of the tree, which have no neighbour besides the predecessor, as for example the vertices x and y in Figure 6. All interior (non terminal) vertices will have forward degree ≥ 2.

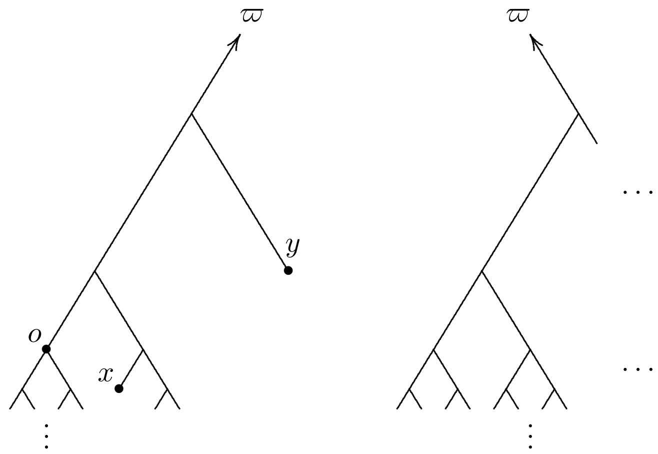

Figure 6

In the compact case, the boundary ∂T of that tree consists of the terminal vertices together with the space of ends. In the non-compact case, we will again have a reference end ϖ as above, and $\partial^* T$ consists of all ends except ϖ, plus the terminal vertices. The definition of an ultra-metric element remains the same, but we only need to define it on interior vertices. In this general setting, the construction of (ϕ, μ, σ)-processes remains unchanged.

Even in presence of isolated points, the duality between (ϕ, μ, σ)-processes and random walks on the associated tree remains as explained here. The random walk should then be such that the terminal vertices are absorbing, and that the Green kernel tends to 0 at infinity. The Doob-Naïm formula extends readily to that setting.

Remark 7.9 Let us again consider the general situation when we start with a transient random walk on a locally finite, rooted tree T.

The limit distribution ν_o will in general not be supported by the whole of ∂T. The boundary process can of course still be constructed, see [36], but will evolve naturally on $\text{supp}(\nu_o)$ only. Thus, we can consider our ultra-metric space to be just $\text{supp}(\nu_o)$. The tree associated with this ultra-metric space will in general not be the tree we started with, nor its *transient skeleton* as defined in [63, (9.27)] (the subtree induced by o and all $v \in T \setminus \{o\}$ with $F(v, v^-) < 1$, where $v^- = v_o^-$).

The reasons are twofold. First, the construction of the tree associated with $\text{supp}(\nu_o)$ will never give back vertices with forward degree 1. Second, some end contained in $\text{supp}(\nu_o)$ may be isolated within that set, while not being isolated in ∂T. But then this element will become a terminal vertex in the tree associated with the ultra-metric (sub)space $\text{supp}(\nu_o)$. This occurs precisely when the transient skeleton has isolated ends.

Thus, one should work with a modified "reduced" tree plus random walk in order to maintain the duality between random walks and isotropic jump processes. The same observations apply to the non-compact case, with a reference end in the place of the root and the measure μ of Lemma 7.6 in the place of ν_o.

Remark 7.10 Given a transient random walk on the rooted tree T, [36] also recovers an *intrinsic metric* of the boundary process on ∂T (compact case !) in terms of what is called an ultra-metric element in the present paper. This is of course $\phi(x) = G(x, o)$, denoted D_x in [36], where it is shown that for ν_o-almost every $\xi \in \partial T$, $D_x \to 0$ along the geodesic ray $\pi(o, \xi)$. This has the following potential theoretic interpretation.

A point $x \in \partial T$ is called *regular for the Dirichlet problem,* if for every $\varphi \in C(\partial T)$, its Poisson transform h_φ satisfies

$$\lim_{v \to x} h_\varphi(v) = \varphi(x).$$

It is known from Cartwright, Soardi and Woess [14, Remark 2] that x is regular if and only if $\lim_{u \to x} G(u, o) = 0$ (as long as T has at least 2 ends), see also [63, Theorem 9.43]. By the latter theorem, the set of regular points has ν_o-measure 1. That is, the Green kernel vanishes at ν_o-almost every boundary point.

Remark 7.11 In the proof of Theorem 7.5, we have reconstructed random walk transition probabilities from $C \cdot \phi(u) = G(u, o)$ and $\mu = \nu_o$.

A similar (a bit simpler) question was addressed by Vondraček [60]: how to reconstruct the transition probabilities from *all* limit distributions ν_u, $u \in T$, on the boundary. The method of [60] as well as our method basically come from (6.11) and (7.7)-(7.8), which can be traced back to Cartier [12].

8 Random walk associated with p-adic fractional derivative

In this section we consider a two-fold specific example which unites the approaches of Section 5 and Sections 6–7. We start with the compact case.

8.1 The p-adic fractional derivative on $\mathbb{Z}_p$

Let $\mathbb{Z}_p \subset \mathbb{Q}_p$ be the group of p-adic integers. As a counterpart of the operator $\mathfrak{D}^\alpha$ we introduce the operator $\mathbb{D}^\alpha$ of fractional derivative on $\mathbb{Z}_p$. We show that it is the Laplacian of an appropriate isotropic Markov semigroup. Then we construct a random walk associated with $\mathbb{D}^\alpha$ in the sense of Sections 6–7.

Since $\mathbb{Z}_p$ is a compact Abelian group, its dual $\widehat{\mathbb{Z}_p}$ is a discrete Abelian group. It is known that the group $\widehat{\mathbb{Z}_p}$ can be identified with the group

$$Z(p^\infty) = \{p^{-n}m : 0 \le m < p^n, n = 1, 2, ...\}$$

equipped with addition of numbers mod 1 as the group operation. As sets (but not as groups) $Z(p^\infty) \subset \mathbb{Q}_p$, whence the function $\xi \mapsto \|\xi\|_p$ is well-defined on the group $Z(p^\infty)$.

Definition 8.1 The operator $(\mathbb{D}^\alpha, \mathcal{V}_c)$, $\alpha > 0$, is defined via the Fourier transform on the compact Abelian group $\mathbb{Z}_p$ by

$$\widehat{\mathbb{D}^\alpha f}(\xi) = \|\xi\|_p^\alpha \, \widehat{f}(\xi), \ \xi \in Z(p^\infty),$$

where $\mathcal{V}_c$ is the space of locally constant functions on $\mathbb{Z}_p$.

Compare with the Definition 5.1 of the operator $\mathfrak{D}^\alpha$.

An immediate consequence is that the operator $\mathbb{D}^\alpha$ is a non-negative definite self-adjoint operator whose spectrum coincides with the range of the function

$$\xi \mapsto \|\xi\|_p^\alpha : Z(p^\infty) \to \mathbb{R}_+,$$

that is,

$$\operatorname{spec}\mathbb{D}^{\alpha} = \{0, p^{\alpha}, p^{2\alpha}, ...\}.$$

The eigenspace $\mathcal{H}(\lambda)$ of the operator $\mathbb{D}^{\alpha}$ corresponding to the eigenvalue $\lambda = p^{k\alpha}$, $k \geq 1$, is spanned by the function

$$f_k = \frac{1}{\mu_p(p^k\mathbb{Z}_p)}\mathbf{1}_{p^k\mathbb{Z}_p} - \frac{1}{\mu_p(p^{k-1}\mathbb{Z}_p)}\mathbf{1}_{p^{k-1}\mathbb{Z}_p}$$

and its shifts $f_k(\cdot + a)$ with any $a \in \mathbb{Z}_p/p^k\mathbb{Z}_p$.

Indeed, computing the Fourier transform of the function f_k,

$$\widehat{f_k}(\xi) = \mathbf{1}_{\{\|\xi\|_p \leq p^k\}} - \mathbf{1}_{\{\|\xi\|_p \leq p^{k-1}\}} = \mathbf{1}_{\{\|\xi\|_p = p^k\}},$$

we obtain

$$\widehat{\mathbb{D}^{\alpha} f_k}(\xi) = \|\xi\|_p^{\alpha}\, \widehat{f_k}(\xi) = p^{k\alpha}\widehat{f_k}(\xi).$$

The maximal number of linearly independent functions in the set $\{f_k(\cdot + a) : a \in \mathbb{Z}_p/p^k\mathbb{Z}_p\}$ is $p^{k-1}(\,p-1)$, whence

$$\dim \mathcal{H}(\lambda) = p^{k-1}(p-1).$$

All the above shows that $\mathbb{D}^{\alpha}$ coincides with the Laplacian of some isotropic Markov semigroup $(\mathbb{P}^t_{\alpha})_{t>0}$ on the ultra-metric measure space $(\mathbb{Z}_p, d_p, \mu_p)$. In particular, using the complete description of $\operatorname{spec}\mathbb{D}^{\alpha}$ we compute the intrinsic distance, call it $d_{p,\alpha}(x,y)$,

$$d_{p,\alpha}(x,y) = \left(\frac{\|x-y\|_p}{p}\right)^{\alpha}.$$

It is now straightforward to compute the spectral distribution function $\mathbb{N}_{\alpha}(x,\tau) \equiv \mathbb{N}_{\alpha}(\tau)$ and then the jump-kernel $\mathbb{J}_{\alpha}(x,y) \equiv \mathbb{J}_{\alpha}(x-y)$ of the operator $\mathbb{D}^{\alpha}$. We claim that

$$\mathbb{J}_{\alpha}(x,y) = \frac{p^{\alpha}-1}{1-p^{-\alpha-1}}\left(\frac{p^{-\alpha}-p^{-\alpha-1}}{1-p^{-\alpha}} + \frac{1}{\|x-y\|_p^{1+\alpha}}\right). \tag{8.1}$$

Recall for comparison that according to (5.5) the jump-kernel $J_{\alpha}(x,y)$ of the operator $\mathfrak{D}^{\alpha}$ is given by

$$J_{\alpha}(x,y) = \frac{p^{\alpha}-1}{1-p^{-\alpha-1}}\frac{1}{\|x-y\|_p^{1+\alpha}}.$$

To prove (8.1), we compute $\mathbb{J}_{\alpha}(z)$. Let $\|z\|_p = p^{-l}$, then $d_{p,\alpha}(0,z) = p^{-(l+1)\alpha}$ and

$$\mathbb{J}_{\alpha}(z) = \int_0^{1/d_{p,\alpha}(0,z)} \mathbb{N}_{\alpha}(\tau)d\tau = \int_0^{p^{(l+1)\alpha}} \mathbb{N}_{\alpha}(\tau)d\tau.$$

The function $\mathbb{N}_{\alpha}(\tau)$ is a non-decreasing, left-continuous staircase function having jumps at the points $\tau_k = p^{k\alpha}$, $k = 1, 2, ...$, and taking values at these points $\mathbb{N}_{\alpha}(\tau_k) = p^{k-1}$, whence

$$\begin{aligned}
\mathbb{J}_{\alpha}(z) &= 1\cdot p^{\alpha} + p(p^{2\alpha}-p^{\alpha}) + p^2(p^{3\alpha}-p^{2\alpha}) + ... + p^l(p^{(l+1)\alpha}-p^{l\alpha}) \\
&= \frac{1-p^{-1}}{1-p^{-\alpha-1}} + \frac{p^{\alpha}-1}{1-p^{-\alpha-1}}p^{l(\alpha+1)} \\
&= \frac{p^{\alpha}-1}{1-p^{-\alpha-1}}\left(\frac{p^{-\alpha}-p^{-\alpha-1}}{1-p^{-\alpha}} + \frac{1}{\|z\|_p^{1+\alpha}}\right)
\end{aligned}$$

as desired. Next, we apply Theorem 3.12 and obtain

$$\mathbb{D}^{\alpha} f(x)=\int_{\mathbb{Z}_p}(f(x)-f(y))\,\mathbb{J}_{\alpha}(x-y)\,d\mu_p(y). \tag{8.2}$$

The equations (8.1)–(8.2) and (5.1) now yield the following result.

Corollary 8.2 *For any function f defined on $\mathbb{Z}_p \subset \mathbb{Q}_p$ we set $\widetilde{f}=f$ on $\mathbb{Z}_p$ and 0, otherwise. Then*

$$f\in \operatorname{dom}(\mathbb{D}^{\alpha}) \Longrightarrow \widetilde{f}\in \ \operatorname{dom}(\mathfrak{D}^{\alpha}),$$

$$\mathbb{D}^{\alpha} f(x)=\mathfrak{D}^{\alpha}\widetilde{f}(x) \quad \text{and} \quad (\mathbb{D}^{\alpha} f,f)=\left(\mathfrak{D}^{\alpha}\widetilde{f}\ ,\widetilde{f}\ \right) \tag{8.3}$$

whenever $x\in\mathbb{Z}_p$, $f\in\operatorname{dom}(\mathbb{D}^{\alpha})$ and $(\mathbf{1},f)=0$.

8.2 Nearest neighbour random walk on the rooted tree $\mathbb{T}_p^o$

As an illustration of Theorem 7.5 we construct a random walk on the rooted tree associated with $\mathbb{Z}_p$ whose boundary process coincides with the isotropic process driven by the operator $C\cdot\mathbb{D}^{\alpha}$, where $C=p^{-\alpha}(1-p^{-\alpha})$.

The Abelian group $\mathbb{Z}_p$ can be identified with the boundary of the tree $\mathbb{T}_p^o$ with root o where every vertex v has p forward neighbours. In our identification, this is the tree of balls of the ultra-metric space $(\mathbb{Z}_p,d_p)$ with root o corresponding to the whole of $\mathbb{Z}_p$ and the ultra-metric $d_p(x,y)=\|x-y\|_p$. See Figure 4 above, where $p=2$. We fix a constant $c\in(0,1)$ and consider the nearest neighbour random walk on $\mathbb{T}_p^o$ with

$$p(v,v^-)=1-c \quad \text{and} \quad p(v^-,v)=\begin{cases} 1/p & \text{if } v^-=o \\ c/p & \text{otherwise}\end{cases}. \tag{8.4}$$

Using [63, Thm. 1.38 and Prop. 9.3] one can compute precisely the Green function $G(v,o)$, the hitting probability $F(v,o)$ and other quantities associated with our random walk. In particular, choosing $c=(1+p^{-\alpha})^{-1}$, we obtain

$$F(v,o)=p^{-\alpha|v|} \quad \text{and} \quad G(v,o)=\frac{p^{-\alpha|v|}}{1-p^{-\alpha}}, \tag{8.5}$$

where $|v|$ is the graph distance from v to o. We see that the Green function vanishes at infinity, whence the random walk is Dirichlet regular.

The transition probabilities are invariant with respect to all automorphisms of the tree. Every such automorphism must fix o and every level of the tree. Let $\nu=\nu_o$ be the limit distribution on $\partial\mathbb{T}_p^o$ of the random walk starting at o. Then also ν is invariant under the automorphism group of the tree (whose action extends to the boundary). In particular it is invariant under the action of $\mathbb{Z}_p$. Thus, under the identification of $\partial\mathbb{T}_p^o$ with $\mathbb{Z}_p$, we have that $\nu=\mu_p$, the normalized Haar measure of $\mathbb{Z}_p$.

We now look at the boundary process induced by our random walk as a jump process on $\mathbb{Z}_p$. By Theorem 7.1, the boundary process arises as an isotropic jump process with the reference measure μ_p. Let $\mathcal{L}$ be its Laplacian. By Corollary 7.2, the set $\operatorname{spec}\mathcal{L}$ coincides with the range of the function $v\mapsto 1/G(v,o)$, $v\in\mathbb{T}_p^o$, together with $\{0\}$. In view of the above formula for $G(v,o)$ this means that

$$\operatorname{spec}\mathcal{L}=\{0,(1-p^{-\alpha}),p^{\alpha}(1-p^{-\alpha}),p^{2\alpha}(1-p^{-\alpha}),\dots\}.$$

Remember that

$$\operatorname{spec}\mathbb{D}^{\alpha}=\{0,p^{\alpha},p^{2\alpha},\dots\}=\frac{p^{\alpha}}{1-p^{-\alpha}}\ \operatorname{spec}\mathcal{L}.$$

Since both $\mathbb{D}^\alpha$ and $\mathcal{L}$ have the same orthonormal basis of eigenfunctions, we conclude that they are proportional, that is,

$$\mathbb{D}^\alpha = \frac{p^\alpha}{1-p^{-\alpha}}\ \mathcal{L}. \tag{8.6}$$

Thus, finally we come to the following conclusion

Proposition 8.3 *The boundary process $\{X_t\}_{t>0}$ associated with the random walk defined in (8.4) with parameter $c = (1+p^{-\alpha})^{-1}$ and the isotropic jump process $\{X_t^\alpha\}_{t>0}$ driven by the operator $\mathbb{D}^\alpha$ are related by the linear time change $X_{t/C} = X_t^\alpha$, where $C = p^{-\alpha}(1-p^{-\alpha})$.*

The equation (8.6) implies that the jump kernel $\mathbb{J}_\alpha(x,y)$ of operator $\mathbb{D}^\alpha$ and the Doob-Naïm kernel $\Theta_o(x,y)$ of operator $\mathcal{L}$ are related by

$$\mathbb{J}_\alpha(x,y) = \frac{p^\alpha}{1-p^{-\alpha}}\ \Theta_o(x,y). \tag{8.7}$$

We now show how to compute the Doob-Naïm kernel

$$\Theta_o(x,y) = \frac{1}{G(o,o)F(o,v)F(v,o)}, \quad \text{where } v = x\wedge y,$$

directly, using the data of (8.5). We do not yet have $F(o,v)$. We shall compute

$$N(v) = \frac{1}{F(o,v)F(v,o)}.$$

Since it depends only on the level k of v, we consider an arbitrary geodesic ray $[o = v_0, v_1, ...]$ and set up a linear recursion for $N(v_k)$. Denoting by w_1 an arbitrary neighbour of o different from v_1 and applying [63, Prop. 9.3(b)] and (8.5), we obtain

$$F(o,v_1) = \frac{1}{p} + \frac{p-1}{p}F(w_1,o)F(o,v_1) = \frac{1}{p} + \frac{(p-1)p^{-\alpha}}{p}F(o,v_1),$$

whence

$$F(o,v_1) = \frac{p^\alpha}{p^{\alpha+1}-p+1}.$$

Thus, we get the initial values

$$N(v_0) = 1 \quad \text{and} \quad N(v_1) = p^{\alpha+1} - p + 1.$$

Next, for $k \geq 1$, we let w_{k+1} be a forward neighbour of v_k different from v_{k+1}. Applying once again [63, Prop. 9.3(b)] and (8.5), we obtain

$$\begin{aligned} F(v_k,v_{k+1}) \quad &= \quad \frac{p^\alpha}{p(p^\alpha+1)} + \frac{(p-1)p^\alpha}{p(p^\alpha+1)}F(w_{k+1},v_k)F(v_k,v_{k+1}) \\ &\quad + \frac{1}{p^\alpha+1}F(v_{k-1},v_k)F(v_k,v_{k+1}). \end{aligned}$$

We insert the value $F(w_{k+1},v_k) = p^{-\alpha}$ and divide by

$$F(o,v_{k+1}) = F(o,v_k)F(v_k,v_{k+1}) = F(o,v_{k-1})F(v_{k-1},v_k)F(v_k,v_{k+1}).$$

Then we get

$$\frac{1}{F(o,v_k)} = \frac{p^\alpha}{p(p^\alpha+1)}\frac{1}{F(o,v_{k+1})} + \frac{p-1}{p(p^\alpha+1)}\frac{1}{F(o,v_k)} + \frac{1}{p^\alpha+1}\frac{1}{F(o,v_{k-1})}.$$

Now we multiply both sides with $1/F(v_k,o) = p^{\alpha k}$ and get

$$N(v_k) = \frac{1}{p(p^\alpha+1)} N(v_{k+1}) + \frac{p-1}{p(p^\alpha+1)} N(v_k) + \frac{p^\alpha}{p^\alpha+1} N(v_{k-1}).$$

This is a homogeneous second order linear recursion with constant coefficients. Its characteristic polynomial has roots 1 and $p^{\alpha+1}$. Therefore

$$N(v_k) = A + Bp^{(\alpha+1)k}.$$

Inserting the initial values we easily find the values of A and B. In order to get the Naïm kernel, we have to multiply by $1/G(o,o) = 1 - p^{-\alpha}$. Thus, we get

$$\Theta_o(x,y) = \frac{(1-p^{-\alpha})(p-1)}{p^{\alpha+1}-1} + \frac{(1-p^{-\alpha})(p^{\alpha+1}-p)}{p^{\alpha+1}-1} p^{(\alpha+1)k} = \frac{1-p^{-\alpha}}{p^\alpha} \mathbb{J}_\alpha(x,y)\,,$$

as desired.

8.3 The random walk corresponding to $\mathfrak{D}^\alpha$ on $\mathbb{Q}_p$

Now we construct the random walk corresponding to the fractional derivative on the whole of $\mathbb{Q}_p$ using Lemma 7.6 and Theorem 7.7 concerning the duality between isotropic processes and random walks in the non-compact case.

The tree associated with $\mathbb{Q}_p$ is the homogenous tree $T = \mathbb{T}_p$ with degree $p+1$. We have to choose a reference end ϖ. Then we can identify its lower boundary $\partial^*\mathbb{T}_p$ with the field of p-adic numbers. With respect to ϖ, every vertex v has its predecessor v^- and p successors. Every subtree $T_v = T_{\varpi,v}$ is isomorphic with the rooted tree $\mathbb{T}_p^o$ considered above in the compact case of the p-adic integers. In particular, we choose the root vertex o such that $\partial T_o = \mathbb{Z}_p$. See Figure 5 above, where $p=2$.

We now define the random walk on $\mathbb{T}_p$ as in (8.4), but with predecessors referring to ϖ:

$$p(v,v^-) = 1-c \quad \text{and} \quad p(v^-,v) = c/p\,, \quad \text{where} \quad c = (1+p^{-\alpha})^{-1}. \tag{8.8}$$

For the following quantities, see e.g. [62, pp. 423-424]. For all $v \in \mathbb{T}_p$,

$$F(v,v^-) = p^{-\alpha}\,, \quad F(v^-,v) = p^{-1}\,, \quad G(v,v) = \frac{1+p^{-\alpha}}{1-p^{-\alpha-1}} \quad \text{and} \quad \nu_v(\partial T_v) = \frac{1-p^{-\alpha}}{1-p^{-\alpha-1}}.$$

This yields that the reference measure μ of the boundary process with respect to ϖ, as given by Lemma 7.6, is the standard Haar measure of $\mathbb{Q}_p$.

We compute $\vartheta = (1-p^{-\alpha})/(1+p^{-\alpha})$. Furthermore, let us set $\mathfrak{h}(v) = d(v, v \curlywedge o) - d(o, v \curlywedge o)$ (where d is the graph metric). This is the *horocycle number* of v. That is, the vertices with $\mathfrak{h}(v) = k$, $k \in \mathbb{Z}$, are the elements in the k-th generation H_k of the tree (see Figure 5), and ∂T_v corresponds to a ball with radius p^{-k} in the standard ultra-metric of $\mathbb{Q}_p$. Then

$$K(v,\varpi) = p^{\alpha\,\mathfrak{h}(v)} \quad \text{and} \quad \mathsf{m}(v) = p^{(\alpha-1)\mathfrak{h}(v)}.$$

Putting things together, we get

$$\phi(v) = \frac{1+p^{-\alpha}}{1-p^{-\alpha}}\, p^{-\alpha\,\mathfrak{h}(v)} \quad \text{and} \quad \mathsf{j}(v) = \frac{(1-p^{-\alpha})^2}{(1+p^{-\alpha})(1-p^{-\alpha-1})}\, p^{(\alpha+1)\,\mathfrak{h}(v)}$$

Retranslating this into p-adic notation, we conclude that the intrinsic metric and jump kernel of the boundary process with respect to ϖ are given by

$$\begin{aligned} d_\phi(x,y) &= \frac{1-p^{-\alpha}}{1+p^{-\alpha}}\, \|x-y\|_p^\alpha \quad \text{and} \\ J(x,y) &= \frac{(1-p^{-\alpha})^2}{(1+p^{-\alpha})(1-p^{-\alpha-1})}\, \frac{1}{\|x-y\|_p^{\alpha+1}} = \frac{1-p^{-\alpha}}{p^\alpha+1}\, J_\alpha(x,y)\,, \end{aligned}$$

where J_α is the jump kernel associated with $\mathfrak{D}^\alpha$. So at last, we get the following.

Proposition 8.4 *The boundary process $\{X_t\}_{t>0}$ with respect to the reference end ϖ associated with the random walk (8.8) on $\mathbb{T}_p$ and the isotropic jump process $\{X_t^\alpha\}_{t>0}$ driven by the operator $\mathfrak{D}^\alpha$ on $\mathbb{Q}_p$ are related by the linear time change $X_{t/C^*} = X_t^\alpha$, where $C^* = (1 - p^{-\alpha})/(p^\alpha + 1)$.*

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