Analysis on ultra-metric spaces and heat kernels

Alexander Grigor'yan
University of Bielefeld and Nankai University

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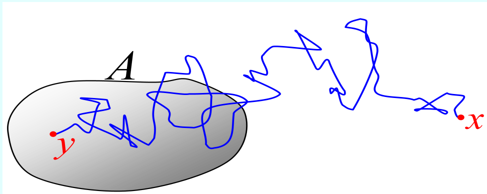
Heat kernels in $\mathbb{R}^n$

The heat equation $\frac{\partial u}{\partial t} = \Delta u$ in $\mathbb{R}^n$ has the following fundamental solution

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) \tag{1}$$

also called the *heat kernel*. This function (also known as the Gauss-Weierstrass function) coincides with the transition density for Brownian motion $\{X_t\}_{t\geq 0}$ in $\mathbb{R}^n$: for any Borel set $A \subset \mathbb{R}^n$,

$$\mathbb{P}_x(X_t \in A) = \int_A p_t(x,y) d\mu(y)$$



The operator $-\Delta$ can be extended to a self-adjoint non-negative definite operator in $L^2(\mathbb{R}^n)$, which allows to define the *heat semigroup* $\left\{e^{t\Delta}\right\}_{t\geq 0}$. Then the operator $e^{t\Delta}$ is an integral operator with the integral kernel $p_t(x,y)$.

For any $\beta \in (0,2)$ the operator $(-\Delta)^{\beta/2}$ is also a self-adjoint non-negative definite operator, and the associated the heat semigroup $\left\{e^{-t(-\Delta)^{\beta/2}}\right\}_{t\geq 0}$ also has the integral kernel $p_t^{(\beta)}(x,y)$ that is the transition density of a *symmetric stable Levy process* of index β that is a Markov jump process.

It is known that in the case $\beta = 1$

$$p_t^{(1)}(x,y) = \frac{c_n t}{\left(t^2 + |x-y|^2\right)^{\frac{n+1}{2}}} \tag{2}$$

with some $c_n > 0$, which is the Cauchy distribution with the parameter t.

For any $\beta \in (0,2)$, the heat kernel of $-(-\Delta)^{\beta/2}$ satisfies the estimate

$$p_t^{(\beta)}(x,y) \simeq \frac{t}{\left(t^{1/\beta} + |x-y|\right)^{n+\beta}} = \frac{1}{t^{n/\beta}} \left(1 + \frac{|x-y|}{t^{1/\beta}}\right)^{-(n+\beta)}. \tag{3}$$

The sign $\simeq$ means that the ratio of two sides is bounded between two positive constants.

The formulas (2) and (3) are obtained from the heat kernel of Δ by using subordination techniques.

The theory of Dirichlet forms of M.Fukushima provides the following method of construction of Markov jump processes. Let (M,d) be a locally compact separable metric space and μ be a Radon measure on M with full support. Fix a non-negative symmetric function $J(x,y)$ on $M\times M$ and consider in $L^2\left(M,\mu\right)$ the following quadratic form

$$\mathcal{E}\left(f,f\right)=\iint_{M\times M}\left(f(x)-f(y)\right)^2 J(x,y)d\mu(x)d\mu(y).$$

Assume that $\mathcal{E}$ extends to a regular Dirichlet form with a domain $\mathcal{F}\subset L^2\left(M,\mu\right)$. Then it has a generator

$$\mathcal{L}f\left(x\right)=\int_M\left(f\left(y\right)-f\left(x\right)\right)J(x,y)d\mu\left(y\right)$$

that is a self-adjoint non-positive definite operator in L^2 with an appropriate domain. The associated heat semigroup $\left\{e^{t\mathcal{L}}\right\}_{t\geq 0}$ determines a jump Markov process $\left\{X_t\right\}_{t\geq 0}$ such that $\mathbb{E}.f(X_t)=e^{t\mathcal{L}}f$.

The *heat kernel* $p_t(x,y)$ of $(\mathcal{E},\mathcal{F})$ is the integral density of the heat semigroup:

$$e^{t\mathcal{L}}f\left(x\right)=\int_M p_t(x,y)f\left(y\right)d\mu\left(y\right).$$

Equivalently, $p_t(x,y)$ is the transition density of the jump process $\left\{X_t\right\}$.

For example, in $\mathbb{R}^n$ the jump process generated by $-(-\Delta)^{\beta/2}$ has the jump kernel

$$J(x,y) = c_{n,\beta} \left| x - y \right|^{-(n+\beta)}.$$

provided $0 < \beta < 2$. If $\beta \geq 2$ then $\mathcal{E}$ with this jump kernel does not extend to a Dirichlet form.

Question: Under what conditions on a metric measure space (M, d, μ) and a jump kernel J, the heat kernel of the associated Dirichlet form exists and satisfies for all $x, y \in M$ and $t > 0$ the *stable-like estimate*

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \tag{4}$$

with some positive parameters α, β? This question is motivated by the following theorem.

Theorem 1 (AG and T.Kumagai 2008). *Assume that the heat kernel of a jump process is conservative and satisfies the estimate*

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \Phi \left(c \frac{d(x,y)}{t^{1/\beta}} \right)$$

for some function Φ *and for all* $x, y \in M$ *and* $t > 0$. *Then it has to be* (4).

The following *necessary* conditions for (4) are known:

- the α-regularity: for any metric ball $B_r(x)$, we have

$$\mu\left(B_r(x)\right) \simeq r^\alpha \tag{V}$$

(consequently, $\alpha = \dim_H M$ and $\mu \simeq \mathcal{H}_\alpha$).

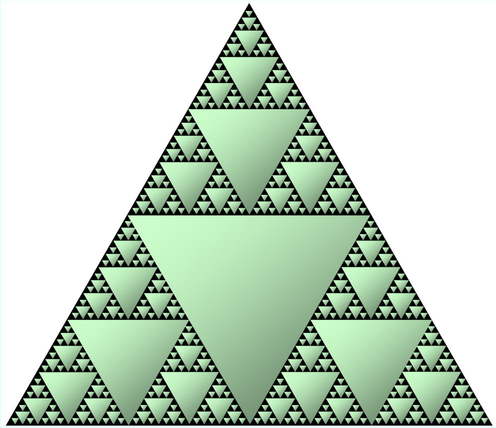
- the jump kernel estimate: for all $x, y \in M$,

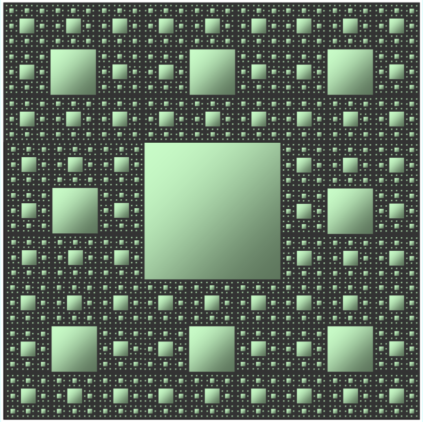
$$J(x,y) \simeq d(x,y)^{-(\alpha+\beta)}. \tag{J}$$

Theorem 2 *(Z.-Q.Chen and T.Kumagai 2003) If* $0 < \beta < 2$ *then*

$$(V)+(J) \Leftrightarrow (4).$$

However, there are many examples of *fractal* spaces where jump kernels (J) generate regular Dirichlet forms even with $\beta > 2$.





It was proved in 1990s by M.Barlow et al., that on a large class of fractals (like unbounded *Sierpinski gasket* and *carpet*) there is a diffusion process whose heat kernel satisfies a *sub-Gaussian* estimate

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta^*}} \exp\left(-c\left(\frac{d^{\beta^*}(x,y)}{t}\right)^{\frac{1}{\beta^*-1}}\right), \tag{5}$$

where $\alpha = \dim_H M$ and β^* is the *walk dimension* that is an invariant of (M,d). If (5) is satisfied on some space then necessarily $\beta^* \geq 2$. Conversely, for any $\alpha > 0$ and $\beta^* \geq 2$ there exists a heat kernel on some space satisfying (5) for these α and β^* (M.Barlow '04).

In $\mathbb{R}^n$ we have $\alpha = n$ and $\beta^* = 2$ but "typically" $\beta^* > 2$! For example, on the Sierpinski gasket $\alpha = \log 3/\log 2$ and $\beta^* = \log 5/\log 2$.

By subordinating such a diffusion, one obtains a jump process with the heat kernel satisfying the stable-like estimate

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \tag{6}$$

and the jump kernel satisfying (J), where the index β can take any value from $(0, \beta^*)$. In particular, β can be larger than 2.

Coming back to a general setting, assume that (V) and (J) are satisfied with $\beta > 2$. Then, in order to obtain the heat kernel estimates (4), one needs on top of (V) and (J) one more quite complicated condition that was established in 2016 independently by

- Z.-Q. Chen, T. Kumagai, J. Wang: condition SCJ (cutoff Sobolev inequality);
- AG, Jiaxin Hu, Eryan Hu: condition $Gcap$ (generalized capacity condition).

A common result of these works: $(V)+(J)+(Gcap) \Leftrightarrow (V)+(J)+(SCJ) \Leftrightarrow (6)$.

The purpose of this talk to discuss question of obtaining heat kernel bounds for jump processes in the setting of *ultra-metric* spaces and, in particular, to show that one can manage *without* the third condition.

Ultra-metric spaces

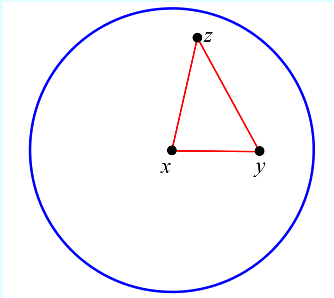
Let (M, d) be a metric space. The metric d is called an *ultra-metric* if it satisfies the ultra-metric inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}. \tag{7}$$

In this case (M, d) is called an ultra-metric space.

Consider the metric balls $B_r(x) = \{y \in M : d(x, y) \leq r\}$. The ultra-metric property (7) implies that *any two metric balls of the same radius are either disjoint or identical.* Indeed, let two balls $B_r(x)$ and $B_r(y)$ have a non-empty intersection, say $z \in B_r(x) \cap B_r(y)$. Then $d(x, z) \leq r$ and $d(y, z) \leq r$ whence it follows $d(x, y) \leq r$.

For any point $z \in B_r(x)$ we have $d(x, z) \leq r$, which together with $d(x, y) \leq r$ implies $d(y, z) \leq r$ so that $z \in B_r(y)$. Therefore, $B_r(x) \subset B_r(y)$ and, similarly, $B_r(y) \subset B_r(x)$ whence $B_r(x) = B_r(y)$.



Consequently, the collection of all distinct balls of the same radius r forms a partition of M.

Another consequence: *every point inside a ball is its center.* Indeed, if $y \in B_r(x)$ then the balls $B_r(y)$ and $B_r(x)$ have a non-empty intersection whence $B_r(x) = B_r(y)$.

Therefore, all balls are closed and open sets, and M is totally disconnected. In particular, an ultra-metric space cannot carry a non-trivial diffusion process.

A well-known example of an ultra-metric space is the field $\mathbb{Q}_p$ of p-adic numbers, where p is a prime. It is defined as the closure of $\mathbb{Q}$ with respect to the p-adic norm: if $x = p^n \frac{a}{b}$, where a, b are integers not divisible by p, then

$$\|x\|_p := p^{-n}.$$

If $x = 0$ then $\|x\|_p := 0$. The p-adic norm satisfies the ultra-metric inequality: if $y = p^m \frac{c}{d}$ and $m \leq n$ then

$$x + y = p^m \left(\frac{p^{n-m}a}{b} + \frac{c}{d} \right)$$

whence

$$\|x + y\|_p \leq p^{-m} = \max \left\{ \|x\|_p , \|y\|_p \right\}.$$

Hence, $\mathbb{Q}$ with the metric $\|x - y\|_p$ is an ultra-metric space, and so is its completion $\mathbb{Q}_p$.

Every p-adic number $x \in \mathbb{Q}_p$ has the form

$$x = \sum_{k=-N}^{\infty} a_k p^k = ...a_k...a_2 a_1 a_0.a_{-1} a_{-2}....a_{-N}$$

where $N \in \mathbb{N}$ and each a_k is a p-adic digit: $a_k \in \{0, 1, ..., p-1\}$. Then $\|x\|_p = p^{-m}$, where

$$m = \min \{k \in \mathbb{Z} : a_k \neq 0\} .$$

It follows that the ball $B_r(x)$ of radius $r = p^{-m}$ (where $m \in \mathbb{Z}$) consists of all numbers

$$y = \sum_{k=-N}^{\infty} b_k p^k = ...b_k...b_2 b_1 b_0.b_{-1}b_{-2}....b_{-N}$$

such that $b_k = a_k$ for $k < m$ and b_k are arbitrary for $k \geq m$; that is,

$$y = ...b_{m+1} b_m a_{m-1} a_{m-2}...$$

Consequently, any ball $B_r(x)$ of radius $r = p^{-m}$ consists of p disjoint balls of radii $p^{-(m+1)}$ that are determined by the value of b_m.

Let μ be the Haar measure on $\mathbb{Q}_p$ with the normalization condition

$$\mu\left(B_1(x)\right) = 1.$$

Then we obtain that

$$\mu\left(B_{p^{-m}}(x)\right) = p^{-m}.$$

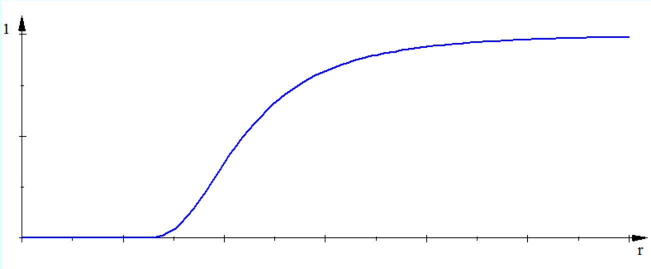
If $p^{-m} \leq r < p^{-(m-1)}$ then $B_r(x) = B_{p^{-m}}(x)$ which implies

$$\mu\left(B_r(x)\right) = p^{-m} \simeq r.$$

Isotropic Dirichlet forms

Let (M, d) be an ultra-metric space where all balls are compact, and μ be a Radon measure on M with full support.

Let $\sigma(r)$ be a cumulative probability distribution function on $(0, \infty)$ that is strictly monotone increasing.



Consider on M a jump kernel

$$J(x, y) = \int_{d(x,y)}^{\infty} \frac{d \log \sigma(r)}{\mu(B_r(x))}. \tag{8}$$

This jump kernel determines a regular Dirichlet form that is referred to as an *isotropic Dirichlet form*. Its heat kernel admits the explicit formula

$$p_t(x, y) = \int_{d(x,y)}^{\infty} \frac{d\sigma^t(r)}{\mu(B_r(x))} \tag{9}$$

(A.Bendikov, AG, Ch.Pittet, W.Woess 2014).

For example, consider $M = \mathbb{Q}_p^n$ with the ultra-metric

$$d(x,y) = \max_{1 \le i \le n} \|x_i - y_i\|_p \,.$$

Let μ be the Haar measure on $\mathbb{Q}_p^n$. If $p^{-m} \le r < p^{-(m-1)}$ then

$$\mu\left(B_r(x)\right) = p^{-nm} \simeq r^n.$$

Fix any $\beta > 0$ and consider the distribution function

$$\sigma\left(r\right) = \exp\left(-\left(\frac{p}{r}\right)^{\beta}\right) \tag{10}$$

(Fréchet distribution). Computing $J(x,y)$ from (8), one obtains

$$J(x,y) = c_{p,n,\beta} d(x,y)^{-(n+\beta)}. \tag{11}$$

We have shown that the generator $\mathcal{L}$ of the Dirichlet form in $\mathbb{Q}_p^n$ with the jump kernel (8) coincides with the *Taibleson operator* $\mathcal{T}^\beta$ defined by Taibleson in '75 using the Fourier transform in $\mathbb{Q}_p^n$.

The Fourier transform for functions f on $\mathbb{Q}_p^n$ is defined by

$$\widehat{f}\left(\xi\right) = \int_{\mathbb{Q}_p^n} e^{2\pi i\langle x,\xi\rangle} f\left(x\right) d\mu\left(x\right),$$

where $\xi \in \mathbb{Q}_p^n$ and

$$\langle x, \xi \rangle = \sum_{k=1}^{n} \{x_k \xi_k\} .$$

Taibleson has defined $\mathcal{T}^\beta$ by means of its Fourier transform:

$$\widehat{\mathcal{T}^\beta f}(\xi) = \|\xi\|_p^\beta \, \widehat{f}(\xi).$$

Substituting (10) to the identity (9), we obtain that the heat kernel of $\mathcal{T}^\beta$ satisfies the estimate

$$p_t(x,y) \simeq \frac{1}{t^{n/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(n+\beta)},$$

that is similar to the stable-like estimate in $\mathbb{R}^n$.

Let us now consider a more general jump kernel on $\mathbb{Q}_p^n$ satisfying for some $\beta > 0$ the estimate

$$J(x,y) \simeq d(x,y)^{-(n+\beta)}.$$

Then the associated Dirichlet form is not necessarily isotropic, and the above method of computing the heat kernel does not work.

In the rest of this talk, we discuss the problem of estimating of the heat kernel under and milder assumptions.

Jump kernels on α-regular ultra-metric spaces

Assume for the rest of the talk that the ultra-metric space (M, d, μ) satisfies the hypothesis (V) for some $\alpha > 0$, that is, $\mu(B_r(x)) \simeq r^\alpha$ for all $x \in M$ and $r > 0$.

The results below are proved by A.Bendikov, AG, Eryan Hu (yet to be published).

Theorem 3 *Let J be a symmetric non-negative function on $M \times M$ such that*

$$J(x, y) \simeq d(x, y)^{-(\alpha+\beta)} \tag{J}$$

for some $\beta > 0$. Then the quadratic form

$$\mathcal{E}(f, f) = \iint_{M \times M} (f(x) - f(y))^2 J(x, y) d\mu(x) d\mu(y)$$

determines a regular Dirichlet form in $L^2(M, \mu)$. Its heat kernel $p_t(x, y)$ exists, is continuous in (t, x, y), Hölder continuous in (x, y) and satisfies the stable-like estimate

$$p_t(x, y) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \tag{12}$$

for all $x, y \in M$ and $t > 0$. Consequently, (V)+$(J)$$\Leftrightarrow$$(12)$.

Next, let us relax the pointwise upper and lower estimates of $J(x,y)$ in (J). We consider two types of conditions instead.

Definition. We say that J satisfies the β-*Poincaré inequality* if, for any ball $B = B_r(x_0)$ and any function $f \in L^2(B, \mu)$,

$$\int_B \left|f - \overline{f}\right|^2 d\mu \le Cr^\beta \iint_{B\times B} (f(x) - f(y))^2 J(x,y) d\mu(x) d\mu(y) \tag{PI}$$

where $\overline{f} = \fint_B f d\mu$ and the constant C is the same for all balls and all functions f.

Definition. We say that J satisfies the β-*tail condition* if, for any ball $B_r(x)$,

$$\int_{M\setminus B_r(x)} J(x,y) d\mu(y) \le Cr^{-\beta}. \tag{TJ}$$

It is easy to verify that

$$J(x,y) \ge cd(x,y)^{-(\alpha+\beta)} \Rightarrow (PI)$$

and

$$J(x,y) \le Cd(x,y)^{-(\alpha+\beta)} \Rightarrow (TJ),$$

so that (PI) and (TJ) can be regarded as relaxed (integral) versions of the lower resp. upper bounds of $J(x,y)$. In fact, both (PI) and (TJ) can be stated for *jump measures*.

Theorem 4 *If (TJ) and (PI) are satisfied then the heat kernel $p_t(x,y)$ exists, is continuous in (t,x,y), Hölder continuous in (x,y) and satisfies the following weak upper bound*

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta} \quad \forall x,y \in M \text{ and } \forall t > 0, \tag{WUE}$$

and the near-diagonal lower bound

$$p_t(x,y) \ge \frac{c}{t^{\alpha/\beta}} \quad \forall x,y \in M \text{ and } \forall t > d(x,y)^{\beta}. \tag{NLE}$$

Moreover, under the standing assumption (TJ), we have

$$(PI) \Leftrightarrow (WUE) + (NLE).$$

Note that the exponent $-\beta$ in (WUE) does not match the exponent $-(\alpha+\beta)$ in the optimal heat kernel bound (12). There are examples showing that, under (TJ) and (PI), one cannot guarantee any exponent better than $-\beta$. In the same way, the lower bound (NLE) cannot be improved to any estimate of the form

$$p_t(x,y) \ge \frac{c}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-N}.$$

An example

Let us give an example showing that the estimates of Theorem 4 cannot be improved. Let $\{(M_i, d_i, \mu_i)\}_{i=1}^n$ be a sequence of ultra-metric measure spaces such that M_i is α_i-regular. Consider on M_i the jump kernel

$$J_i(x, y) = d_i(x, y)^{-(\alpha_i+\beta)}$$

where $\beta > 0$ is the same for all i. By Theorem 3, the heat kernel $p_t^{(i)}(x, y)$ on M_i satisfies

$$p_t^{(i)}(x, y) \simeq \frac{1}{t^{\alpha_i/\beta}} \left(1 + \frac{d_i(x, y)}{t^{1/\beta}}\right)^{-(\alpha_i+\beta)}.$$

Consider now the product space $M = M_1 \times ... \times M_n$ with the ultra-metric

$$d(x, y) = \max_{1\leq i\leq n} d_i(x_i, y_i)$$

and with the product measure $\mu = \mu_1 \times ... \times \mu_n$. Then M is α-regular with

$$\alpha = \alpha_1 + ... + \alpha_n.$$

Consider the following jump measure $J(x, dy)$ (not kernel!) on M:

$$J(x, dy) = \sum_{i=1}^{n} \delta_{x_1}(dy_1) \dots \delta_{x_{i-1}}(dy_{i-1})\, J_i(x_i, y_i) d\mu_i(y_i) \delta_{x_{i+1}}(dy_{i+1}) \dots \delta_{x_n}(dy_n),$$

It induces a Dirichlet form on M with the generator

$$\mathcal{L} = \mathcal{L}_1 + ... + \mathcal{L}_n$$

where $\mathcal{L}_i$ acts on the coordinate x_i. The heat kernel on M is given by

$$p_t(x,y) = \prod_{i=1}^{n} p_t^{(i)}(x_i, y_i) \simeq \frac{1}{t^{\alpha/\beta}} \prod_{i=1}^{n} \left(1 + \frac{d_i(x_i, y_i)}{t^{1/\beta}}\right)^{-(\alpha_i+\beta)}. \tag{13}$$

It is easy to show that J satisfies (TJ) and that $p_t(x,y)$ satisfies both (WUE) and (NLE). By extension of Theorem 4 to jump measures, the Poincaré inequality (PI) is also satisfied.

Consider the range of x, y such that

$$d_1(x_1, y_1) > t^{1/\beta} \text{ and } d_i(x_i, y_i) \leq t^{1/\beta} \text{ for } i = 2, ..., n.$$

Then (13) yields

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d_1(x_1, y_1)}{t^{1/\beta}}\right)^{-(\alpha_1+\beta)} = \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha_1+\beta)}.$$

Since α_1 can be chosen arbitrarily small, we see that (WUE) is optimal.

Similarly, consider the range of x, y such that

$$d_i\left(x_i, y_i\right) \simeq d_j\left(x_j, y_j\right) \quad \text{for all } i, j.$$

Then $d(x, y) \simeq d_i\left(x_i, y_i\right)$ and

$$\begin{aligned} p_t(x, y) &\simeq \frac{1}{t^{\alpha/\beta}} \prod_{i=1}^{n}\left(1+\frac{d_i\left(x_i, y_i\right)}{t^{1/\beta}}\right)^{-(\alpha_i+\beta)} \\ &\simeq \frac{1}{t^{\alpha/\beta}}\left(1+\frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+n\beta)} \end{aligned}.$$

Since n can be chosen arbitrarily large, while α and β are fixed, we see that no lower bound of the form

$$p_t(x, y) \geq \frac{c}{t^{\alpha/\beta}}\left(1+\frac{d(x, y)}{t^{1/\beta}}\right)^{-N}$$

can be guaranteed.

Semi-bounded kernels

In the above setting of an α-regular ultra-metric space (M, d, μ), consider the following two conditions: the pointwise upper bound of the jump kernel

$$J(x,y) \le Cd(x,y)^{-(\alpha+\beta)} \tag{$J_\le$}$$

and the pointwise lower bound:

$$J(x,y) \ge cd(x,y)^{-(\alpha+\beta)}. \tag{$J_\ge$}$$

Theorem 5 *If* $(J_\le)$ *and* (PI) *are satisfied then the heat kernel satisfies for all* $x, y \in M$ *and* $t > 0$ *the optimal upper bound*

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \tag{UE}$$

and the near-diagonal lower bound

$$p_t(x,y) \ge \frac{c}{t^{\alpha/\beta}} \quad \forall x, y \in M \quad \text{and } \forall t > d(x,y)^\beta. \tag{NLE}$$

In fact, we have

$$(J_\le) + (PI) \Leftrightarrow (UE) + (NLE).$$

Theorem 6 *If ($J_{\geq}$) and (TJ) are satisfied then the heat kernel satisfies for all $x, y \in M$ and $t > 0$ the optimal lower bound*

$$p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \tag{LE}$$

and the weak upper bound

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta}. \tag{WUE}$$

Moreover, under the standing assumption (TJ), we have

$$(J_{\geq}) \Leftrightarrow (WUE) + (LE).$$

Clearly, Theorems 5 and 6 imply that

$$(J) \Leftrightarrow (UE) + (LE),$$

which is equivalent to Theorem 3.