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FUNDAMENTAL SOLUTION OF THE HEAT EQUATION ON AN ARBITRARY RIEMANNIAN MANIFOLD

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We consider upper estimates for the Green function of the heat equation on an arbitrary smooth connected Riemannian manifold M. We define the Green function G(t, x; y) as the limit of the Green functions G_Ω for the precompact domains $\Omega \subset M$ as $\Omega \to M$. If the manifold M has boundary, then we will always assume the Neumann homogeneous condition to be fulfilled on the boundary and also consider only those Ω for which $\partial\Omega$ is transversed to ∂M.

Let us denote the geodesic distance between two points x, $y \in M$ by $|x - y|$ and the geodesic ball of radius r with center at x by B_r^x. If N is a submanifold, then we will denote its volume, corresponding to the dimension, by $|N|$.

THEOREM 1. Let the following isoperimetric inequality be fulfilled in a precompact geodesic ball B_ρ^x: for each domain $Q \Subset B_\rho^x$ that has smooth boundary ∂Q, transversal to ∂M,

$$|\partial Q| \geqslant k\,|Q|^{(n-1)/n}, \tag{1}$$

where $n \geq \dim M$ and $K > 0$.

Then

$$\int_M G^2(t, x; y)\, e^{|x-y|^2/2\beta t}\, dy \leqslant \frac{ck^{-n}}{\min(t^{n/2}, \rho^n)}, \tag{2}$$

where β is an arbitrary number greater than 1 and c depends on n and β.

Remarks. 1. We can consider the following arbitrary parabolic equation on the manifold M:

$$p(x)\frac{\partial u}{\partial t} = \operatorname{div}(a(t, x)\nabla u), \tag{3}$$

where p(x) is a smooth positive function on M and $a(t, x)$ is a positive self-adjoint operator $T_xM \to T_xM$ that depends smoothly on t and x. In particular, if $p \equiv 1$ and $a \equiv \mathrm{id}$, then we get the heat equation.

The estimate (2) is also valid for the Green function of Eq. (3) with the only difference that the parabolicity constant of (3) occurs in the exponent.

2. In the case where M is a domain in R^n, Gushchin [1] has obtained an estimate, similar to (2). However, Gushchin's proof is highly complicated, uses a stronger isoperimetric inequality, and does not give the exact exponent.

3. We obtain the following point estimate from the estimate (2) by standard arguments, using the semigroup property of the Green function:

If the isoperimetric inequalities with the constant k_1 and k_2 are fulfilled in the balls $B_{\rho_1}^x$ and $B_{\rho_2}^x$, respectively, then

$$G(t, x; y) \leqslant \frac{c\,(k_1k_2)^{-n/2}}{f(t)} \exp\left(-\frac{|x-y|^2}{4\beta t}\right), \tag{4}$$

where $f(t) = \min(t^{n/4}, \rho_1{}^{n/2})\min(t^{n/4}, \rho_2{}^{n/2})$; in particular, $f(t) = t^{n/2}$ for small t.

Proof of Theorem 1. It is sufficient to prove the estimate (2) for the Green function G_Ω of a precompact domain $\Omega \supset B_\rho^x$. We use the following facts.

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1. Integral Principle of Maximum. If u(t, y) is the solution of the equation $\partial u/\partial t = \Delta u$ in the cylinder $\Omega \times [0, +\infty)$, such that $u|_{\partial\Omega} = 0$, then the integral

$$\int_\Omega u^2(t, y) \exp \frac{|x-y|^2}{2\beta(t-s)} dy$$

is a nonincreasing function of t in the intervals (0, s) and (s, +∞). Here $x \in \Omega$, $s > 0$, $\beta \geq 1$ are arbitrary and fixed. See [2, 3] for the proof.

2. Moser's Lemma. If u is a positive solution of the heat equation in the cylinder $B^x_{\sqrt{t}} \times [0, t]$, and if the isoperimetric inequality (1) is fulfilled in the ball $B^x_{\sqrt{t}}$, then

$$u^2(t, x) \leqslant \frac{ck^{-n}}{t^{(n+2)/2}} \int_0^t \int_{B^x_{\sqrt{t}}} u^2(\tau, y)\, d\tau\, dy.$$

Moser's proof [4] for the case of R^n uses only the isoperimetric inequality (or rather, a consequence of it – the Sobolev inequality) and passes to our case (see also [3, 5, 6]).

At first, we prove (2) for $\sqrt{t} < \rho$. Let u be the function from the condition of the integral principle of maximum. Since $\sqrt{t} < \rho$, we can apply Moser's lemma and get

$$u^2(t, x) \leqslant \frac{ck^{-n}}{t^{n/2}} \sup_{0 \leqslant \tau \leqslant t} \int_{B^x_{\sqrt{t}}} u^2(\tau, y)\, dy.$$

Observing that

$$\exp \frac{|x-y|^2}{2\tau - 2\beta t} \geqslant \text{const} > 0$$

for $|x - y| \leq \sqrt{t}$, $\tau \leq t$, and $\beta > 1$, and using the integral principle of maximum with respect to τ, we get

$$u^2(t, x) \leqslant \frac{ck^{-n}}{t^{n/2}} \sup_{0 \leqslant \tau \leqslant t} \int_{B^x_{\sqrt{t}}} u^2(\tau, y) \exp \frac{|x-y|^2}{2\tau - 2\beta t} dy \leqslant \frac{ck^{-n}}{t^{n/2}} \int_\Omega u^2(0, y) \exp\left(-\frac{|x-y|^2}{2\beta t}\right) dy. \quad (5)$$

On the other hand, by the definition of the Green function,

$$u(t, x) = \int_\Omega G_\Omega(t, x; y)\, u(0, y)\, dy. \quad (6)$$

The initial function u(0, y) can be chosen arbitrarily. Let us set $u(0, y) = G_\Omega(t, x; y) \cdot \exp \frac{|x-y|^2}{2\beta t}$. Then the integrals in the right-hand sides of (5) and (6) coincide and we get

$$u^2(t, x) \leqslant \frac{ck^{-n}}{t^{n/2}} \int_\Omega G^2_\Omega(t, x; y) \exp \frac{|x-y|^2}{2\beta t} dy = \frac{ck^{-n}}{t^{n/2}} u(t, x).$$

Hence

$$\int_\Omega G^2_\Omega(t, x; y) \exp \frac{|x-y|^2}{2\beta t} dy = u(t, x) \leqslant \frac{ck^{-n}}{t^{n/2}}. \quad (7)$$

The estimate (2) for $\sqrt{t} \geq \rho$ follows from the fact that the integral (7) is a nonincreasing function of t. The theorem is proved.

Let us observe that to obtain the point estimate of the Green function, we traditionally consider the integral

$$\int_{|y-x|>\rho} G^2(t, x; y)\, dy. \quad (8)$$

Although it can be estimated for $\sqrt{t} < \rho$ by the Aronson–Serrin method [7, 8], a variant of which we have used in the proof of Theorem 1, great difficulties arise for $\sqrt{t} > \rho$ (see, e.g., [1]). We have easily avoided them by using an integral with exponential weight in place of (8).

In spite of the simplicity of proof, majority of the known upper estimates of the Green function follow from Theorem 1.

Here are some of them.

1. If the isoperimetric inequality (1) is fulfilled everywhere in M, then it follows from (4) that

$$G(t,x;y)\leqslant\frac{c}{t^{n/2}}\exp\left(-\frac{|x-y|^2}{4\beta t}\right).$$

In the case of R^n with a metric that is quasiisometric with the Euclidean metric we get the classical estimate of Aronson [7] and, in the case of a domain in R^n we get Gushchin's estimate [1].

2. Let M be a complete manifold with the sectional curvature bounded on both sides. As shown in [2], for a fixed ρ_2 the isoperimetric constant $k_2 \geq C^{-1}e^{-Cr}$, where $r = |x - y|$, and $C > 0$; whence we get

$$G(t,x;y)\leqslant\frac{c}{(\operatorname{th} t)^{n/2}}\exp\left(cr-\frac{r^2}{4\beta t}\right).$$

This estimate is suitable for all $t > 0$, in distinction from the estimate [2]

$$G(t,x;y)\leqslant\frac{C(T)}{t^{n/2}}\exp\left(-\frac{r^2}{4\beta t}\right)\quad\text{for } t\in[0,T]$$

[the constant C(T) can grow exponentially with respect to T].

3. Let the numbers ρ and k be fixed for all $x \in M$. Then it follows from (4) that

$$G(t,x;y)\leqslant\frac{c}{(\operatorname{th} t)^{n/2}}\exp\left(-\frac{r^2}{4\beta t}\right).$$

Under more restrictive conditions on M, this estimate was obtained by S. A. Molchanov (a report at the Landis–Kondrat'ev seminar, 1984) without the exact value of the constant in the exponent.

4. If the isoperimetric inequality (1) with the constant $k=C\frac{|B_R^x|^{1/n}}{R}$, is fulfilled in the ball B_R^x for each point x, then it follows from (4) that (for $\rho = \sqrt{t}$)

$$G(t,x;y)\leqslant\frac{c}{\sqrt{|B^x_{\sqrt{t}}|\cdot|B^y_{\sqrt{t}}|}}\exp\left(-\frac{|x-y|^2}{4\beta t}\right).$$

A similar estimate has been proved by A. K. Gushchin, V. P. Mikhailov, and Yu. A. Mikhailov (a report at the joint session of the I. G. Petrovskii seminar and the Moscow Mathematical Society, 1985) for a stronger isoperimetric inequality and an additional condition on the growth of the volume $|B_R^x|$.

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