PDEs on manifolds and volume growth

Alexander Grigor'yan
University of Bielefeld and CUHK

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0 Setup

Let M be a Riemannian manifold that is geodesically complete and non-compact. Let $d(x,y)$ denote the geodesic distance on M and μ be the Riemannian measure. Consider geodesic balls

$$B(x,r) = \{y \in M : d(x,y) < r\},$$

that are necessarily precompact, and their volumes:

$$V(x,r) = \mu(B(x,r)).$$

In this survey we collect some results relating the rate growth of $V(x,r)$ as $r \to \infty$ to the properties of elliptic and parabolic PDEs on M.

Recall that the *Laplace-Beltrami operator* Δ on M is given in the local coordinates $x_1, ..., x_n$ as follows:

$$\Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i}\left(\sqrt{\det g} g^{ij} \frac{\partial}{\partial x_j}\right)$$

where $g = (g_{ij})$ is the Riemannian metric tensor and $(g^{ij}) = (g_{ij})^{-1}$. Equivalently, we have $\Delta = \operatorname{div} \circ \nabla$ where ∇ is the Riemannian gradient and div – the corresponding divergence.

The Laplace operator Δ satisfies the Green formula: for all functions $u \in C^2(M)$ and $v \in C_0^1(M)$, we have

$$\int_M v\,\Delta u\,d\mu = -\int_M \langle \nabla u, \nabla v\rangle\,d\mu, \tag{0.1}$$

where $\langle , \rangle$ denotes the g-inner product in tangent spaces.

The *heat kernel* $p_t(x,y)$ of M is the minimal positive fundamental solution of the heat equation

$$\frac{\partial}{\partial t} u = \Delta u$$

on $M \times \mathbb{R}_+$. It is known that the heat kernel exists on any manifold and is a smooth, positive function of $x, y \in M$ and $t > 0$ ([19]). For example, in $\mathbb{R}^n$ we have

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{0.2}$$

The heat kernel on an arbitrary manifold satisfies the following conditions: the symmetry $p_t(x,y) = p_t(y,x)$, the total mass condition:

$$\int_M p_t(x,y) d\mu(y) \le 1,$$

and the semigroup identity

$$p_{t+s}(x,y) = \int_M p_t(x,z)\,p_s(z,y)\,d\mu(z).$$

Consequently, the heat kernel can be used as a transition density for constructing a diffusion process $\{X_t\}_{t\geq 0}$ on M (see [18]). This diffusion process is called *Brownian motion* on M (see Fig. 1).

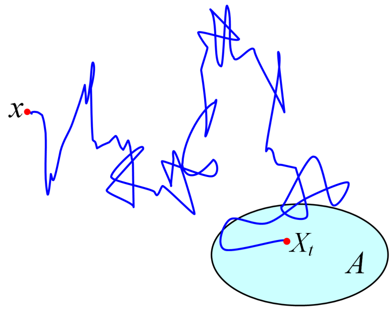

Figure 1:

More precisely, the relation between the heat kernel and Brownian motion is given by the identity

$$\mathbb{P}_x(X_t \in A) = \int_A p_t(x,y)d\mu(y)$$

for any Borel set $A \subset M$ (see Fig. 1). Here $\mathbb{P}_x$ denotes the probability measure in the space Ω_x of all continuous paths starting at the point $x \in M$. If $M = \mathbb{R}^n$ then one obtains in this way the classical Brownian motion in $\mathbb{R}^n$ with the time scaled by the factor 2.

1 Parabolicity and recurrence

A function $u \in C^2(M)$ is called *superharmonic* if $\Delta u \leq 0$.

Definition. *A manifold M is called* parabolic *if any positive superharmonic function on M is constant, and* non-parabolic *otherwise.*

The motivation behind this definition is as follows. By the celebrated uniformization theorem of Koebe-Poincaré, any simply connected Riemann surface M is conformally equivalent to one of the following Riemannian manifolds:

1. $\mathbb{S}^2$ (in this case one says that M is of elliptic type);
2. $\mathbb{R}^2$ (in this case one says that M is of parabolic type);
3. $\mathbb{H}^2$ (in this case one says that M is of hyperbolic type).

In the elliptic case, M is compact, while in the second and third cases M is non-compact. In order to distinguish the parabolic and hyperbolic types intrinsically, one can ask about existence of positive superharmonic functions on M. The point is that

- the superharmonicity is preserved by conformal transformations;

- $\mathbb{H}^2$ has plenty of non-trivial positive (and bounded) superharmonic functions, while in $\mathbb{R}^2$ any positive superharmonic function is constant.

Hence, a Riemann surface of the parabolic type is also a parabolic manifold in the sense of the above definition, and a Riemann surface of the hyperbolic type is a non-parabolic[1] manifold.

For any compact set $K \subset M$ define its capacity by

$$\operatorname{cap}(K) = \inf_{\varphi \in C_0^\infty(M),\ \varphi|_K \equiv 1} \int_M |\nabla \varphi|^2 \, d\mu.$$

The following theorem gives equivalent characterizations of the parabolicity

Theorem 1.1 *([17, Thm. 5.1]) The following properties are equivalent:*

1. *M is parabolic.*
2. *Any bounded superharmonic function on M is constant.*
3. *There exists no positive fundamental solution of $-\Delta$ on M.*
4. *For all/some $x, y \in M$ we have*

$$\int_1^\infty p_t(x, y) \, dt = \infty. \tag{1.1}$$

5. *For any compact set $K \subset M$, we have* $\operatorname{cap}(K) = 0$.
6. *Brownian motion on M is recurrent.*

The *Green function* of Δ is defined by

$$g(x, y) = \int_0^\infty p_t(x, y) \, dt.$$

The condition (1.1) is equivalent to the fact that $g(x, y) \equiv \infty$. If M is non-parabolic then $g(x, y) < \infty$ for all $x \neq y$ and, moreover, $g(x, y)$ is the minimal positive fundamental solution of $-\Delta$.

A celebrated theorem of Polya (1921) says that Brownian motion in $\mathbb{R}^n$ is recurrent if and only if $n \leq 2$. Indeed, one can see from the explicit formula (0.2) for the heat kernel that the condition (1.1) holds if and only if $n \leq 2$.

Surprisingly enough, there exists a rather good sufficient condition for the recurrence of Brownian motion in terms of the volume function. Let us fix a reference point x_0 and set

$$V(r) = V(x_0, r).$$

Theorem 1.2 ([9], [31], [40]) *If*

$$\int_1^\infty \frac{r dr}{V(r)} = \infty \tag{1.2}$$

then M is parabolic.

[1] We use the adjective "non-parabolic" rather than "hyperbolic" because the notion of "hyperbolic manifold" is used in Differential Geometry in a different sense.

For example, (1.2) is satisfied provided $V(r) \leq Cr^2$ for large r or $V(r) \leq Cr^2 \log r$. In particular, this theorem recovers and explains the aforementioned theorem of Polya.

Proof. Let $u \in C^2(M)$ be a positive superharmonic function on M. Choose any Lipschitz function f on M with compact support. Multiplying the inequality $-\Delta u \geq 0$ by $\frac{f^2}{u}$ and integrating using the Green formula (0.1) and the product rule for ∇, we obtain

$$\begin{aligned}
0 &\leq -\int_M \frac{f^2}{u} \Delta u \, d\mu \\
&= \int_M \langle \nabla \frac{f^2}{u}, \nabla u \rangle d\mu \\
&= \int_M \frac{\langle \nabla f^2, \nabla u \rangle}{u} d\mu + \int_M \langle \nabla \frac{1}{u}, \nabla u \rangle f^2 d\mu \\
&= 2\int_M \frac{\langle \nabla f, \nabla u \rangle}{u} f \, d\mu - \int_M \frac{|\nabla u|^2}{u^2} f^2 d\mu,
\end{aligned}$$

whence

$$\begin{aligned}
\int_M \frac{|\nabla u|^2}{u^2} f^2 d\mu &\leq 2\int_M \frac{\langle \nabla f, \nabla u \rangle}{u} f \, d\mu \\
&\leq 2\left(\int_M |\nabla f|^2 \, d\mu\right)^{1/2} \left(\int_M \frac{|\nabla u|^2}{u^2} f^2 d\mu\right)^{1/2}.
\end{aligned}$$

It follows that

$$\int_M \frac{|\nabla u|^2}{u^2} f^2 d\mu \leq 4\int_M |\nabla f|^2 \, d\mu. \tag{1.3}$$

Set $\rho(x) = d(x, x_0)$ and choose $f(x)$ in the form $f(x) = \varphi(\rho(x))$ where $\varphi(r)$ is a function of $r \in [0, +\infty)$ yet to be defined (see Fig. 2).

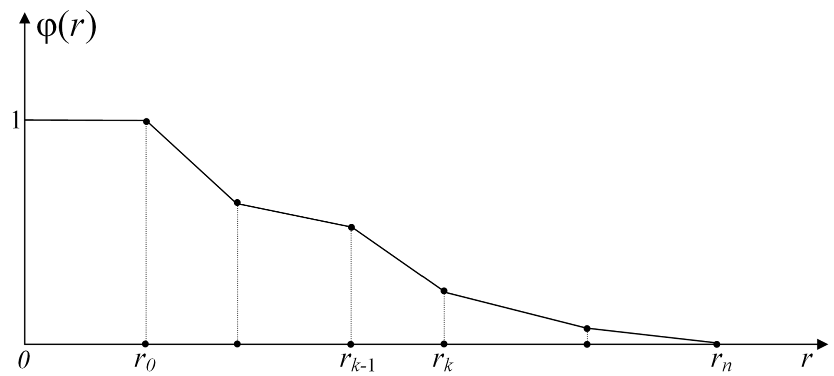

Figure 2: Function $\varphi(r)$

Fix a finite sequence

$$0 < r_0 < r_1 < ... < r_n < \infty$$

and define function φ by the conditions that it is continuous and piecewise linear on $[0, +\infty)$,

$$\varphi(r) = 1 \text{ if } 0 \leq r \leq r_0, \quad \varphi(r) = 0 \text{ if } r \geq r_n, \tag{1.4}$$

and, for any $k = 1, ..., n$,

$$\varphi'(r) = -a\frac{r_k - r_{k-1}}{V(r_k)} \quad \text{if } r_{k-1} < r < r_k, \tag{1.5}$$

where a is a positive constant that is chosen to be compatible with (1.5).

Indeed, we have

$$1 = \varphi(r_n) - \varphi(r_0) = \int_{r_0}^{r_n} \varphi'(r)\,dr = \sum_{k=1}^{n} \int_{r_{k-1}}^{r_k} \varphi'(r)\,dr = -a\sum_{k=1}^{n} \frac{(r_k - r_{k-1})^2}{V(r_k)},$$

whence we obtain the following value for a:

$$a = \left(\sum_{k=1}^{n} \frac{(r_k - r_{k-1})^2}{V(r_k)}\right)^{-1}.$$

Clearly, $\varphi(r)$ is a Lipschitz function, which implies that $f = \varphi \circ \rho$ is Lipschitz on M.

Denote for simplicity $B_r = B(x_0, r)$. By (1.4), $\operatorname{supp} f \subset \overline{B}_{r_n}$ and, since the balls are relatively compact, $f \in Lip_0(M)$. Obviously, $\nabla\varphi = 0$ in B_{r_0} and outside B_{r_n}. Since $\nabla f = \varphi'\nabla\rho$ and $|\nabla\rho| \le 1$ a.e. we have[2] for any $k = 1, ..., n$

$$|\nabla f| \le a\frac{r_k - r_{k-1}}{V(r_k)} \text{ in } B_{r_k} \setminus B_{r_{k-1}} \text{ a.e.}, \tag{1.6}$$

which implies

$$\begin{aligned}\int_M |\nabla f|^2\,d\mu &= \sum_{k=1}^{n} \int_{B_{r_k} \setminus B_{r_{k-1}}} |\nabla f|^2\,d\mu \\ &\le a^2 \sum_{k=1}^{n} \frac{(r_k - r_{k-1})^2}{V(r_k)^2} V(r_k) = a. \end{aligned} \tag{1.7}$$

On the other hand, using the monotonicity of $V(r)$, we obtain

$$\begin{aligned}\int_{r_1}^{r_n} \frac{rdr}{V(r)} &= \sum_{k=1}^{n-1} \int_{r_k}^{r_{k+1}} \frac{rdr}{V(r)} \\ &\le \sum_{k=1}^{n-1} \frac{1}{V(r_k)} \int_{r_k}^{r_{k+1}} rdr\end{aligned}$$

[2]Strictly speaking, we can apply the chain rule $\nabla v = \varphi'(\rho)\nabla\rho$ and, hence, obtain (1.6) only in the open set $B_{r_k} \setminus \overline{B}_{r_{k-1}}$. Then (1.6) in $B_{r_k} \setminus B_{r_{k-1}}$ follows from the fact that the boundary of any geodesic ball has measure zero. However, the proof of this fact requires more Riemannian geometry than we would like to use here. Without this fact, one can argue as follows. The volume function $V(r)$ is monotone and, hence, the set S of the points of discontinuity of $V(r)$ is at most countable. We can choose the sequence $\{r_k\}$ to avoid S, which implies that

$$\mu(\partial B_{r_k}) = \lim_{\varepsilon \to 0} (V(r_k + \varepsilon) - V(r_k)) = 0.$$

$$= \frac{1}{2} \sum_{k=1}^{n-1} \frac{r_{k+1}^2 - r_k^2}{V(r_k)}.$$

Let us now specify $\{r_k\}$ to be a geometric sequence: $r_k = 2^k r_0$, that is, $r_{k+1} = 2r_k$. We obtain

$$r_{k+1}^2 - r_k^2 = 3r_k^2 = 12 (r_k - r_{k-1})^2,$$

which implies

$$\int_{r_1}^{r_n} \frac{rdr}{V(r)} \le 6 \sum_{k=1}^{n-1} \frac{(r_k - r_{k-1})^2}{V(r_k)} \le 6a^{-1}.$$

Comparing with (1.7), we conclude that

$$\int_M |\nabla f|^2 \, d\mu \le 6 \left(\int_{r_1}^{r_n} \frac{rdr}{V(r)} \right)^{-1}.$$

Returning to (1.3) and using the fact that $f = 1$ on B_{r_0}, we obtain

$$\int_{B_{r_0}} \frac{|\nabla u|^2}{u^2} d\mu \le 24 \left(\int_{r_1}^{r_n} \frac{rdr}{V(r)} \right)^{-1}.$$

We can still choose r_0 and n. By the hypothesis (1.2), for any $r_0 > 0$ and $\varepsilon > 0$, there exists n so big that

$$\int_{r_1}^{r_n} \frac{rdr}{V(r)} = \int_{2r_0}^{2^n r_0} \frac{rdr}{V(r)} > \varepsilon^{-1},$$

which implies

$$\int_{B_{r_0}} \frac{|\nabla u|^2}{u^2} d\mu \le 24\varepsilon.$$

Since r_0 and ε are arbitrary, we conclude $\nabla u \equiv 0$ and $u = \text{const}$, which was to be proved. ∎

The condition (1.2) is sharp: if $f(r)$ is a smooth convex function on $(0, +\infty)$ such that $f'(r) > 0$ and

$$\int_1^\infty \frac{rdr}{f(r)} < \infty,$$

then there is a non-parabolic manifold such that $V(r) = f(r)$ for large r. On the other hand, the condition (1.2) is not necessary for parabolicity: there exist parabolic manifolds with arbitrarily large volume function $V(r)$ as it follows from [17, Prop. 3.1].

Theorem 1.3 (Cheng-Yau [5]) *If there exists a sequence $r_k \to \infty$ such that, for some $C > 0$ and all k*

$$V(r_k) \le Cr_k^2, \tag{1.8}$$

then M is parabolic.

Proof. It suffices to show that (1.8) implies (1.2) so that Theorem 1.3 follows from Theorem 1.2. More precisely, let us prove that if $f(r)$ is a positive increasing function on $(0, +\infty)$ and there exists a sequence $\{r_k\}_{k=1}^\infty$ such that $r_k \to \infty$ and

$$f(r_k) \le Cr_k^2 \text{ for all } k \ge 1,$$

then

$$\int_{r_1}^{\infty} \frac{rdr}{f(r)} = \infty.$$

Without loss of generality, we can assume that $\frac{r_{k+1}}{r_k} \to \infty$. We have

$$\begin{aligned} \int_{r_1}^{\infty} \frac{rdr}{f(r)} &= \sum_{k=1}^{\infty} \int_{r_k}^{r_{k+1}} \frac{rdr}{f(r)} \\ &\geq \sum_{k=1}^{\infty} \frac{1}{f(r_{k+1})} \int_{r_k}^{r_{k+1}} rdr \\ &\geq \frac{1}{2C} \sum_{k=1}^{\infty} \frac{r_{k+1}^2 - r_k^2}{r_{k+1}^2}. \end{aligned} \tag{1.9}$$

We are left to observe that the series

$$\sum_{k=1}^{\infty} \frac{r_{k+1}^2 - r_k^2}{r_{k+1}^2} = \sum_{k=1}^{\infty} \left(1 - \frac{r_k^2}{r_{k+1}^2}\right)$$

diverges as $\frac{r_k}{r_{k+1}} \to 0$. ∎

03.04.25 **Lecture 2**

2 Stochastic completeness

A manifold M is called *stochastically complete* if for all $x \in M$ and $t > 0$

$$\int_M p_t(x, y)\, d\mu(y) = 1.$$

Here are some equivalent characterizations of the stochastic completeness.

Theorem 2.1 ([17, Thm. 6.2]) *The following conditions are equivalent.*

1. *M is stochastically complete.*
2. *For some/any $\lambda > 0$, any bounded solution v to $\Delta v - \lambda v = 0$ on M is identical zero.*
3. *For some/any $T \in (0, \infty]$, the Cauchy problem*

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } M \times (0, T) \\ u|_{t=0} = 0 \end{cases} \tag{2.1}$$

   has the only bounded solution $u \equiv 0$.

4. *The lifetime of Brownian motion $\{X_t\}$ on M is equal to ∞ almost surely.*

As above, we fix a reference point $x_0 \in M$ and denote for simplicity $B_r = B(x_0, r)$ and $V(r) = \mu(B_r)$.The following theorem provides a volume test for stochastic completeness.

Theorem 2.2 ([10]) *If for some* $r_0 > 0$

$$\int_{r_0}^{\infty} \frac{rdr}{\log V(r)} = \infty \tag{2.2}$$

then M *is stochastically complete.*

In particular, M is stochastically complete provided

$$V(r) \leq \exp\left(Cr^2\right) \tag{2.3}$$

or even if

$$V(r_k) \leq \exp\left(Cr_k^2\right) \tag{2.4}$$

for a sequence $r_k \to \infty$. That (2.3) implies the stochastic completeness was also proved by different methods also in [7], [32], [39].

The condition (2.2) is sharp: if $f(r)$ is a smooth convex function such that $f'(r) > 0$ and

$$\int_{r_0}^{\infty} \frac{rdr}{f(r)} < \infty$$

then there exists a stochastically incomplete manifold with $V(r) = \exp(f(r))$. On there other hand, there are stochastically complete manifolds with arbitrarily large volume function as it follows from [17, Prop. 3.2].

Theorem 2.2 follows from the following more general fact.

Theorem 2.3 ([10]) *Let* M *be a geodesically complete manifold, and let* $u(t,x)$ *be a solution to the Cauchy problem (2.1) in* $M \times (0,T)$. *Assume that, for some* $x_0 \in M$ *and for all* R *large enough,*

$$\int_0^T \int_{B_R} u^2(t,x)\, d\mu(x)\, dt \leq \exp\left(f(R)\right), \tag{2.5}$$

where $f(r)$ *is a positive monotone increasing function on* $(0, +\infty)$ *such that*

$$\int^{\infty} \frac{rdr}{f(r)} = \infty. \tag{2.6}$$

Then $u \equiv 0$ *in* $M \times (0,T)$.

Theorem 2.3 provides the uniqueness class (2.5) for the Cauchy problem. The condition (2.6) holds if, for example, $f(r) = Cr^2$, but fails for $f(r) = Cr^{2+\varepsilon}$ when $\varepsilon > 0$.

Let us show how Theorem 2.3 implies Theorem 2.2. By Theorem 2.1, it suffices to verify that the only bounded solution to the Cauchy problem (2.1) is $u \equiv 0$. Indeed, if u is a bounded solution of (2.1), then setting

$$S := \sup |u| < \infty$$

we obtain

$$\int_0^T \int_{B_R} u^2(t,x)d\mu(x) \le S^2TV(R) = \exp\left(f\left(R\right)\right),$$

where $f(r) := \log\left(S^2TV(r)\right)$. It follows from the hypothesis (2.2) that the function f satisfies (2.6). Hence, we conclude by Theorem 2.3 that $u \equiv 0$.

Before we embark on the proof of Theorem 2.3, let us mention the following consequence of it for $M = \mathbb{R}^n$: if $u\left(t,x\right)$ is a solution to (2.1) in $\mathbb{R}^n \times (0,T)$ satisfying the condition

$$|u(t,x)| \le C \exp\left(C\left|x\right|^2\right), \tag{2.7}$$

then $u \equiv 0$. Moreover, the same is true if u satisfies instead of (2.7) a more general condition

$$|u(t,x)| \le C \exp\left(f\left(|x|\right)\right), \tag{2.8}$$

where $f\left(r\right)$ is a convex increasing function on $(0, +\infty)$ satisfying (2.6).

Indeed, it suffices to treat the condition (2.8). In $\mathbb{R}^n$ set $x_0 = 0$. Since $V\left(R\right) = cR^n$, (2.8) implies that

$$\int_0^T \int_{B_R} u^2(t,x)\, d\mu(x)dt \le CR^n \exp\left(f\left(R\right)\right) = C\exp(\widetilde{f}\left(R\right)),$$

where $\widetilde{f}\left(r\right) := f\left(r\right) + n\log r$. The convexity of f implies that $\log r \le Cf\left(r\right)$ for large enough r. Hence, $\widetilde{f}\left(r\right) \le Cf\left(r\right)$ and function $\widetilde{f}$ also satisfies the condition (2.6). By Theorem 2.3, we conclude $u \equiv 0$.

The class of functions u satisfying (2.7) is called the *Tikhonov class*, and the conditions (2.8) and (2.6) define the *Täcklind class*. The uniqueness of the Cauchy problem in $\mathbb{R}^n$ in each of these classes is a classical result.

Proof of Theorem 2.3. The main technical part of the proof is the following claim.

Claim. *Let $u\left(t,x\right)$ solve the heat equation in $M \times (0,T)$, and assume that u satisfies (2.5) with a function f as in (2.6). Then, for any $R > 0$ and $a, b \in (0,T)$, satisfying the condition*

$$0 < b - a \le \frac{R^2}{8f(4R)}, \tag{2.9}$$

the following inequality holds:

$$\int_{B_R} u^2(b,\cdot)d\mu \le \int_{B_{4R}} u^2(a,\cdot)d\mu + \frac{4}{R^2}. \tag{2.10}$$

Let us first show how this Claim allows to prove that any solution u to (2.1), satisfying (2.5), is identical 0. Fix $R > 0$ and $t \in (0,T)$. For any non-negative integer k, set

$$R_k = 4^k R$$

and, for any $k \ge 1$, choose (so far arbitrarily) a number τ_k to satisfy the condition

$$0 < \tau_k \le c\frac{R_k^2}{f(R_k)}, \tag{2.11}$$

where $c = \frac{1}{128}$. Then define a decreasing sequence of times $\{t_k\}$ inductively by $t_0 = t$ and $t_k = t_{k-1} - \tau_k$ (see Fig. 3).

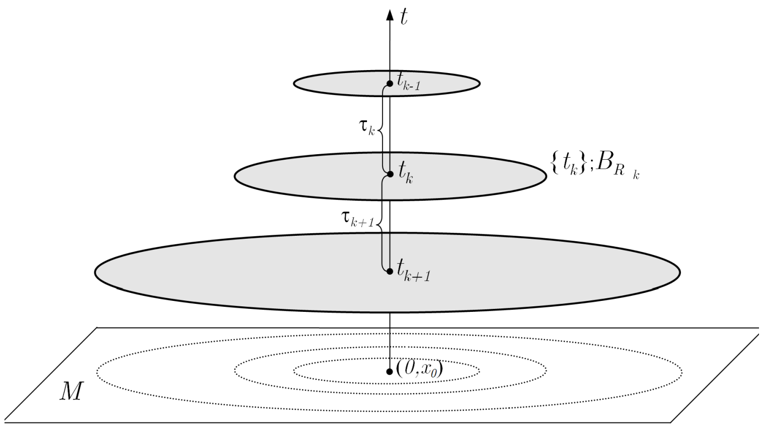

Figure 3: The sequence of the balls B_{R_k} and the time moments t_k.

If $t_k \geq 0$ then function u satisfies all the conditions of the Claim with $a = t_k$ and $b = t_{k-1}$ because

$$t_{k-1} - t_k = \tau_k \leq c \frac{R_k^2}{f(R_k)} = \frac{1}{128} \frac{(4R_{k-1})^2}{f(4R_{k-1})} = \frac{1}{8} \frac{R_{k-1}^2}{f(4R_{k-1})}.$$

Hence, we obtain from (2.10)

$$\int_{B_{R_{k-1}}} u^2(t_{k-1}, \cdot) d\mu \leq \int_{B_{R_k}} u^2(t_k, \cdot) d\mu + \frac{4}{R_{k-1}^2}, \tag{2.12}$$

which implies by induction that

$$\int_{B_R} u^2(t, \cdot) d\mu \leq \int_{B_{R_k}} u^2(t_k, \cdot) d\mu + \sum_{i=1}^{k} \frac{4}{R_{i-1}^2}. \tag{2.13}$$

If it happens that $t_k = 0$ for some k then, by the initial condition in (2.1),

$$\int_{B_{R_k}} u^2(t_k, \cdot) d\mu = 0.$$

In this case, it follows from (2.13) that

$$\int_{B_R} u^2(t, \cdot) d\mu \leq \sum_{i=1}^{\infty} \frac{4}{R_{i-1}^2} = \frac{C}{R^2},$$

which implies by letting $R \to \infty$ that $u(\cdot, t) \equiv 0$.

Hence, to finish the proof, it suffices to construct, for any $R > 0$ and $t \in (0, T)$, a sequence $\{t_k\}$ as above that vanishes at some finite k. The condition $t_k = 0$ is equivalent to

$$t = \tau_1 + \tau_2 + \ldots + \tau_k\,. \tag{2.14}$$

The only restriction on τ_k is the inequality (2.11). The hypothesis that $f(r)$ is an increasing function implies that

$$\int_R^\infty \frac{rdr}{f(r)} \le \sum_{k=0}^\infty \int_{R_k}^{R_{k+1}} \frac{rdr}{f(r)} \le \sum_{k=0}^\infty \frac{R_{k+1}^2}{f(R_k)},$$

which together with (2.6) yields

$$\sum_{k=1}^\infty \frac{R_k^2}{f(R_k)} = \infty.$$

Therefore, the sequence $\{\tau_k\}_{k=1}^\infty$ can be chosen to satisfy simultaneously (2.11) and

$$\sum_{k=1}^\infty \tau_k = \infty.$$

By reducing some of τ_k, we can achieve (2.14) for any finite t, which finishes the proof.

Proof of the above Claim. Let $\rho(x)$ be a Lipschitz function on M (to be specified below) with the Lipschitz constant 1, that is, $|\nabla\rho| \le 1$. Fix a real $s \notin [a,b]$ (also to be specified below) and consider the following the function

$$\xi(t,x) := \frac{\rho^2(x)}{4(t-s)},$$

that is well defined on $M\times [a,b]$. Since $|\nabla\rho| \le 1$, we have, for all $t \ne s$,

$$|\nabla\xi(t,x)| \le \frac{\rho(x)}{2(t-s)}.$$

Since

$$\frac{\partial\xi}{\partial t} = -\frac{\rho^2(x)}{4(t-s)^2},$$

we obtain

$$\boxed{\frac{\partial\xi}{\partial t} + |\nabla\xi|^2 \le 0.} \tag{2.15}$$

For a given $R > 0$, define a continuous function $\varphi(x)$ on M by

$$\varphi(x) = \begin{cases} 1, & x \in B_{2R}, \\ 0, & x \notin B_{3R}, \\ \text{linear in } d(x,x_0), & x \in B_{3R} \setminus B_{2R} \end{cases}$$

(see Fig. 4).

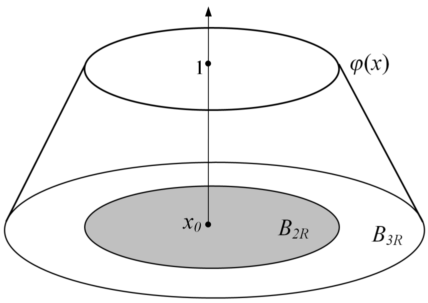

Figure 4: Function $\varphi(x)$

Since the function $d(x, x_0)$ is Lipschitz with the Lipschitz constant 1, we obtain that φ is also Lipschitz and $|\nabla\varphi| \leq 1/R$. Since all the balls in M are relatively compact sets, we have $\varphi \in Lip_0(M)$.

Consider the function $u\varphi^2 e^{\xi}$ as a function of x for any fixed $t \in [a, b]$. This function also belongs to $Lip_0(M)$. Multiplying the heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

by $u\varphi^2 e^{\xi}$ and integrating it over $M \times [a, b]$, we obtain

$$\int_a^b \int_M \frac{\partial u}{\partial t} u\varphi^2 e^{\xi} d\mu dt = \int_a^b \int_M (\Delta u)\, u\varphi^2 e^{\xi} d\mu dt. \tag{2.16}$$

The time integral on the left hand side is equal to:

$$\frac{1}{2}\int_a^b \frac{\partial (u^2)}{\partial t}\varphi^2 e^{\xi} dt = \frac{1}{2}\left[u^2\varphi^2 e^{\xi}\right]_a^b - \frac{1}{2}\int_a^b \frac{\partial \xi}{\partial t} u^2\varphi^2 e^{\xi} dt. \tag{2.17}$$

Using the Green formula to evaluate the spatial integral on the right hand side of (2.16), we obtain

$$\int_M (\Delta u)\, u\varphi^2 e^{\xi} d\mu = -\int_M \langle \nabla u, \nabla(u\varphi^2 e^{\xi})\rangle d\mu.$$

Applying the product rule and the chain rule to compute $\nabla(u\varphi^2 e^{\xi})$, we obtain

$$\begin{aligned}
-\langle \nabla u, \nabla(u\varphi^2 e^{\xi})\rangle &= -\left|\nabla u\right|^2 \varphi^2 e^{\xi} - \langle \nabla u, \nabla \xi\rangle u\varphi^2 e^{\xi} - 2\langle \nabla u, \nabla\varphi\rangle u\varphi e^{\xi} \\
&\leq -\left|\nabla u\right|^2 \varphi^2 e^{\xi} + \left|\nabla u\right|\left|\nabla \xi\right|\left|u\right| \varphi^2 e^{\xi} + \left(\frac{1}{2}\left|\nabla u\right|^2 \varphi^2 + 2\left|\nabla\varphi\right|^2 u^2\right) e^{\xi} \\
&= \left(-\frac{1}{2}\left|\nabla u\right|^2 + \left|\nabla u\right|\left|\nabla \xi\right|\left|u\right|\right)\varphi^2 e^{\xi} + 2\left|\nabla\varphi\right|^2 u^2 e^{\xi}.
\end{aligned}$$

Combining with (2.16), (2.17), and using (2.15) in the form $\frac{\partial\xi}{\partial t} \leq -\left|\nabla\xi\right|^1$, we obtain

$$\begin{aligned}
\left[\int_M u^2\varphi^2 e^{\xi} d\mu\right]_a^b &= \int_a^b \int_M \frac{\partial \xi}{\partial t} u^2\varphi^2 e^{\xi}\, d\mu dt + 2\int_a^b \int_M (\Delta u)\, u\varphi^2 e^{\xi} d\mu dt \\
&\leq \int_a^b \int_M \left(-\left|\nabla\xi\right|^2 u^2 - \left|\nabla u\right|^2 + 2\left|\nabla u\right|\left|\nabla\xi\right|\left|u\right|\right)\varphi^2 e^{\xi} d\mu dt + 4\int_a^b \int_M \left|\nabla\varphi\right|^2 u^2 e^{\xi} d\mu dt \\
&= -\int_a^b \int_M \left(\left|\nabla\xi\right|\left|u\right| - \left|\nabla u\right|\right)^2 \varphi^2 e^{\xi} d\mu dt + 4\int_a^b \int_M \left|\nabla\varphi\right|^2 u^2 e^{\xi} d\mu dt,
\end{aligned}$$

whence

$$\int_M u^2(b,\cdot)\varphi^2 e^{\xi(b,\cdot)} d\mu - \int_M u^2(a,\cdot)\varphi^2 e^{\xi(a,\cdot)} d\mu \leq 4\int_a^b \int_M \left|\nabla\varphi\right|^2 u^2 e^{\xi} d\mu dt. \tag{2.18}$$

Using $\varphi|_{B_R} = 1$, $\varphi \leq \mathbf{1}_{B_{4R}}$ and $|\nabla\varphi| \leq \frac{1}{R}\mathbf{1}_{B_{4R}\setminus B_{2r}}$, we obtain from (2.18)

$$\int_{B_R} u^2(b,\cdot)e^{\xi(b,\cdot)}d\mu \leq \int_{B_{4R}} u^2(a,\cdot)e^{\xi(a,\cdot)}d\mu + \frac{4}{R^2}\int_a^b \int_{B_{4R}\setminus B_{2R}} u^2 e^{\xi} d\mu dt. \tag{2.19}$$

Let us now specify $\rho(x)$ and s. Set $\rho(x)$ to be the distance function from the ball B_R, that is,

$$\rho(x) = (d(x, x_0) - R)_+$$

(see Fig. 5).

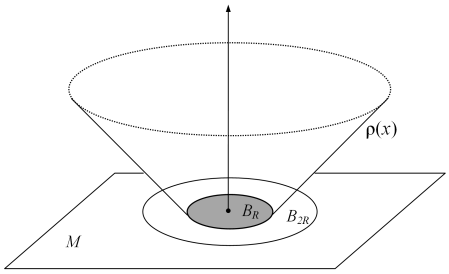

Figure 5: Function $\rho(x)$.

Set $s = 2b - a$ so that, for all $t \in [a,b]$,

$$b - a \leq s - t \leq 2(b-a),$$

whence

$$\xi(t,x) = -\frac{\rho^2(x)}{4(s-t)} \leq -\frac{\rho^2(x)}{8\,(b-a)} \leq 0. \tag{2.20}$$

Consequently, we can drop the factor e^{ξ} on the left hand side of (2.19) because $\xi = 0$ in B_R, and drop the factor e^{ξ} in the first integral on the right hand side of (2.19) because $\xi \leq 0$. Clearly, if $x \in B_{4R}\setminus B_{2R}$ then $\rho(x) \geq R$, which together with (2.20) implies that in $B_{4R}\setminus B_{2R} \times [a,b]$

$$\xi(t,x) \leq -\frac{R^2}{8\,(b-a)}.$$

Hence, we obtain from (2.19)

$$\int_{B_R} u^2(b,\cdot)d\mu \leq \int_{B_{4R}} u^2(a,\cdot)d\mu + \frac{4}{R^2}\exp\left(-\frac{R^2}{8\,(b-a)}\right)\int_a^b\int_{B_{4R}} u^2 d\mu dt.$$

By (2.5) we have

$$\int_a^b\int_{B_{4R}} u^2 d\mu dt \leq \exp\left(f(4R)\right)$$

whence

$$\int_{B_R} u^2(b,\cdot)d\mu \leq \int_{B_{4R}} u^2(a,\cdot)d\mu + \frac{4}{R^2}\exp\left(-\frac{R^2}{8\,(b-a)} + f(4R)\right).$$

Finally, applying the hypothesis (2.9), we obtain (2.10). ■

Remark 2.4 Using of the factor e^ξ in the above proof is motivated by the following observation that goes back to Aronson [1] in the case of $\mathbb{R}^n$: if $u(t,x)$ is a semigroup solution to the heat equation, that is,

$$u(t,x) = \int_M p_t(x,y) f(y) d\mu(y)$$

with $f \in L^2(M)$, and $\xi(t,x)$ is a locally Lipschitz function on $M \times (0,\infty)$ such that

$$\frac{\partial \xi}{\partial t} + \frac{1}{2} |\nabla \xi|^2 \le 0, \tag{2.21}$$

then the function

$$J(t) := \int_M u(t,x)^2 e^{\xi(t,x)} d\mu(x)$$

is monotone decreasing. For the proof first observe that *it* suffices to prove a similar statement when $u(t,x)$ is a semigroup solution in a precompact open set $\Omega \subset M$, that is,

$$u(t,x) = \int_\Omega p_t^\Omega(x,y) f(y) d\mu(y)$$

where $p_t^\Omega(x,y)$ is the heat kernel in Ω with the Dirichlet boundary condition. Indeed, having proved that in Ω, we can choose then an exhausting sequence of Ω_k and pass to the limit as $\Omega_k \to M$.

Since in Ω the function ξ is bounded and $u(t,\cdot) \in W_0^{1,2}(\Omega)$, we can apply the Green formula in the next computation. We have

$$\begin{aligned}
\frac{d}{dt} J(t) &= \int_\Omega \partial_t \left(u^2 e^\xi\right) d\mu = \int_\Omega \left(2u_t u e^\xi + u^2 \xi_t e^\xi\right) d\mu \\
&\le 2 \int_\Omega \left(\Delta u \, u e^\xi - \frac{1}{2} u^2 |\nabla \xi|^2 e^\xi\right) d\mu \\
&= -2 \int_\Omega \left(\left\langle \nabla u, \nabla \left(u e^\xi\right)\right\rangle + \frac{1}{4} u^2 |\nabla \xi|^2 e^\xi\right) d\mu \\
&= -2 \int_\Omega \left(|\nabla u|^2 e^\xi + \langle \nabla u, \nabla \xi \rangle u e^\xi + \frac{1}{4} u^2 |\nabla \xi|^2 e^\xi\right) d\mu \\
&= -2 \int_\Omega \left(\nabla u + \frac{1}{2} u \nabla \xi\right)^2 e^\xi d\mu
\end{aligned}$$

whence $\frac{dJ}{dt} \le 0$ follows. See [19, Thm 12.1] for more details and justification.

3 Escape rate of Brownian motion

Fix a reference point $x_0 \in M$ and set $|x| = d(x, x_0)$. Let $\{X_t\}_{t\geq 0}$ be Brownian motion on M. An increasing positive function $R(t)$ of $t \in \mathbb{R}_+$ is called *an upper rate function* for Brownian motion if we have $|X_t| < R(t)$ for all t large enough with probability 1, that is,

$$\mathbb{P}_{x_0}\left(\exists T \ \ \forall t > T \quad |X_t| < R(t)\right) = 1.$$

Hence, for large enough t, X_t is contained in the ball $B(x_0, R(t))$ almost surely, as on Fig. 6.

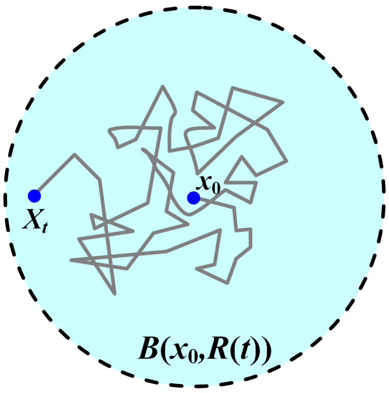

Figure 6: An upper radius $R(t)$

Note that an upper rate function may exist only if M is stochastically complete. For example, in $\mathbb{R}^n$ the following function

$$R(t) = \sqrt{(4+\varepsilon)\, t \ln \ln t}$$

is an upper rate function for any $\varepsilon > 0$, which follows from Khinchin's law of iterated log that says

$$\limsup_{t\to\infty} \frac{|X_t|}{\sqrt{4t \ln \ln t}} = 1 \quad \text{a.s.}$$

Theorem 3.1 ([22], [16]) *Assume that, for all r large enough,*

$$V(r) \leq Cr^N, \tag{3.1}$$

with some $N, C > 0$. Then the following function is an upper rate function:

$$R(t) = \sqrt{2Nt \log t}. \tag{3.2}$$

A similar result holds for simple random walks on graphs: it was proved by Hardy and Littlewood in 1914 for $\mathbb{Z}$ and in [4] for arbitrary graphs.

Under assumption (3.1), the upper rate function (3.2) is almost optimal (cf. [23]), in particular, $\log t$ here cannot be replaced by $\log \log t$.

In this example, M is a *model manifold*, that is, $\mathbb{R}^n$ with the following spherically symmetric metric given in the polar coordinates (r, θ):

$$ds^2 = dr^2 + h^2(r)d\theta^2$$

where $d\theta^2$ is the standard metric on $\mathbb{S}^{n-1}$ and $h(r)$ is a smooth positive function on $(0, \infty)$, such that $h(r) = r$ for $r \leq 1$. In this case

$$V(r) = \omega_n \int_0^r h^{n-1}(s)ds.$$

The function h is chosen so that $V(r)$ is as on Fig. 7:

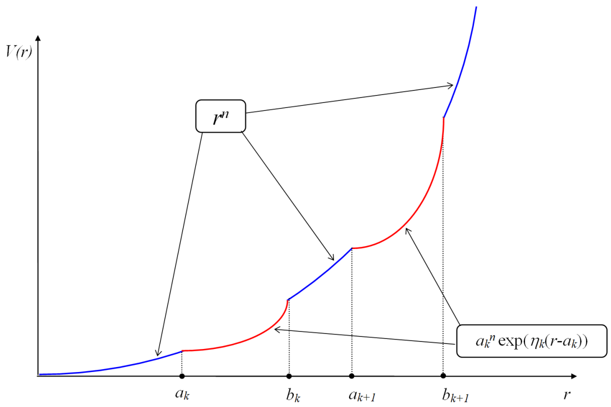

Figure 7: Function $V(r)$

In particular, $V(r)$ satisfies (3.1) with $N = n$ but small inclusions of negative curvature (red areas on the picture) speed up Brownian motion to make the rate function (3.2) almost optimal. More precisely, on this manifold we have

$$R(t) \geq \sqrt{ct \log^{1-\frac{2}{n}} t}\,. \tag{3.3}$$

The next result is a generalization of Theorem 3.1 an for arbitrary volume function.

Theorem 3.2 ([21], [30]) *Assume that M is geodesically complete and that*

$$\int^{\infty} \frac{rdr}{\log V(r)} = \infty. \tag{3.4}$$

Define a function $\varphi(t)$ by:

$$t = \int_{r_0}^{\varphi(t)} \frac{rdr}{\log V(r) + \log \log r}. \tag{3.5}$$

Then $R(t) = C\varphi(Ct)$ is an upper rate function.

If $V(r) \geq c \log r$ for some $c > 0$ then (3.5) can be replaced by

$$t = \int_{r_0}^{\varphi(t)} \frac{rdr}{\log V(r)}. \tag{3.6}$$

In other words, the rate of divergence of the integral (3.4) determines an upper rate function!

Example. If $V(r) = Cr^N$ then we obtain from (3.6)

$$t \simeq \frac{\varphi^2(t)}{\log \varphi(t)}$$

whence $R(t) \simeq \varphi(t) \simeq \sqrt{t \log t}$ that matches (3.2).

Example. If $V(r) = \exp(r^\alpha)$ where $0 < \alpha < 2$ then

$$t \simeq \varphi(t)^{2-\alpha}$$

whence $R(t) = Ct^{\frac{1}{2-\alpha}}$.

Example. If $V(r) = \exp(r^2)$ then

$$t \simeq \log \varphi(t)$$

whence $R(t) = \exp(Ct)$.

Example. Let $V(r) \leq C \log r$. Then

$$t \simeq \frac{\varphi^2(t)}{\log \log \varphi(t)}$$

and we obtain an upper rate function

$$R(t) = C\sqrt{t \log \log t}.$$

4 Heat kernel lower bounds

Here we show some results on pointwise lower bounds of the heat kernel that use only the volume function.

Theorem 4.1 ([6]) *Assume that, for some $x \in M$ and all $r \geq r_0 > 0$,*

$$V(x,r) \leq Cr^N, \tag{4.1}$$

for some $C, N > 0$. Then, for all large enough t,

$$p_t(x,x) \geq \frac{1/4}{V(x, \sqrt{Kt \log t})}, \tag{4.2}$$

where $K = K(x, r_0, C, N) > 0$. Consequently, for some $c > 0$,

$$p_t(x,x) \geq \frac{c}{(t \log t)^{N/2}}. \tag{4.3}$$

If M has non-negative Ricci curvature then by the theorem of Li-Yau [34] the heat kernel satisfies on the diagonal the following two-sided estimate

$$p_t(x,x) \simeq \frac{1}{V(x,\sqrt{t})} \tag{4.4}$$

for all $x \in M$ and $t > 0$. Hence, the lower bound (4.3) differs from the best possible estimate (4.4) by the log-factor. However, under the hypothesis (4.1) alone, the lower bound (4.2) is optimal and cannot be essentially improved, which is the case for the model manifold from Section 3 (cf. [23]).

Proof. For the fixed point x, set $B_r = B(x,r)$ and $V(r) = V(x,r)$. Using the semigroup identity and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} p_{2t}(x,x) &= \int_M p_t^2(x,\cdot)d\mu \geq \int_{B_r} p_t^2(x,\cdot)d\mu \\ &\geq \frac{1}{V(r)}\left(\int_{B_r} p_t(x,\cdot)d\mu\right)^2. \end{aligned} \tag{4.5}$$

Since M is complete and the condition (4.1) obviously implies (3.4), we obtain by Theorem 2.2 that M is stochastically complete, that is

$$\int_M p_t(x,\cdot)d\mu = 1.$$

Using also that $p_t(x,x) \geq p_{2t}(x,x)$ we obtain from (4.5)

$$p_t(x,x) \geq \frac{1}{V(r)}\left(1 - \int_{M\setminus B_r} p_t(x,\cdot)d\mu\right)^2. \tag{4.6}$$

Choose $r = r(t)$ so that

$$\int_{M\setminus B_{r(t)}} p_t(x,\cdot)d\mu \leq \frac{1}{2}. \tag{4.7}$$

Assuming that (4.7) holds, we obtain from (4.6)

$$p_t(x,x) \geq \frac{1/4}{V(r(t))}.$$

To match (4.2), we need the following estimate of $r(t)$:

$$r(t) \leq \sqrt{Kt\log t}. \tag{4.8}$$

Let us prove that (4.7) holds with a function $r(t)$ satisfying (4.8) with some K. Setting $\rho = d(x,\cdot)$ and fixing some $D > 2$, we obtain by the Cauchy-Schwarz inequality

$$\begin{aligned} \left(\int_{M\setminus B_r} p_t(x,\cdot)d\mu\right)^2 &\leq \int_M p_t^2(x,\cdot)\exp\left(\frac{\rho^2}{Dt}\right)d\mu \int_{M\setminus B_r} \exp\left(-\frac{\rho^2}{Dt}\right)d\mu \\ &= E_D(t)\int_{M\setminus B_r} \exp\left(-\frac{\rho^2}{Dt}\right)d\mu, \end{aligned} \tag{4.9}$$

where

$$E_D(t) := \int_M p_t^2(x,\cdot)\exp\left(\frac{\rho^2}{Dt}\right)d\mu \tag{4.10}$$

It is known that if $D > 2$ then $E_D(t)$ is finite and monotone decreasing in t ([14], [19, Thm. 12.1 and Cor. 15.9]). The latter follows from the fact that. for the function $\xi(t,\cdot) = \frac{\rho^2}{Dt}$, we have

$$|\nabla\xi| \leq 2\frac{\rho}{Dt} \quad \text{and} \quad \partial_t\xi = -\frac{\rho^2}{Dt^2}$$

whence

$$\partial_t\xi + \frac{1}{2}|\nabla\xi|^2 \leq -\frac{\rho^2}{Dt^2} + \frac{1}{2}\left(2\frac{\rho}{Dt}\right)^2 = \left(-\frac{1}{D}+\frac{2}{D^2}\right)\frac{\rho^2}{t^2} \leq 0$$

as $D \geq 2$, which matches (2.21).

In particular, we have, for all $t > t_0$,

$$E_D(t) \leq E_D(t_0) < \infty. \tag{4.11}$$

Since x is fixed, we can consider $E_D(t_0)$ as a constant.

Let us now estimate the integral in (4.9) assuming that

$$r = r(t) \geq r_0. \tag{4.12}$$

By splitting the complement of B_r into the union of the annuli

$$B_{2^{k+1}r} \setminus B_{2^k r}, \quad k = 0, 1, 2, ...,$$

and using the hypothesis (4.1), we obtain

$$\begin{aligned} \int_{M\setminus B_r} \exp\left(-\frac{\rho^2}{Dt}\right)d\mu &= \sum_{k=0}^{\infty}\int_{B_{2^{k+1}r}\setminus B_{2^k r}} \exp\left(-\frac{\rho^2}{Dt}\right)d\mu \\ &\leq \sum_{k=0}^{\infty}\exp\left(-\frac{4^k r^2}{Dt}\right)V\left(2^{k+1}r\right) \\ &\leq Cr^N\sum_{k=0}^{\infty}2^{N(k+1)}\exp\left(-\frac{4^k r^2}{Dt}\right). \end{aligned} \tag{4.13}$$

Assuming

$$\frac{r^2}{Dt} \geq 1, \tag{4.14}$$

the sum in (4.13) is majorized by a geometric series whence

$$\int_{M\setminus B_r}\exp\left(-\frac{\rho^2}{Dt}\right)d\mu \leq Cr^N\exp\left(-\frac{r^2}{Dt}\right). \tag{4.15}$$

Set

$$r(t) = \sqrt{Kt\log t}, \tag{4.16}$$

where the constant K will be chosen below; in any case, it will be larger than D. If so then assuming that

$$t \geq t_0 := \max\left(r_0^2, 3\right)$$

we obtain that both conditions (4.12) and (4.14) are satisfied.

Substituting (4.16) into (4.15), we obtain

$$\int_{M\setminus B_{r(r)}} \exp\left(-\frac{\rho^2}{Dt}\right) d\mu \le C(Kt\log t)^{\frac{N}{2}} \exp\left(-\frac{K\log t}{D}\right) = CK^{N/2}\frac{(\log t)^{\frac{N}{2}}}{t^{\frac{K}{D}-\frac{N}{2}}}. \quad (4.17)$$

If K is large enough then the function in the right hand side of (4.17) is monotone decreasing in $t > t_0$, whence by (4.9) and (4.11)

$$\left(\int_{M\setminus B_{r(t)}} p_t(x,\cdot)d\mu\right)^2 \le CK^{N/2}\frac{(\log t_0)^{N/2}}{t_0^{\frac{K}{D}-\frac{N}{2}}} E_D(t_0). \quad (4.18)$$

Finally, choosing K even larger, we can make the right hand side arbitrarily small, which finishes the proof. ■

Theorem 4.2 ([6]) *Fix $x \in M$ and assume that the function $V(x,r)$ is doubling, that is, for all $r > 0$,*

$$V(x,2r) \le CV(x,r).$$

Assume also that, for all $t > 0$,

$$p_t(x,x) \le \frac{C}{V(x,\sqrt{t})}.$$

Then, for all $t > 0$,

$$p_t(x,x) \ge \frac{c}{V(x,\sqrt{t})}.$$

5 Recurrence of subordinated process

For any $\alpha \in (0,2)$, the operator $(-\Delta)^{\alpha/2}$ is the generator of a jump process $\{X_t^{(\alpha)}\}$ on M that is called *the α-process*. It is a natural generalization of the symmetric stable Levy process of index α in $\mathbb{R}^n$. By a general semigroup theory, the Green function $g^{(\alpha)}(x,y)$ of $(-\Delta)^{\alpha/2}$ is given by

$$g^{(\alpha)}(x,y) = \int_0^\infty t^{\alpha/2-1}p_t(x,y)dt,$$

and the recurrence of $\{X_t^{(\alpha)}\}$ is equivalent to $g^{(\alpha)} \equiv \infty$, that is, to

$$\int^\infty t^{\alpha/2-1}p_t(x,x)dt = \infty. \quad (5.1)$$

Theorem 5.1 ([17, Thm. 16.2]) *If, for some $x \in M$ and all large enough r*

$$V(x,r) \le Cr^\alpha, \quad (5.2)$$

then $\{X_t^{(\alpha)}\}$ is recurrent.

Proof. Indeed, by Theorem 4.1 we have

$$p_t(x,x) \geq \frac{c}{t^{\alpha/2}\log^{\alpha/2} t}.$$

Substituting into (5.1) we see that the integral diverges. ■

Conjecture. *The process* $\{X_t^{(\alpha)}\}$ *is recurrent provided*

$$\int_1^\infty \frac{r^{\alpha-1}dr}{V(x,r)} = \infty.$$

The answer is positive under the hypotheses of Theorem 4.2. For example, this is the case when M has non-negative Ricci curvature.

6 Bounded solutions of Schrödinger equations

Let $Q(x)$ be a nonnegative continuous function on M, $Q \not\equiv 0$. Consider the equation

$$\Delta u - Qu = 0 \tag{6.1}$$

and ask if any bounded solution to (6.1) is identical zero. In this case we say that the *Liouville property* holds for (6.1). In fact, one can prove that (6.1) has a non-zero bounded solution if and only if it has a positive solution.

By Theorem 2.1, if $Q = \mathrm{const} > 0$ then the Liouville property for (6.1) is equivalent to the stochastic completeness of M. If Q is a compactly supported function then one can show that the Liouville property for (6.1) is equivalent to the parabolicity of M. Hence, it is interesting to find conditions for Liouville property also for a general function Q.

Fix a reference point $x_0 \in M$, set $|x| = d(x,x_0)$ and denote

$$q(r) = \inf_{|x|=r} Q(x) \quad \text{and} \quad F(r) = \int_0^{r/2} \sqrt{q(t)}dt.$$

Theorem 6.1 ([12]) *If there is a sequence $r_k \to \infty$ such that for some $C > 0$ and all k*

$$V(r_k) \leq C r_k^2 \exp\left(CF(r_k)^2\right) \tag{6.2}$$

then the Liouville property is satisfied for (6.1).

Example. Let $Q \equiv 1$.Then we have $q \equiv 1$, $F(r) = r/2$, and (6.2) becomes $V(r_k) \leq \exp(Cr_k^2)$, which coincides with the condition (2.4) for the stochastic completeness.

Example. Let Q have a compact support. Since $q(r) = 0$ for large enough r, we obtain that $F(r) = \mathrm{const}$ for large r, and (6.2) becomes $V(r_l) \leq Cr_k^2$, which coincides with the sufficient condition (1.8) for the parabolicity.

Example. Assume that, for all large $|x|$ and some $c > 0$,

$$Q(x) \geq \frac{c}{|x|^2 \log |x|}.$$

Then

$$F(r) \geq \int_2^{r/2} \frac{c}{t\sqrt{\log t}} dt \simeq \sqrt{\log r}$$

so that (6.2) is satisfied provided

$$V(r) \leq Cr^N$$

for some $C, N > 0$ and all large r. Hence, in this case (6.1) has no positive solution. For example, this is the case for $M = \mathbb{R}^n$.

On the other hand, if in $\mathbb{R}^n$

$$Q(x) \leq \frac{C}{|x|^2 \log^{1+\varepsilon} |x|}$$

then (6.1) has a positive bounded solution in $\mathbb{R}^n$.

Problem. *Find conditions for the Liouville property for (6.1) without using pointwise information about Q.*

7 Semilinear PDEs

Consider on M the inequality

$$\Delta u + u^\sigma \leq 0 \tag{7.1}$$

and ask if it has a non-negative solution u on M except for $u \equiv 0$. Here $\sigma > 1$ is a given parameter. Note that any non-negative solution of (7.1) is superharmonic. Hence, if M is parabolic then u must be identical zero. In particular, this is the case if $V(r) \leq Cr^2$.

Otherwise (7.1) may have positive solutions. For example, in $\mathbb{R}^n$ with $n > 2$ the inequality (7.1) has a positive solution if and only if $\sigma > \frac{n}{n-2}$ (cf. [35]). As above, fix $x_0 \in M$ and set $V(r) = V(x_0, r)$.

Theorem 7.1 ([26]) *Assume that, for all large r,*

$$V(r) \leq Cr^p \log^q r, \tag{7.2}$$

where

$$p = \frac{2\sigma}{\sigma - 1} \quad \text{and} \quad q = \frac{1}{\sigma - 1}. \tag{7.3}$$

Then any nonnegative solution of (7.1) is identical zero.

The values of the exponents p and q in (7.3) are sharp: if either $p > \frac{2\sigma}{\sigma-1}$ or $p = \frac{2\sigma}{\sigma-1}$ and $q > \frac{1}{\sigma-1}$ then there is a manifold satisfying (7.2) where the inequality (7.1) has a positive solution.

Theorem 7.1 can be equivalently reformulated as follows: if, for some $\alpha > 2$

$$V(r) \leq Cr^\alpha \log^{\frac{\alpha-2}{2}} r, \tag{7.4}$$

then, for any $\sigma \leq \frac{\alpha}{\alpha-2}$, any nonnegative solution of (7.1) is identical zero. In this form it contains the aforementioned result for $\mathbb{R}^n$ as in $\mathbb{R}^n$ (7.4) is satisfied with $\alpha = n$.

Conjecture. ([27]) *If*

$$\int^{\infty}\frac{r^{2\sigma-1}dr}{V\left(r\right)^{\sigma-1}}=\infty \tag{7.5}$$

then any nonnegative solution of (7.1) is identical zero.

In particular, the function (7.4) satisfies (7.5) with $\sigma=\frac{\alpha}{\alpha-2}$.

Similar results for a more general inequality $\Delta u+Qu^{\sigma}\leq 0$ with $Q\left(x\right)\geq 0$ were obtained in [37]. In the view of results of Section 6, it may be interesting to investigate the question of existence of positive solutions for a semilinear equation $\Delta u-Qu^{\sigma}=0$.

Analogous problems for semilinear heat equation were addressed in [38].

8 Biparabolic manifolds

A function $u\in C^{4}(M)$ is called *bi-superharmonic* if $\Delta u\leq 0$ and $\Delta^{2}u\geq 0$.

For example, let M be nonparabolic and consider the Green operator

$$Gf=\int_{0}^{\infty}g\left(x,y\right)f\left(y\right)d\mu\left(y\right),$$

where

$$g\left(x,y\right)=\int_{0}^{\infty}p_{t}(x,y)dt$$

is the Green function of Δ. If f is non-negative and superharmonic then the function $u=Gf$ is bi-superharmonic (provided it is finite) because

$$\Delta u=-f\leq 0\ \text{ and }\ \Delta^{2}u=-\Delta f\geq 0.$$

Here is another example of bi-superharmonic functions in a precompact domain $\Omega\subset M$. Let τ_{Ω} be the first exit time from Ω of Brownian motion X_{t}. If f is a non-negative continuous function on $\partial\Omega$ then the function

$$u\left(x\right)=\mathbb{E}_{x}\left(\tau_{\Omega}f\left(X_{\tau_{\Omega}}\right)\right)$$

solves the following boundary value problem

$$\left\{\begin{array}{l}\Delta^{2}u=0\ \text{ in }\Omega\\ \Delta u|_{\partial\Omega}=-f,\\ u|_{\partial\Omega}=0,\end{array}\right.$$

and, hence, is bi-superharmonic in Ω.

Definition. *A manifold M is called* biparabolic*, if any positive bi-superharmonic function on M is harmonic, that is $\Delta u=0$.*

Note that the notion of parabolicity also admits a similar equivalent definition: M is parabolic if and only if any positive superharmonic function on M is harmonic.

Recall that the parabolicity of M is equivalent to $g(x,,y)\equiv\infty$. One can prove that M is biparabolic if and only if $g^{(2)}(x,y)\equiv\infty$ where

$$g^{(2)}(x,y):=\int_{M}g(x,z)g\left(z,y\right)d\mu(z).$$

Using the Green function $g(x,y) = c_n |x-y|^{2-n}$ in $\mathbb{R}^n$ $(n>2)$, one can show that $\mathbb{R}^n$ is biparabolic if and only if $n \leq 4$. If $n > 4$ then $u(x) = |x|^{-(n-4)}$ is an example of a (weakly) bi-superharmonic but not harmonic function.

Theorem 8.1 ([28]) *If, for all large enough r,*

$$V(r) \leq C \frac{r^4}{\log r} \tag{8.1}$$

then M is biparabolic.

The condition (8.1) is not far from optimal in the following sense: for any $\beta > 1$ there exists a manifold M with

$$V(r) \leq C\, r^4 \log^\beta r$$

that is not biparabolic.

Conjecture. *If*

$$V(r) \leq Cr^4 \log r \quad \text{or even} \quad \int^\infty \frac{r^3 dr}{V(r)} = \infty,$$

then M is biparabolic.

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