ON POSITIVE SOLUTIONS OF SEMI-LINEAR ELLIPTIC INEQUALITIES ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We determine the critical exponent for certain semi-linear elliptic problem on a Riemannian manifold assuming the volume regularity and Green function estimates.

CONTENTS

1. INTRODUCTION

Let (M, g) be a complete non-compact Riemannian manifold and Δ be the Laplace-Beltrami operator on M. Consider the following differential inequality

$$\Delta u + \Phi(x) u^\sigma \leq 0, \tag{1.1}$$

where Φ is a given positive function, $\sigma > 1$ is a given exponent, and u is an unknown non-negative C^2-function.

In this paper we discuss the problem of existence of a non-trivial solution u of (1.1) in a connected exterior domain $M \setminus K$, where K is a compact subset of M. The minimum principle for superharmonic functions implies that either $u \equiv 0$ or $u > 0$ in $M \setminus K$. The existence or non-existence of a positive solution depend on the value of σ as well as on the geometric properties of M.

The question of existence of a positive solution of (1.1) in $\mathbb{R}^n$ $(n > 2)$ has a long history. It originated from the work [7] of Gidas and Spruck who considered the equation

$$\Delta u + u^\sigma = 0 \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

and proved that if

$$1 < \sigma < \frac{n+2}{n-2}.$$

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then the only non-negative solution of (1.2) is zero. On the contrary, if $\sigma \geq \frac{n+2}{n-2}$ then there exists a positive solution; for example, in the case $\sigma = \frac{n+2}{n-2}$ it is

$$u(x) = \frac{c_n}{(1+|x|^2)^{\frac{n-2}{2}}},$$

with some $c_n > 0$. Hence, the critical value of the exponent σ for the problem (1.2) is $\frac{n+2}{n-2}$.

However, the critical value of σ changes if we consider the equation in an exterior domain:

$$\Delta u + u^\sigma = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}\,. \tag{1.3}$$

It was proved by Bidaut-Véron [2] that if

$$1 < \sigma \leq \frac{n}{n-2}, \tag{1.4}$$

then any non-negative solution of (1.3) is zero, while for $\sigma > \frac{n}{n-2}$ a positive solution exists, for example,

$$u(x) = c_{n,\sigma}|x|^{-\frac{2}{\sigma-1}},$$

with some $c_{n,\sigma} > 0$. Hence, the critical value of the exponent σ for the problem (1.3) is $\frac{n}{n-2}$.

Consider now the inequality (1.1) in $\mathbb{R}^n$ with $\Phi \equiv 1$, that is

$$\Delta u + u^\sigma \leq 0 \ \text{ in } \mathbb{R}^n. \tag{1.5}$$

Mitidieri and Pohozaev [26] proved that any non-negative solution to (1.5) is zero if and only if σ satisfies (1.4). Hence, the critical value of σ in this case is again $\frac{n}{n-2}$. The sharpness of $\frac{n}{n-2}$ can be seen as follows: if $\sigma > \frac{n}{n-2}$ then the function

$$u(x) = \frac{c}{(1+|x|^2)^{\frac{1}{\sigma-1}}}.$$

is a solution to (1.5) for small enough $c > 0$. Mitidieri and Pohozaev investigated in [27] the inequality (1.1) in $\mathbb{R}^n$ with $\Phi(x) \geq C|x|^m$ for large enough $|x|$, and obtained that if

$$1 < \sigma \leq \frac{n+m}{n-2}, \tag{1.6}$$

then any non-negative solution to (1.1) is zero. Later Bidaut-Véron [3] showed that, for the inequality (1.1) with $\Phi(x) = |x|^m$ ($m > -2$) in the exterior domain $\mathbb{R}^n \setminus \{0\}$, the critical value of σ is equal to $\frac{n+m}{n-2}$.

A rich class of problems generalizing (1.1) in $\mathbb{R}^n$ was systematically studied by Mitidieri and Pohozaev, who developed test function techniques (or nonlinear capacity approach). Their approach can be systematically applied to many types of differential inequalities especially in Euclidean space, such as quasilinear elliptic and even parabolic differential inequalities. Let us refer the readers to [4, 5, 21, 26, 27, 28] for details.

Let us return to a Riemannian manifold M and define for a fixed K the critical value σ of the problem (1.1) as follows:

$$\sigma^* = \sup\{\sigma > 1 : (1.1) \text{ has no positive solution in } M \setminus K\}.$$

Let $d(x,y)$ be the geodesic distance on M. Denote by $B(x,r)$ geodesic balls, that is,

$$B(x,r) = \{y \in M : d(x,y) < r\}\,.$$

Let μ be the Riemannian measure on M, and set

$$V(x,r) = \mu(B(x,r)).$$

In [16], the authors investigated problem (1.1) on M with $\Phi(x) \equiv 1$, that is

$$\Delta u + u^\sigma \leq 0 \ \text{ in } M, \tag{1.7}$$

and proved that if, for some reference point $o \in M$ and $\alpha > 2$,

$$V(o,r) \leq Cr^{\alpha} \ln^{\frac{\alpha-2}{2}} r, \quad \text{for all large enough } r, \tag{1.8}$$

then, for any $\sigma \leq \frac{\alpha}{\alpha-2}$, the inequality (1.7) has no positive solution. Consequently, we have

$$\sigma^{*} \geq \frac{\alpha}{\alpha-2}. \tag{1.9}$$

In the paper [29], the second author investigated inequality (1.1) on M with

$$\Phi(x) \geq Cd(x,o)^{m}, \quad \text{for large } d(x,o)$$

where $m > -2$, and proved that if, for some $\alpha > 2$,

$$V(o,r) \leq Cr^{\alpha} \ln^{\frac{\alpha-2}{m+2}} r, \quad \text{for all large enough } r, \tag{1.10}$$

then

$$\sigma^{*} \geq \frac{\alpha+m}{\alpha-2}.$$

The idea of using upper bounds of the volume function of balls in order to restrict the set of solutions to certain differential inequalities has been widely used in the literature. It originated from the pioneering work of Cheng and Yau [6] who proved that if, on a geodesically complete Riemannian manifold M,

$$V(o,r) \leq Cr^{2}, \quad \text{for all large enough } r, \tag{1.11}$$

then any non-negative superharmonic function on M is identical constant. Since any non-negative solution of the inequality (1.1) on M is superharmonic, it follows that under the hypothesis (1.11) the inequality (1.7) has only trivial solution 0 for any σ. Note also that in the aforementioned hypotheses (1.8) and (1.10) the volume function may grow faster than Cr^2.

Further results of in this direction were obtained by Wang and Xiao [30] and Mastrolia, Monticelli and Punzo [24]. For other related studies in this area we refer the readers to [9, 12, 25, 23].

In this paper, we investigate the problem of existence of a positive solution to (1.1) using apart from the volume growth condition also some bounds of the Green function $G(x,y)$ of the Laplacian. Although the latter is a stronger restriction on the class of manifolds in question, it still allows us to obtain a better lower bound for the critical exponent σ^* and even to compute σ^* exactly.

By definition, the Green function $G(x,y)$ is the minimal positive fundamental solution of the Laplace operator on M. The existence of $G(x,y)$ is equivalent to the existence of a non-constant positive superharmonic function on M. Hence, the existence of the Green function is a necessary condition for the existence of a positive solution of (1.1).

Assume that, for some $o \in M$ and all large enough r,

$$V(o,r) \simeq r^{\alpha} \tag{V_o}$$

and, for all $x, y \in M$ with large enough $d(x,y)$,

$$G(x,y) \simeq d(x,y)^{-\gamma}, \tag{G}$$

where α, γ are positive parameters and the sign $\simeq$ means that the relation of the both sides is bounded from above and below by positive constants.

Our first main result (Theorem 2.1 and Corollary 2.3) says, in particular, that if (V_o) and (G) and satisfied and $\sigma \leq \frac{\alpha}{\gamma}$, then the inequality

$$\Delta u + u^{\sigma} \leq 0 \quad \text{in } M \setminus K \tag{1.12}$$

has the only non-negative solution $u \equiv 0$. Consequently, the critical exponent for the problem (1.12) admits the estimate

$$\sigma^* \geq \frac{\alpha}{\gamma}.$$

Of course, if $\gamma < \alpha - 2$ (that can actually occur – see Section 3), then this lower bound of σ^* is better than (1.9).

Replace now (V_o) by a stronger condition: for all large enough r and for all $x \in M$,

$$V(x,r) \simeq r^\alpha. \tag{V}$$

Our second main result Theorem 2.6 says that if M satisfies (V) and (G) as well as has bounded geometry then, for any $\sigma > \frac{\alpha}{\gamma}$, the inequality (1.12) has a positive solution in $M \setminus K$. Consequently, the critical exponent for the problem (1.12) in $M \setminus K$ has the value

$$\sigma^* = \frac{\alpha}{\gamma}.$$

The aforementioned Corollary 2.3 is a particular case of Theorem 2.1 that says the following: if M satisfies (V_o) and (G) , while Φ satisfies, for some $m > \gamma - \alpha$, the condition

$$\Phi(x) \geq d(x,o)^{-m}, \text{ for all } x \in M \text{ with } d\left(x,o\right) \geq r_0, \tag{Φ}$$

then, for any $\sigma \leq \frac{\alpha+m}{\gamma}$, the inequality (1.1) in $M \setminus K$ has the only solution $u = 0$. Consequently, the critical value of σ in this case admits the lower bound

$$\sigma^* \geq \frac{\alpha+m}{\gamma}.$$

The methods of proofs of Theorems 2.1 and 2.6 are based on some new ideas. Let u be a positive solution of (1.1) with $\sigma \leq \frac{\alpha+m}{\gamma}$. Assume without loss of generality that $o \in K$. For any precompact open set $U \supset K$ and for all $x \in U^c$, we obtain from (1.1) that

$$u\left(x\right) \geq \int_{U^c} G_{\overline{U}^c}\left(x,y\right) \Phi\left(y\right) u^\sigma\left(y\right) d\mu\left(y\right), \tag{1.13}$$

where $G_{\overline{U}^c}\left(x,y\right)$ is the Green function of Δ in $\overline{U}^c$ with the Dirichlet boundary condition. The superharmonicity of u implies that

$$u\left(y\right) \geq cG\left(y,o\right) \tag{1.14}$$

for some $c > 0$ and for all $y \in U^c$. On the other hand, by Lemma 4.1 we have, for any precompact open set $\Omega \subset M$,

$$\sup_\Omega (\Delta u + \lambda_1(\Omega) u) \geq 0,$$

where $\lambda_1\left(\Omega\right)$ is the first Dirichlet eigenvalue of Δ in Ω. It follows that

$$\lambda_1\left(\Omega\right) \geq \inf_{x\in\Omega} \Phi\left(x\right) u^{\sigma-1}\left(x\right).$$

Combining this with (1.13) and (1.14), we obtain for $\Omega \subset \overline{U}^c$ that

$$\lambda_1\left(\Omega\right)^{\frac{1}{\sigma-1}} \geq c \inf_{x\in\Omega} \Phi^{\frac{1}{\sigma-1}} \int_{U^c} G_{\overline{U}^c}\left(x,y\right) \Phi\left(y\right) G^\sigma\left(y,o\right) d\mu\left(y\right).$$

Then we bring this inequality to contradiction by choosing Ω large enough and by applying the hypotheses (V_o), (G), (Φ) and $\sigma \leq \frac{\alpha}{\gamma}$ to estimate all the quantities involved.

To prove Theorem 2.6, we construct for any $\sigma > \alpha/\gamma$ a positive solution to the equation

$$\Delta u + u^\sigma + \lambda^\sigma f^\sigma = 0 \ \text{ in } M,$$

where f is a specifically chosen decreasing function and $\lambda > 0$ is small enough (see Theorem 7.6). This differential equation amounts to the integral equation

$$u(x) = \int_M G(x,y)\,(u^\sigma(y) + \lambda^\sigma f(y)^\sigma)\,d\mu(y),$$

and the latter is solved in a certain closed subset of $L^\infty(M)$ by observing that the operator in the right hand side is a contraction for small enough λ. Next, we improve the regularity properties of u in two steps: first show that u is Hölder and then $u \in C^2$.

The paper is organized as follows: In Section 2, we state our results in the setting of a weighted manifold M with an arbitrary distance function. In Section 3, we give examples of manifolds, satisfying the conditions (V_o), (V), (G). Section 4 contains some technical preparation for the proof of the uniqueness Theorem 2.1, and the latter is then proved in Section 5. In Section 6 we obtain some applications of Theorem 2.1 to the uniqueness problem for some Schrödinger operators. In Section 7 we prove the existence Theorems 2.6, 7.6.

Notation. The letters $C, C', C_0, C_1, c_0, c_1 ...$ denote positive constants whose values are unimportant and may vary at different occurrences.

2. Statements

A *weighted manifold* is a couple (M,μ), where M is a connected Riemannian manifold and μ is a measure on M with a smooth positive density with respect to the Riemannian measure μ_0. Assume that $d\mu = \omega d\mu_0$, where ω is a positive C^∞ function on M. The *weighted Laplacian* Δ of (M,μ) is defined by

$$\Delta u = \frac{1}{\omega}\operatorname{div}(\omega\nabla u).$$

In particular, if $\omega \equiv 1$ then Δ is the Laplace-Beltrami operator on M.

Fix a compact set K (may be empty) such that $M \setminus K$ is connected. Given a non-negative function Φ on $M \setminus K$ and a constant $\sigma > 1$, consider the following differential inequality in $M \setminus K$

$$\Delta u + \Phi(x)u^\sigma \le 0, \tag{2.1}$$

where u is an unknown non-negative C^2-function in $M \setminus K$, and set

$$\sigma^* = \sup\{\sigma > 1 : \text{any non-negative solution of (2.1) in } M \setminus K \text{ is identical zero}\}.$$

Let d be a distance function on M (not necessarily geodesic). We always assume that the metric balls

$$B(x,r) = \{y \in M, \quad d(x,y) < r\}.$$

are precompact open subsets of M. If d is the geodesic distance, then the latter assumption is equivalent to the geodesic completeness of M. Set also

$$V(x,r) = \mu(B(x,r)). \tag{2.2}$$

Let $G(x,y)$ be the Green function of Δ, that is, the smallest positive fundamental solution of Δ. We always assume that G exists.

Fix a reference point $o \in K$ (when K is empty, o can be any point on M), positive reals α, γ, R_0 and introduce the following hypotheses.

(V_o) There exist positive constants c and C such that, for all $r > R_0$,

$$cr^\alpha \le V(o,r) \le Cr^\alpha.$$

(G) There exist positive constants c and C such that, for all $x, y \in M$ with $d(x,y) > R_0$,

$$cd(x,y)^{-\gamma} \le G(x,y) \le Cd(x,y)^{-\gamma}. \tag{2.3}$$

(Φ) There exist reals $m > \gamma - \alpha$ and $c > 0$ such that, for all $x \in M$ with $d(x, o) > R_0$,

$$\Phi(x) \geq cd(x, o)^m. \tag{2.4}$$

Our first main result is the following theorem.

Theorem 2.1. *Assume that the hypotheses* (V_o) , (G) *and* (Φ) *are satisfied. If*

$$1 < \sigma \leq \frac{\alpha + m}{\gamma},$$

then any non-negative solution u *of* (2.1) *in* $M \setminus K$ *is identical zero. Consequently, we have*

$$\sigma^* \geq \frac{\alpha + m}{\gamma}.$$

Remark 2.2. As we will see from proof of Theorem 2.1 in Section 5, the condition (V_o) here can be replaced by a weaker condition $(V_{\geq})$ as follows.

$(V_{\geq})$ There exist $\tau \in (0, 1)$ and $c > 0$, such that for all large enough r

$$V(o, r) - V(o, \tau r) \geq cr^{\alpha}. \tag{2.5}$$

Clearly, (V_o) implies $(V_{\geq})$.

If $\Phi(x) \equiv 1$, then Theorem 2.1 implies the following.

Corollary 2.3. *Assume that conditions* (V_0) *and* (G) *are satisfied with* $\alpha > \gamma$. *If*

$$1 < \sigma \leq \frac{\alpha}{\gamma},$$

then any non-negative solution u *of*

$$\Delta u + u^{\sigma} \leq 0 \quad \text{in } M \setminus K,$$

is identical zero. Consequently, we have

$$\sigma^* \geq \frac{\alpha}{\gamma}.$$

Example 2.4. Let $M = \mathbb{R}^n$ $(n > 2)$, μ be the Lebesgue measure and $d(x, y) = |x - y|$. Then Δ is the classical Laplacian, and its Green function is given by

$$G(x, y) = \frac{c_n}{|x - y|^{n-2}},$$

with $c_n > 0$. It follows that (V_o) and (G) are satisfied with $\alpha = n$ and $\gamma = n - 2$. If, for some $m > -2$,

$$\Phi(x) \geq c|x|^m \quad \text{for large } |x|,$$

then we conclude by Theorem 2.1, that

$$\sigma^* \geq \frac{n + m}{n - 2}.$$

Recall that, by a result of [27], if $\Phi(x) = c|x|^m$ for large $|x|$, then

$$\sigma^* = \frac{n + m}{n - 2}.$$

Example 2.5. Let M be a geodesic complete Riemannian manifold, μ be the Riemannian measure and d be the geodesic distance. Set $\Phi(x) \equiv 1$. Then, by a result of [16], if

$$V(o, r) \leq Cr^{\alpha}, \tag{2.6}$$

then

$$\sigma^* \geq \frac{\alpha}{\alpha - 2}.$$

Although the assumption (V_0) and (G) are stronger than (2.6), the estimate $\sigma^* \geq \frac{\alpha}{\gamma}$ may be sharper than $\sigma^* \geq \frac{\alpha}{\alpha-2}$ provided that $\gamma < \alpha - 2$. The latter can actually occur in a number of situations (see Section 3).

Now we provide sufficient conditions for the existence of a positive solution u of

$$\Delta u + u^\sigma \leq 0 \quad \text{in } M. \tag{2.7}$$

For that, we need to slightly strengthen our assumptions on M. We restrict our setting as follows: assume that M is a connected non-compact geodesically complete Riemannian manifold, d is the geodesic distance on M, and μ is the Riemannian measure of M.

Assume in addition that M has *bounded geometry*, that is, there exists $r_0 > 0$ such that the geodesic balls $B(x, r_0)$ on M are uniformly quasi-isometric to the Euclidean ball $B_{r_0}(0)$ in $\mathbb{R}^n$. Consider the following stronger version of the hypothesis (V_o).

(V) There exist positive constants c and C such that, for all $x \in M$ and $r \geq R_0$,

$$cr^\alpha \leq V(x, r) \leq Cr^\alpha.$$

Our existence result is stated in the following theorem.

Theorem 2.6. *Assume that* $\dim M > 2$ *and that* M *has bounded geometry. Assume also that* (V) *and* (G) *are satisfied with* $\alpha > \gamma$. *Then, for any*

$$\sigma > \frac{\alpha}{\gamma},$$

the inequality (2.7) *admits a positive solution on* M. *Consequently, in this case we have*

$$\sigma^* = \frac{\alpha}{\gamma}.$$

3. Examples in the case $\Phi(x) \equiv 1$

We present here some examples of application of Corollary 2.3 and Theorem 2.6.

Example 3.1. Let Γ be an infinite, locally finite, connected graph. Let $d(x, y)$ be the graph distance on Γ and μ be the degree measure. Define the volume function $V(x, r)$ on Γ by (2.2). Assume that the discrete Laplace operator on Γ has a positive finite Green function $G(x, y)$. Barlow constructed in [1] a family of fractal graphs such that each graph from this family satisfies the conditions (V) and (G) for some α and γ. Moreover, such a graph exists for any pair (α, γ) of reals satisfying the following restrictions:

$$0 < \gamma \leq \alpha - 2.$$

Now let us inflate Γ into a 2-dim manifold M by replacing the edges of Γ by 2-dim cylinders. Then the Riemannian manifold M with the geodesic distance d and the Riemannian measure μ will satisfy conditions (V) and (G) with the same values of α and γ as above (which can be proved by using the techniques from Kanai's papers [18, 19]). For example, if we take $\Gamma = \mathbb{Z}^n$, $n > 2$, then we obtain a 2-dim manifold M satisfying (V) and (G) with $\alpha = n$ and $\gamma = n - 2$.

Since the manifold M obtained in this way from a graph Γ has bounded geometry, we obtain by Theorem 2.6 that on M

$$\sigma^* = \frac{\alpha}{\gamma}.$$

Hence, if $\gamma < \alpha - 2$ then $\sigma^* > \frac{\alpha}{\alpha-2}$, which improves the estimate of Example 2.5.

Example 3.2. Assume that (M, g) is a 2-dim Riemannian manifold with the Riemannian measure μ and some distance function d. We refer to the triple (M, g, d) as a *metric manifold*. Assume that (M, g, d) satisfies (V_o) and (G) with some positive reals α, γ. For example, (M, g, d) could be one of the manifolds, constructed in the previous example.

Fix a smooth positive function φ on M and consider a conformal change of metric

$$\tilde{g} = \varphi g.$$

Consider now the metric manifold $(M, \tilde{g}, d)$ (with the same distance d as before, but with the Riemannian measure $\tilde{\mu}$ of $\tilde{g}$). Denote by $\tilde{V}$ and $\tilde{G}$ the volume and Green functions on $(M, \tilde{g}, d)$, respectively.

It is well known that on 2-dim manifolds the Green function does not change after a conformal transformation of the Riemannian metric, hence $\tilde{G} = G$. Assume in addition that

$$\varphi(x) \simeq d(x, o)^{\delta}, \quad \text{for large } d(x, o),$$

with some real δ. Since $V(o, r)$ satisfies (V_o) and $d\tilde{\mu} = \varphi d\mu$, a simple computation shows that

$$\tilde{V}(o, r) \simeq r^{\alpha+\delta}, \quad \text{for large } r,$$

provided $\alpha + \delta > 0$. Thus, $(M, \tilde{g}, d)$ satisfies (V_0) and (G) with parameters

$$\tilde{\alpha} = \alpha + \delta \text{ and } \tilde{\gamma} = \gamma.$$

Obviously, $\tilde{\alpha}$ and $\tilde{\gamma}$ can be arbitrary positive numbers. Under the above hypotheses, we obtain by Corollary 2.3 that the critical value σ^* on $(M, \tilde{g}, d)$ satisfies

$$\sigma^* \geq \frac{\alpha + \delta}{\gamma}.$$

Example 3.3. Let (M, g) be a Riemannian manifold. Assume that the Green function $G(x, y)$ on M exists and satisfies the following $3G$-inequality

$$\min\left(G(x, z), G(z, y)\right) \leq CG\left(x, y\right),$$

for all $x, y, z \in M$ and some $C > 1$ (cf. [13, Sect. 4,5]). Then the function

$$\rho(x, y) := \frac{1}{G\left(x, y\right)}$$

is a quasi-metric on M. It is well-known that, for any quasi-metric ρ, there exist a distance function $d(x, y)$ and a real $\gamma > 0$ such that

$$\rho(x, y) \simeq d(x, y)^{\gamma}.$$

Hence, on the metric manifold (M, g, d), we have

$$G(x, y) \simeq d(x, y)^{-\gamma}, \quad \text{for all } x, y \in M.$$

In particular, (M, g, d) satisfies (G).

Assume that (M, g) satisfies (V_o) in terms of the quasi-metric ρ, that is

$$\mu\left\{y : \rho(o, y) < r\right\} = \mu\left\{y : G(o, y)^{-1} < r\right\} \simeq r^{\alpha}, \quad \text{for large } r. \tag{3.1}$$

Then for metric balls $B(o, r)$ with respect to d, we have

$$\begin{aligned} \mu(B(o, r)) &= \mu(\{y : d(o, y) < r\}) \\ &= \mu\left\{y : d(o, y)^{\gamma} < r^{\gamma}\right\} \simeq r^{\alpha\gamma}. \end{aligned}$$

Hence, (M, g, d) satisfies (V_o) with $\tilde{\alpha} := \alpha\gamma$. Assuming in addition that all balls $B(o, r)$ are precompact, we obtain by Corollary 2.3 that the critical value σ^* on (M, g, d) satisfies

$$\sigma^* \geq \frac{\tilde{\alpha}}{\gamma} = \alpha.$$

Example 3.4. Fix an integer $\alpha > 2$, a compact Riemannian manifold N, and consider the Riemannian manifold $M = \mathbb{R}^\alpha \times N$ with the product metric. It is well known that on M

$$V(x,r) \simeq r^\alpha, \quad \text{for large } r.$$

and

$$G(x,y) \simeq d(x,y)^{-(\alpha-2)}, \quad \text{for large } d(x,y),$$

By Theorem 2.6, we obtain $\sigma^* = \frac{\alpha}{\alpha-2}$.

Example 3.5. Assume that a Riemannian manifold M admits a discrete group of isometries Γ such that the fundamental domain M/Γ is compact. Assume also that the group Γ has a polynomial volume growth r^n for some $n > 2$ (with respect to some generating set). In this case the volume growth function $V(x,r)$ of M satisfies (V) with $\alpha = n$, and the Green function on M satisfies (G) with $\gamma = n-2$ (cf. [17]). By Theorem 2.6, we conclude that $\sigma^* = \frac{n}{n-2}$.

4. Preliminaries

We start the proof of Theorem 2.1 with some preliminary results. For any precompact open domain Ω in M, denote by $\lambda_1(\Omega)$ the first Dirichlet eigenvalue for Δ in Ω, and by $G_\Omega(x,y)$ – the Green function of Δ on Ω with the Dirichlet boundary condition.

Lemma 4.1. *For any non-negative function $f \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ we have*

$$\sup_\Omega (\Delta f + \lambda_1(\Omega) f) \geq 0. \tag{4.1}$$

Proof. Without loss of generality, we can assume that Ω has smooth boundary (otherwise consider approximation of Ω from inside by an increasing sequence $\{\Omega_n\}_{n=1}^\infty$ of domains with smooth boundaries using $\lambda_1(\Omega_n) \to \lambda_1(\Omega)$). Let v be the Dirichlet eigenfunction of Δ in Ω with the eigenvalue $\lambda = \lambda_1(\Omega)$. Since v does not change sign in any connected component of Ω, we can assume that $v > 0$ in Ω. Using the equation $\Delta v + \lambda v = 0$ and the Green formula, we obtain

$$\begin{aligned}\int_\Omega (\Delta f + \lambda f) v \, d\mu &= \int_{\partial\Omega} \left(\frac{\partial f}{\partial \nu} v - \frac{\partial v}{\partial \nu} f\right) dS + \int_\Omega f(\Delta v + \lambda v) d\mu \\ &= -\int_{\partial\Omega} \frac{\partial v}{\partial \nu} f \, dS,\end{aligned}$$

where ν is the outward normal unit vector field on $\partial\Omega$ and S is the surface measure on $\partial\Omega$. Since $\frac{\partial v}{\partial \nu}\big|_{\partial\Omega} \leq 0$, it follows that

$$\int_\Omega (\Delta f + \lambda f) v \, d\mu \geq 0.$$

whence the claim follows. ■

Lemma 4.2. *Assume that M satisfies (G). Choose $R > R_0$ and set $\Omega = B(o, NR)$, where $N > 1$ is a large enough constant depending on the constants in (G). Then, for all $x \in B(o,R) \setminus B(o,R_0)$, we have*

$$G_\Omega(x,o) \simeq d(x,o)^{-\gamma}. \tag{4.2}$$

Proof. We will prove that if N is large enough then, for all $x \in B(o, R)$,

$$\frac{1}{2} G(x, o) \leq G_\Omega(x, o) \leq G(x, o), \tag{4.3}$$

which together with (2.3) will imply (4.2). The upper bound in (4.3) is trivially satisfied for all x. It suffices to prove the lower bound for a smaller domain Ω. Let now Ω be a domain with smooth boundary between $B\left(o, \frac{1}{2}NR\right)$ and $B(o, NR)$. Consider the function

$$h(x) = G(x, o) - G_\Omega(x, o)$$

that is non-negative and harmonic in Ω. Since $\partial\Omega$ is smooth, $G_\Omega(x, o)$ vanishes at any point $x \in \partial\Omega$. Hence, we obtain by (2.3)

$$h(x)|_{\partial\Omega} = G(x, o)|_{\partial\Omega} \leq C(\tfrac{1}{2}NR)^{-\gamma}. \tag{4.4}$$

The maximum principle implies then that

$$h(x) \leq C(\tfrac{1}{2}NR)^{-\gamma} \text{ for all } x \in \Omega.$$

On the other hand, by the lower bound of $G(x, o)$ in (2.3), we have, for all $x \in B(o, R)$,

$$G(x, o) \geq \inf_{\partial B(o,R)} G(x, o) \geq cR^{-\gamma}.$$

If N is large enough, then this implies by (4.4) that, for all $x \in B(o, R)$,

$$h(x) \leq \frac{1}{2} G(x, o).$$

It follows that in $B(o, R)$

$$G_\Omega(x, o) = G(x, o) - h(x) \geq \frac{1}{2} G(x, o),$$

which finishes the proof. ■

Proposition 4.3. *Assume that* $(V_{\geq})$ *and* (G) *hold on* M*. Let*

$$U = B(o, \tau R), \qquad \Omega = B(o, N^2 R),$$

where N *is the constant from* Lemma 4.2*, and* τ *is from* $(V_{\geq})$*. Then, for large enough* R*, we have*

$$\lambda_1(\Omega \setminus \overline{U}) \leq CR^{-(\alpha-\gamma)},$$

where the constant C *depends on the constants in the hypotheses.*

For the proof we need the notion of capacity. For any open set $\Omega \subset M$ and any precompact open set $U \Subset \Omega$, the capacity of the pair (U, Ω) defined by

$$\operatorname{cap}(U, \Omega) = \inf_{\substack{\varphi \in C_0^\infty(\Omega) \\ \varphi|_U \equiv 1}} \int_\Omega |\nabla\varphi|^2 \, d\mu. \tag{4.5}$$

If Ω is precompact and if the both boundaries ∂U and $\partial\Omega$ are smooth then it is known that

$$\operatorname{cap}(U, \Omega) = \int_{\Omega \setminus \overline{U}} |\nabla u|^2 \, d\mu,$$

where u is the solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \overline{U} \\ u|_{\partial U} = 1, & u|_{\partial\Omega} = 0 \end{cases}$$

(see [9, Prop. 1], [22, Thm. 4.1,4.2]).

The capacity $\operatorname{cap}(U,\Omega)$ is monotone increasing with respect to U and decreasing with respect to Ω. For any exhaustion $\Omega \to M$ of M by increasing sequence of open sets Ω, we have

$$\lim_{\Omega\to M} \operatorname{cap}(U,\Omega) = \operatorname{cap}(U,M) =: \operatorname{cap}(U).$$

Proof of Proposition 4.3. Recall that, for a precompact open set $E \subset M$,

$$\lambda_1(E) = \inf_{\varphi\in C_0^\infty(E)\setminus\{0\}} \frac{\int_E |\nabla\varphi|^2 \, d\mu}{\int_E \varphi^2 d\mu}. \tag{4.6}$$

Since for any function $\varphi \in C_0^\infty(E)$ with $\varphi|_D \equiv 1$ we have

$$\int_E \varphi^2 d\mu \ge \mu(D),$$

we obtain from comparison of (4.6) and (4.5) that

$$\lambda_1(E) \le \frac{\operatorname{cap}(D,E)}{\mu(D)}. \tag{4.7}$$

Let us introduce the following sets (see Fig. 1):

$$U_1 := B(o,\tau NR), \quad U_2 := B(o,NR), \quad D = U_2 \setminus \overline{U_1}, \quad E = \Omega \setminus \overline{U}.$$

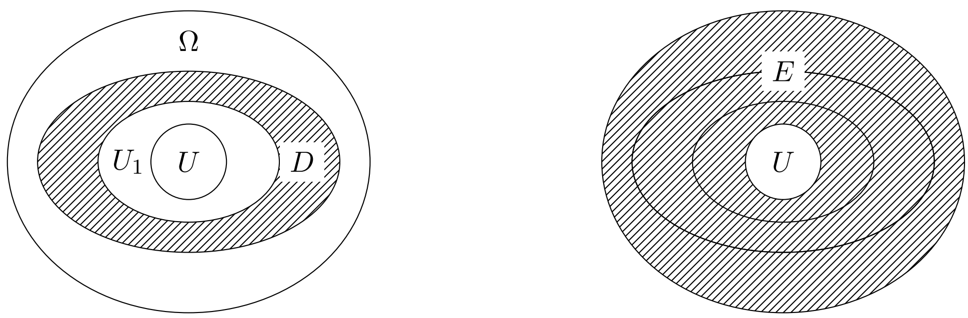

FIGURE 1. Sets $D = U_2 \setminus \overline{U}_1$ and $E = \Omega \setminus \overline{U}$

By [15, Remark 2.4], we have

$$\operatorname{cap}(D,E) \le \operatorname{cap}(U,U_1) + \operatorname{cap}(U_2,\Omega). \tag{4.8}$$

Applying [10, Proposition 4.1], we obtain

$$\begin{aligned}
\operatorname{cap}(U,U_1) &\le \left(\inf_{x\in\partial U} G_{U_1}(x,o)\right)^{-1}, \\
\operatorname{cap}(U_2,\Omega) &\le \left(\inf_{x\in\partial U_2} G_{\Omega}(x,o)\right)^{-1}.
\end{aligned}$$

Combining these estimates with (4.8), and applying Lemma 4.2 with a large enough R, we obtain

$$\operatorname{cap}(D,E) \le CR^\gamma. \tag{4.9}$$

By the condition $(V_\ge)$, we have

$$\mu(D) = \mu(U_2) - \mu(U_1) = V(o,NR) - V(o,\tau NR) \ge cR^\alpha. \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.7), we obtain

$$\lambda_1(E) \le CR^{\gamma-\alpha},$$

which finishes our proof. ∎

Lemma 4.4. *Let U and Ω be two precompact open subsets of M with smooth boundaries and such that $U \Subset \Omega$. Let η be a harmonic function in $\Omega \setminus \overline{U}$ satisfying $\eta = 0$ on $\partial\Omega$ and $\eta = 1$ on ∂U. Then we have*

$$\eta(x) \le C \operatorname{cap}(U, \Omega) d(x, U)^{-\gamma} \quad \textit{for large enough } d(x, U), \tag{4.11}$$

where the constant C depends on the constants in (G).

Proof. The harmonic function η admits at any $x \in \Omega \setminus \overline{U}$ the following representation via its boundary values and the Green function:

$$-\eta(x) = \int_{\partial(\Omega\setminus\overline{U})} \left(\frac{\partial G_\Omega}{\partial \nu}(x, y)\eta(y) - G_\Omega(x, y)\frac{\partial \eta}{\partial \nu} \right) dS(y),$$

where ν is the outward normal unit vector field to $\partial\left(\Omega \setminus \overline{U}\right)$. Since $\eta = 0$ on $\partial\Omega$, $\eta = 1$ on ∂U, we have

$$\int_{\partial(\Omega\setminus\overline{U})} \frac{\partial G_\Omega}{\partial \nu}(x, y)\eta(y) dS(y) = \int_{\partial U} \frac{\partial G_\Omega}{\partial \nu}(x, y) dS(y) = 0,$$

where we have used that the function $G_\Omega(x, \cdot)$ is harmonic in U. Since $G_\Omega(x, \cdot)$ vanishes on $\partial\Omega$, we obtain

$$\eta(x) = \int_{\partial U} G_\Omega(x, y)\frac{\partial \eta}{\partial \nu} dS(y).$$

By the maximum principle, we have $\frac{\partial \eta}{\partial \nu} \ge 0$, whence it follows that

$$\eta(x) \le \sup_{y\in\partial U} G(x, y) \int_{\partial U} \frac{\partial \eta}{\partial \nu} dS(y).$$

Applying (G) and observing that

$$\int_{\partial U} \frac{\partial \eta}{\partial \nu} dS(y) = \int_{\partial(\Omega\setminus\overline{U})} \frac{\partial \eta}{\partial \nu}\eta dS(y) = \int_{\Omega\setminus\overline{U}} |\nabla\eta|^2 d\mu = \operatorname{cap}(U, \Omega),$$

we obtain (4.11). ■

5. Proof of uniqueness of solutions

Proof of Theorem 2.1. Let u be a non-negative solution of (2.1) in $M \setminus K$ that is not identical zero. Since u is superharmonic outside K, it follows from the strong minimum principle that u is strictly positive.

Let U be a fixed precompact neighborhood of K with smooth boundary. Then $u \ge \rho > 0$ on ∂U for small enough constant ρ. Let w be the equilibrium potential of the capacitor (U, M), that is the smallest positive harmonic function outside U such that $w|_{\partial U} = 1$. By the maximum principle we have

$$u(x) \ge \rho w(x) \quad \text{for all } x \in U^c. \tag{5.1}$$

Since $-\Delta u \ge \Phi u^\sigma$ and $u \ge 0$ on $\overline{U}^c$, we have

$$u(x) \ge \int_{U^c} G_{\overline{U}^c}(x, y)\Phi(y)u^\sigma(y) d\mu(y).$$

Substituting here (5.1), we obtain

$$u(x) \ge \rho^\sigma \int_{U^c} G_{\overline{U}^c}(x, y)\Phi(y)w^\sigma(y) d\mu(y) \quad \text{for all } x \in U^c. \tag{5.2}$$

On the other hand, we claim that there exists some $\theta > 0$ such that

$$w(x) \ge \theta G(x, o) \quad \text{for all } x \in U^c. \tag{5.3}$$

Indeed, set

$$\theta := \frac{1}{\sup_{x\in\partial U} G(x,o)} > 0.$$

For any precompact open set Ω with smooth boundary containing $\overline{U}$, the function $\theta G_\Omega(\cdot,0)$ is bounded by 1 on ∂U and vanishes on $\partial\Omega$, which implies by the maximum principle that it is bounded by w in $\Omega\setminus U$. Exhausting M by a sequence of such sets Ω, we obtain (5.3).

From (5.2) and (5.3) we obtain

$$u(x) \geq \rho^\sigma\theta^\sigma \int_{U^c} G_{\overline{U}^c}(x,y)\Phi(y)G^\sigma(y,o)d\mu(y) \quad \text{for all } x\in U^c. \tag{5.4}$$

Next, choose two precompact open sets U_0 and Ω with smooth boundaries such that

$$U \Subset U_0 \Subset \Omega. \tag{5.5}$$

By Lemma 4.1 we have

$$\sup_{x\in\Omega\setminus U_0}\left(\Delta u + \lambda_1(\Omega\setminus\overline{U_0})u\right) \geq 0,$$

which together with (2.1) implies

$$\inf_{x\in\Omega\setminus U_0}\left(\Phi u^\sigma - \lambda_1(\Omega\setminus\overline{U_0})u\right) \leq 0$$

and, hence,

$$\inf_{x\in\Omega\setminus U_0} \Phi(x)^{\frac{1}{\sigma-1}}u(x) \leq \lambda_1(\Omega\setminus\overline{U_0})^{\frac{1}{\sigma-1}}.$$

From (5.4), we obtain

$$\rho^\sigma\theta^\sigma \inf_{x\in\Omega\setminus U_0}\Phi(x)^{\frac{1}{\sigma-1}}\int_{U^c} G_{\overline{U}^c}(x,y)\Phi(y)G^\sigma(y,o)d\mu(y) \leq \lambda_1(\Omega\setminus\overline{U_0})^{\frac{1}{\sigma-1}}.$$

A desired contradiction will be obtained if we show that, for any $\varepsilon>0$, there exist U_0 and Ω as in (5.5) and such that

$$\lambda_1(\Omega\setminus\overline{U_0})^{\frac{1}{\sigma-1}} < \varepsilon \inf_{x\in\Omega\setminus U_0}\Phi(x)^{\frac{1}{\sigma-1}}\int_{U^c} G_{\overline{U}^c}(x,y)\Phi(y)G^\sigma(y,o)d\mu(y). \tag{5.6}$$

Our next goal is to estimate $G_{\overline{U}^c}(x,y)$ from below away from ∂U. For any $x\in\Omega\setminus\overline{U}$ consider in $\Omega\setminus\overline{U}$ the function

$$\eta_x^\Omega(\cdot) = G_\Omega(x,\cdot) - G_{\overline{U}^c}(x,\cdot). \tag{5.7}$$

Clearly, $\eta_x^\Omega(\cdot)$ is harmonic in $\Omega\setminus\overline{U}$ and

$$\eta_x^\Omega(y)\big|_{y\in\partial\Omega} = -G_{\overline{U}^c}(x,y)\big|_{y\in\partial\Omega} \leq 0,$$

while

$$\eta_x^\Omega(y)\big|_{y\in\partial U} = G_\Omega(x,y)|_{y\in\partial U} \leq \sup_{y\in\partial U} G(x,y) \leq Cd(x,U)^{-\gamma},$$

provided $d(x,U)$ is large enough. By Lemma 4.4, we conclude that

$$\eta_x^\Omega(y) \leq C\,\mathrm{cap}(U,\Omega)d(x,U)^{-\gamma}d(y,U)^{-\gamma}, \tag{5.8}$$

provided both $d(x,U)$ and $d(y,U)$ are large enough. Let

$$\eta_x(y) := \lim_{\Omega\to M}\eta_x^\Omega(y) = G(x,y) - G_{\overline{U}^c}(x,y). \tag{5.9}$$

If follows from (5.8) that

$$\eta_x(y) \leq C\,\mathrm{cap}(U)d(x,U)^{-\gamma}d(y,U)^{-\gamma}, \tag{5.10}$$

for large enough $d(x,U)$ and $d(y,U)$.

Now we specify U_0 and Ω as follows. Using the constant $N > 2$ from Lemma 4.2 and $\tau \in (0,1)$ from (2.5), set

$$\Omega = B(o, N^2 R) \quad \text{and} \quad U_0 = B(o, \tau R),$$

where R is large enough and will tend to $+\infty$ in what follows. Define also the balls

$$\Omega_1 = B(o, 2N^2 R) \quad \text{and} \quad U_1 = B(o, r),$$

where r is a fixed but large enough number to be specified below. At the moment we assume that

$$r > 2 \operatorname{diam} U,$$

so that $U \Subset U_1$. Also, let r be so large that (5.10) holds for all x, y outside $B(o, r/2)$. Later on we will impose one more restriction on r (see (5.14)). We always assume that $R > \tau^{-1} r$ so that $U_1 \Subset U_0$ (see Fig. 2).

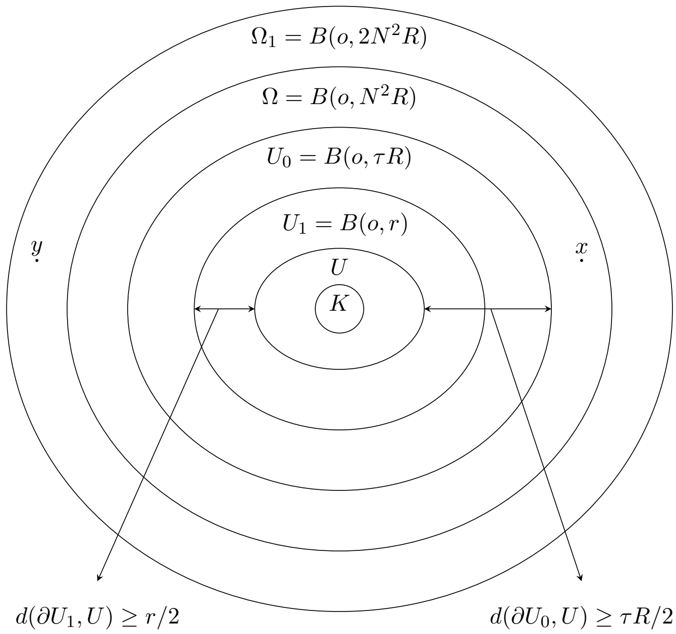

Figure 2. Points x and y

In order to estimate the integral in (5.6) from below, let us restrict the integration domain to $\Omega_1 \setminus U_1$. Observe that, for all large enough R and for

$$x \in \Omega \setminus U_0 \quad \text{and} \quad y \in \Omega_1 \setminus U_1, \tag{5.11}$$

we have

$$d(x, U) \geq \tau R - \operatorname{diam} U \geq \frac{\tau}{2} R \quad \text{and} \quad d(y, U) \geq r - \operatorname{diam} U \geq \frac{r}{2}.$$

Hence, we obtain from (5.10)

$$\eta_x(y) \leq C_1 (\tau R)^{-\gamma} r^{-\gamma}, \tag{5.12}$$

where $\operatorname{cap}(U)$ is absorbed into the constant C_1. Next, for all

$$x \in \Omega \setminus U_0 \quad \text{and} \quad y \in \partial \Omega_1,$$

we obtain

$$N^2 R \leq d(x, y) \leq 3N^2 R$$

whence

$$G(x, y) \geq c d(x, y)^{-\gamma} \geq c_1 R^{-\gamma}.$$

Since $G(x,\cdot)$ is superharmonic, the above inequality holds also for all $y\in\Omega_1$. In particular, for all x,y as in (5.11), we have

$$G(x,y)\geq c_1 R^{-\gamma}. \tag{5.13}$$

Therefore, for x,y as in (5.11), we obtain from (5.12) and (5.13) that

$$G_{\overline{U}^c}(x,y)=G(x,y)-\eta_x(y)\geq c_1R^{-\gamma}-C_1(\tau R)^{-\gamma}r^{-\gamma}\geq c_2R^{-\gamma}, \tag{5.14}$$

where $c_2=c_1/2$ and r is chosen to be large enough.

Hence, we obtain that, for all $x\in\Omega\setminus U_0$,

$$\int_{U^c}G_{\overline{U}^c}(x,y)\Phi(y)G^\sigma(y,o)d\mu(y)\geq c_2R^{-\gamma}\int_{\Omega\setminus U_1}\Phi(y)G^\sigma(y,o)d\mu(y). \tag{5.15}$$

Next, we claim that, for some $c>0$,

$$\int_{\Omega\setminus U_1}\Phi(y)G^\sigma(y,o)d\mu(y)\geq c\begin{cases}R^{\alpha+m-\sigma\gamma}, & \text{if } \alpha+m>\sigma\gamma,\\ \ln R, & \text{if } \alpha+m=\sigma\gamma.\end{cases} \tag{5.16}$$

Assuming that R is large enough, let us choose a positive integer k such that

$$\tau^{k+1}\geq\frac{r}{N^2R}\geq\tau^{k+2}. \tag{5.17}$$

Since

$$\Omega\setminus U_1\supset B\left(o,N^2R\right)\setminus B\left(o,\tau^{k+1}N^2R\right)$$

we obtain, using (G) and $(V_\geq)$, that

$$\begin{aligned}\int_{\Omega\setminus U_1}\Phi(y)G^\sigma(y,o)d\mu(y) &\geq \sum_{i=0}^{k}\int_{B(o,\tau^iN^2R)\setminus B(o,\tau^{i+1}N^2R)}\Phi(y)G^\sigma(y,o)d\mu(y)\\ &\geq C\sum_{i=0}^{k}\int_{B(o,\tau^iN^2R)\setminus B(o,\tau^{i+1}N^2R)}d(y,o)^{m-\sigma\gamma}d\mu(y)\\ &\geq C'\sum_{i=0}^{k}(\tau^iN^2R)^{m-\sigma\gamma}\mu\left(B(o,\tau^iN^2R)\setminus B(o,\tau^{i+1}N^2R)\right)\\ &\geq C''\sum_{i=0}^{k}(\tau^iN^2R)^{\alpha+m-\sigma\gamma}. \end{aligned}\tag{5.18}$$

If $\alpha+m>\sigma\gamma$ then, using $\tau\in(0,1)$, (5.17) and assuming that R is large enough, we obtain from (5.18)

$$\begin{aligned}\int_{\Omega\setminus U_1}\Phi(y)G^\sigma(y,o)d\mu(y) &\geq C(N^2R)^{\alpha+m-\sigma\gamma}\frac{1-\tau^{(k+1)(\alpha+m-\sigma\gamma)}}{1-\tau^{\alpha+m-\sigma\gamma}}\\ &\geq C'R^{\alpha+m-\sigma\gamma}.\end{aligned}\tag{5.19}$$

If $\alpha+m=\sigma\gamma$ then by (5.17)

$$k\geq c\ln R, \tag{5.20}$$

where $c=c(N,r,\tau)>0$. We obtain from (5.18) and (5.20) that

$$\int_{\Omega\setminus U_1}\Phi(y)G^\sigma(y,o)d\mu(y)\geq Ck\geq C'\ln R. \tag{5.21}$$

Clearly, (5.19) and (5.21) contain the claim (5.16).

Combining (5.15), (5.16) and (Φ), we obtain, for large enough R,

$$\inf_{\Omega\setminus U_0}\Phi^{\frac{1}{\sigma-1}}\int_{U^c}G_{\overline{U}^c}(x,\cdot)\Phi(\cdot)G^{\sigma}(\cdot,o)d\mu\geq c\begin{cases}R^{\frac{m\sigma}{\sigma-1}+\alpha-(\sigma+1)\gamma}, & \text{if } \alpha+m>\sigma\gamma,\\ R^{\frac{m}{\sigma-1}-\gamma}\ln R, & \text{if } \alpha+m=\sigma\gamma.\end{cases} \tag{5.22}$$

On the other hand, by Proposition 4.3, we have

$$\lambda_1(\Omega\setminus\overline{U_0})^{\frac{1}{\sigma-1}}\leq CR^{-\frac{\alpha-\gamma}{\sigma-1}}. \tag{5.23}$$

Let us verify that, under the hypothesis,

$$1<\sigma\leq\frac{\alpha+m}{\gamma} \tag{5.24}$$

the estimates (5.22) and (5.23) imply (5.6).

Indeed, consider first the case of equality in (5.24), that is,

$$\alpha+m=\sigma\gamma.$$

It follows that $\alpha-\gamma=(\sigma-1)\,\gamma-m$ and

$$\frac{\alpha-\gamma}{\sigma-1}=\gamma-\frac{m}{\sigma-1}.$$

Therefore, thanks to the factor $\ln R$ in (5.22), we obtain, as $R\to\infty$,

$$\begin{aligned}\lambda_1(\Omega\setminus\overline{U_0})^{\frac{1}{\sigma-1}} &= O(R^{-\frac{\alpha-\gamma}{\sigma-1}})=o\left(R^{\frac{m}{\sigma-1}-\gamma}\ln R\right)\\ &= o\left(\inf_{\Omega\setminus U_0}\Phi^{\frac{1}{\sigma-1}}\int_{U^c}G_{\overline{U}^c}(x,\cdot)\Phi(\cdot)G^{\sigma}(\cdot,o)d\mu\right),\end{aligned}$$

whence (5.6) follows.

Consider now the case of a strict inequality in (5.24), that is,

$$\alpha+m>\sigma\gamma. \tag{5.25}$$

Let us verify that in this case

$$\frac{\alpha-\gamma}{\sigma-1}>-\left[\frac{m\sigma}{\sigma-1}+\alpha-(\sigma+1)\gamma\right].$$

Indeed, this inequality is equivalent to

$$\begin{aligned}m &> \left[(\sigma+1)\,\gamma-\alpha-\frac{\alpha-\gamma}{\sigma-1}\right]\frac{\sigma-1}{\sigma}\\ &= \left[\left(\sigma^2-1\right)\gamma-(\sigma-1)\,\alpha-(\alpha-\gamma)\right]\frac{1}{\sigma}\\ &= \sigma\gamma-\alpha,\end{aligned}$$

which is equivalent to (5.25). Hence, comparing (5.22) and (5.23), we obtain, as $R\to\infty$,

$$\begin{aligned}\lambda_1(\Omega\setminus\overline{U_0})^{\frac{1}{\sigma-1}} &= O(R^{-\frac{\alpha-\gamma}{\sigma-1}})=o\left(R^{\frac{m\sigma}{\sigma-1}+\alpha-(\sigma+1)\gamma}\right)\\ &= o\left(\inf_{\Omega\setminus U_0}\Phi^{\frac{1}{\sigma-1}}\int_{U^c}G_{\overline{U}^c}(x,\cdot)\Phi(\cdot)G^{\sigma}(\cdot,o)d\mu\right),\end{aligned}$$

which yields again (5.6) and thus finishes proof. ∎

6. Application to Schrödinger operator $\Delta - \Psi$

As in Section 2, let (M,μ) be a weighted manifold, d be a distance function on M with precompact balls, and Δ be the weighted Laplace operator on (M,μ). Let Ψ,Φ be smooth non-negative functions on M.

We investigate the problem of uniqueness of a non-negative solution to the following non-linear Schrödinger problem

$$\Delta u - \Psi(x)\,u + \Phi(x)\,u^{\sigma} \le 0 \quad \text{in } M\setminus K. \tag{6.1}$$

Let us introduce the following conditions. As before, we assume that Δ has a finite positive Green function $G(x,y)$.

(Ψ) The function Ψ is Green bounded, that is,

$$\sup_{x\in M}\int_M G(x,y)\Psi(y)d\mu(y) < \infty.$$

(G_+) The Green function satisfies the following estimates:

$$G(x,y) \simeq \begin{cases} d(x,y)^{-\gamma'}, & d(x,y)\le 1,\\ d(x,y)^{-\gamma}, & d(x,y) > 1,\end{cases}$$

for some $\gamma,\gamma' > 0$.

Clearly, $(G_+) \Rightarrow (G)$.

Our uniqueness result for (6.1) is as follows.

Theorem 6.1. *Assume that conditions (V_o), (G_+), (Φ) and (Ψ) are satisfied. If*

$$1 < \sigma \le \frac{\alpha+m}{\gamma},$$

then any non-negative C^2 solution u of the inequality (6.1) is identically equal to zero.

Remark 6.2. In the case $M=\mathbb{R}^n$ $(n>2)$ this result was obtained by Kondratiev, Liskevich and Sobol [20].

Let us cite some previous results that we need for the proof.

Lemma 6.3. [11, Lemmas 10.1, 10.3] *For any smooth non-negative function Ψ on M, there exists a smooth positive function h such that*

$$\Delta h = \Psi h \quad \text{on } M. \tag{6.2}$$

If in addition Ψ is Green bounded, then the equation (6.2) has a solution $h\simeq 1$ on M.

Let us fixed h as in Lemma 6.3 for the rest of this section. Consider the weighted manifold $(M,\tilde\mu)$, where $\tilde\mu$ is a measure on M defined by $d\tilde\mu = h^2 d\mu$. Let $\tilde\Delta$ be the weighted Laplace operator of $(M,\tilde\mu)$. By [11, Lemma 4.3], the operators $\tilde\Delta$ and $\Delta-\Psi$ are related by the Doob transform

$$\tilde\Delta = \frac{1}{h}\circ(\Delta-\Psi)\circ h. \tag{6.3}$$

Let $\tilde V$ be the volume function on $(M,\tilde\mu)$, that is,

$$\tilde V(x,r) := \tilde\mu(B(x,r)) = \int_{B(x,r)} h^2 d\mu.$$

If (Ψ) is satisfied, then we obtain by Lemma 6.3 that

$$\tilde V(x,r) \simeq V(x,r). \tag{6.4}$$

Lemma 6.4. [11, Lemma 4.7] *The Green function $G^{\Psi}(x,y)$ of the operator $\Delta-\Psi$ on (M,μ) and the Green function $\tilde{G}(x,y)$ of the operator $\tilde{\Delta}$ on $(M,\tilde{\mu})$ are related by the identity*

$$G^{\Psi}(x,y)=\tilde{G}(x,y)h(x)h(y).$$

Lemma 6.5. [13, Prop. 5.1] *If (G_+) is satisfied then there exists some constant $C>0$, such that, for all $x,y,z\in M$,*

$$\min\{G(x,z),G(z,y)\}\leq CG(x,y). \tag{6.5}$$

In other words, function $\frac{1}{G(x,y)}$ is a quasi-metric on M.

Lemma 6.6. [11, Remark 10.8] *If conditions (Ψ) and (6.5) are satisfied, then, for all $x\neq y$.*

$$G^{\Psi}(x,y)\simeq G(x,y).$$

Combining Lemmas 6.3, 6.4, 6.5, 6.6, we obtain that, under (G_+) and (Ψ),

$$\tilde{G}(x,y)\simeq G(x,y). \tag{6.6}$$

Proof of Theorem 6.1. Since Ψ is Green bounded, we obtain by Lemma 6.3, that there exists a solution h of (6.2) that satisfies $h\simeq 1$. Set

$$v=u/h,$$

so that v satisfies in $M\setminus K$ the inequality

$$(\Delta-\Psi)(hv)+\Phi h^{\sigma}v^{\sigma}\leq 0.$$

Multiplying the both sides by $\frac{1}{h}$, we obtain by means of (6.3) that

$$\tilde{\Delta}v+\Phi h^{\sigma-1}v^{\sigma}\leq 0.$$

Setting

$$c:=\min_{x\in M}h^{\sigma-1}>0.$$

we obtain

$$\tilde{\Delta}v+c\Phi(x)v^{\sigma}\leq 0. \tag{6.7}$$

Since (V_o) and (G_+) are satisfied on (M,μ), we see from (6.4) and (6.6) that the conditions (V_o) and (G_+) are satisfied also on $(M,\tilde{\mu})$. In particular, we have on $(M,\tilde{\mu})$ all the hypotheses (V_o), (G), (Φ). Applying Theorem 2.1 to the inequality (6.7), we obtain $v\equiv 0$ in $M\setminus K$, which implies $u=hv\equiv 0$. ∎

7. Proof of existence of solutions

In this section we prove Theorem 2.6. Let $\dim M=n>2$.

Lemma 7.1. *If M has bounded geometry and (G) is satisfied, then*

$$V(x,r)\simeq r^{n},\quad \text{for } r<R_0, \tag{7.1}$$

and

$$G(x,y)\simeq d(x,y)^{2-n},\quad \text{for } d(x,y)<R_0. \tag{7.2}$$

Proof. If $r\leq r_0$, then $V(x,r)\simeq r^n$ by definition of bounded geometry. For any $r_0\leq r<R_0$, we have

$$V(x,r)\geq V(x,r_0)\simeq r_0^n\simeq R_0^n\geq cr^n,$$

and

$$V(x,r)\leq V(x,R_0)\simeq R_0^{\alpha}\simeq r_0^n\leq Cr^n,$$

which yields (7.1).

Let us fix $x \in M$ and set $\Omega = B(x, r_0)$. By [22], we have

$$G_\Omega(x,y) \simeq d(x,y)^{2-n}, \quad \text{for } y \in B(x, \tfrac{r_0}{2}).$$

Since $G \geq G_\Omega$, it follows that for any $y \in B(x, \frac{r_0}{2})$

$$G(x,y) \geq c d\,(x,y)^{2-n}. \tag{7.3}$$

Let $y \in B(x, R_0) \setminus B(x, \frac{r_0}{2})$. Then connect x to y by a shortest geodesic that intersects $\partial B(x, \frac{r_0}{2})$ at a point z. Connecting z and y along the geodesic by a sequence of balls of radii $\frac{r_0}{2}$ (the number of such balls is obviously uniformly bounded), and applying a uniform Harnack inequality in such balls, we obtain

$$G(x,y) \geq cG(x,z) \geq cd(x,z)^{2-n} \geq cd(x,y)^{2-n}. \tag{7.4}$$

Thus, we have proved the lower bound in (7.2).

To prove the upper bound for $G(x,y)$, observe that the function

$$G(x,\cdot) - G_\Omega(x,\cdot)$$

is harmonic in Ω and, therefore, it is bounded in Ω by its maximum on $\partial\Omega$. This maximum is equal to $G(x,z)$ for some $z \in \partial\Omega$. Hence, for any $y \in \Omega$,

$$G(x,y) \leq G_\Omega(x,y) + G(x,z).$$

Connecting z to a point $z' \in \partial B(x, R_0)$ by a chain of balls of radii $\frac{1}{2}r_0$ and using again a uniform Harnack inequality, we obtain

$$G(x,z) \leq CG(x,z') \simeq R_0^{-\gamma},$$

whence it follows for $y \in B(x, \frac{r_0}{2})$

$$G(x,y) \leq Cd(x,y)^{2-n} + CR_0^{-\gamma} \leq C'd(x,y)^{2-n}.$$

It remains to prove the same inequality also for $y \in B(x, R_0) \setminus B(x, \frac{r_0}{2})$. Connecting y to a point $y' \in \partial B(x, \frac{r_0}{2})$ by a chain of balls of radii of $\frac{1}{4}r_0$, and applying again a uniform Harnack inequality, we obtain

$$G(x,y) \leq CG(x,y') \leq Cd(x,y')^{2-n} \leq C'd(x,y)^{2-n},$$

which finishes the proof. ■

In the rest of this section we give the proof of Theorem 2.6. We always assume that the hypotheses of this theorem are satisfied, that is, M is a manifold of bounded geometry satisfying (V) and (G).

Without loss of generality, let us take $R_0 = 1$. We introduce two functions

$$v(r) = \begin{cases} r^n, & r \leq 1, \\ r^\alpha, & r \geq 1. \end{cases} \tag{7.5}$$

and

$$g(r) = \begin{cases} r^{2-n}, & r \leq 1, \\ r^{-\gamma}, & r \geq 1. \end{cases} \tag{7.6}$$

It follows easily from (V), (G) and Lemma 7.1 that, for all $x, y \in M$ and $r > 0$,

$$V(x,r) \simeq v(r), \quad G(x,y) \simeq g(d\,(x,y)). \tag{7.7}$$

In what follows we use the notation

$$|x| := d(x,o).$$

Lemma 7.2. *Set $\rho = d(x,y)$. If $\rho \geq |y|$, then*

$$\frac{|x| + |y|}{3} \leq \rho \leq |x| + |y|.$$

Proof. The upper bound is true by the triangle inequality. Since $\rho \geq |y|$, we have

$$\rho \geq |x| - |y| \geq |x| - \rho,$$

whence $2\rho \geq |x|$. Since $\rho \geq |y|$, we obtain $3\rho \geq |x| + |y|$, which was to be proved. ∎

Lemma 7.3. *If F is a non-negative monotone decreasing function on $(0, \infty)$. Then, for any $x_0 \in M$ and $0 \leq a < b \leq +\infty$,*

$$\int_{B(x_0,b)\setminus B(x_0,a)} F(|x|)d\mu(x) \leq C \int_{\frac{1}{4}a}^{b} F(r)v(r)\frac{dr}{r},$$

where the constant C depends only on the constant c in $V(x,r) \leq cv(r)$ as well as on α and n.

Proof. This follows by a standard argument decomposing the integral over M into a sum of the integrals over the annuli $B\left(x_0, 2^{i+1}\right) \setminus B\left(x_0, 2^i\right)$ and then using $V(x_0, r) \leq cv(r)$ and the monotonicity of F. ∎

Consider the following function on $\mathbb{R}_+$:

$$f(r) = g(r) \wedge 1 = \begin{cases} 1, & r \leq 1, \\ r^{-\gamma}, & r \geq 1. \end{cases} \tag{7.8}$$

Proposition 7.4. *If $\sigma > \frac{\alpha}{\gamma}$ then, for the function f from (7.8), we have*

$$\int_M G(x,y)f(|y|)^\sigma d\mu(y) \leq Cf(|x|) \tag{7.9}$$

for all $x \in M$, where the constant C depends only on the constants in the hypotheses of Theorem 2.6.

Proof. Fix $x \in M$ and write for simplicity

$$\rho = \rho(y) = d(x,y).$$

We split the domain of integration in (7.9) into two parts:

$$\{y \in M : |y| \leq \rho(y)\} \quad \text{and} \quad \{y \in M : |y| > \rho(y)\},$$

and estimate separately each of the two integrals.

Step 1. Let us prove that

$$\int_{\{|y| \leq \rho(y)\}} G(x,y)f(|y|)^\sigma d\mu(y) \leq Cf(|x|). \tag{7.10}$$

By Lemma 7.2, we have in the domain of integration $\rho \simeq |x| + |y|$. By (7.7), we have

$$G(x,y) \simeq g(\rho) \simeq g(|x| + |y|),$$

whence from Lemma 7.3

$$\begin{aligned} \int_{\{|y| \leq \rho(y)\}} G(x,y)f(|y|)^\sigma d\mu(y) &\leq C \int_M g(|x| + |y|)f(|y|)^\sigma d\mu(y) \\ &\leq C \int_0^\infty g(|x| + r)f(r)^\sigma v(r)\frac{dr}{r}, \end{aligned}$$

It remains to verify that

$$\int_0^\infty g(|x| + r)f(r)^\sigma v(r)\frac{dr}{r} \leq Cf(|x|). \tag{7.11}$$

Consider further two cases: $|x| \leq 1$ and $|x| > 1$.

Case $|x| \leq 1$. We have

$$\begin{aligned}\int_0^\infty g(|x|+r)f(r)^\sigma v(r)\frac{dr}{r} &\leq \int_0^\infty g(r)f(r)^\sigma v(r)\frac{dr}{r} \\ &\leq \int_0^1 g(r)v(r)\frac{dr}{r} + \int_1^\infty g(r)^{\sigma+1}v(r)\frac{dr}{r} < \infty, \quad (7.12)\end{aligned}$$

where the both integrals in (7.12) converge due to (7.6) and $\sigma\gamma > \alpha$. Hence, we obtain (7.11).

Case $|x| \geq 1$. We have

$$\begin{aligned}\int_0^\infty g(|x|+r)f(r)^\sigma v(r)\frac{dr}{r} &= \int_0^\infty \frac{1}{(|x|+r)^\gamma} f(r)^\sigma v(r)\frac{dr}{r} \\ &\leq \frac{1}{|x|^\gamma}\int_0^\infty \left(\frac{|x|}{|x|+r}\right)^\gamma f(r)^\sigma v(r)\frac{dr}{r} \\ &\leq \frac{1}{|x|^\gamma}\int_0^\infty f(r)^\sigma v(r)\frac{dr}{r} \qquad (7.13) \\ &= Cf(|x|),\end{aligned}$$

because the integral in (7.13) converges at ∞ due to $\sigma\gamma > \alpha$. Hence, we finish the proof of (7.11) and (7.10).

Step 2. Let us prove that

$$\int_{\{|y|>\rho(y)\}} G(x,y)f(|y|)^\sigma d\mu(y) \leq Cf(|x|). \tag{7.14}$$

By Lemma 7.2, we have $|y| \simeq |x| + \rho$. Since $G(x,y) \simeq g(\rho)$, we obtain by Lemma 7.3 that

$$\begin{aligned}\int_{\{|y|>\rho(y)\}} G(x,y)f(|y|)^\sigma d\mu(y) &\leq C\int_M g(\rho)\, f(|x|+\rho)^\sigma\, d\mu(y) \\ &\leq C\int_0^\infty g(r)f(|x|+r)^\sigma v(r)\frac{dr}{r},\end{aligned}$$

so that it remains to prove that

$$\int_0^\infty g(r)f(|x|+r)^\sigma v(r)\frac{dr}{r} \leq Cf(|x|). \tag{7.15}$$

We also consider two cases: $|x| \leq 1$ and $|x| > 1$.

Case $|x| \leq 1$. Since

$$\int_0^\infty g(r)f(|x|+r)^\sigma v(r)\frac{dr}{r} \leq \int_0^\infty g(r)f(r)^\sigma v(r)\frac{dr}{r},$$

the estimate (7.15) follows from (7.12).

Case $|x| \geq 1$. We have

$$\begin{aligned}\int_0^\infty g(r)f(|x|+r)^\sigma v(r)\frac{dr}{r} &= \int_0^\infty g(r)\frac{1}{(|x|+r)^{\sigma\gamma}}v(r)\frac{dr}{r} \\ &= \int_0^\infty g(r)\frac{1}{(|x|+r)^\gamma}\frac{1}{(|x|+r)^{(\sigma-1)\gamma}}v(r)\frac{dr}{r} \\ &\leq \int_0^\infty f(|x|)\, g(r)\frac{1}{(1+r)^{(\sigma-1)\gamma}}r^\alpha\frac{dr}{r} \\ &= Cf(|x|),\end{aligned}$$

where the last integral converges at ∞ by $\sigma\gamma > \alpha$.

Combining the two steps, we complete the proof. ■

Proposition 7.5. *If $\sigma > \frac{\alpha}{\gamma}$, then*

$$\sup_{x\in M}\int_M G(x,y)f(|y|)^{\sigma-1}d\mu(y) < \infty.$$

Proof. Since

$$\begin{aligned}\int_M G(x,y)f(|y|)^{\sigma-1}d\mu(y) &= \int_{\{|y|\le 2|x|\}} G(x,y)f(|y|)^{\sigma-1}d\mu(y)\\ &\quad+\int_{\{|y|>2|x|\}} G(x,y)f(|y|)^{\sigma-1}d\mu(y).\end{aligned}$$

If $|y|\le 2|x|$ then we have by (7.8)

$$\frac{1}{f(|y|)}\le \frac{C}{f(|x|)},$$

and Proposition 7.4 implies

$$\int_{\{|y|\le 2|x|\}} G(x,y)f(|y|)^{\sigma-1}d\mu(y) \le C\int_{|y|\le 2|x|} G(x,y)\frac{f(|y|)^{\sigma}}{f(|x|)}d\mu(y)\le C' < \infty. \tag{7.16}$$

If $|y| > 2|x|$ then by Lemma 7.2, we have $d(x,y)\simeq |y|$ and, hence,

$$G(x,y)\simeq g(|y|).$$

By Lemma 7.3, we obtain

$$\begin{aligned}\int_{\{|y|>2|x|\}} G(x,y)f(|y|)^{\sigma-1}d\mu(y) &\le C\int_M g(|y|)f(|y|)^{\sigma-1}d\mu(y)\\ &\le C\int_0^\infty g(r)f(r)^{\sigma-1}v(r)\frac{dr}{r}\le C', \end{aligned}\tag{7.17}$$

where the last integral converges due to (7.6) and $\sigma\gamma > \alpha$. Combining (7.16) and (7.17), we finish the proof. ■

Theorem 2.6 is contained in the following theorem.

Theorem 7.6. *Assume that M has bounded geometry, and assume also that (V) and (G) are satisfied. If*

$$\sigma > \frac{\alpha}{\gamma},$$

then, for small enough $\lambda > 0$, there exists a positive solution $u\in C^2(M)$ to the equation

$$\Delta u + u^\sigma + \lambda^\sigma f^\sigma = 0 \quad \text{in } M, \tag{7.18}$$

where f is defined as in (7.8). In particular, u is also a solution to differential inequality

$$\Delta u + u^\sigma < 0 \quad \text{in } M.$$

Proof. Define the operator

$$Tu(x) = \int_M G(x,y)\left(u^\sigma(y)+\lambda^\sigma f(|y|)^\sigma\right)d\mu(y), \tag{7.19}$$

acting on the space

$$S_\lambda = \{u\in L^\infty(M) | 0\le u(x)\le \lambda f(|x|).\}$$

where λ is a small enough constant.

It is easy to see that S_λ is a closed set of $L^\infty(M)$. Let us show that

$$TS_\lambda \subset S_\lambda.$$

By Proposition 7.4, we have

$$\begin{aligned} Tu &= \int_M G(x,y)\left(u^\sigma + \lambda^\sigma f(|y|)^\sigma\right) d\mu(y) \\ &\leq 2\lambda^\sigma \int_M G(x,y) f(|y|)^\sigma d\mu(y) \\ &\leq 2C\lambda^\sigma f(|x|). \end{aligned}$$

By choosing λ small enough, we obtain $Tu \in S_\lambda$ and hence $TS_\lambda \subset S_\lambda$.

Let us show that T is a contraction map. For $u_1, u_2 \in S_\lambda$, we have

$$|Tu_1 - Tu_2| \leq \int_M G(x,y)|u_1^\sigma - u_2^\sigma| d\mu(y).$$

Noting that

$$|u_1^\sigma - u_2^\sigma| \leq \sigma \sup\{u_1^{\sigma-1}, u_2^{\sigma-1}\}|u_1 - u_2|,$$

we obtain

$$\begin{aligned} |Tu_1 - Tu_2| &\leq \lambda^{\sigma-1}\sigma \left\|u_1 - u_2\right\|_{L^\infty} \int_M G(x,y) f(|y|)^{\sigma-1} d\mu(y) \\ &\leq C\lambda^{\sigma-1}\sigma \left\|u_1 - u_2\right\|_{L^\infty} \int_M G(x,y) f(|y|)^{\sigma-1} d\mu(y). \end{aligned}$$

Applying Proposition 7.5, we obtain

$$\left\|Tu_1 - Tu_2\right\|_{L^\infty} \leq C\lambda^{\sigma-1}\sigma \left\|u_1 - u_2\right\|_{L^\infty}.$$

Choosing λ small enough we obtain that $C\lambda^{\sigma-1}\sigma < 1$, and hence T is a contraction map. By the Banach fixed point theorem, T has a fixed point u.

In the rest of the proof, we verify that the fixed point u of T belongs to $C^2(M)$ and satisfies (7.18). Denote

$$w := u^\sigma + \lambda^\sigma f^\sigma. \tag{7.20}$$

so that by (7.19)

$$u(x) = \int_M G(x,y) w(y) d\mu(y).$$

Since $u \in S_\lambda$, we have

$$w \leq 2\lambda^\sigma f^\sigma,$$

which implies

$$w(x) \leq C(1 + |x|)^{-\sigma\gamma}. \tag{7.21}$$

Let us first prove that u is locally Hölder, that is, there exists $\theta \in (0,1)$ (depending on n, α, γ and the bounded geometry constants) and $C > 0$ (depending on the all the hypotheses) such that

$$|u(x) - u(x')| \leq Cd(x,x')^\theta,$$

provided $d(x,x')$ is small enough (depending on $|x|$). Set

$$\varepsilon := d(x,x')^{1/N},$$

where $N \geq 2$ is a large enough positive real that will be specified below depending on the constants in the hypotheses (see (7.26)). Assume that $d(x,x')$ is so small that

$$\varepsilon < \frac{1}{2} \min\left(1, r_0, r_0^{-1}, |x|^{-1}\right).$$

It follows that $d(x,x') = \varepsilon^N < \frac{1}{2}\varepsilon$ so that $x' \in B(x, \frac{1}{2}\varepsilon)$. Set

$$R := \varepsilon^{-1} > 2\max(1, r_0, |x|) \tag{7.22}$$

and observe that

$$
\begin{aligned}
|u(x)-u(x')| \leq & \int_{B(x,2\varepsilon)} |G(x,y)-G(x',y)|w(y)d\mu(y) && (7.23)\\
& + \int_{M\setminus B(x,R)} |G(x,y)-G(x',y)|w(y)d\mu(y) && (7.24)\\
& + \int_{B(x,R)\setminus B(x,2\varepsilon)} |G(x,y)-G(x',y)|w(y)d\mu(y). && (7.25)
\end{aligned}
$$

For the integral in (7.23), we have by the boundedness of w and by (7.7),

$$
\begin{aligned}
\int_{B(x,2\varepsilon)} |G(x,y)-G(x',y)|w(y)d\mu(y) &\leq C\int_{B(x,2\varepsilon)} \big(G(x,y)+G(x',y)\big)\,d\mu(y)\\
&\leq C\varepsilon^2.
\end{aligned}
$$

In order to estimate the integral in (7.24), observe that, for $y \in M\setminus B(x,R)$, we have by (7.7)

$$
|G(x,y)-G(x',y)| \leq G(x,y)+G(x',y) \leq CR^{-\gamma}
$$

and by (7.21) and (7.22)

$$
w(y) \leq C|y|^{-\sigma\gamma} \leq C\left(d(x,y)-|x|\right)^{-\sigma\gamma} \leq C\left(\frac{1}{2}d(x,y)\right)^{-\sigma\gamma}.
$$

Using also that $\sigma\gamma > \alpha$, we obtain by Lemma 7.3

$$
\begin{aligned}
\int_{M\setminus B(x,R)} |G(x,y)-G(x',y)|w(y)d\mu(y) &\leq CR^{-\gamma}\int_{M\setminus B(x,R)} d(x,y)^{-\sigma\gamma}\,d\mu(y)\\
&\leq CR^{-\gamma}\int_{\frac{1}{4}R}^{\infty} r^{-\sigma\gamma}r^{\alpha-1}dr\\
&\leq CR^{-\gamma-\sigma\gamma+\alpha} \leq CR^{-\gamma} = C\varepsilon^{\gamma}.
\end{aligned}
$$

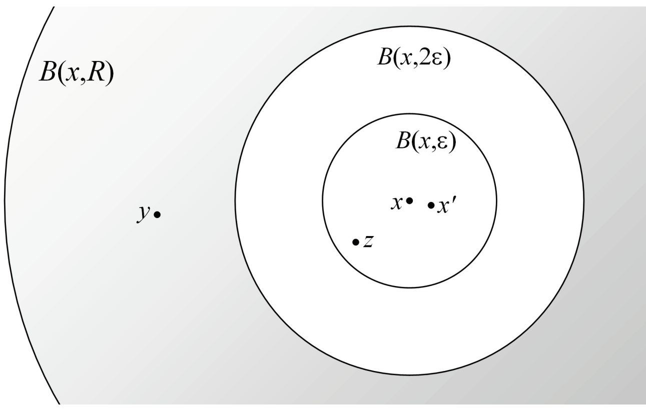

FIGURE 3.

If $y \in B(x,R)\setminus B(x,2\varepsilon)$ then the function $G(\cdot,y)$ is harmonic in $B(x,\varepsilon)$ (see Fig. 3). Applying de Giorgi's theorem in the ball $B(x,r_0)$ that is quasi-isometric to a Euclidean ball (cf. [8, Theorem 8.22]), we obtain

$$|G(x,y)-G(x',y)| \le C\left(\frac{d(x,x')}{\varepsilon}\right)^{\eta} \sup_{z\in B(x,\varepsilon)} G(z,y),$$

where $\eta, C > 0$ depend on the bounded geometry constants and on n. Since $d(z,y) \ge \varepsilon$, by (7.7) we have

$$\sup_{z\in B(x,\varepsilon)} G(z,y) \le C\varepsilon^{2-n}.$$

Using also $d(x,x') = \varepsilon^N$, we obtain, for the integral in (7.25),

$$\begin{aligned}\int_{B(x,R)\setminus B(x,2\varepsilon)} |G(x,y)-G(x',y)|w(y)d\mu(y) &\le CR^{\alpha}\varepsilon^{(N-1)\eta+2-n} \\ &= C\varepsilon^{(N-1)\eta+2-n-\alpha} \\ &= C\varepsilon^2,\end{aligned}$$

where in the last step we choose N from the equation

$$(N-1)\eta = n+\alpha. \tag{7.26}$$

Combing all the above estimates, we obtain from (7.23)-(7.25) that

$$|u(x)-u(x')| \le C\left(\varepsilon^2+\varepsilon^{\gamma}+\varepsilon^2\right) \le C'd\left(x,x'\right)^{\theta},$$

with $\theta = \min(2,\gamma)/N$.

Since f is also locally Hölder continuous, we obtain from (7.20) that w is locally Hölder on M. For any precompact domain $\Omega \subset M$, we obtain by Lemma 8.1 in Appendix that the function

$$u_\Omega(x) := \int_\Omega G_\Omega(x,y)\,w(y)\,d\mu(y)$$

belongs to $C^2(\Omega)$. Since the difference $u-u_\Omega$ is harmonic in Ω in the distributional sense, it follows that $u-u_\Omega$ has a smooth modification in Ω. Therefore, u has a C^2-modification in Ω. Since u is continuous, we conclude that $u \in C^2(\Omega)$. Since Ω is arbitrary, it follows that $u \in C^2(M)$. By [12, Lemma 13.1], the function u solves $\Delta u = -w$, which is equivalent to (7.18). ■

8. Appendix

The following statement is well-know for domains in $\mathbb{R}^n$, but we need it for an arbitrary manifold. We use the notation

$$G_\Omega h(x) = \int_\Omega G_\Omega(x,y)\,h(y)\,d\mu(y).$$

Lemma 8.1. *Let M be an arbitrary weighted manifold and let f be a locally Hölder continuous function on M with some Hölder exponent $\theta \in (0,1)$. Then, for any precompact domain Ω, the function $u := G_\Omega f$ belongs to $C^2(\Omega)$.*

Proof. We use in the proof the fact that the Green function $G_\Omega(x,y)$ has a uniformly bounded integral

$$\int_\Omega G_\Omega(x,y)\,d\mu(y),$$

which implies that, for any bounded function h in Ω,

$$\sup_{x\in\Omega}|G_\Omega h(x)| \le C\sup_{x\in\Omega}|h(x)|. \tag{8.1}$$

Choose a sequence $f_k \in C_0^\infty(M)$ such that

$$f_k \overset{C^{0,\theta}(\Omega)}{\longrightarrow} f$$

and set

$$u_k := G_\Omega f_k. \tag{8.2}$$

By [12, Lemma 13.1], we have $u_k \in C^\infty(\Omega)$ and

$$-\Delta u_k = f_k \quad \text{in } \Omega.$$

Since $f_k \rightrightarrows f$ in Ω as $k \to \infty$ (where $\rightrightarrows$ means the uniform convergence), it follows from (8.1) that

$$u_k \rightrightarrows u \text{ in } \Omega. \tag{8.3}$$

In any small enough precompact chart $U \Subset \Omega$, that is contained in a ball of radius r_0, we can apply the Schauder estimates (cf. [8, Theorem 6.2]), to obtain

$$\|u_k\|_{C^{2,\theta}(U)} \le C(\|u_k\|_{C(\Omega)} + \|f_k\|_{C^{0,\theta}(\Omega)}), \tag{8.4}$$

(where C depends on U). Since the sequence of norms $\|f_k\|_{C(\Omega)}$ is uniformly bounded, we obtain by (8.1) that also the sequence $\|u_k\|_{C(\Omega)}$ is uniformly bounded. By (8.4) we conclude that the sequence $\|u_k\|_{C^{2,\theta}(U)}$ is uniformly bounded. By the Arzelà-Ascoli theorem, there exists a subsequence $\{u_{k_i}\}$ that converges in $C^2(U)$. By (8.3) we conclude that the limit function is u and, hence, $u \in C^2(U)$. It follows that $u \in C^2(\Omega)$, which finishes the proof. ∎

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References

[1] M. T. Barlow, Which values of the volume growth and escape time exponent are possible for a graph? *Rev. Mat. Iberoamericana*, **20** (2004) no. 1, 1–31.
[2] M. F. Bidaut-Véron, Local and global behavior of solutions of quasilinear equations of Emden-Fowler type, *Arch. Rat. Mech. Anal.* **107** (1989) no. 4, 293–324.
[3] M. F. Bidaut-Véron, Local behaviour of solutions of a class of nonlinear elliptic systems, *Adv. Diff. Eq.* **5** (2000) 147–192.
[4] G. Caristi, L. D'Ambrosio, and E. Mitidieri, Liouville Theorems for some nonlinear inequalities, *Proc. Steklov Inst. Math.* **260** (2008) 90–111.
[5] G. Caristi, E. Mitidieri, S. I. Pohozaev, Some Liouville theorems for quasilinear elliptic inequalities, *Dokl. Math.* **79** (2009) 118–124.
[6] S. Y. Cheng, S.-T. Yau, Differential equations on Riemannian manifolds and their geometric applications, *Comm. Pure Appl. Math.* **28** (1975) 333–354.
[7] B. Gidas, J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.* **34** (1981) 525–598.
[8] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer, 1998.
[9] A. Grigor'yan, On the existence of positive fundamental solution of the Laplace equation on Riemannian manifolds (in Russian), *Matem. Sbornik* **128** (1985) no.3, 354–363. Engl. transl.: *Math. USSR Sb.* **56** no.2 (1987) 349–358.
[10] A. Grigor'yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, *Bull. Amer. Math. Soc.* **36** (1999) 135–249.
[11] A. Grigor'yan, Heat kernels on weighted manifolds and applications, *Contemp. Math.* **398** (2006) 93–191.
[12] A. Grigor'yan, *Heat Kernel and Analysis on Manifolds*, AMS/IP, 2009.
[13] A. Grigor'yan, W. Hansen, Lower estimates for a perturbed Green function, *J. Anal. Math.* **104** (2008) 25–58.

[14] A. Grigor'yan, V. A. Kondratiev, On the existence of positive solutions of semi-linear elliptic inequalities on Riemannian manifolds, *Inter. Math. Series* **12** (2010) 203–218.
[15] A. Grigor'yan, Y. Netrusov, S.-T. Yau, Eigenvalues of elliptic operators and geometric applications, *Surveys in Diff. Geom.* **IX** (2004) 147–218.
[16] A. Grigor'yan, Y. Sun, On non-negative of the inequality $\Delta u + u^{\sigma} \leq 0$ on Riemannian manifolds, *Comm. Pure Appl. Math.* **67** (2014) no. 8, 1336–1352.
[17] W. Hebisch, L. Saloff-Coste, Gaussian estimates for Markov chains and random walks on groups, *Ann. Prob.* **21** (1993) 673–709.
[18] M. Kanai, Rough isometries, and combinatorial approximations of geometries of non-compact Riemannian manifolds, *J. Math. Soc. Japan*, **37** (1985) no. 3, 391–413.
[19] M. Kanai, Rough isometries and the parabolicity of Riemannian manifolds, *J. Math. Soc. Japan*, **38** (1986) no. 2, 227–238.
[20] V. Kondratiev, V. Liskevich, Z. Sobol, Second-order semilinear elliptic inequalities in exterior domains, *J. Diff. Eq.* **187** (2003) 429–455.
[21] V. V. Kurta, On the absence of positive solutions to semilinear elliptic equations (in Russian), *Tr. Mat. Inst. Steklova*, **227** (1999) 162–169. Engl. transl.: *Proc. Steklov Inst. Math.* **227** (1999) no.4, 155–162.
[22] W. Littman, G. Stampacchia, H. F. Weinberger, Regular points for elliptic equations with discontinuous coefficients, *Ann. Scuola Norm. Sup. Pisa* (3) **17** (1963) 43–77.
[23] Y. Liu, Y. Wang, J. Xiao, Nonnegative solutions of a fractional sub-Laplacian differential inequality on Heisenberg group. *Dyn. Partial Differ. Eq.* **12** (2015) no. 4, 379–403.
[24] P. Mastrolia, D. D. Monticelli, F. Punzo, Nonexistence results for elliptic differential inequalities with a potential on Riemannian manifolds, *Calc. Var. PDEs*, **54** (2015) no. 2, 1345–1372.
[25] P. Mastrolia, D. D. Monticelli, F. Punzo, Nonexistence of solutions to parabolic differential inequalities with a potential on Riemannian manifolds, *Math. Ann.* **367** (2017) no. 3–4, 929–963.
[26] E. Mitidieri, S. I. Pohozaev, Absence of global positive solutions of quasilinear elliptic inequalities (in Russian), *Dokl. Akad. Nauk,* **359** (1998) no. 4, 456–460.
[27] E. Mitidieri, S. I. Pohozaev, Nonexistence of positive solutions for quasilinear elliptic problems on $\mathbb{R}^N$, *Proc. Steklov Inst. Math.* **227** (1999) 186-216.
[28] E. Mitidieri, S. I. Pohozaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities (in Russian), *Tr. Math. Inst. Steklova*, **234** (2001) 1–384. Engl. transl.: *Proc. Steklov Inst. Math.* **234** (2001) no.3, 1–362.
[29] Y. Sun, Uniqueness result for non-negative solutions of semi-linear inequalities on Riemannian manifolds, *J. Math. Anal. Appl.* **419** (2014) 643–661.
[30] Y. Wang, J. Xiao, A constructive approach to positive solutions of $\Delta_p u + f(u, \nabla u) \leq 0$ on Riemannian manifolds, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **33** (2016) no. 6, 1497–1507.

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