An extension of Dwyer's and Palmieri's proof of Ohkawa's theorem on Bousfield classes

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Abstract

We give a proof that in any compactly generated triangulated category with a biexact coproduct preserving symmetric monoidal structure the collection of Bousfield classes forms a set.

In [Ohk89] Ohkawa proved that the collection of Bousfield classes (i.e. kernels of the smash product with some fixed object) in the stable homotopy category of spectra is a set. Dwyer and Palmieri presented in [DP01] a slick proof of Ohkawa's result that works for any Brown category. However, this is still a restrictive hypothesis; Neeman has shown in [Nee97] that the unbounded derived category of $\mathbb{C}[x, y]$ is not a Brown category. In this note we show Ohkawa's theorem extends to any compactly generated triangulated category equipped with a biexact and coproduct preserving tensor product. In particular, this answers Question 5.9 of [DP08] in the affirmative: for any commutative ring R the Bousfield lattice of $D(R)$ forms a set.

The idea of the proof is to extend the argument given by Dwyer and Palmieri in [DP01]. Instead of working with weak colimits in the triangulated category $\mathcal{T}$ we pass to the category of abelian presheaves over the compact objects of $\mathcal{T}$. There is a restricted Yoneda functor from $\mathcal{T}$ to this presheaf category and the image of every object of $\mathcal{T}$ is canonically a filtered colimit of representable functors. Section 1 is the technical part of the paper where we show these colimit representations are compatible with the monoidal structure on $\mathcal{T}$. In Section 2 we use this compatibility to adapt the proof of Dwyer and Palmieri to this more general situation, proving in Theorem 2.7 our claim that the collection of Bousfield classes always forms a set in such a category.

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1 Modules over the compacts

Throughout we denote by $\mathcal{T} = (\mathcal{T}, \otimes, \mathbf{1})$ a compactly generated triangulated category equipped with a symmetric monoidal structure which is exact and commutes with coproducts in each variable. Let us make it clear what we do *not* assume: the compact objects are not assumed to form a tensor subcategory and so in particular we are not assuming $\mathcal{T}$ rigidly-compactly generated.
We denote by Mod-$\mathcal{T}^c$ the Grothendieck abelian category of additive presheaves of abelian groups on the subcategory of compact objects $\mathcal{T}^c$. We let

$$H_{(-)} \colon \mathcal{T} \longrightarrow \text{Mod-}\mathcal{T}^c$$

denote the restricted Yoneda functor sending X to $H_X = \mathrm{Hom}(-, X)|_{\mathcal{T}^c}$ and note that it preserves coproducts.
We now recall the definition of a certain comma category associated to objects of $\mathcal{T}$ which will be central to our argument.

Definition 1.1. Let X be an object of $\mathcal{T}$ and denote by $\mathcal{T}^c/X$ the *slice category over* X (this is a slight abuse of terminology as X is not necessarily an object of $\mathcal{T}^c$) whose objects are maps

$$x \xrightarrow{f} X$$

with x compact, which we will sometimes denote by (x, f) if the object X is clear, and whose morphisms are commutative diagrams

$$\begin{tikzcd} x \arrow[r] \arrow[d] & X \\ y \arrow[ur] & \end{tikzcd}$$

Lemma 1.2. *For every object X of $\mathcal{T}$ the slice category $\mathcal{T}^c/X$ is filtered.*

Proof. This is well known, see for instance [Nee92] Lemma 2.1. □

Lemma 1.3. *For every X in $\mathcal{T}$ there is a natural isomorphism*

$$H_X \cong \operatorname{colim}_{\mathcal{T}^c/X} H_{(-)} \circ Q$$

where $Q\colon \mathcal{T}^c/X \longrightarrow \mathcal{T}^c$ is the projection $(x,f) \mapsto x$. In particular, the functor H_X can be written canonically as a filtered colimit of representable functors.

Proof. This is essentially just a restatement of the Yoneda lemma. □

Given objects X and Y of $\mathcal{T}$ there is a natural map

$$\operatorname{colim}_{\mathcal{T}^c/X} H_{(-\otimes Y)} \circ Q \longrightarrow H_{X\otimes Y}$$

induced by composition. To be explicit the component of this morphism at the image of (x,f) is

$$H_{f\otimes Y}\colon H_{x\otimes Y} \longrightarrow H_{X\otimes Y}.$$

We consider the full subcategory $\mathcal{L} \subseteq \mathcal{T}$ given by the objects for which this natural map is always an isomorphism

$$\mathcal{L} = \left\{ X \in \mathcal{T} \;\middle|\; \begin{array}{c} \operatorname{colim}_{\mathcal{T}^c/X} H_{(-\otimes Y)} \circ Q \xrightarrow{\sim} H_{X\otimes Y} \\ \text{all } Y \in \mathcal{T} \end{array} \right\}.$$

We prove in Proposition 1.11 that $\mathcal{L} = \mathcal{T}$. From this we deduce there is a set of Bousfield classes in $\mathcal{T}$ by adapting the argument of [DP01] (this is done in our Lemma 2.6 and Theorem 2.7).

Convention 1.4. Throughout whenever we write $H_X \cong \operatorname{colim}_I H_{x_i}$ for an object X of $\mathcal{T}$ it is understood that this is the canonical representation in terms of the slice category over X. That is, the diagram over which we are taking the colimit is the image of $H_{(-)} \circ Q\colon \mathcal{T}^c/X \longrightarrow \text{Mod-}\mathcal{T}^c$. Similarly $\operatorname{colim}_I H_{x_i\otimes Y}$ always means the colimit of the functor $H_{(-\otimes Y)} \circ Q$ with source $\mathcal{T}^c/X$.

Lemma 1.5. *The subcategory $\mathcal{L}$ is closed under arbitrary small coproducts in $\mathcal{T}$.*

Proof. Suppose $\{X_\alpha\}_{\alpha\in A}$ is a collection of objects from $\mathcal{L}$ indexed by a set A and let $X = \coprod_\alpha X_\alpha$. We wish to show that the natural map

$$\gamma\colon \operatorname{colim}_I H_{x_i\otimes Y} \longrightarrow H_{X\otimes Y}$$

is an isomorphism.

First we show γ is surjective. Let c be a compact object of $\mathcal{T}$ and $f\colon H_c \longrightarrow H_{X\otimes Y}$ an element of $H_{X\otimes Y}(c)$. As c is compact there are isomorphisms

$$\begin{aligned} \mathrm{Hom}(c, X\otimes Y) &= \mathrm{Hom}(c, (\coprod_\alpha X_\alpha)\otimes Y) \\ &\cong \mathrm{Hom}(c, \coprod_\alpha (X_\alpha \otimes Y)) \\ &\cong \bigoplus_\alpha \mathrm{Hom}(c, X_\alpha\otimes Y). \end{aligned}$$

Thus in $\mathcal{T}$, identifying maps in $\mathcal{T}$ and Mod-$\mathcal{T}^c$ via Yoneda, the map f factors as

$$\begin{tikzcd} c \arrow[r, "f'"] \arrow[d, "f"] & \oplus_{j=1}^n X_{\alpha_j}\otimes Y \arrow[dl] \\ X\otimes Y & \end{tikzcd}$$

where f' is determined by its components $f'_j\colon c \longrightarrow X_{\alpha_j}\otimes Y$.
By hypothesis for each α_j the natural map

$$\mathrm{colim}_{\mathcal{T}^c/X_{\alpha_j}} H_{(-\otimes Y)}\circ Q \longrightarrow H_{X_{\alpha_j}\otimes Y}$$

is an isomorphism. As H_c is finitely presented in Mod-$\mathcal{T}^c$ this isomorphism gives rise to a factorization

$$\begin{tikzcd} H_c \arrow[r] \arrow[d, "f'_j"'] & H_{x_{\alpha_j}\otimes Y} \arrow[dl] \\ H_{X_{\alpha_j}\otimes Y} & \end{tikzcd}$$

for each j where $H_{x_{\alpha_j}\otimes Y}$ is in the image of $\mathcal{T}^c/X$ under $H_{(-\otimes Y)}\circ Q$. In particular we get a factorization

$$H_c \longrightarrow \oplus_{j=1}^n H_{x_{\alpha_j}\otimes Y} \longrightarrow \oplus_{j=1}^n H_{X_{\alpha_j}\otimes Y}$$

of the element of $\oplus_{j=1}^n H_{X_{\alpha_j}\otimes Y}(c)$ corresponding to f'. Consider the diagram

$$\begin{tikzcd} H_c \arrow[r] & \oplus_{j=1}^n H_{x_{\alpha_j}\otimes Y} \arrow[r] \arrow[d] & \oplus_{j=1}^n H_{X_{\alpha_j}\otimes Y} \arrow[d] \\ & \mathrm{colim}_I H_{x_i\otimes Y} \arrow[r, "\gamma"] & H_{X\otimes Y} \end{tikzcd}$$

where both vertical maps are induced by the fact that X is the coproduct of the X_α. This diagram commutes as going down and then right in the square gives the other composite by definition of γ. This shows γ is surjective: both composites $H_c \longrightarrow H_{X\otimes Y}$ give the element f so it is in the image of γ.

Now let us show γ is injective. Consider a compact object c together with a map $g\colon H_c \longrightarrow \operatorname{colim}_I H_{x_i\otimes Y}$ representing an element in the kernel of γ. The object H_c is finitely presented so g factors as

$$\begin{tikzcd} H_c \arrow[r] \arrow[dr] & \operatorname{colim}_I H_{x_i\otimes Y} \\ & H_{x_j\otimes Y} \arrow[u] \end{tikzcd}$$

where x_j is compact and the map from $H_{x_j\otimes Y}$ to the colimit is the structure map corresponding to $h\colon x_j \longrightarrow X$. As x_j is compact h factors via some finite sum $\oplus_{k=1}^n X_{\alpha_k}$. We thus have a diagram

$$\begin{tikzcd} H_c \arrow[r, "g"] \arrow[d] & \operatorname{colim}_I H_{x_i\otimes Y} \arrow[rr, "\gamma"] & & H_{X\otimes Y} \\ H_{x_j\otimes Y} \arrow[r] \arrow[ur] & \oplus_{k=1}^n \operatorname{colim}_{\mathcal{T}^c/X_{\alpha_k}} H_{(-\otimes Y)} \circ Q \arrow[u] \arrow[rr, "\sim"] & & \oplus_{k=1}^n H_{X_{\alpha_k}\otimes Y} \arrow[u] \end{tikzcd}$$

where the two triangles on the left commute as they are induced by the factorizations we have observed above, the righthand square commutes by naturality of the horizontal comparison maps, and the bottom right map is an isomorphism as the X_{α_k} lie in $\mathcal{L}$. It follows that $g = 0$: if not the composite

$$H_c \longrightarrow \oplus_{k=1}^n \operatorname{colim}_{\mathcal{T}^c/X_{\alpha_k}} H_{(-\otimes Y)} \circ Q$$

would be non-zero, but then $\gamma(g)$ would also necessarily be non-zero on some component of $H_{X\otimes Y} \cong \oplus_\alpha H_{X_\alpha\otimes Y}$. $\square$

Lemma 1.6. *The subcategory $\mathcal{L}$ is closed under suspension.*

Proof. Let us suppose we are given an object X of $\mathcal{L}$ and an object Y of $\mathcal{T}$. For any compact object c of $\mathcal{T}$ and any $Z \in \mathcal{T}$ there is an isomorphism

$$H_{\Sigma Z}(c) \xrightarrow{\sim} H_Z(\Sigma^{-1}c).$$

In particular there is such an isomorphism for each $x_i \otimes Y$ occurring in $\operatorname{colim}_I H_{x_i \otimes Y}$. Taking colimits and considering the comparison maps we get a commutative square

$$\begin{tikzcd} \operatorname{colim}_I H_{\Sigma x_i \otimes Y}(c) \arrow[r, "\sim"] \arrow[d] & \operatorname{colim}_I H_{x_i \otimes Y}(\Sigma^{-1} c) \arrow[d, "\wr"] \\ H_{\Sigma X \otimes Y}(c) \arrow[r, "\sim"'] & H_{X \otimes Y}(\Sigma^{-1} c) \end{tikzcd}$$

where the right vertical map is an isomorphism by assumption. Thus the left vertical map must also be an isomorphism and ΣX is an object of $\mathcal{L}$ as claimed. $\square$

Lemma 1.7. *The subcategory $\mathcal{L}$ contains the subcategory $\mathcal{T}^c$ of compact objects.*

Proof. Suppose that z is a compact object of $\mathcal{T}$. Then the slice category over z has a terminal object, namely $z \xrightarrow{1_z} z$. Thus the functor $i_z \colon \bullet \longrightarrow \mathcal{T}^c/z$, where $\bullet$ is the category with one object and no non-identity morphisms, sending the object of $\bullet$ to $z \xrightarrow{1_z} z$ is final. So for any object Y of $\mathcal{T}$ the canonical map

$$H_{-\otimes Y} \circ Q \circ i_z \longrightarrow \operatorname{colim}_{\mathcal{T}^c/z} H_{-\otimes Y} \circ Q$$

is an isomorphism from which it follows that z is an object of $\mathcal{L}$. $\square$

We next wish to show the subcategory $\mathcal{L}$ is closed under the formation of triangles. Let us begin by observing that morphisms in $\mathcal{T}$ induce natural maps between the relevant colimits.

Suppose X, Y, and F are objects of $\mathcal{T}$ and let $\phi \in \operatorname{Hom}(X, Y)$. Then there is an induced functor

$$\overline{\phi} \colon \mathcal{T}^c/X \longrightarrow \mathcal{T}^c/Y$$

sending an object $x \longrightarrow X$ to the composite $x \longrightarrow X \xrightarrow{\phi} Y$. This induces, via abstract nonsense, a natural map

$$\operatorname{colim}_{\mathcal{T}^c/X} H_{-\otimes F} \circ Q = \operatorname{colim}_{\mathcal{T}^c/X} H_{-\otimes F} \circ Q \circ \overline{\phi} \longrightarrow \operatorname{colim}_{\mathcal{T}^c/Y} H_{-\otimes F} \circ Q.$$

This construction is clearly functorial i.e., given $\psi \colon Y \longrightarrow Z$ we have

$$\overline{\psi \circ \phi} = \overline{\psi} \circ \overline{\phi}.$$

The following lemma is an immediate consequence of this functoriality.

Lemma 1.8. *Let*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

be a triangle in $\mathcal{T}$. Given an object F of $\mathcal{T}$ form the induced sequence

$$\operatorname{colim}_I H_{x_i \otimes F} \xrightarrow{f} \operatorname{colim}_J H_{y_j \otimes F} \xrightarrow{g} \operatorname{colim}_K H_{z_k \otimes F}$$

of $\mathcal{T}^c$-modules. In this sequence the image of f is contained in the kernel of g.

Proof. Given the above this is simply the statement that the zero map $X \longrightarrow Z$ induces the zero map on the relevant colimits. □

We now prove the sequence of the last lemma is in fact exact.

Lemma 1.9. *Let*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

be a triangle in $\mathcal{T}$ and let F be an object of $\mathcal{T}$. In the induced sequence

$$\operatorname{colim}_I H_{x_i \otimes F} \xrightarrow{f} \operatorname{colim}_J H_{y_j \otimes F} \xrightarrow{g} \operatorname{colim}_K H_{z_k \otimes F}$$

of $\mathcal{T}^c$-modules the kernel of g is contained in the image of f.

Proof. Suppose a is a compact object of $\mathcal{T}$ and let the morphism $\alpha\colon H_a \longrightarrow \operatorname{colim}_J H_{y_j \otimes F}$ correspond to an element in the kernel of g. As H_a is finitely presented the map α factors through some $H_{y_\lambda \otimes F}$ and the composite $g\alpha$ factors through some $H_{z_\gamma \otimes F}$. We thus deduce a commutative diagram

$$\begin{tikzcd}
 & H_a \arrow[rr, "\alpha'"] \arrow[dl, "(g\alpha)'"'] \arrow[dr, "\alpha"] & & H_{y_\lambda \otimes F} \arrow[dl, "s_\lambda"'] \arrow[dr, "g'"] & \\
H_{z_\gamma \otimes F} \arrow[drr, "t_\gamma"'] & & \operatorname{colim}_J H_{y_j \otimes F} \arrow[d, "g"] & & H_{z_{\gamma'} \otimes F} \arrow[dll, "t_{\gamma'}"] \\
 & & \operatorname{colim}_K H_{z_k \otimes F} & &
\end{tikzcd}$$

of factorizations; one obtains the factorization of gs_λ by noting that the composite

$$H_{y_\lambda} \longrightarrow \operatorname{colim}_J H_{y_j} \longrightarrow \operatorname{colim}_K H_{z_k}$$

factors via some $H_{z_{\gamma'}}$ as H_{y_λ} is finitely presented and that this factorization remains valid when one instead applies $H_{-\otimes F}$ in which case the corresponding composite is gs_λ. In particular this line of argument shows that g' has a canonical lift to $\mathcal{T}$ of the form $\tilde{g}' \otimes 1_F$.

As K is filtered we can without loss of generality take $\gamma = \gamma'$. We thus have

$$t_\gamma(g\alpha)' = g\alpha = gs_\lambda\alpha' = t_\gamma g'\alpha' = 0.$$

Again using the fact that K is filtered we may assume $g'\alpha' = 0$. We observed above that g' has a canonical lift to $\mathcal{T}$ which we denoted $\tilde{g}' \otimes 1_F$. Thus we can use this lift to obtain a diagram in $\mathcal{T}$

$$\begin{tikzcd}
 & a \arrow[dl, "\exists"'] \arrow[d, "\alpha'"] & \\
w \otimes F \arrow[r] & y_\lambda \otimes F \arrow[r, "\tilde{g}' \otimes 1_F"] & z_\gamma \otimes F
\end{tikzcd}$$

where we have completed $\tilde{g}' \otimes 1_F$ to a triangle. The object w is compact as both y_λ and z_γ are and completing a morphism to a triangle commutes with tensoring with F by exactness of the tensor product. The factorization exists as the composite $(\tilde{g}' \otimes 1_F)\tilde{\alpha}'$ vanishes. Now consider the diagram of triangles

$$\begin{tikzcd}
w \arrow[r] \arrow[d, "\exists"'] & y_\lambda \arrow[r] \arrow[d] & z_\gamma \arrow[r] \arrow[d] & \Sigma w \arrow[d, "\exists"] \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r] & \Sigma X
\end{tikzcd}$$

where the outer maps exist as the middle square commutes in $\mathcal{T}$. Indeed its image commutes in Mod-$\mathcal{T}^c$ so it commutes up to phantoms and y_λ is compact so it commutes on the nose. As the index category I in $\operatorname{colim}_I H_{x_i}$ is the slice category $\mathcal{T}^c/X$ the object H_w together with the map in this diagram occurs in $\operatorname{colim}_I H_{x_i}$. Thus combining the last two diagrams gives a commutative diagram of $\mathcal{T}^c$-modules

$$\begin{tikzcd}
H_{w\otimes F} \arrow[dr] \arrow[dd] & & H_a \arrow[ll] \arrow[dl] \arrow[dd, "\alpha"] \\
 & H_{y_\lambda \otimes F} \arrow[dr] & \\
\operatorname{colim}_I H_{x_i \otimes F} \arrow[rr, "f"'] & & \operatorname{colim}_J H_{y_j \otimes F}
\end{tikzcd}$$

proving α is in the image of f. Thus $\ker g \subseteq \operatorname{im} f$ as claimed. $\square$

Lemma 1.10. *Suppose $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ is a triangle in $\mathcal{T}$ where X and Y are objects of $\mathcal{L}$. Then the object Z also lies in $\mathcal{L}$.*

Proof. Let F be an object of $\mathcal{T}$ and consider the sequence

$$\operatorname{colim}_I H_{x_i \otimes F} \xrightarrow{f} \operatorname{colim}_J H_{y_j \otimes F} \xrightarrow{g} \operatorname{colim}_K H_{z_k \otimes F}.$$

Combining Lemma 1.9 and Lemma 1.8 we see that any such sequence arising from a triangle is exact. Consider the diagram

$$\begin{tikzcd}
\operatorname{colim}_I H_{x_i\otimes F} \arrow[r] \arrow[d, "\wr"] & \operatorname{colim}_J H_{y_j\otimes F} \arrow[r] \arrow[d, "\wr"] & \operatorname{colim}_K H_{z_k\otimes F} \arrow[r] \arrow[d] & {} \\
H_{X\otimes F} \arrow[r] & H_{Y\otimes F} \arrow[r] & H_{Z\otimes F} \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \operatorname{colim}_I H_{\Sigma x_i\otimes F} \arrow[r] \arrow[d, "\wr"] & \operatorname{colim}_J H_{\Sigma y_j\otimes F} \arrow[d, "\wr"] \\
{} \arrow[r] & H_{\Sigma X\otimes F} \arrow[r] & H_{\Sigma Y\otimes F}
\end{tikzcd}$$

which commutes by naturality of the vertical morphisms. The two rightmost isomorphisms follow from suspension closure of $\mathcal{L}$ which was proved in Lemma 1.6. The bottom row is exact as H is homological and we have just seen the top row is also exact. So by the 5-lemma the natural morphism $\operatorname{colim}_K H_{z_k\otimes F} \longrightarrow H_{Z\otimes F}$ is an isomorphism proving Z is in $\mathcal{L}$ as claimed. □

Proposition 1.11. *There is an equality of categories $\mathcal{L} = \mathcal{T}$.*

Proof. We have seen in Lemma 1.5 that $\mathcal{L}$ is coproduct closed, in Lemma 1.6 that it is suspension closed, in Lemma 1.7 that $\mathcal{T}^c \subseteq \mathcal{L}$, and in Lemma 1.10 that it is closed under the formation of triangles. Now the localization theorem of Neeman-Ravenel-Thomason-Trobaugh-Yao (see for example [Nee96] Theorem 2.1) tells us we must have $\mathcal{L} = \mathcal{T}$ as claimed. □

2 Extending Ohkawa's Theorem

We use compatibility of the restricted Yoneda functor, the tensor product, and filtered colimits to extend Dwyer and Palmieri's [DP01] reformulation of the proof of Ohkawa's theorem to any compactly generated tensor triangulated category. As in the first section $(\mathcal{T}, \otimes, \mathbf{1})$ is a compactly generated triangulated category equipped with a biexact and coproduct preserving symmetric monoidal structure. Fix a generating set of compact objects $G = \Sigma G$ for $\mathcal{T}$ and denote by $[\mathcal{T}^c]$ a set of compact objects meeting each isomorphism class in $\mathcal{T}^c$ precisely once.

Definition 2.1. Given an object E of $\mathcal{T}$ we denote by $\langle E \rangle$ the *Bousfield class* of E

$$\langle E \rangle = \{X \in \mathcal{T} \mid E \otimes X = 0\}.$$

Definition 2.2. For $f \in \mathrm{Hom}(g, E \otimes c)$ with $g \in G$ and $c \in \mathcal{T}^c$ we define the *annihilator of f with respect to g* to be

$$\mathrm{ann}_{g,c}(f) = \{\alpha \in \mathrm{Hom}(c, d) \mid d \in [\mathcal{T}^c], \mathrm{Hom}(g, E \otimes \alpha)(f) = 0\}.$$

For any $g \in G$, $c \in \mathcal{T}^c$ and $f \in \mathrm{Hom}(g, E \otimes c)$ the set of morphisms $\mathrm{ann}_{g,c}(f)$ is a left ideal in $\mathcal{T}^c$ based at c. In other words every $(\alpha\colon c \longrightarrow d) \in \mathrm{ann}_{g,c}(f)$ has source c and for any $\beta\colon d \longrightarrow d'$ in $\mathcal{T}^c$ it holds that $\beta\alpha \in \mathrm{ann}_{g,c}(f)$. The statement that the annihilators are left ideals is immediate from functoriality of $\mathrm{Hom}(g, E \otimes (-))$.

Definition 2.3. The *Ohkawa class* of $E \in \mathcal{T}$ is defined to be the set of left ideals

$$\langle\langle E \rangle\rangle = \{\mathrm{ann}_{g,c}(f) \mid g \in G, c \in [\mathcal{T}^c], f \in \mathrm{Hom}(g, E \otimes c)\}$$

in $\mathcal{T}^c$.

Notation 2.4. We write $\mathbb{O}$ for the collection of Ohkawa classes and $\mathbb{B}$ for the collection of Bousfield classes.

Lemma 2.5. *The collection $\mathbb{O}$ of all Ohkawa classes forms a set.*

Proof. Since $\mathcal{T}^c$ is essentially small and $\mathcal{T}$ is locally small there is a set of morphisms between objects of $[\mathcal{T}^c]$. Thus there is a set of left ideals of maps between compacts. Each Ohkawa class is a set of such left ideals so $\mathbb{O}$ forms a set. □

We partially order the Bousfield classes by reverse inclusion i.e., for $\langle E \rangle, \langle F \rangle \in \mathbb{B}$ we declare $\langle E \rangle \geq \langle F \rangle$ if for every $X \in \mathcal{T}$ we have $E \otimes X = 0$ implies $F \otimes X = 0$. We partially order the Ohkawa classes by inclusion i.e., $\langle\langle E \rangle\rangle \geq \langle\langle F \rangle\rangle$ if for every annihilator ideal $\mathrm{ann}_{g,c}(f) \in \langle\langle F \rangle\rangle$ there exists $g' \in G$ and $f' \in \mathrm{Hom}(g', E \otimes c)$ such that $\mathrm{ann}_{g,c}(f) = \mathrm{ann}_{g',c}(f')$.

Lemma 2.6. *Suppose E and F are objects of $\mathcal{T}$. Then if $\langle\langle E\rangle\rangle \geq \langle\langle F\rangle\rangle$ it follows that $\langle E\rangle \geq \langle F\rangle$.*

Proof. Suppose $X \in \langle E\rangle$ that is, $E \otimes X = 0$. We wish to demonstrate that $F \otimes X = 0$ as well. By Proposition 1.11 the subcategory

$$\mathcal{L} = \left\{ \begin{array}{c|c} X \in \mathcal{T} & \begin{array}{c}\operatorname{colim}_{\mathcal{T}^c/X} H_{(-\otimes Y)} \circ Q \xrightarrow{\sim} H_{X\otimes Y} \\ \text{all } Y \in \mathcal{T}\end{array} \end{array} \right\}$$

is equal to $\mathcal{T}$. Thus the compact objects x_i indexed by the slice category $I = \mathcal{T}^c/X$ give

$$H_{X\otimes E} \cong \operatorname{colim}_I H_{x_i\otimes E} \quad \text{and} \quad H_{X\otimes F} \cong \operatorname{colim}_I H_{x_i\otimes F}.$$

To see $X \otimes F$ is zero it is thus sufficient to check for every $f \in H_{x_i\otimes F}(g)$ with $g \in G$ that f goes to zero in the colimit. Indeed, if this is the case then

$$\begin{aligned} 0 = \operatorname{colim}_I \operatorname{Hom}(H_g, H_{x_i\otimes F}) &\cong \operatorname{Hom}(H_g, \operatorname{colim}_I H_{x_i\otimes F}) \\ &\cong \operatorname{Hom}(H_g, H_{X\otimes F}) \\ &\cong \operatorname{Hom}(g, X \otimes F), \end{aligned}$$

where we get the first isomorphism as H_g is finitely presented in Mod-$\mathcal{T}^c$, so $X \otimes F$ must be zero as G is a generating set.
By assumption $\langle\langle E\rangle\rangle \geq \langle\langle F\rangle\rangle$ so for each such f we have

$$\operatorname{ann}_{g,x_i}(f) = \operatorname{ann}_{g',x_i}(f')$$

for some $g' \in G$ and $f' \in H_{x_i\otimes E}(g')$. As $X \otimes E = 0$ we must have $\operatorname{colim}_I H_{x_i\otimes E} = 0$ so the structure maps for the colimit are eventually in $\operatorname{ann}_{g',x_i}(f')$. Hence they are also in $\operatorname{ann}_{g,x_i}(f)$ showing that f is killed. This works for every $g \in G$ and every map to $X \otimes F$ as both were arbitrary. Thus $X \otimes F = 0$ as claimed so $\langle E\rangle \geq \langle F\rangle$. $\square$

Theorem 2.7. *The collection of Bousfield classes $\mathbb{B}$ forms a set.*

Proof. By Lemma 2.6 the assignment

$$\langle\langle E\rangle\rangle \mapsto \langle E\rangle$$

is a well defined morphism of "posets" $\mathbb{O} \longrightarrow \mathbb{B}$ and it is clearly surjective. In Lemma 2.5 we saw that $\mathbb{O}$ is a set so that $\mathbb{B}$ must also be a set. $\square$

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