Tensor Actions and Locally Complete Intersections

Gregory Steuart Douglas Stevenson

January 2011

A thesis submitted for the degree of Doctor of Philosophy
of the Australian National University

Declaration

Except where acknowledged in the customary manner, the material contained in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university or other tertiary education institute.

Greg Stevenson

Acknowledgements

First and foremost I would like to thank my supervisor Amnon Neeman for his support, both intellectual and financial, his enthusiasm, his incredible patience, and for his role in making the Mathematical Sciences Institute at the ANU a stimulating environment in which to do algebra. I'd like to thank James Borger and Lance Gurney for many helpful and interesting conversations concerning algebraic geometry, category theory, homological algebra, and many other topics. Finally I want to express my gratitude to Paul Balmer for his interest in my research, for many valuable comments on a preliminary version of this work, and for some good advice.

Abstract

We introduce a relative version of Balmer's tensor triangular geometry by considering the action of a tensor triangulated category on another triangulated category. Several of Balmer's results are extended to this relative setting giving rise to, among other things, a theory of supports for objects of a category upon which a tensor triangulated category acts.

In the case that a rigidly-compactly generated tensor triangulated category acts on a compactly generated category we describe a version of the local-to-global principle of Benson, Iyengar, and Krause, and a relative version of the telescope conjecture. We prove the local-to-global principle holds quite generally which is new even in the case that a tensor triangulated category acts on itself as in Balmer's theory. We are also able to give sufficient conditions for the relative telescope conjecture to hold.

As an application we study the stable injective category of a noetherian separated scheme X, as introduced by Krause, in terms of an action of the derived category $D(X)$. We give a complete classification of the localizing subcategories of this category in the case that X is the spectrum of a hypersurface ring and prove that the telescope conjecture holds. Our methods allow us to extend these results, suitably modified, to certain complete intersection schemes of arbitrary codimension.

Contents

Notation and terminology

Unless explicitly mentioned otherwise all rings are commutative, unital and noetherian.

We index our complexes cohomologically so a complex X of R-modules is of the form

$$\cdots \longrightarrow X^{-2} \longrightarrow X^{-1} \longrightarrow X^{0} \longrightarrow X^{1} \longrightarrow X^{2} \longrightarrow \cdots$$

the exception being when speaking of complete resolutions as in Definition 4.1.1 where mixed indexing is used.

For a ring R we denote by R-Mod the category of R-modules and by R-mod the category of finitely generated R-modules. For a scheme X we denote by $\operatorname{QCoh} X$ and $\operatorname{Coh} X$ the categories of quasi-coherent and coherent $\mathcal{O}_X$-modules respectively. For an abelian category $\mathcal{A}$ we follow standard notation and denote by $D(\mathcal{A})$ the unbounded derived category of $\mathcal{A}$, by $D^b(\mathcal{A})$ the bounded derived category, by $D^+(\mathcal{A})$ the derived category of cohomologically bounded below complexes, and by $D^-(\mathcal{A})$ the derived category of cohomologically bounded above complexes. We denote by $D^{\mathrm{perf}}(R) \subseteq D(R\text{-Mod})$ the subcategory of perfect complexes i.e., the compact objects of the unbounded derived category.

We denote by $\operatorname{Inj} R$, $\operatorname{Proj} R$, and $\operatorname{Flat} R$ respectively the full subcategories of injective, projective, and flat R-modules, and for a scheme X by $\operatorname{Inj} X$ and $\operatorname{Flat} X$ the categories of injective and flat quasi-coherent $\mathcal{O}_X$-modules.

For an R-module M we use $\operatorname{pd}_R M$ and $\operatorname{id}_R M$ to denote the projective and injective dimension of M.

We will denote the suspension functor of a triangulated category by Σ; we use this notation for the suspension functor of all triangulated categories concerned but this should not cause any confusion.

Chapter 1

Introduction

Triangulated categories, introduced by Verdier [70] and by Dold and Puppe [31] (but without Verdier's octahedral axiom), permeate modern mathematics. Their utility has been demonstrated in algebraic geometry, motivic theory, homotopy theory, modular representation theory, and noncommutative geometry: the theory of Grothendieck duality ([38], [43], [57], [60], [62]), Voevodsky's motivic category ([54], [6]), Devinatz, Hopkins, and Smith's work on tensor nilpotence [30], support varieties and the extension of complexity to infinitely generated representations ([27], [14], [15]), and recent work on the Baum-Connes conjecture [29] respectively are striking examples of the applications of triangulated categories in these areas.

In each of these areas one often has the good fortune to have more than just a triangulated category. Indeed, usually the triangulated categories arising are naturally *tensor triangulated categories*: we say $(\mathcal{T}, \otimes, \mathbf{1})$ is tensor triangulated if $\mathcal{T}$ is a triangulated category and $(\otimes, \mathbf{1})$ is a symmetric monoidal structure on $\mathcal{T}$ such that $\otimes$ is exact in each variable and preserves any coproducts $\mathcal{T}$ might possess. This is a very rich structure and exploiting the monoidal product leads to many beautiful results such as the work of Neeman [59] and Thomason [69] on the classification of thick subcategories of derived categories of perfect complexes in algebraic geometry.

Tensor triangular geometry, developed by Paul Balmer [7], [9], [8], [11], associates to any essentially small tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$ a topological space $\operatorname{Spc} \mathcal{T}$, the *spectrum* of $\mathcal{T}$. The spectrum comes with a universal, tensor compatible, support theory which assigns to objects of $\mathcal{T}$ closed subsets of the spectrum. This generalizes the homological support for derived categories of sheaves in algebraic geometry and the support varieties attached to representations in modular representation theory. One obtains from this support theory a classification of certain $\otimes$-ideals which unifies classifications occurring in algebraic geometry, modular representation theory, and algebraic topology.

So far we have only mentioned the topology of $\operatorname{Spc}\mathcal{T}$. In the case that $\mathcal{T}$ is *rigid* i.e., every object of $\mathcal{T}$ admits a strong dual, the spectrum can be endowed with a sheaf of rings making it a locally ringed space. This construction essentially embeds algebraic geometry into tensor triangular geometry; the derived category of quasi-coherent sheaves on a quasi-compact quasi-separated scheme X does not generally contain enough information to reconstruct X (cf. [24] and [19]), but together with the left derived tensor product one can recover X via the spectrum of the perfect complexes.

Now suppose $(\mathcal{T}, \otimes, \mathbf{1})$ is a compactly generated tensor triangulated category and the compact objects form a tensor subcatgory. In [11] Balmer and Favi have used tensor idempotents built from support data on the spectrum $\operatorname{Spc}\mathcal{T}^c$ of the compact objects $\mathcal{T}^c$ to extend Balmer's notion of supports to $\mathcal{T}$. A related construction due to Benson, Iyengar, and Krause [13] takes as input an R-linear compactly generated triangulated category $\mathcal{K}$, where R is a (graded) commutative noetherian ring, and assigns supports valued in $\operatorname{Spec} R$ to objects of $\mathcal{K}$. In the first part of this thesis we develop relative tensor triangular geometry by allowing a tensor triangulated category $\mathcal{T}$ to act on $\mathcal{K}$ i.e., there is a biexact functor $\mathcal{T} \times \mathcal{K} \longrightarrow \mathcal{K}$ which is compatible with the monoidal structure on $\mathcal{T}$ and associative and unital in the appropriate senses. This can be viewed as a categorification of the work of Benson, Iyengar, and Krause; for instance, letting R be a commutative noetherian ring, an action of the unbounded derived category $D(R)$ yields the same support theory as the support construction of [13]. By construction it specializes to the theory of Balmer and Favi when a tensor triangulated category acts on itself in the obvious way. Thus the notion of action provides a link between these two theories of supports and we are able to extend many of the important results of both theories to the case of actions.

Let us fix compactly generated triangulated categories $\mathcal{T}$ and $\mathcal{K}$. Furthermore, suppose $\mathcal{T}$ carries a compatible symmetric monoidal structure $(\mathcal{T}, \otimes, \mathbf{1})$ so that the compact objects form a rigid tensor triangulated subcategory $(\mathcal{T}^c, \otimes, \mathbf{1})$ whose spectrum $\operatorname{Spc}\mathcal{T}^c$ is a noetherian topological space (these hypotheses are not necessary for all of the results we quote but are chosen for simplicity). We recall that $\mathcal{T}^c$ is rigid if for all x and y in $\mathcal{T}^c$, setting $x^\vee = \hom(x, \mathbf{1})$, the natural map

$$x^\vee \otimes y \longrightarrow \hom(x, y)$$

is an isomorphism, where $\hom(-,-)$ denotes the internal hom which is guaranteed to exist in this case by Brown representability. In Chapter 2 we give a definition of a left action of $\mathcal{T}$ on $\mathcal{K}$. This amounts to giving a functor $\mathcal{T}\times\mathcal{K} \xrightarrow{*} \mathcal{K}$ satisfying certain compatibility conditions. To each specialization closed subset $\mathcal{V} \subseteq \operatorname{Spc}\mathcal{T}^c$ and each point $x \in \operatorname{Spc}\mathcal{T}^c$ we associate $\otimes$-idempotent objects $\Gamma_{\mathcal{V}}\mathbf{1}$ and $\Gamma_x\mathbf{1}$ of $\mathcal{T}$ as in [11]. The object $\Gamma_{\mathcal{V}}\mathbf{1}$ is the idempotent corresponding to acyclization with respect to the smashing subcategory generated by the compact objects supported in $\mathcal{V}$ and we denote by $L_{\mathcal{V}}\mathbf{1}$ the idempotent corresponding to localization at this category. Then $\Gamma_x\mathbf{1}$ is defined to be $\Gamma_{\mathcal{V}(x)}\mathbf{1} \otimes L_{\mathcal{Z}(x)}\mathbf{1}$ where

$$\mathcal{V}(x) = \overline{\{x\}} \quad \textit{and} \quad \mathcal{Z}(x) = \{y \in \operatorname{Spc}\mathcal{T}^c \mid x \notin \mathcal{V}(y)\}.$$

We prove in Lemmas 2.2.5 and 2.2.6 that each specialization closed subset $\mathcal{V}$ yields a localization sequence

$$\Gamma_{\mathcal{V}}\mathcal{K} \rightleftarrows \mathcal{K} \rightleftarrows L_{\mathcal{V}}\mathcal{K}$$

where $\Gamma_{\mathcal{V}}\mathcal{K}$ is the essential image of $\Gamma_{\mathcal{V}}\mathbf{1} * (-)$. Furthermore, $\Gamma_{\mathcal{V}}\mathcal{K}$ is generated by objects of $\mathcal{K}^c$ by Corollary 2.2.13. The idempotents $\Gamma_x\mathbf{1}$ give rise to supports on $\mathcal{K}$ with values in $\operatorname{Spc}\mathcal{T}^c$: for an object A of $\mathcal{K}$ we set

$$\operatorname{supp} A = \{x \in \operatorname{Spc}\mathcal{T}^c \mid \Gamma_x\mathbf{1} * A \neq 0\}.$$

In good situations the subcategories $\Gamma_{\mathcal{V}}\mathcal{K}$ and $L_{\mathcal{V}}\mathcal{K}$ consist precisely of those objects whose support is in $\mathcal{V}$ and $\operatorname{Spc}\mathcal{T}^c \setminus \mathcal{V}$ respectively and the associated localization triangles decompose objects into a piece supported in each of these subsets; this last fact is proved in Proposition 2.2.20 together with other desirable properties of the support.

The local-to-global principle, originally introduced in [17] in the context of ring actions on triangulated categories, allows one to reduce classification problems to considering local pieces of a triangulated category. We introduce the following version for actions of triangulated categories:

Definition (2.3.1)**.** We say $\mathcal{T} \times \mathcal{K} \xrightarrow{*} \mathcal{K}$ satisfies the *local-to-global principle* if for each A in $\mathcal{K}$

$$\langle A \rangle_* = \langle \Gamma_x A \mid x \in \operatorname{Spc}\mathcal{T}^c \rangle_*$$

where $\langle A \rangle_*$ and $\langle \Gamma_x A \mid x \in \operatorname{Spc}\mathcal{T}^c \rangle_*$ are the smallest localizing subcategories of $\mathcal{K}$ containing A or the $\Gamma_x A$ respectively and closed under the action of $\mathcal{T}$.

Our main result concerning the local-to-global principle is that, assuming $\mathcal{T}$ is sufficiently nice, it is only a property of $\mathcal{T}$ not of the action and it always holds.

Theorem (2.3.9)**.** *Suppose $\mathcal{T}$ is a rigidly-compactly generated tensor triangulated category with a model and that $\operatorname{Spc} \mathcal{T}^c$ is noetherian. Then the following statements hold:*

(i) The local-to-global principle holds for the action of $\mathcal{T}$ on itself;

(ii) The associated support function detects vanishing of objects i.e., $X \in \mathcal{T}$ is zero if and only if $\operatorname{supp} X = \varnothing$;

(iii) For any chain $\{\mathcal{V}_i\}_{i \in I}$ of specialization closed subsets of $\operatorname{Spc} \mathcal{T}^c$ with union $\mathcal{V}$ there is an isomorphism

$$\Gamma_{\mathcal{V}} \mathbf{1} \cong \operatorname{hocolim} \Gamma_{\mathcal{V}_i} \mathbf{1}$$

where the structure maps are the canonical ones.

Furthermore, the relative versions of (i) and (ii) hold for any action of $\mathcal{T}$ on a compactly generated triangulated category $\mathcal{K}$.

In the penultimate section of Chapter 2 we explore a relative version of the telescope conjecture. The telescope conjecture states that if $\mathcal{L}$ is a localizing subcategory of a compactly generated triangulated category $\mathcal{T}$ such that the inclusion of $\mathcal{L}$ admits a coproduct preserving right adjoint i.e., $\mathcal{L}$ is smashing, then $\mathcal{L}$ is generated by compact objects of $\mathcal{T}$. This is a general version of the conjecture originally made for the stable homotopy category of spectra by Bousfield [22] and Ravenel [65]. It is still open for the stable homotopy category, it is known to be true for certain categories such as the derived category of a noetherian ring (by [59]), and in the generality we have stated it the conjecture is actually false. For instance Keller has given a counterexample in [45], although Krause in [47] shows that a slightly weaker version of the conjecture does hold. Our version in the relative setting is as follows:

Definition (2.4.1)**.** We say the *relative telescope conjecture* holds for $\mathcal{K}$ with respect to the action of $\mathcal{T}$ if every smashing $\mathcal{T}$-submodule $\mathcal{S} \subseteq \mathcal{K}$ (this means $\mathcal{S}$ is a localizing subcategory with an associated coproduct preserving localization functor such that $\mathcal{T} \times \mathcal{S} \xrightarrow{*} \mathcal{K}$ factors via $\mathcal{S}$) is generated by compact objects of $\mathcal{K}$.

We give sufficient conditions for the relative telescope conjecture to hold for the action of $\mathcal{T}$ on $\mathcal{K}$. In order to state one of our results let us introduce the following assignments relating subsets of $\operatorname{Spc} \mathcal{T}^c$ and localizing submodules of $\mathcal{K}$ i.e., those localizing subcategories of $\mathcal{K}$ stable under the action of $\mathcal{T}$.

Definition (2.2.22)**.** There are order preserving assignments

$$\left\{ \text{ subsets of } \operatorname{Spc}\mathcal{T}^c \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \text{ localizing submodules of } \mathcal{K} \right\}$$

where for a localizing submodule $\mathcal{L}$ we set

$$\sigma(\mathcal{L}) = \operatorname{supp}\mathcal{L} = \{x \in \operatorname{Spc}\mathcal{T}^c \mid \Gamma_x\mathbf{1} * \mathcal{L} \neq 0\}$$

for a subset W of $\operatorname{Spc}\mathcal{T}^c$

$$\tau(W) = \{A \in \mathcal{K} \mid \operatorname{supp} A \subseteq W\}$$

and both the subsets and subcategories are ordered by inclusion.

Our theorem is:

Theorem (2.4.14)**.** *Suppose $\mathcal{T}$ is rigidly-compactly generated and has a model. Let $\mathcal{T}$ act on a compactly generated triangulated category $\mathcal{K}$ so that the support of any compact object of $\mathcal{K}$ is a specialization closed subset of $\sigma\mathcal{K}$ and for each irreducible closed subset $\mathcal{V}$ in $\sigma\mathcal{K}$ there exists a compact object whose support is precisely $\mathcal{V}$. Furthermore, suppose the assignments σ and τ give a bijection between localizing submodules of $\mathcal{K}$ and subsets of $\sigma\mathcal{K}$. Then the relative telescope conjecture holds for $\mathcal{K}$ i.e., every smashing $\mathcal{T}$-submodule of $\mathcal{K}$ is generated by objects compact in $\mathcal{K}$.*

Studying schemes via derived categories of sheaves has an auspicious history. The theory of Grothendieck duality, which we have already mentioned, semiorthogonal decompositions, Fourier-Mukai transforms and applications to birational geometry [21], [24], the Riemann-Hilbert correspondence [55], and the study of singularities [26], [63] all give examples of important work couched in the language of derived categories. It is this last example, the study of singularities, which will be of most interest to us. Suppose X is a noetherian separated scheme. Then one defines a category

$$D_{\mathrm{Sg}}(X) := D^b(\operatorname{Coh} X)/D^{\mathrm{perf}}(X)$$

where $D^b(\operatorname{Coh} X)$ is the bounded derived category of coherent sheaves on X and $D^{\mathrm{perf}}(X)$ is the full subcategory of complexes locally isomorphic to bounded complexes of finitely generated projectives, which measures the singularities of X. In particular, $D_{\mathrm{Sg}}(X)$ vanishes if and only if X is regular, it is related to other measures of the singularities of X for example maximal Cohen-Macaulay modules (see [26]), and its properties reflect the severity of the singularities of X. The particular category which will concern us is the stable injective category of Krause [48], namely

$$S(X) := K_{\mathrm{ac}}(\operatorname{Inj} X)$$

the homotopy category of acyclic complexes of injective quasi-coherent $\mathcal{O}_X$-modules. We call $S(X)$ the *singularity category* of X. The singularity category is a compactly generated triangulated category whose compact objects are equivalent to $D_{\mathrm{Sg}}(X)$ up to summands.

We show that the unbounded derived category of quasi-coherent sheaves of $\mathcal{O}_X$-modules, which we denote $D(X)$, acts on the singularity category $S(X)$. Given an object E in $D(X)$ one can replace E by a K-flat resolution and tensoring this resolution with an acyclic complex of injectives again gives an acyclic complex of injectives; as X is noetherian the tensor product of a flat quasi-coherent sheaf and an injective quasi-coherent sheaf is injective and preservation of acyclicity can be taken as the defining property of K-flat complexes. This gives rise to a theory of supports for objects of $S(X)$ and $D_{\mathrm{Sg}}(X)$ taking values in X.

In Chapter 3 we treat the case $X = \operatorname{Spec} R$ the spectrum of a noetherian ring. We first verify that the claimed action of $D(R)$, the unbounded derived category of R-modules, on $S(R) = K_{\mathrm{ac}}(\operatorname{Inj} R)$ is in fact an action and demonstrate its basic properties. In particular we use work of Greenlees [34] to give a concrete description for the action of certain objects.

We next consider the assignments

$$\sigma(\mathcal{L}) = \operatorname{supp} \mathcal{L} = \{\mathfrak{p} \in \operatorname{Spec} R \mid \Gamma_{\mathfrak{p}}\mathcal{L} \neq 0\}$$

and

$$\tau(W) = \{A \in S(R) \mid \operatorname{supp} A \subseteq W\}$$

for a localizing subcategory $\mathcal{L} \subseteq S(R)$ and a subset $W \subseteq \operatorname{Spec} R$. It is proved that, as one would expect, the support actually takes values in $\operatorname{Sing} R$ the singular locus of $\operatorname{Spec} R$. The behaviour of the action of $D(R)$ with respect to the various functors connecting $D(R)$, $S(R)$, and $K(\operatorname{Inj} R)$ is also discussed. The main theorem of this chapter is a technical one proving that σ and τ restrict to bijections between certain localizing subcategories of $S(R)$ and subsets of $\operatorname{Sing} R$; we do not state it here as the remainder of the thesis is dedicated to improving this result as well as extending it to the non-affine case and sharper results are obtained.

The focus of the fourth chapter is on proving the assignments σ and τ give a complete classification of the localizing subcategories of $S(R)$ when the ring R is locally a hypersurface. This extends work of Takahashi [68] who has classified thick subcategories of $D_{\mathrm{Sg}}(R)$ when R is a local hypersurface. We are able to extend this result to cover all localizing subcategories of $S(R)$ as well as removing the hypothesis that R be local by using the action of $D(R)$:

Theorem (4.2.13)**.** *If R is a noetherian ring which is locally a hypersurface then there is an order preserving bijection*

$$\left\{ \text{subsets of } \operatorname{Sing} R \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \text{localizing subcategories of } S(R) \right\}.$$

It follows that there are also order preserving bijections

$$\left\{ \begin{matrix} \text{specialization closed} \\ \text{subsets of } \operatorname{Sing} R \end{matrix} \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \begin{matrix} \text{localizing subcategories of } S(R) \\ \text{generated by objects of } S(R)^c \end{matrix} \right\}$$

and

$$\left\{ \begin{matrix} \text{specialization closed} \\ \text{subsets of } \operatorname{Sing} R \end{matrix} \right\} \rightleftarrows \left\{ \text{thick subcategories of } D_{\mathrm{Sg}}(R) \right\}.$$

Using the machinery of Chapter 2 we are also able to deduce the telescope conjecture for $S(R)$ when R is locally a hypersurface.

In Chapter 5 we approach the problem of understanding the structure of $S(X)$ where X is any noetherian separated scheme. We prove there is an action of $D(X)$ on $S(X)$ and by working locally we extend our main results from the affine case. In particular, as we show in Theorem 5.2.7 our result for hypersurface rings extends to classify certain localizing subcategories of $S(X)$ where X is a noetherian separated scheme with only hypersurface singularities. For such X subsets of the singular locus correspond to localizing subcategories of $S(X)$ which are stable under the action of $D(X)$. As a corollary (5.2.9) we obtain a complete classification of the localizing subcategories of $S(X)$ when X can be expressed as the zero scheme of a section of an ample line bundle on an ambient regular scheme. As consequences we obtain proofs of the relative telescope conjecture and the telescope conjecture respectively.

We end by considering locally complete intersection schemes which are not necessarily hypersurfaces. By a theorem of Orlov [64], working over some fixed base field, if X is a noetherian separated locally complete intersection scheme, admitting a suitable embedding into a regular scheme with enough locally frees, the category $D_{\mathrm{Sg}}(X)$ is equivalent to $D_{\mathrm{Sg}}(Y)$ for a hypersurface Y which can be given explicitly. We prove that this equivalence extends to the level of Krause's stable injective categories and are thus able to employ it in Theorem 5.3.8 to reduce an aspect of the classification problem for such local complete intersection schemes to the corresponding classification problem for hypersurfaces which we have already solved. As a special case we are able to completely classify the localizing subcategories of $S(R)$ when R is a local (non-abstract) complete intersection ring over a field

Theorem (5.3.16)**.** *Suppose $(R, \mathfrak{m}, k)$ is a local complete intersection of finite type over a field. Then there are order preserving bijections*

$$\left\{ \begin{array}{c} \text{subsets of} \\ \coprod_{\mathfrak{p} \in \operatorname{Sing} R} \mathbb{P}^{c_\mathfrak{p}-1}_{k(\mathfrak{p})} \end{array} \right\} \xrightleftharpoons[\sigma]{\tau} \left\{ \text{localizing subcategories of } S(R) \right\}$$

where $c_\mathfrak{p}$ is the codimension of the singularity at the closed point of $R_\mathfrak{p}$. Furthermore, the telescope conjecture holds for $S(R)$.

This gives a new proof of certain cases of a similar result for local complete intersections announced by Iyengar [42].

Chapter 2

Actions in Tensor Triangular Geometry

In recent work Paul Balmer has developed a notion of *tensor triangular geometry* ([7], [9], [11]) associating to an essentially small triangulated category with a compatible tensor product $(\mathcal{T}, \otimes, \mathbf{1})$ a topological space $\operatorname{Spc} \mathcal{T}$ called the *spectrum* of $\mathcal{T}$. The spectrum of $\mathcal{T}$ classifies certain $\otimes$-ideals of $\mathcal{T}$ in a spirit similar to the classification of Thomason [69] in the case of the left derived tensor product on $D^{\mathrm{perf}}(X)$ for a quasi-compact quasi-separated scheme X.

Our aim here is to develop the corresponding theory of 'modules' in tensor triangular geometry. Rather than working only with tensor triangulated categories we consider actions of such categories on other triangulated categories. From this we deduce structural information about the target. This can be regarded as a 'categorified' version of some recent work of Benson, Iyengar, and Krause ([13], [16], [17]).

2.1 A Candidate Definition

Before beginning we owe it to the reader to make explicit exactly what we mean by tensor triangulated category. A *tensor triangulated category* $(\mathcal{T}, \otimes, \mathbf{1})$ is a triangulated category $\mathcal{T}$ together with a symmetric monoidal structure such that the monoidal product $\otimes$ is an exact functor in each variable. We also require that $\otimes$ preserves whatever coproducts $\mathcal{T}$ might have. We do not assume any further compatibility between the monoidal structure and the triangulation. We also do not assume, unless explicitly stated, that the triangulated categories we deal with are essentially small.

Let us propose a definition of what it means for a tensor triangulated category to act on another triangulated category. We define here the notion of left action and express a sinistral bias by only considering left actions and referring to them just as actions.

Definition 2.1.1. Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a tensor triangulated category and $\mathcal{K}$ a triangulated category. A *left action* of $\mathcal{T}$ on $\mathcal{K}$ is a functor

$$*\colon \mathcal{T} \times \mathcal{K} \longrightarrow \mathcal{K}$$

which is exact in each variable, i.e. for all $X \in \mathcal{T}$ and $A \in \mathcal{K}$ the functors $X * (-)$ and $(-) * A$ are exact (such a functor is called *biexact*), together with natural isomorphisms

$$a\colon\ * \ (\otimes \times \mathrm{id}_{\mathcal{K}}) \xrightarrow{\sim} * \ (\mathrm{id}_{\mathcal{T}} \times *)$$

and

$$l\colon \mathbf{1}* \xrightarrow{\sim} \mathrm{id}_{\mathcal{K}}$$

compatible with the biexactness of $(-) * (-)$ and satisfying the following conditions:

(1) The associator a satisfies the pentagon condition which asserts that the following diagram commutes for all X, Y, Z in $\mathcal{T}$ and A in $\mathcal{K}$

$$\begin{array}{ccccc}
 & & X * (Y * (Z * A)) & & \\
 & \overset{X * a_{Y,Z,A}}{\nearrow} & & \overset{a_{X,Y,Z*A}}{\nwarrow} & \\
X * ((Y \otimes Z) * A) & & & & (X \otimes Y) * (Z * A) \\
\uparrow{\scriptstyle a_{X,Y\otimes Z,A}} & & & & \uparrow{\scriptstyle a_{X\otimes Y,Z,A}} \\
(X \otimes (Y \otimes Z)) * A & & \longleftarrow & & ((X \otimes Y) \otimes Z) * A
\end{array}$$

where the bottom arrow is the associator of $(\mathcal{T}, \otimes, \mathbf{1})$.

(2) The unitor l makes the following squares commute for every X in $\mathcal{T}$ and A in $\mathcal{K}$

$$\begin{array}{ccc}
X * (\mathbf{1} * A) & \xrightarrow{1_X * l_A} & X * A \\
\uparrow{\scriptstyle a_{X,\mathbf{1},A}} & & \downarrow{\scriptstyle 1_{X*A}} \\
(X \otimes \mathbf{1}) * A & \longrightarrow & X * A
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{1} * (X * A) & \xrightarrow{l_{X*A}} & X * A \\
\uparrow{\scriptstyle a_{\mathbf{1},X,A}} & & \downarrow{\scriptstyle 1_{X*A}} \\
(\mathbf{1} \otimes X) * A & \longrightarrow & X * A
\end{array}$$

where the bottom arrows are the right and left unitors of $(\mathcal{T}, \otimes, \mathbf{1})$.

(3) For every A in $\mathcal{K}$ and $r, s \in \mathbb{Z}$ the diagram

$$\begin{array}{ccc} \Sigma^r \mathbf{1} * \Sigma^s A & \xrightarrow{\sim} & \Sigma^{r+s} A \\ \wr\downarrow & & \downarrow (-1)^{rs} \\ \Sigma^r(\mathbf{1} * \Sigma^s A) & \xrightarrow[\sim]{} & \Sigma^{r+s} A \end{array}$$

is commutative, where the left vertical map comes from exactness in the second variable of the action, the bottom horizontal map is the unitor, and the top map is given by the composite

$$\Sigma^r \mathbf{1} * \Sigma^s A \longrightarrow \Sigma^s(\Sigma^r \mathbf{1} * A) \longrightarrow \Sigma^{r+s}(\mathbf{1} * A) \xrightarrow{l} \Sigma^{r+s} A$$

whose first two maps use exactness in both variables of the action.

(4) The functor $*$ distributes over coproducts whenever they exist i.e., for families of objects $\{X_i\}_{i \in I}$ in $\mathcal{T}$ and $\{A_j\}_{j \in J}$ in $\mathcal{K}$, and X in $\mathcal{T}$, A in $\mathcal{K}$ there are natural isomorphisms

$$\coprod_i (X_i * A) \xrightarrow{\sim} (\coprod_i X_i) * A$$

and

$$\coprod_j (X * A_j) \xrightarrow{\sim} X * (\coprod_j A_j)$$

whenever the coproducts concerned exist.

Remark 2.1.2. Given composable morphisms f, f' in $\mathcal{T}$ and g, g' in $\mathcal{K}$ one has

$$(f' * g')(f * g) = (f'f * g'g)$$

by functoriality of $\mathcal{T} \times \mathcal{K} \xrightarrow{*} \mathcal{K}$.

Remark 2.1.3. It should be possible (at least in the essentially small case and in the presence of an enhancement) to give a more natural definition in terms of strong monoidal triangulated functors. This point of view will be pursued further elsewhere.

We begin with a simple observation which we will use freely from now on without reference.

Lemma 2.1.4. *There are natural isomorphisms*

$$0_{\mathcal{T}} * (-) \cong (-) * 0_{\mathcal{K}} \cong 0$$

where 0 *denotes the zero functor* $\mathcal{K} \longrightarrow \mathcal{K}$

Proof. For any A in $\mathcal{K}$ the functor $(-) * A$ is exact and so in particular it is additive. Thus we must have $0_\mathcal{T} * A \cong 0_\mathcal{K}$. Similarly $X * (-)$ is also additive for each X in $\mathcal{T}$ so $X * 0_\mathcal{K} \cong 0_\mathcal{K}$. □

We now give an indication of the sense in which our actions may be regarded as an enhancement of the actions introduced in [13]. In order to do this we first need to recall the definition of the graded centre of a triangulated category.

Definition 2.1.5. Let $\mathcal{T}$ be a triangulated category. The *graded centre* (or *central ring*) of $\mathcal{T}$ is the graded abelian group

$$Z^*(\mathcal{T}) = \bigoplus_n Z^n(\mathcal{T}) = \bigoplus_n \{\alpha\colon \operatorname{id}_\mathcal{T} \longrightarrow \Sigma^n \mid \alpha\Sigma = (-1)^n \Sigma\alpha\}$$

where n ranges over the integers, which is given the structure of a graded commutative ring by composition of natural transformations.

Remark 2.1.6. Using the words ring and group above is somewhat abusive as the centre of $\mathcal{T}$ may not form a set (we do not assume $\mathcal{T}$ essentially small). However, this is not a problem if one only wishes to consider the image of genuine rings in the centre.

Lemma 2.1.7. *An action $\mathcal{T} \times \mathcal{K} \xrightarrow{*} \mathcal{K}$ induces a morphism of rings*

$$\operatorname{End}^*_\mathcal{T}(\mathbf{1}) \longrightarrow Z^*(\mathcal{K}).$$

Proof. Given $f \in \operatorname{Hom}(\mathbf{1}, \Sigma^i \mathbf{1})$ we send it to the natural transformation whose component at $A \in \mathcal{K}$ is

$$A \xrightarrow{\sim} \mathbf{1} * A \xrightarrow{f * 1_A} \Sigma^i \mathbf{1} * A \xrightarrow{\sim} \Sigma^i A.$$

This is natural by our coherence conditions. It is a standard fact that the graded endomorphism ring of the unit is graded commutative from which it is straightforward that this is a map of graded commutative rings. □

Thus provided $\operatorname{End}^*_\mathcal{T}(\mathbf{1})$ is noetherian one is in a position to apply the machinery of Benson, Iyengar, and Krause. In fact this is discussed in Section 8 of [13] for the case of tensor triangulated categories acting on themselves and it is shown in Section 9 that for the derived category of a noetherian ring one recovers the classical notion of supports from their construction. Thus it agrees with the action of $D(R)$ on itself which also gives the usual supports.

We would like to view $\mathcal{K}$ as a module over $\mathcal{T}$ and from now on we will use the terms module and action interchangeably. There are of course, depending on the context, natural notions of $\mathcal{T}$-submodule.

Definition 2.1.8. Let $\mathcal{L} \subseteq \mathcal{K}$ be a thick (localizing) subcategory. We say $\mathcal{L}$ is a (localizing) $\mathcal{T}$*-submodule* of $\mathcal{K}$ if the functor

$$\mathcal{T} \times \mathcal{L} \xrightarrow{*} \mathcal{K}$$

factors via $\mathcal{L}$ i.e., $\mathcal{L}$ is closed under the action of $\mathcal{T}$. We note that in the case $\mathcal{K} = \mathcal{T}$ acts on itself by $\otimes$ this gives the notion of a (localizing) $\otimes$-ideal of $\mathcal{T}$. By a smashing or compactly generated submodule we mean the obvious things.

Notation 2.1.9. For a collection of objects $\{A_\lambda \mid \lambda \in \Lambda\}$ in $\mathcal{K}$ we denote by

$$\langle A_\lambda \mid \lambda \in \Lambda \rangle_*$$

the smallest (localizing) $\mathcal{T}$-submodule of $\mathcal{K}$ containing the A_λ.

For (localizing) thick subcategories $\mathcal{L} \subseteq \mathcal{T}$and $\mathcal{M} \subseteq \mathcal{K}$ we set

$$\mathcal{L} * \mathcal{M} = \langle X * A \mid X \in \mathcal{L}, A \in \mathcal{M} \rangle_*.$$

There is some ambiguity in the notation as we do not clutter it by distinguishing between the localizing and thick cases and so we take care to make it clear which we mean. However, this is not a serious problem in any case as in general if there are sufficient coproducts submodules will always be localizing.

The operation of forming such submodules is well behaved. The results below show that it commutes with the action in an appropriate sense. Most important for us is the fact that given generating sets for $\mathcal{L} \subseteq \mathcal{T}$ and $\mathcal{M} \subseteq \mathcal{K}$ we obtain a generating set for $\mathcal{L} * \mathcal{M}$ as a submodule.

Lemma 2.1.10. *Suppose $\mathcal{I} \subseteq \mathcal{T}$ is a thick $\otimes$-ideal. Then there is an equality of subcategories of $\mathcal{K}$*

$$\mathcal{I} * \mathcal{K} = \langle X * A \mid X \in \mathcal{I}, A \in \mathcal{K} \rangle_{\mathrm{thick}}.$$

The obvious analogue holds for localizing submodules.

Proof. Let us set

$$\langle X * A \mid X \in \mathcal{I}, A \in \mathcal{K} \rangle_{\mathrm{thick}} = \mathcal{L}$$

and denote by $\mathcal{M} \subseteq \mathcal{L}$ the full subcategory of $\mathcal{L}$ consisting of those objects B such that $Y * B \in \mathcal{L}$ for every $Y \in \mathcal{T}$. Then for every $X \in \mathcal{I}$ and $A \in \mathcal{K}$ the object $X * A$ lies in $\mathcal{M}$. Indeed, we have for any object Y of $\mathcal{T}$

$$Y * (X * A) \cong (Y \otimes X) * A$$

which is one of the generators given for $\mathcal{L}$ as $Y \otimes X$ is an object of $\mathcal{I}$. To see that $\mathcal{M}$ is closed under suspension observe that for $B \in \mathcal{M}$ and $Y \in \mathcal{T}$ it holds by the exactness properties of the action that

$$Y * \Sigma B \cong \Sigma Y * B$$

where $\Sigma Y * B \in \mathcal{L}$ by the defining property of $\mathcal{M}$. It is closed under finite biproducts as the action commutes with the biproduct in $\mathcal{K}$ and by assumption $\mathcal{L}$ is closed under biproducts. Similar considerations show that $\mathcal{M}$ is closed under summands and triangles. Thus $\mathcal{M} = \mathcal{L}$ as we have shown $\mathcal{M}$ is a thick subcategory containing the generators of $\mathcal{L}$. By construction $\mathcal{T} * \mathcal{M} \subseteq \mathcal{L}$ and combining this with the equality $\mathcal{M} = \mathcal{L}$ shows $\mathcal{L}$ is a $\mathcal{T}$-submodule of $\mathcal{K}$. In particular, it is already the smallest thick submodule containing the specified objects so agrees with $\mathcal{I} * \mathcal{K}$.

In the case of localizing ideals and submodules coproduct closure is easy to deduce from the fact that we require the action to commute with coproducts in both $\mathcal{T}$ and $\mathcal{K}$. □

Remark 2.1.11. We only state and prove Lemmas 2.1.12, 2.1.13, and 2.1.14 for localizing subcategories and modules as we will mostly be concerned with categories having enough coproducts. Of course one can replace localizing by thick everywhere and the corresponding results hold.

Lemma 2.1.12. *Formation of localizing subcategories commutes with the action, i.e., given a set of objects $\{X_i\}_{i \in I}$ of $\mathcal{T}$ and a set of objects $\{A_j\}_{j \in J}$ of $\mathcal{K}$*

$$\langle X * A \mid X \in \langle X_i \mid i \in I\rangle_{\mathrm{loc}}, A \in \langle A_j \mid j \in J\rangle_{\mathrm{loc}}\rangle_{\mathrm{loc}} = \langle X_i * A_j \mid i \in I, j \in J\rangle_{\mathrm{loc}}.$$

Proof. Denote the category on the left by $\mathcal{L}$ and the one on the right by $\mathcal{M}$. It is clear $\mathcal{M} \subseteq \mathcal{L}$ as each $X_i * A_j$ is in $\mathcal{L}$. For the converse it is sufficient to check that $\mathcal{M}$ contains generators for $\mathcal{L}$. For each $j \in J$ define a subcategory

$$\mathcal{T}_j = \{X \in \mathcal{T} \mid X * A_j \in \mathcal{M}\}.$$

The subcategory $\mathcal{T}_j$ is localizing as $(-) * A_j$ is an exact coproduct preserving functor and the subcategory $\mathcal{M}$ is localizing. As, by definition, $X_i * A_j$ is in $\mathcal{M}$

for all $i \in I$ each X_i lies in $\mathcal{T}_j$. So for any X in $\langle X_i \mid i \in I \rangle_{\mathrm{loc}}$ we have X in $\mathcal{T}_j$. In particular, $X * A_j$ lies in $\mathcal{M}$ for each such X and all $j \in J$.

Now consider the subcategory

$$\{A \in \mathcal{K} \mid X * A \in \mathcal{M} \ \text{ for all } \ X \in \langle X_i \mid i \in I \rangle_{\mathrm{loc}}\}.$$

It is localizing as $\mathcal{M}$ is so and by what we have just seen it contains the A_j for $j \in J$. Thus it contains $\langle A_j \mid j \in J \rangle_{\mathrm{loc}}$ so for every X in $\langle X_i \mid i \in I \rangle_{\mathrm{loc}}$ and every A in $\langle A_j \mid j \in J \rangle_{\mathrm{loc}}$ we have $X * A$ in $\mathcal{M}$. Hence $\mathcal{M}$ contains generators for $\mathcal{L}$ which gives the equality $\mathcal{L} = \mathcal{M}$. □

Lemma 2.1.13. *Given collections of objects $\{X_i\}_{i \in I}$ of $\mathcal{T}$ and $\{A_j\}_{j \in J}$ of $\mathcal{K}$ there is an equality of submodules*

$$\langle X_i \mid i \in I \rangle_{\mathrm{loc}} * \langle A_j \mid j \in J \rangle_{\mathrm{loc}} = \langle X_i \mid i \in I \rangle_{\otimes} * \langle A_j \mid j \in J \rangle_{\mathrm{loc}}.$$

Proof. It is clear that

$$\langle X_i \mid i \in I \rangle_{\mathrm{loc}} * \langle A_j \mid j \in J \rangle_{\mathrm{loc}} \subseteq \langle X_i \mid i \in I \rangle_{\otimes} * \langle Y_j \mid j \in J \rangle_{\mathrm{loc}}.$$

To see there is an inclusion in the other direction note that by definition and Lemma 2.1.10 the subcategory $(\mathcal{T} \otimes \langle X_i \rangle_{\mathrm{loc}}) * \langle A_j \rangle_{\mathrm{loc}}$ can be written as

$$\langle W * A' \mid W \in \langle Z \otimes X' \mid Z \in \mathcal{T}, X' \in \langle X_i \rangle_{\mathrm{loc}} \rangle_{\mathrm{loc}}, A' \in \langle A_j \rangle_{\mathrm{loc}} \rangle_{*}$$

where we drop the indexing sets for brevity of notation. By Lemma 2.1.12 we can rewrite this as

$$\langle (Z \otimes X') * A_j \mid Z \in \mathcal{T}, X' \in \langle X_i \rangle_{\mathrm{loc}} \rangle_{*}.$$

Each of the generators in the above presentation can be rewritten in the form $Z * (X' * A_j)$ via the associator. In particular each of the generators is an object of the localizing submodule $\langle X_i \rangle_{\mathrm{loc}} * \langle A_j \rangle_{\mathrm{loc}}$ so

$$(\mathcal{T} \otimes \langle X_i \rangle_{\mathrm{loc}}) * \langle A_j \rangle_{\mathrm{loc}} \subseteq \langle X_i \rangle_{\mathrm{loc}} * \langle A_j \rangle_{\mathrm{loc}}.$$

It just remains to observe that since $(\mathcal{T} \otimes \langle X_i \rangle_{\mathrm{loc}})$ is a localizing $\otimes$-ideal of $\mathcal{T}$ containing the X_i it must contain the $\otimes$-ideal they generate. This gives the desired containment

$$\langle X_i \mid i \in I \rangle_{\mathrm{loc}} * \langle A_j \mid j \in J \rangle_{\mathrm{loc}} \supseteq \langle X_i \mid i \in I \rangle_{\otimes} * \langle Y_j \mid j \in J \rangle_{\mathrm{loc}}$$

and completes the proof. □

We can now give a version of Lemma 2.1.12 for submodules.

Lemma 2.1.14. *Formation of localizing $\mathcal{T}$-submodules commutes with the action i.e., given a set of objects $\{X_i\}_{i\in I}$ of $\mathcal{T}$ and a set of objects $\{A_j\}_{j\in J}$ of $\mathcal{K}$ we have*

$$\begin{aligned}\langle X_i \mid i \in I\rangle_{\otimes} * \langle A_j \mid j \in J\rangle_{\mathrm{loc}} &= \langle X_i \mid i \in I\rangle_{\mathrm{loc}} * \langle A_j \mid j \in J\rangle_{\mathrm{loc}} \\ &= \langle X_i * A_j \mid i \in I, j \in J\rangle_{*}.\end{aligned}$$

Proof. The first equality is Lemma 2.1.13. The second follows from Lemma 2.1.12 as it identifies the smallest localizing subcategories containing generators (as submodules) for the submodules in question and hence the smallest submodules containing these generating sets. □

We record here the following trivial observation which turns out to be quite useful.

Lemma 2.1.15. *If $\mathcal{T}$ is generated as a localizing subcategory by the tensor unit $\mathbf{1}$ then every localizing subcategory of $\mathcal{K}$ is a $\mathcal{T}$-submodule.*

Proof. Let $\mathcal{L} \subseteq \mathcal{K}$ be a localizing subcategory and set

$$\mathcal{T}^{\mathcal{L}} = \{X \in \mathcal{T} \mid X * \mathcal{L} \subseteq \mathcal{L}\}.$$

Now note that as $\mathcal{T}^{\mathcal{L}}$ contains $\mathbf{1}$ and is localizing $\mathcal{T}^{\mathcal{L}}$ must be equal to $\mathcal{T}$. □

2.2 The Case of Rigidly-Compactly Generated Tensor Triangulated Categories

We now restrict ourselves to the case that $(\mathcal{T}, \otimes, \mathbf{1})$ is a rigidly-compactly generated tensor triangulated category (unless explicitly mentioned otherwise) acting on a compactly generated $\mathcal{K}$. Actions of such categories have several desirable properties and we can extend much of the machinery developed in [11], [13], and [17] to this setting. First let us make explicit our hypotheses on $\mathcal{T}$.

Definition 2.2.1. A *rigidly-compactly generated tensor triangulated category* is a compactly generated tensor triangulated category (as usual the monoidal structure is assumed to be symmetric, biexact, and preserve coproducts so that $\mathcal{T}$ has an internal hom by Brown representability which we denote by $\mathrm{hom}(-,-)$) such that $\mathcal{T}^{\mathrm{c}}$, the (essentially small) subcategory of compact objects, is a rigid tensor triangulated subcategory. We recall that $\mathcal{T}^c$ is a rigid tensor triangulated subcategory if the monoidal structure and internal hom restrict to $\mathcal{T}^c$ (in particular the unit object $\mathbf{1}$ must be compact), and for all x and y in $\mathcal{T}^c$, setting $x^\vee = \hom(x, \mathbf{1})$, the natural map

$$x^\vee \otimes y \longrightarrow \hom(x, y)$$

is an isomorphism. In particular such categories are unital algebraic stable homotopy categories in the sense of [40] Definition 1.1.4.

In the case that $\mathcal{T}$ is rigidly-compactly generated we can use $\operatorname{Spc} \mathcal{T}^c$, as defined in [7], in order to define a theory of supports by using the $\otimes$-ideals of $\mathcal{T}$ generated by objects of $\mathcal{T}^c$ which provide us with many Rickard idempotents as in [11].

Our first task is to show that if such a $\mathcal{T}$ acts on a compactly generated triangulated category $\mathcal{K}$ that we can transfer compactly generated subcategories across this action: from Rickard idempotents on $\mathcal{T}$ we can obtain localization sequences on $\mathcal{K}$ where each of the categories involved is compactly generated by compact objects of $\mathcal{K}$. Before proceeding let us fix some notation and recall the definition of Thomason subsets.

Convention 2.2.2. Throughout this subsection all submodules will be localizing unless explicitly mentioned otherwise.

Definition 2.2.3. Suppose that X is a topological space. A subset $V \subseteq X$ is a *Thomason subset* if it is of the form $V = \cup_i V_i$ where each V_i is a closed subset of X with quasi-compact complement.

Notation 2.2.4. Given a Thomason subset $\mathcal{V} \subseteq \operatorname{Spc} \mathcal{T}^c$ we denote by $\mathcal{T}^c_\mathcal{V}$ the thick subcategory of compact objects supported, in the sense of [7], on $\mathcal{V}$. We let $\mathcal{T}_\mathcal{V}$ be the localizing subcategory generated by $\mathcal{T}^c_\mathcal{V}$ and note that $\mathcal{T}_\mathcal{V}$ is smashing as it is generated by compact objects of $\mathcal{T}$. In particular there are associated Rickard idempotents which we denote by $\Gamma_\mathcal{V}\mathbf{1}$ and $L_\mathcal{V}\mathbf{1}$ with the property that under the tensor product they give rise to the smashing acyclization and localization functors corresponding to $\mathcal{T}_\mathcal{V}$ (see for example [11] Theorem 2.13). It follows that they are $\otimes$-orthogonal by the usual properties of localization and acyclization functors. We will also sometimes write $\Gamma_\mathcal{V}\mathcal{T}$ for the category associated to $\mathcal{V}$.

We now prove that from a Thomason subset of $\operatorname{Spc} \mathcal{T}^c$ we can produce a pair of compactly generated subcategories of $\mathcal{K}$. We do this via a series of relatively straightforward lemmas.

Lemma 2.2.5. *Suppose $\mathcal{V} \subseteq \operatorname{Spc} \mathcal{T}^c$ is a Thomason subset. Then the subcategory*

$$\Gamma_{\mathcal{V}}\mathcal{K} := \{A \in \mathcal{K} \mid \exists A' \text{ with } A \cong \Gamma_{\mathcal{V}}\mathbf{1} * A'\}$$

is a localizing $\mathcal{T}$-submodule.

Proof. We begin by showing $\Gamma_{\mathcal{V}}\mathcal{K}$ is localizing. It is sufficient to show that

$$\Gamma_{\mathcal{V}}\mathcal{K} = \ker L_{\mathcal{V}}\mathbf{1} * (-),$$

as the kernel of any exact coproduct preserving functor is a localizing subcategory. By [11] Theorem 3.5 the subcategory $\Gamma_{\mathcal{V}}\mathcal{T}$ of $\mathcal{T}$ is precisely the essential image, $\operatorname{im}(\Gamma_{\mathcal{V}}\mathbf{1} \otimes (-))$, of tensoring with $\Gamma_{\mathcal{V}}\mathbf{1}$ and the corresponding idempotents are tensor orthogonal i.e., $\Gamma_{\mathcal{V}}\mathbf{1} \otimes L_{\mathcal{V}}\mathbf{1} = 0$. So if A is in $\Gamma_{\mathcal{V}}\mathcal{K}$ then

$$\begin{aligned} L_{\mathcal{V}}\mathbf{1} * A &\cong L_{\mathcal{V}}\mathbf{1} * (\Gamma_{\mathcal{V}}\mathbf{1} * A') \\ &\cong (L_{\mathcal{V}}\mathbf{1} \otimes \Gamma_{\mathcal{V}}\mathbf{1}) * A' \\ &\cong 0 \end{aligned}$$

showing

$$\Gamma_{\mathcal{V}}\mathcal{K} \subseteq \ker L_{\mathcal{V}}\mathbf{1} * (-).$$

Conversely, suppose $L_{\mathcal{V}}\mathbf{1} * A = 0$. Then applying $(-) * A$ to the localization triangle

$$\Gamma_{\mathcal{V}}\mathbf{1} \longrightarrow \mathbf{1} \longrightarrow L_{\mathcal{V}}\mathbf{1} \longrightarrow \Sigma\Gamma_{\mathcal{V}}\mathbf{1}$$

in $\mathcal{T}$ we deduce an isomorphism $\Gamma_{\mathcal{V}}\mathbf{1} * A \xrightarrow{\sim} A$. Thus A is in $\Gamma_{\mathcal{V}}\mathcal{K}$ so the two subcategories of $\mathcal{K}$ in question are equal as claimed. As stated above this proves $\Gamma_{\mathcal{V}}\mathcal{K}$ is localizing as the kernel of any exact coproduct preserving functor is a localizing subcategory.

To see it is a submodule note that for X in $\mathcal{T}$ and A in $\Gamma_{\mathcal{V}}\mathcal{K}$ we have

$$\begin{aligned} X * A &\cong X * (\Gamma_{\mathcal{V}}\mathbf{1} * A') \\ &\cong (X \otimes \Gamma_{\mathcal{V}}\mathbf{1}) * A' \\ &\cong (\Gamma_{\mathcal{V}}\mathbf{1} \otimes X) * A' \\ &\cong \Gamma_{\mathcal{V}}\mathbf{1} * (X * A'). \end{aligned}$$

□

Lemma 2.2.6. *Suppose $\mathcal{V}$ is a Thomason subset of $\operatorname{Spc} \mathcal{T}^c$. The subcategory $\Gamma_{\mathcal{V}}\mathcal{K}$ and the subcategory*

$$L_{\mathcal{V}}\mathcal{K} := \{A \in \mathcal{K} \mid \exists A' \text{ with } A \cong L_{\mathcal{V}}\mathbf{1} * A'\}$$

give rise to a localization sequence

$$\Gamma_{\mathcal{V}}\mathcal{K} \rightleftarrows \mathcal{K} \rightleftarrows L_{\mathcal{V}}\mathcal{K}$$

and $L_{\mathcal{V}}\mathcal{K}$ is also a localizing $\mathcal{T}$-submodule.

Proof. The statement that $L_{\mathcal{V}}\mathcal{K}$ is a submodule follows in exactly the same way as for $\Gamma_{\mathcal{V}}\mathcal{K}$ in the proof of Lemma 2.2.5.

So let us demonstrate we have the claimed localization sequence. There is a triangle in $\mathcal{T}$

$$\Gamma_{\mathcal{V}}\mathbf{1} \longrightarrow \mathbf{1} \longrightarrow L_{\mathcal{V}}\mathbf{1} \longrightarrow \Sigma\Gamma_{\mathcal{V}}\mathbf{1}$$

associated to $\mathcal{V}$ by definition (see Notation 2.2.4). For any A in $\mathcal{K}$ the action thus gives us functorial triangles

$$\Gamma_{\mathcal{V}}\mathbf{1} * A \longrightarrow A \longrightarrow L_{\mathcal{V}}\mathbf{1} * A \longrightarrow \Sigma\Gamma_{\mathcal{V}}\mathbf{1} * A.$$

So to prove we have the desired localization sequence it is sufficient to demonstrate

$$L_{\mathcal{V}}\mathcal{K} = \Gamma_{\mathcal{V}}\mathcal{K}^{\perp}$$

by Lemma 3.1 of [20].

We first show $L_{\mathcal{V}}\mathcal{K} \supseteq \Gamma_{\mathcal{V}}\mathcal{K}^{\perp}$. Suppose $A \in \Gamma_{\mathcal{V}}\mathcal{K}^{\perp}$ and consider the triangle

$$\Gamma_{\mathcal{V}}\mathbf{1} * A \longrightarrow A \longrightarrow L_{\mathcal{V}}\mathbf{1} * A \longrightarrow \Sigma\Gamma_{\mathcal{V}}\mathbf{1} * A.$$

By hypothesis the morphism $\Gamma_{\mathcal{V}}\mathbf{1} * A \longrightarrow A$ must be zero so the triangle splits yielding

$$L_{\mathcal{V}}\mathbf{1} * A \cong A \oplus \Sigma\Gamma_{\mathcal{V}}\mathbf{1} * A.$$

As $L_{\mathcal{V}}\mathcal{K}$ is localizing it must contain $\Gamma_{\mathcal{V}}\mathbf{1} * A$ i.e., there is some A' in $\mathcal{K}$ such that $\Gamma_{\mathcal{V}}\mathbf{1} * A \cong L_{\mathcal{V}}\mathbf{1} * A'$. Hence there are isomorphisms

$$\Gamma_{\mathcal{V}}\mathbf{1} * A \cong \Gamma_{\mathcal{V}}\mathbf{1} * (\Gamma_{\mathcal{V}}\mathbf{1} * A) \cong \Gamma_{\mathcal{V}}\mathbf{1} * (L_{\mathcal{V}}\mathbf{1} * A') \tag{2.1}$$
$$\cong (\Gamma_{\mathcal{V}}\mathbf{1} \otimes L_{\mathcal{V}}\mathbf{1}) * A' \cong 0 \tag{2.2}$$

where we have used tensor orthogonality of the Rickard idempotents. Thus $L_{\mathcal{V}}\mathbf{1} * A \cong A$ is in $L_{\mathcal{V}}\mathcal{K}$.

It remains to check the containment $L_{\mathcal{V}}\mathcal{K} \subseteq \Gamma_{\mathcal{V}}\mathcal{K}^{\perp}$. Let A be an object of $\Gamma_{\mathcal{V}}\mathcal{K}$ and B an object of $L_{\mathcal{V}}\mathcal{K}$. Observe that as A is in $\Gamma_{\mathcal{V}}\mathcal{K}$ and B is in $L_{\mathcal{V}}\mathcal{K}$ we have $L_{\mathcal{V}}\mathbf{1} * A \cong 0$ and $\Gamma_{\mathcal{V}}\mathbf{1} * B \cong 0$. Indeed, by symmetry of the monoidal structure on $\mathcal{T}$ the objects $L_{\mathcal{V}}\mathbf{1} * A$ and $\Gamma_{\mathcal{V}}\mathbf{1} * B$ lie in both $\Gamma_{\mathcal{V}}\mathcal{K}$ and $L_{\mathcal{V}}\mathcal{K}$. It follows they must vanish by orthogonality of the tensor idempotents $\Gamma_{\mathcal{V}}\mathbf{1}$ and $L_{\mathcal{V}}\mathbf{1}$ as in (2.1) and (2.2) above. So for $f \in \mathrm{Hom}(A, B)$ we obtain via functoriality a map of triangles

$$\begin{array}{ccccc} \Gamma_{\mathcal{V}}\mathbf{1} * A & \xrightarrow{\sim} & A & \longrightarrow & 0 \\ \downarrow & & \downarrow f & & \downarrow \\ 0 & \longrightarrow & B & \xrightarrow[\sim]{} & L_{\mathcal{V}}\mathbf{1} * B \end{array}$$

which shows $f = 0$. Hence

$$L_{\mathcal{V}}\mathcal{K} \subseteq \Gamma_{\mathcal{V}}\mathcal{K}^{\perp}$$

proving the equality of these two subcategories. As stated above this yields the desired localization sequence by Lemma 3.1 of [20].

□

Notation 2.2.7. We will be somewhat slack with notation and often write, for A in $\mathcal{K}$, $\Gamma_{\mathcal{V}}A$ rather than $\Gamma_{\mathcal{V}}\mathbf{1} * A$ when it is clear from the context what we mean. When working with objects X of $\mathcal{T}$ we will use the idempotent notation for the localization and acyclization functors, e.g. $\Gamma_{\mathcal{V}}\mathbf{1} \otimes X$, so no confusion should be possible.

The next lemma is the first of several results showing rigidly-compactly generated tensor triangulated categories are not just lovely categories in their own right, but they also act well on other compactly generated categories.

Lemma 2.2.8. *Suppose $\mathcal{T} \times \mathcal{K} \xrightarrow{*} \mathcal{K}$ is an action where $\mathcal{T}$ is rigidly-compactly generated and $\mathcal{K}$ is compactly generated. Then the action restricts to an action at the level of compact objects $\mathcal{T}^c \times \mathcal{K}^c \xrightarrow{*} \mathcal{K}^c$.*

Proof. Let t be a compact object of $\mathcal{T}$. As $\mathcal{T}^c$ is rigid the object t admits a strong dual i.e., there is an object $t^{\vee}$ together with morphisms

$$\eta_t \colon \mathbf{1} \longrightarrow t^{\vee} \otimes t \quad \text{and} \quad \epsilon_t \colon t \otimes t^{\vee} \longrightarrow \mathbf{1}$$

such that the composite

$$t \xrightarrow{\rho_t^{-1}} t \otimes \mathbf{1} \xrightarrow{t \otimes \eta_t} t \otimes (t^{\vee} \otimes t) \xrightarrow{\alpha} (t \otimes t^{\vee}) \otimes t \xrightarrow{\epsilon_t \otimes t} \mathbf{1} \otimes t \xrightarrow{\lambda_t} t$$

where ρ_t, λ_t, and α are the right and left unitors and the associator for $\mathcal{T}$, is the identity and similarly for $t^{\vee}$. Using these maps together with the unitor l and associator a for the action we define natural transformations

$$\eta'_t \colon \ \mathrm{id}_{\mathcal{K}} \xrightarrow{l^{-1}} \mathbf{1} * \xrightarrow{\eta_t *} (t^{\vee} \otimes t) * \longrightarrow t^{\vee} * t *$$

and

$$\epsilon'_t\colon\ t * t^\vee * \longrightarrow (t \otimes t^\vee)* \xrightarrow{\epsilon_t *} \mathbf{1}* \xrightarrow{l} \mathrm{id}_\mathcal{K}$$

which we claim are the unit and counit of an adjunction. In order to prove this it is sufficient to verify that the composites

$$t* \xrightarrow{t*\eta'_t} t * t^\vee * t* \xrightarrow{\epsilon'_t t*} t* \quad \text{and} \quad t^\vee * \xrightarrow{\eta'_t t^\vee *} t^\vee * t * t^\vee \xrightarrow{t^\vee * \epsilon'_t} t^\vee *$$

are the respective identity natural transformations (see for instance [49] IV.1 Theorem 2). In fact these are precisely the identity composites corresponding to the existence of strong duals in $\mathcal{T}$ applied to $\mathcal{K}$. This is easily checked using the compatibility conditions required for $\mathcal{T}$ to act on $\mathcal{K}$.

Thus η'_t and ϵ'_t give the desired adjunction. In particular, $t*$ has a coproduct preserving right adjoint and so by [60] Theorem 5.1 it must send compact objects to compact objects. □

Of course there are other situations in which this is true, although one has to assume more.

Lemma 2.2.9. *Let $\mathcal{T}$ be a (not necessarily rigidly) compactly generated tensor triangulated category acting on a compactly generated triangulated category $\mathcal{K}$. If there exists a set of compact generators $\{x_i\}_{i\in I}$ for $\mathcal{T}$ such that $x_i * \mathcal{K}^c \subseteq \mathcal{K}^c$ for each $i \in I$ then the action of $\mathcal{T}$ on $\mathcal{K}$ restricts to an action of $\mathcal{T}^c$ on $\mathcal{K}^c$. In particular, if the unit object $\mathbf{1}$ of $\mathcal{T}$ is compact and generates $\mathcal{T}$ the action restricts.*

Proof. The argument is standard: as the action is exact in each variable the subcategory of $\mathcal{T}^c$ which acts on $\mathcal{K}^c$ is thick and by assumption it contains a generating set. □

Our next lemma is a relative version of Miller's Theorem (see [56] or [40] Theorem 3.3.3).

Lemma 2.2.10. *Suppose $\mathcal{T}$ is a (not necessarily rigidly) compactly generated tensor triangulated category which acts on a compactly generated triangulated category $\mathcal{K}$ and that $\mathcal{C}$ is a thick $\mathcal{T}^c$-submodule of $\mathcal{K}^c$. Then $\langle \mathcal{C} \rangle_{\mathrm{loc}}$ is a localizing $\mathcal{T}$-submodule of $\mathcal{K}$.*

Proof. We first show $\langle \mathcal{C} \rangle_{\mathrm{loc}}$ is a $\mathcal{T}^c$-submodule. Let $\mathcal{L}$ be the full subcategory of objects A of $\langle \mathcal{C} \rangle_{\mathrm{loc}}$ such that $\mathcal{T}^c * A \subseteq \langle \mathcal{C} \rangle_{\mathrm{loc}}$. Then $\mathcal{C} \subseteq \mathcal{L}$ as it is a $\mathcal{T}^c$-submodule by hypothesis. Since $\langle \mathcal{C} \rangle_{\mathrm{loc}}$ is localizing and $*$ is biexact and preserves coproducts in the second variable it is straightforward to see $\mathcal{L}$ is a localizing subcategory. Thus, as it contains $\mathcal{C}$, we have $\mathcal{L} = \langle \mathcal{C} \rangle_{\mathrm{loc}}$ which proves the claim.

We now complete the proof by showing $\langle \mathcal{C} \rangle_{\mathrm{loc}}$ is also closed under the action of $\mathcal{T}$. Consider $\mathcal{M}$ the full subcategory of objects X of $\mathcal{T}$ such that $X * \langle \mathcal{C} \rangle_{\mathrm{loc}} \subseteq \langle \mathcal{C} \rangle_{\mathrm{loc}}$. We have just seen $\langle \mathcal{C} \rangle_{\mathrm{loc}}$ is a $\mathcal{T}^c$-submodule so $\mathcal{T}^c \subseteq \mathcal{M}$. As above, since $*$ is biexact and coproduct preserving in the first variable and $\langle \mathcal{C} \rangle_{\mathrm{loc}}$ is localizing, it follows that $\mathcal{M}$ is a localizing subcategory. Hence $\mathcal{M} = \mathcal{T}$ as it contains the compacts. Thus $\langle \mathcal{C} \rangle_{\mathrm{loc}}$ is a localizing $\mathcal{T}$-submodule as claimed. □

We are now ready to demonstrate a general result (we do not assume $\mathcal{T}$ rigidly-compactly generated) on compact generation of subcategories produced via actions. It implies compact generation of subcategories of the form $\Gamma_{\mathcal{V}}\mathcal{K}$ for $\mathcal{V}$ a Thomason subset of $\operatorname{Spc} \mathcal{T}^c$.

Proposition 2.2.11. *Suppose $\mathcal{T}$ acts on $\mathcal{K}$, with both $\mathcal{T}$ and $\mathcal{K}$ compactly generated, in such a way that the action restricts to one of $\mathcal{T}^c$ on $\mathcal{K}^c$ (e.g., $\mathcal{T}$ is rigidly-compactly generated). Then given a $\otimes$-ideal $\mathcal{L} \subseteq \mathcal{T}$ generated (as a localizing subcategory) by compact objects of $\mathcal{T}$ and a subcategory $\mathcal{M} \subseteq \mathcal{K}$ generated by objects of $\mathcal{K}^c$ the subcategory $\mathcal{L} * \mathcal{M}$ is also generated, as a localizing subcategory, by compact objects of $\mathcal{K}$.*

Proof. Let us fix generating sets $\{x_i\}_{i\in I}$ for $\mathcal{L}$ and $\{a_j\}_{j\in J}$ for $\mathcal{M}$ where the x_i and a_j lie in $\mathcal{T}^c$ and $\mathcal{K}^c$ respectively. By Lemma 2.1.14 we have equalities of subcategories of $\mathcal{K}$

$$\mathcal{L} * \mathcal{M} = \langle x_i \mid i \in I \rangle_{\otimes} * \langle a_j \mid j \in J \rangle_{\mathrm{loc}} = \langle x_i * a_j \mid i \in I, j \in J \rangle_{*}$$

where by hypothesis each $x_i * a_j$ is a compact object of $\mathcal{K}$.

Let us denote by $\mathcal{G}$ the smallest thick $\mathcal{T}^c$-submodule of $\mathcal{K}^c$ containing the set of objects $\{x_i * a_j\}_{i\in I, j\in J}$. Lemma 2.2.10 tells us the localizing subcategory $\mathcal{N} = \langle \mathcal{G} \rangle_{\mathrm{loc}}$ is a $\mathcal{T}$-submodule. We claim that $\mathcal{L} * \mathcal{M} = \mathcal{N}$. Since $\mathcal{L} * \mathcal{M}$ contains $\{x_i * a_j\}_{i\in I, j\in J}$ and is a localizing and hence thick $\mathcal{T}$-submodule it contains $\mathcal{G}$. Thus $\mathcal{N} \subseteq \mathcal{L} * \mathcal{M}$.

On the other hand $\mathcal{N}$ is a $\mathcal{T}$-submodule containing $\mathcal{G}$ and so certainly contains the set of objects $\{x_i * a_j\}_{i\in I, j\in J}$. Hence it is a localizing $\mathcal{T}$-submodule containing a generating set (as a localizing $\mathcal{T}$-submodule) for $\mathcal{L} * \mathcal{M}$ and so contains $\mathcal{L} * \mathcal{M}$. It follows that $\mathcal{N} = \mathcal{L} * \mathcal{M}$. In particular, $\mathcal{L} * \mathcal{M}$ has a generating set of objects compact in $\mathcal{K}$ obtained by taking a skeleton for $\mathcal{G} \subseteq \mathcal{K}^c$. □

Remark 2.2.12. We get more from the proof of this proposition when $\mathcal{T}$ is generated by the tensor unit. In this case all localizing and thick subcategories are closed under the action of $\mathcal{T}$ and $\mathcal{T}^c$ respectively so given compact generating sets for $\mathcal{L}$ and $\mathcal{M}$, we get an explicit generating set for $\mathcal{L} * \mathcal{M}$. Indeed we showed that if $\mathcal{L}$ is generated by objects $\{x_i\}_{i\in I}$ of $\mathcal{T}^c$ and $\mathcal{M}$ is generated by objects $\{a_j\}_{j\in J}$ of $\mathcal{K}^c$ then $\mathcal{L} * \mathcal{M}$ has a generating set $\{x_i * a_j\}_{i\in I, j\in J}$ of objects compact in $\mathcal{K}$.

Corollary 2.2.13. *Suppose $\mathcal{T}$ is a rigidly-compactly generated tensor triangulated category acting on a compactly generated triangulated category $\mathcal{K}$ and that $\mathcal{V}$ is a Thomason subset of* $\operatorname{Spc} \mathcal{T}^c$*. Then the subcategory*

$$\Gamma_{\mathcal{V}}\mathcal{K} = \{A \in \mathcal{K} \mid \exists A' \textit{ with } A \cong \Gamma_{\mathcal{V}}\mathbf{1} * A'\}$$

is generated by compact objects of $\mathcal{K}$.

Proof. By the proposition we have just proved it is sufficient to make the identification $\Gamma_{\mathcal{V}}\mathcal{K} = \Gamma_{\mathcal{V}}\mathcal{T} * \mathcal{K}$. If X is an object of $\Gamma_{\mathcal{V}}\mathcal{T}$ then there is an isomorphism $X \cong \Gamma_{\mathcal{V}}\mathbf{1} \otimes X$. Thus we have

$$\begin{aligned}\Gamma_{\mathcal{V}}\mathcal{T} * \mathcal{K} &= \langle X * A \mid X \in \Gamma_{\mathcal{V}}\mathcal{T}, A \in \mathcal{K}\rangle_* \\ &= \langle \Gamma_{\mathcal{V}}\mathbf{1} * (X * A) \mid X \in \Gamma_{\mathcal{V}}\mathcal{T}, A \in \mathcal{K}\rangle_* \\ &= \langle \Gamma_{\mathcal{V}}\mathbf{1} * A \mid A \in \mathcal{K}\rangle_*.\end{aligned}$$

Closing the generators of this last submodule under isomorphisms gives $\Gamma_{\mathcal{V}}\mathcal{K}$ which, by Lemma 2.2.5, is a localizing $\mathcal{T}$-submodule. Thus $\Gamma_{\mathcal{V}}\mathcal{K} = \Gamma_{\mathcal{V}}\mathcal{T} * \mathcal{K}$ and we can apply the last proposition to complete the proof. □

We now define the functors which give rise to supports on $\mathcal{K}$ relative to $(\mathcal{T}, *)$.

Definition 2.2.14. For every $x \in \operatorname{Spc} \mathcal{T}^c$ we define subsets of the spectrum

$$\mathcal{V}(x) = \overline{\{x\}}$$

and

$$\mathcal{Z}(x) = \{y \in \operatorname{Spc} \mathcal{T}^c \mid x \notin \mathcal{V}(y)\}.$$

Both of these subsets are specialization closed but they are not necessarily Thomason. In the case that they are both Thomason we define a $\otimes$-idempotent

$$\Gamma_x\mathbf{1} = (\Gamma_{\mathcal{V}(x)}\mathbf{1} \otimes L_{\mathcal{Z}(x)}\mathbf{1}).$$

In keeping with previous notation we will sometimes write $\Gamma_x A$ instead of $\Gamma_x \mathbf{1} * A$ for objects A of $\mathcal{K}$. We recall from [11] Corollary 7.5 that the idempotent functors $\Gamma_x \mathbf{1} \otimes (-)$ on $\mathcal{T}$ for $x \in \operatorname{Spc} \mathcal{T}^c$ only depend on x and not on the choice of Thomason subsets $\mathcal{W}, \mathcal{V}$ satisfying $\mathcal{V} \setminus \{\mathcal{V} \cap \mathcal{W}\} = \{x\}$ that we use to obtain them via $\Gamma_{\mathcal{V}} \mathbf{1} \otimes L_{\mathcal{W}} \mathbf{1}$ (as in Theorem 6.2 of [13]). Thus, with $\mathcal{T}$ acting on $\mathcal{K}$, the functors $\Gamma_x \colon \mathcal{K} \longrightarrow \mathcal{K}$ also only depend on x. In other words we have:

Lemma 2.2.15. *Let $x \in \operatorname{Spc} \mathcal{T}^c$ and suppose $\mathcal{V}$ and $\mathcal{W}$ are Thomason subsets of $\operatorname{Spc} \mathcal{T}^c$ such that $\mathcal{V} \setminus (\mathcal{V} \cap \mathcal{W}) = \{x\}$. Then there are natural isomorphisms*

$$(L_{\mathcal{W}} \mathbf{1} \otimes \Gamma_{\mathcal{V}} \mathbf{1}) * (-) \cong \Gamma_x \cong (\Gamma_{\mathcal{V}} \mathbf{1} \otimes L_{\mathcal{W}} \mathbf{1}) * (-).$$

If such sets exist for $x \in \operatorname{Spc} \mathcal{T}^c$ let us follow the terminology of [11] and call x *visible*. By [11] Corollary 7.14 every point is visible in our sense if the spectrum of $\mathcal{T}^c$ is noetherian. We denote by $\operatorname{Vis} \mathcal{T}^c$ the spectrum of visible points of $\mathcal{T}$.

Remark 2.2.16. The fact that certain points are "invisible" is rather unsatisfactory. It should be possible to give a refined notion of the spectrum for any compactly generated tensor triangulated category $\mathcal{T}$ which agrees with $\operatorname{Spc} \mathcal{T}^c$ when $\operatorname{Spc} \mathcal{T}^c$ is noetherian. A step toward realizing this is given in [67]; the point is that one would like to work with localizing prime $\otimes$-ideals but first one needs to know (when) there is a set of such.

Notation 2.2.17. Following previous notation we use $\Gamma_x \mathcal{K}$, for $x \in \operatorname{Spc} \mathcal{T}^c$, to denote the essential image of $\Gamma_x \mathbf{1} * (-)$. It is a $\mathcal{T}$-submodule as for any $X \in \mathcal{T}$ and $A \in \Gamma_x \mathcal{K}$

$$X * A \cong X * (\Gamma_x \mathbf{1} * A') \cong \Gamma_x \mathbf{1} * (X * A')$$

for some $A' \in \mathcal{K}$.

We can define supports taking values in the set of visible points of $\operatorname{Spc} \mathcal{T}^c$.

Definition 2.2.18. Given A in $\mathcal{K}$ we define the support of A to be the set

$$\operatorname{supp}_{(\mathcal{T},*)} A = \{x \in \operatorname{Vis} \mathcal{T}^c \mid \Gamma_x A \neq 0\}.$$

When the action in question is clear we will omit the subscript from the notation.

Remark 2.2.19. We are now in a position to make the connection between our machinery and the machinery of Benson, Iyengar, and Krause more transparent. Let us ponder the case $\mathcal{T} = D(R)$ where R is a noetherian ring. In this case

$\mathcal{K}$ is R-linear so the theory developed in [13] applies. By [13] Theorem 6.4 the subcategories giving rise to supports in the sense of Benson, Iyengar, and Krause are generated by certain Koszul objects: if $\mathcal{V} \subseteq \operatorname{Spec} R$ is specialization closed then their subcategory $\mathcal{K}_\mathcal{V}$ is easily seen to be generated by the objects

$$\{K(\mathfrak{p}) * a \mid a \in \mathcal{K}^c,\ \mathfrak{p} \in \mathcal{V}\}.$$

As $\{\Sigma^i K(\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{V}, i \in \mathbb{Z}\}$ is a generating set for $\Gamma_\mathcal{V} D(R)$ we see, by Remark 2.2.12 and the corollary following it, that the localizing subcategories $\mathcal{K}_\mathcal{V}$ and $\Gamma_\mathcal{V}\mathcal{K}$ agree. Thus our support functors are precisely those of Benson, Iyengar, and Krause in the case that the derived category of a noetherian ring acts.

Proposition 2.2.20. *The support assignment* $\operatorname{supp}_{(\mathcal{T},*)}$ *satisfies the following properties:*

(1) *given a triangle*

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

in $\mathcal{K}$ *we have* $\operatorname{supp} B \subseteq \operatorname{supp} A \cup \operatorname{supp} C$*;*

(2) *for any* A *in* $\mathcal{K}$ *and* $i \in \mathbb{Z}$

$$\operatorname{supp} A = \operatorname{supp} \Sigma^i A;$$

(3) *given a set-indexed family* $\{A_\lambda\}_{\lambda\in\Lambda}$ *of objects of* $\mathcal{K}$ *there is an equality*

$$\operatorname{supp} \coprod_\lambda A_\lambda = \bigcup_\lambda \operatorname{supp} A_\lambda;$$

(4) *the support satisfies the separation axiom i.e., for every specialization closed subset* $\mathcal{V} \subseteq \operatorname{Vis} \mathcal{T}^c$ *and every object* A *of* $\mathcal{K}$

$$\begin{aligned} \operatorname{supp} \Gamma_\mathcal{V}\mathbf{1} * A &= \operatorname{supp} A \cap \mathcal{V} \\ \operatorname{supp} L_\mathcal{V}\mathbf{1} * A &= \operatorname{supp} A \cap (\operatorname{Vis} \mathcal{T}^c \setminus \mathcal{V}). \end{aligned}$$

Proof. As $\Gamma_x\mathbf{1} * (-)$ is a coproduct preserving exact functor (1), (2), and (3) are immediate. To see the separation axiom holds suppose $\mathcal{V} \subseteq \operatorname{Vis} \mathcal{T}^c$ is a specialization closed subset and let A be an object of $\mathcal{K}$. Then

$$\begin{aligned} \Gamma_x\mathbf{1} * (\Gamma_\mathcal{V}\mathbf{1} * A) &\cong (\Gamma_x\mathbf{1} \otimes \Gamma_\mathcal{V}\mathbf{1}) * A \\ &= (\Gamma_\mathcal{W}\mathbf{1} \otimes L_\mathcal{Y}\mathbf{1} \otimes \Gamma_\mathcal{V}\mathbf{1}) * A \end{aligned}$$

where $\mathcal{W}$ and $\mathcal{Y}$ are Thomason subsets such that $\mathcal{W} \setminus (\mathcal{W} \cap \mathcal{Y}) = \{x\}$. If $x \in \mathcal{V}$ the subsets $\mathcal{W} \cap \mathcal{V}$ and $\mathcal{Y}$ also satisfy the conditions of Lemma 2.2.15 i.e.,

$$\mathcal{W} \cap \mathcal{V} \setminus (\mathcal{W} \cap \mathcal{V} \cap \mathcal{Y}) = \{x\}.$$

By [11] Proposition 3.11 $\Gamma_{\mathcal{W}}\mathbf{1} \otimes \Gamma_{\mathcal{V}}\mathbf{1} = \Gamma_{\mathcal{W}\cap\mathcal{V}}\mathbf{1}$. So in this case

$$\Gamma_x\mathbf{1} * \Gamma_{\mathcal{V}}\mathbf{1} * A \cong (\Gamma_{\mathcal{W}\cap\mathcal{V}}\mathbf{1} \otimes L_{\mathcal{Y}}\mathbf{1}) * A \cong \Gamma_x\mathbf{1} * A.$$

If $x \notin \mathcal{V}$ then $\mathcal{W} \cap \mathcal{V}$ is contained in $\mathcal{Y}$. It follows that $\Gamma_{\mathcal{W}\cap\mathcal{V}}\mathcal{T} \subseteq \Gamma_{\mathcal{Y}}\mathcal{T}$ so, using standard facts about acyclization and localization functors e.g. [13] Lemma 3.4,

$$\Gamma_x\mathbf{1} * \Gamma_{\mathcal{V}}\mathbf{1} * A \cong 0.$$

This proves $\operatorname{supp} \Gamma_{\mathcal{V}}\mathbf{1} * A = \operatorname{supp} A \cap \mathcal{V}$. One proves the analogue for $L_{\mathcal{V}}\mathbf{1} * A$ similarly. □

Corollary 2.2.21. *Let x be a visible point of* $\operatorname{Spc} \mathcal{T}^c$. *Then, for $\mathcal{T}$ acting on itself,* $\operatorname{supp} \Gamma_x\mathbf{1} = \{x\}$. *We also have that for distinct points x_1, x_2 of* $\operatorname{Vis} \mathcal{T}^c$ *the tensor product $\Gamma_{x_1}\mathbf{1} \otimes \Gamma_{x_2}\mathbf{1}$ vanishes.*

Proof. Let $\mathcal{V}$ and $\mathcal{W}$ be Thomason subsets giving rise to $\Gamma_x\mathbf{1}$. Statement (4) of the proposition implies

$$\begin{aligned} \operatorname{supp} \Gamma_x\mathbf{1} &= \operatorname{supp}(\Gamma_{\mathcal{V}}\mathbf{1} \otimes (L_{\mathcal{W}}\mathbf{1} \otimes \mathbf{1})) \\ &= \mathcal{V} \cap \operatorname{supp}(L_{\mathcal{W}}\mathbf{1} \otimes \mathbf{1}) \\ &= \mathcal{V} \cap (\operatorname{Vis} \mathcal{T}^c \setminus \mathcal{W}) \cap \operatorname{supp} \mathbf{1} \\ &= \mathcal{V} \cap (\operatorname{Vis} \mathcal{T}^c \setminus \mathcal{W}) \cap \operatorname{Vis} \mathcal{T}^c \\ &= \{x\} \end{aligned}$$

which proves the first part of the corollary.

For the second statement recall from [11] Remark 7.6 that $\Gamma_{x_1}\mathbf{1} \otimes \Gamma_{x_2}\mathbf{1}$ is isomorphic to $\Gamma_{\varnothing}\mathbf{1}$. Given any Thomason subset $\mathcal{V}$ we have

$$\Gamma_{\varnothing}\mathbf{1} \cong \Gamma_{\mathcal{V}}\mathbf{1} \otimes L_{\mathcal{V}}\mathbf{1} \cong 0,$$

by [11] Corollary 7.5, which shows the tensor product in question vanishes as claimed. □

Finally we can in this generality define a pair of assignments between visible subsets of $\operatorname{Spc} \mathcal{T}^c$ and localizing submodules of $\mathcal{K}$.

Definition 2.2.22. We say a subset $W \subseteq \operatorname{Spc} \mathcal{T}^c$ is *visible* if every $x \in W$ is a visible point or equivalently if $W \subseteq \operatorname{Vis} \mathcal{T}^c$. There are order preserving assignments

$$\left\{ \begin{matrix} \text{visible} \\ \text{subsets of } \operatorname{Spc} \mathcal{T}^c \end{matrix} \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \text{localizing submodules of } \mathcal{K} \right\}$$

where both collections are ordered by inclusion, for a localizing submodule $\mathcal{L}$ we set

$$\sigma(\mathcal{L}) = \operatorname{supp} \mathcal{L} = \{x \in \operatorname{Vis} \mathcal{T}^c \mid \Gamma_x \mathcal{L} \neq 0\}$$

and

$$\tau(W) = \{A \in \mathcal{K} \mid \operatorname{supp} A \subseteq W\}.$$

Both of these are well defined; this is clear for σ and for τ it follows from Proposition 2.2.20.

2.3 Homotopy Colimits and the Local-to-Global Principle

Throughout this section we fix an action $\mathcal{T} \times \mathcal{K} \xrightarrow{*} \mathcal{K}$ where $\mathcal{T}$ is a rigidly-compactly generated tensor triangulated category and $\mathcal{K}$ is compactly generated. Furthermore, we assume $\operatorname{Spc} \mathcal{T}^c$ is a noetherian topological space so that specialization closed subsets are the same as Thomason subsets. All submodules are again assumed to be localizing.

We begin by generalizing the local-to-global principle of [17].

Definition 2.3.1. We say $\mathcal{T} \times \mathcal{K} \xrightarrow{*} \mathcal{K}$ satisfies the *local-to-global principle* if for each A in $\mathcal{K}$

$$\langle A \rangle_* = \langle \Gamma_x A \mid x \in \operatorname{Spc} \mathcal{T}^c \rangle_*.$$

Remark 2.3.2. In the case that every localizing subcategory is also a $\mathcal{T}$-submodule we recover the Benson-Iyengar-Krause local-to-global principle.

The local-to-global principle has the following rather pleasing consequences for the assignments σ and τ of Definition 2.2.22.

Lemma 2.3.3. *Suppose the local-to-global principle holds for the action of $\mathcal{T}$ on $\mathcal{K}$ and let W be a subset of $\operatorname{Spc} \mathcal{T}^c$. Then*

$$\tau(W) = \langle \Gamma_x \mathcal{K} \mid x \in W \cap \sigma\mathcal{K} \rangle_*.$$

Proof. By the local-to-global principle we have for every object A of $\mathcal{K}$ an equality

$$\langle A \rangle_* = \langle \Gamma_x A \mid x \in \operatorname{Spc} \mathcal{T}^c \rangle_*.$$

Thus

$$\begin{aligned} \tau(W) &= \langle A \mid \operatorname{supp} A \subseteq W \rangle_* \\ &= \langle \Gamma_x A \mid A \in \mathcal{K}, x \in W \rangle_* \\ &= \langle \Gamma_x A \mid A \in \mathcal{K}, x \in W \cap \sigma\mathcal{K} \rangle_* \\ &= \langle \Gamma_x \mathcal{K} \mid x \in W \cap \sigma\mathcal{K} \rangle_*. \end{aligned}$$

□

Proposition 2.3.4. *Suppose the local-to-global principle holds for the action of $\mathcal{T}$ on $\mathcal{K}$ and let W be a subset of $\operatorname{Spc} \mathcal{T}^c$. Then there is an equality of subsets*

$$\sigma\tau(W) = W \cap \sigma\mathcal{K}.$$

In particular, τ is injective when restricted to subsets of $\sigma\mathcal{K}$.

Proof. With $W \subseteq \operatorname{Spc} \mathcal{T}^c$ as in the statement we have

$$\begin{aligned} \sigma\tau(W) &= \operatorname{supp} \tau(W) \\ &= \operatorname{supp}\langle \Gamma_x \mathcal{K} \mid x \in W \cap \sigma\mathcal{K} \rangle_*, \end{aligned}$$

the first equality by definition and the second by the last lemma. Thus $\sigma\tau(W) = W \cap \sigma\mathcal{K}$ as claimed: by the properties of the support (Proposition 2.2.20) we have $\sigma\tau(W) \subseteq W \cap \sigma\mathcal{K}$ and it must in fact be all of $W \cap \sigma\mathcal{K}$ as $x \in \sigma\mathcal{K}$ if and only if $\Gamma_x \mathcal{K}$ contains a non-zero object.

□

We will show that the local-to-global principle holds quite generally. Before proceeding let us fix some terminology we will use throughout the section.

Definition 2.3.5. We will say $\mathcal{T}$ *has a model* when it occurs as the homotopy category of a Quillen model category.

Our main interest in such categories is that the existence of a model provides a good theory of homotopy colimits. For our purposes only directed homotopy colimits are required. We begin by showing that, when $\mathcal{T}$ has a model, taking the union of a chain of specialization closed subsets is compatible with taking the homotopy colimit of the associated idempotents.

Lemma 2.3.6. *Suppose $\mathcal{T}$ has a model. Then for any chain $\{\mathcal{V}_i\}_{i\in I}$ of specialization closed subsets of $\operatorname{Spc}\mathcal{T}^c$ with union $\mathcal{V}$ there is an isomorphism*

$$\Gamma_{\mathcal{V}}\mathbf{1} \cong \operatorname{hocolim} \Gamma_{\mathcal{V}_i}\mathbf{1}$$

where the structure maps are the canonical ones.

Proof. As each $\mathcal{V}_i$ is contained in $\mathcal{V}$ there are corresponding inclusions for $i < j$

$$\mathcal{T}_{\mathcal{V}_i} \subseteq \mathcal{T}_{\mathcal{V}_j} \subseteq \mathcal{T}_{\mathcal{V}}$$

which give rise to commuting triangles of canonical morphisms

$$\begin{array}{ccc} \Gamma_{\mathcal{V}_i}\mathbf{1} & \longrightarrow & \Gamma_{\mathcal{V}}\mathbf{1} \\ & \searrow \quad \nearrow & \\ & \Gamma_{\mathcal{V}_j}\mathbf{1} & \end{array}$$

We thus get an induced morphism from the homotopy colimit of the $\Gamma_{\mathcal{V}_i}\mathbf{1}$ to $\Gamma_{\mathcal{V}}\mathbf{1}$ which we complete to a triangle

$$\operatorname{hocolim}_I \Gamma_{\mathcal{V}_i}\mathbf{1} \longrightarrow \Gamma_{\mathcal{V}}\mathbf{1} \longrightarrow Z \longrightarrow \Sigma \operatorname{hocolim}_I \Gamma_{\mathcal{V}_i}\mathbf{1}.$$

In order to prove the lemma it is sufficient to show that Z is isomorphic to the zero object in $\mathcal{T}$.

The argument in [23] extends to show localizing subcategories are closed under directed homotopy colimits so this triangle consists of objects of $\Gamma_{\mathcal{V}}\mathcal{T}$. By definition $\Gamma_{\mathcal{V}}\mathcal{T}$ is the full subcategory of $\mathcal{T}$ generated by those objects of $\mathcal{T}^c$ whose support (in the sense of [7]) is contained in $\mathcal{V}$. Thus $Z \cong 0$ if for each compact object k with $\operatorname{supp} k \subseteq \mathcal{V}$ we have $\operatorname{Hom}(k, Z) = 0$; we remark that there is no ambiguity here as by [11] Proposition 7.17 the two notions of support, that of [7] and [11], agree for compact objects. In particular the support of any compact object is closed.

Recalling from [25] that $\operatorname{Spc}\mathcal{T}^c$ is spectral in the sense of Hochster [39] we see $\operatorname{supp} k$, by virtue of being closed, is a finite union of irreducible closed subsets. We can certainly find an $i \in I$ so that $\mathcal{V}_i$ contains the generic points of these finitely many irreducible components which implies $\operatorname{supp} k \subseteq \mathcal{V}_i$ by specialization closure of the $\mathcal{V}_i$.

Therefore, by adjunction, it is enough to show

$$\begin{aligned} \operatorname{Hom}(k, Z) &\cong \operatorname{Hom}(\Gamma_{\mathcal{V}_i}k, Z) \\ &\cong \operatorname{Hom}(k, \Gamma_{\mathcal{V}_i}Z) \end{aligned}$$

is zero. The vanishing of this hom-set is clear by construction so $Z \cong 0$ and we get the claimed isomorphism. □

Lemma 2.3.7. *Let $P \subseteq \operatorname{Spc} \mathcal{T}^c$ be given and suppose A is an object of $\mathcal{K}$ such that $\Gamma_x A \cong 0$ for all $x \in (\operatorname{Spc} \mathcal{T}^c \setminus P)$. If $\mathcal{T}$ has a model then A is an object of the localizing subcategory*

$$\mathcal{L} = \langle \Gamma_y \mathcal{K} \mid y \in P \rangle_{\mathrm{loc}}.$$

Proof. Let $\Lambda \subseteq \mathcal{P}(\operatorname{Spc} \mathcal{T}^c)$ be the set of specialization closed subsets $\mathcal{W}$ such that $\Gamma_{\mathcal{W}} A$ is in $\mathcal{L} = \langle \Gamma_y \mathcal{K} \mid y \in P \rangle_{\mathrm{loc}}$. We first note that Λ is not empty. Indeed, as $\mathcal{T}^c$ is rigid the only compact objects with empty support are the zero objects by [8] Corollary 2.5 so

$$\mathcal{T}_\varnothing = \langle t \in \mathcal{T}^c \mid \operatorname{supp}_{(\mathcal{T},\otimes)} t = \varnothing \rangle_{\mathrm{loc}} = \langle 0 \rangle_{\mathrm{loc}}$$

giving $\Gamma_\varnothing A = 0$ and hence $\varnothing \in \Lambda$.

Since $\mathcal{L}$ is localizing, Lemma 2.3.6 shows the set Λ is closed under taking increasing unions: as mentioned above the argument in [23] extends to show that localizing subcategories are closed under directed homotopy colimits in our situation. Thus Λ contains a maximal element Y by Zorn's lemma. We claim that $Y = \operatorname{Spc} \mathcal{T}^c$.

Suppose $Y \neq \operatorname{Spc} \mathcal{T}^c$. Then since $\operatorname{Spc} \mathcal{T}^c$ is noetherian $\operatorname{Spc} \mathcal{T}^c \setminus Y$ contains a maximal element z with respect to specialization. We have

$$L_Y \mathbf{1} \otimes \Gamma_{Y \cup \{z\}} \mathbf{1} \cong \Gamma_z \mathbf{1}$$

as $Y \cup \{z\}$ is specialization closed by maximality of z and Lemma 2.2.15 tells us that we can use any suitable pair of Thomason subsets to define $\Gamma_z \mathbf{1}$. So $L_Y \Gamma_{Y \cup \{z\}} A \cong \Gamma_z A$ and by our hypothesis on vanishing either $\Gamma_z \mathcal{K} \subseteq \mathcal{L}$ or $\Gamma_z A = 0$. Considering the triangle

$$\begin{array}{ccccc} \Gamma_Y \Gamma_{Y \cup \{z\}} A & \longrightarrow & \Gamma_{Y \cup \{z\}} A & \longrightarrow & L_Y \Gamma_{Y \cup \{z\}} A \\ \wr\downarrow & & & & \downarrow\wr \\ \Gamma_Y A & & & & \Gamma_z A \end{array}$$

we see that in either case, since $\Gamma_Y A$ is in $\mathcal{L}$, that $Y \cup \{z\} \in \Lambda$ contradicting maximality of Y. Hence $Y = \operatorname{Spc} \mathcal{T}^c$ and so A is in $\mathcal{L}$. □

Proposition 2.3.8. *Suppose $\mathcal{T}$ has a model. Then the local-to-global principle holds for the action of $\mathcal{T}$ on $\mathcal{K}$. Explicitly for any A in $\mathcal{K}$ there is an equality of $\mathcal{T}$-submodules*

$$\langle A \rangle_* = \langle \Gamma_x A \mid x \in \operatorname{supp} A \rangle_*.$$

Proof. Observe that by Lemma 2.3.7 applied to the action

$$\mathcal{T} \times \mathcal{T} \xrightarrow{\otimes} \mathcal{T}$$

we see $\mathcal{T} = \langle \Gamma_x \mathcal{T} \mid x \in \operatorname{Spc} \mathcal{T}^c \rangle_{\mathrm{loc}}$. Since $\Gamma_x \mathcal{T} = \langle \Gamma_x \mathbf{1} \rangle_\otimes$ it follows that the set of objects $\{\Gamma_x \mathbf{1} \mid x \in \operatorname{Spc} \mathcal{T}^c\}$ generates $\mathcal{T}$ as a localizing $\otimes$-ideal. By Lemma 2.1.14 given an object $A \in \mathcal{K}$ we get a generating set for $\mathcal{T} * \langle A \rangle_{\mathrm{loc}}$:

$$\mathcal{T} * \langle A \rangle_{\mathrm{loc}} = \langle \Gamma_x \mathbf{1} \mid x \in \operatorname{Spc} \mathcal{T} \rangle_\otimes * \langle A \rangle_{\mathrm{loc}} = \langle \Gamma_x A \mid x \in \operatorname{supp} A \rangle_*.$$

But it is also clear that $\mathcal{T} = \langle \mathbf{1} \rangle_\otimes$ so, by Lemma 2.1.14 again,

$$\mathcal{T} * \langle A \rangle_{\mathrm{loc}} = \langle \mathbf{1} \rangle_\otimes * \langle A \rangle_{\mathrm{loc}} = \langle A \rangle_*$$

and combining this with the other string of equalities gives

$$\langle A \rangle_* = \mathcal{T} * \langle A \rangle_{\mathrm{loc}} = \langle \Gamma_x A \mid x \in \operatorname{supp} A \rangle_*$$

which completes the proof.

∎

We thus have the following theorem concerning the local-to-global principle for actions of rigidly-compactly generated tensor triangulated categories.

Theorem 2.3.9. *Suppose $\mathcal{T}$ is a rigidly-compactly generated tensor triangulated category with a model and that $\operatorname{Spc} \mathcal{T}^c$ is noetherian. Then $\mathcal{T}$ satisfies the following properties:*

(*i*) *The local-to-global principle holds for the action of $\mathcal{T}$ on itself;*

(*ii*) *The associated support theory detects vanishing of objects i.e., $X \in \mathcal{T}$ is zero if and only if $\operatorname{supp} X = \varnothing$;*

(*iii*) *For any chain $\{\mathcal{V}_i\}_{i \in I}$ of specialization closed subsets of $\operatorname{Spc} \mathcal{T}^c$ with union $\mathcal{V}$ there is an isomorphism*

$$\Gamma_{\mathcal{V}} \mathbf{1} \cong \operatorname{hocolim} \Gamma_{\mathcal{V}_i} \mathbf{1}$$

where the structure maps are the canonical ones.

Furthermore, the relative versions of (i) and (ii) hold for any action of $\mathcal{T}$ on a compactly generated triangulated category $\mathcal{K}$.

Proof. That (iii) always holds is the content of Lemma 2.3.6 and we have proved in Proposition 2.3.8 that (i) holds. To see (i) implies (ii) observe that if $\operatorname{supp} X = \varnothing$ for an object X of $\mathcal{T}$ then the local-to-global principle yields

$$\langle X \rangle_\otimes = \langle \Gamma_x X \mid x \in \operatorname{Spc} \mathcal{T}^c \rangle_\otimes = \langle 0 \rangle_\otimes$$

so $X \cong 0$.

Finally, we saw in Proposition 2.3.8 that the relative version of (i) holds. This in turn implies (ii) for supports with values in $\operatorname{Spc} \mathcal{T}^c$ by the same argument as we have used in the proof of (i)⇒(ii) above. □

2.4 The Telescope Conjecture

We now explore a relative version of the telescope conjecture. We show that for particularly nice actions $\mathcal{T} \times \mathcal{K} \xrightarrow{*} \mathcal{K}$ we can deduce the relative telescope conjecture for $\mathcal{K}$. We will denote by $\mathcal{T}$ a rigidly-compactly generated tensor triangulated category with noetherian spectrum (although let us note that not all of the results require rigidity or a noetherian spectrum) and by $\mathcal{K}$ a compactly generated triangulated category on which $\mathcal{T}$ acts.

Definition 2.4.1. We say the *relative telescope conjecture* holds for $\mathcal{K}$ with respect to the action of $\mathcal{T}$ if every smashing $\mathcal{T}$-submodule $\mathcal{S} \subseteq \mathcal{K}$ (we recall this means $\mathcal{S}$ is a localizing submodule with an associated coproduct preserving localization functor) is generated by compact objects of $\mathcal{K}$.

Remark 2.4.2. This reduces to the usual telescope conjecture if every localizing subcategory of $\mathcal{K}$ is a submodule. It is also the usual telescope conjecture in the case that a rigidly-compactly generated triangulated category acts on itself (see [40] Definition 3.3.2).

Lemma 2.4.3. *Suppose $\mathcal{S} \subseteq \mathcal{K}$ is a smashing $\mathcal{T}$-submodule. Then $\mathcal{S}^\perp$ is a localizing $\mathcal{T}$-submodule.*

Proof. Let us denote by $\mathcal{L}$ the subcategory of those objects of $\mathcal{T}$ which send $\mathcal{S}^\perp$ to itself

$$\mathcal{L} = \{X \in \mathcal{T} \mid X * \mathcal{S}^\perp \subseteq \mathcal{S}^\perp\}.$$

As $\mathcal{S}$ is smashing the subcategory $\mathcal{S}^\perp$ is a localizing subcategory of $\mathcal{K}$ (see for example [46] Proposition 5.5.1). Thus $\mathcal{L}$ is a localizing subcategory of $\mathcal{T}$ by the standard argument.

If x is a compact object of $\mathcal{T}$ then, as we have assumed $\mathcal{T}$ rigidly-compactly generated, the object x is strongly dualizable. By Lemma 2.2.8 the functor $x*(-)$ has a right adjoint $x^\vee * (-)$ so given B in $\mathcal{S}^\perp$ we have, for every A in $\mathcal{S}$,

$$0 = \mathrm{Hom}(x * A, B) \cong \mathrm{Hom}(A, x^\vee * B),$$

where the first hom-set vanishes due to the fact that $\mathcal{S}$ is a submodule so $x * A$ is an object of $\mathcal{S}$. Hence $x^\vee * B$ is an object of $\mathcal{S}^\perp$ for every x in $\mathcal{T}^c$. As taking duals of compact objects in $\mathcal{T}$ is involutive this implies that every object of $\mathcal{T}^c$ sends $\mathcal{S}^\perp$ to $\mathcal{S}^\perp$. Thus $\mathcal{T}^c$ is contained in the localizing subcategory $\mathcal{L}$ yielding the equality $\mathcal{L} = \mathcal{T}$. Hence every object X of $\mathcal{T}$ satisfies $X * \mathcal{S}^\perp \subseteq \mathcal{S}^\perp$ so that $\mathcal{S}^\perp$ is a localizing $\mathcal{T}$-submodule of $\mathcal{K}$. □

Definition 2.4.4. Let $\mathcal{M}$ be a localizing $\mathcal{T}$-submodule of $\mathcal{K}$. We define a subcategory $\mathcal{T}_\mathcal{M}$ of $\mathcal{T}$ by

$$\mathcal{T}_\mathcal{M} = \{X \in \mathcal{T} \mid X * \mathcal{K} \subseteq \mathcal{M}\}.$$

Lemma 2.4.5. *Suppose $\mathcal{M}$ is a localizing submodule of $\mathcal{K}$. Then the subcategory $\mathcal{T}_\mathcal{M}$ is a localizing $\otimes$-ideal of $\mathcal{T}$.*

Proof. The usual argument shows that $\mathcal{T}_\mathcal{M}$ is a localizing subcategory; as $\mathcal{M}$ is localizing and the action is exact and coproduct preserving in both variables one deduces triangle, suspension, and coproduct closure from the corresponding properties of $\mathcal{M}$.

It is also easily seen that $\mathcal{T}_\mathcal{M}$ is a $\otimes$-ideal. If X is an object of $\mathcal{T}_\mathcal{M}$, Y is any object of $\mathcal{T}$, and A is in $\mathcal{K}$

$$(Y \otimes X) * A \cong (X \otimes Y) * A \cong X * (Y * A)$$

which lies in $\mathcal{M}$ as $X * \mathcal{K} \subseteq \mathcal{M}$. Thus $Y \otimes X$ lies in $\mathcal{T}_\mathcal{M}$. □

Hypotheses 2.4.6. *We now, and for the rest of this section unless otherwise stated, ask more of $\mathcal{T}$ and $\mathcal{K}$: we suppose $\mathcal{T}$ has a model, so Theorem 2.3.9 applies, and that the assignments σ and τ of Definition 2.2.22 provide a bijection between subsets of $\sigma\mathcal{K} \subseteq \mathrm{Spc}\,\mathcal{T}^c$ (which we give the subspace topology throughout) and localizing $\mathcal{T}$-submodules of $\mathcal{K}$. In particular, for any localizing submodule $\mathcal{M}$ of $\mathcal{K}$ there is an equality*

$$\mathcal{M} = \tau(\sigma\mathcal{M}) = \{A \in \mathcal{K} \mid \mathrm{supp}\, A \subseteq \sigma\mathcal{M}\}.$$

Lemma 2.4.7. *Suppose $\mathcal{M}$ is a localizing $\mathcal{T}$-submodule of $\mathcal{K}$. Then there is an equality of subcategories*

$$\mathcal{M} = \mathcal{T}_\mathcal{M} * \mathcal{K}.$$

Proof. By Lemma 2.3.3 and $\tau(\sigma\mathcal{M}) = \mathcal{M}$ we have

$$\mathcal{M} = \langle \Gamma_x\mathcal{K} \mid x \in \sigma\mathcal{M}\rangle_*.$$

So by definition of $\mathcal{T}_\mathcal{M}$ the objects $\Gamma_x\mathbf{1}$ for $x \in \sigma\mathcal{M}$ lie in $\mathcal{T}_\mathcal{M}$. Thus $\mathcal{M} \subseteq \mathcal{T}_\mathcal{M} * \mathcal{K}$. That $\mathcal{T}_\mathcal{M} * \mathcal{K} \subseteq \mathcal{M}$ is immediate from the definition of $\mathcal{T}_\mathcal{M}$ giving the claimed equality. □

Proposition 2.4.8. *Suppose $\mathcal{T}$ satisfies the telescope conjecture and let $\mathcal{S} \subseteq \mathcal{K}$ be a smashing $\mathcal{T}$-submodule. If the inclusion $\mathcal{T}_\mathcal{S} \longrightarrow \mathcal{T}$ admits a right adjoint and*

$$(\mathcal{T}_\mathcal{S})^\perp = \mathcal{T}_{\mathcal{S}^\perp}$$

then $\mathcal{S}$ is generated by compact objects of $\mathcal{K}$.

Proof. The subcategory $\mathcal{S}$ is, by assumption, a localizing submodule and as it is smashing $\mathcal{S}^\perp$ is also a localizing submodule by Lemma 2.4.3. Thus Lemma 2.4.5 yields that both $\mathcal{T}_\mathcal{S}$ and $\mathcal{T}_{\mathcal{S}^\perp}$ are localizing $\otimes$-ideals of $\mathcal{T}$. By hypothesis the $\otimes$-ideals $\mathcal{T}_\mathcal{S}$ and $(\mathcal{T}_\mathcal{S})^\perp = \mathcal{T}_{\mathcal{S}^\perp}$ fit into a localization sequence. Hence $\mathcal{T}_\mathcal{S}$ is a smashing subcategory of $\mathcal{T}$ (this is well known, see for example [11] Theorem 2.13). As the telescope conjecture is assumed to hold for $\mathcal{T}$ the subcategory $\mathcal{T}_\mathcal{S}$ is generated by objects of $\mathcal{T}^c$. By Lemma 2.4.7 there is an equality of submodules

$$\mathcal{S} = \mathcal{T}_\mathcal{S} * \mathcal{K}$$

which implies that $\mathcal{S}$ is generated by compact objects of $\mathcal{K}$: by Proposition 2.2.11, since $\mathcal{T}$ is rigidly-compactly generated and $\mathcal{T}_\mathcal{S}$ is generated by objects of $\mathcal{T}^c$, the subcategory $\mathcal{T}_\mathcal{S} * \mathcal{K}$ is generated by objects of $\mathcal{K}^c$. □

Lemma 2.4.9. *Let $\mathcal{M}$ be a localizing submodule of $\mathcal{K}$ and let W be a subset of* $\operatorname{Spc}\mathcal{T}^c$ *such that $W \cap \sigma\mathcal{K} = \sigma\mathcal{M}$. Then there is a containment of $\otimes$-ideals of $\mathcal{T}$*

$$\mathcal{T}_\mathcal{M} \supseteq \mathcal{T}_W = \{X \in \mathcal{T} \mid \operatorname{supp} X \subseteq W\}$$

and

$$\mathcal{T}_W * \mathcal{K} = \mathcal{M}.$$

Proof. It follows from the good properties of the support that $\mathcal{T}_W$ is a localizing $\otimes$-ideal of $\mathcal{T}$. Let X be an object of $\mathcal{T}_W$, let A be an object of $\mathcal{K}$ and let x be a point in $\operatorname{Spec}\mathcal{T}^c$. We have isomorphisms

$$\Gamma_x\mathbf{1} * (X * A) \cong (\Gamma_x\mathbf{1} \otimes X) * A \cong X * (\Gamma_x\mathbf{1} * A).$$

The object $\Gamma_x\mathbf{1} \otimes X$ is zero if x is not in W and $\Gamma_x\mathbf{1} * A \cong 0$ if $x \notin \sigma\mathcal{K}$ so we see $\operatorname{supp} X * A$ is contained in $\sigma\mathcal{M}$. Thus $X * A$ is an object of $\mathcal{M} = \tau\sigma\mathcal{M}$. It follows that X is in $\mathcal{T}_\mathcal{M}$ and hence $\mathcal{T}_W \subseteq \mathcal{T}_\mathcal{M}$.

As $\operatorname{supp}\Gamma_x\mathbf{1} = \{x\}$ for $x \in \operatorname{Spc}\mathcal{T}^c$ by Corollary 2.2.21 we have $\Gamma_x\mathbf{1} \in \mathcal{T}_W$ for $x \in \sigma\mathcal{M}$. By the local-to-global principle (Theorem 2.3.9) and $\tau(\sigma\mathcal{M}) = \mathcal{M}$ we have

$$\mathcal{M} = \langle \Gamma_x\mathcal{K} \mid x \in \sigma\mathcal{M}\rangle_*$$

so $\mathcal{T}_W * \mathcal{K} \supseteq \mathcal{M}$. We proved above that $\mathcal{T}_W \subseteq \mathcal{T}_\mathcal{M}$ which gives $\mathcal{T}_W * \mathcal{K} \subseteq \mathcal{M}$. Thus $\mathcal{T}_W * \mathcal{K} = \mathcal{M}$. □

Lemma 2.4.10. *Suppose the support of any compact object of $\mathcal{K}$ is a specialization closed subset of $\sigma\mathcal{K}$. Then for any specialization closed subset $\mathcal{V}$ of $\operatorname{Spc}\mathcal{T}^c$, with complement $\mathcal{U}$, the support of every compact object of $L_\mathcal{V}\mathcal{K}$ is specialization closed in the complement $\mathcal{U} \cap \sigma\mathcal{K}$ of $\mathcal{V} \cap \sigma\mathcal{K}$ in $\sigma\mathcal{K}$ (with the subspace topology).*

Proof. Let us denote by π the quotient functor $\mathcal{K} \longrightarrow L_\mathcal{V}\mathcal{K}$. We assert it sends compact objects to compact objects. To see this is the case recall $\Gamma_\mathcal{V}\mathcal{K}$ has a generating set consisting of objects in $\mathcal{K}^c$ by Corollary 2.2.13 so π has a coproduct preserving right adjoint. The functor π thus takes compact objects to compact objects by Theorem 5.1 of [60].

Given any compact object l of $L_\mathcal{V}\mathcal{K}$ there exists an object k in $\mathcal{K}^c$ such that $l \oplus \Sigma l$ is isomorphic to πk by [61] Corollary 4.5.14. Thus

$$\operatorname{supp} l = \operatorname{supp}(l \oplus \Sigma l) = \operatorname{supp} \pi k = \operatorname{supp} L_\mathcal{V} k = \operatorname{supp} k \cap \mathcal{U}$$

where this last equality is (4) of Proposition 2.2.20. Thus $\operatorname{supp} l$ is specialization closed in $\mathcal{U} \cap \sigma\mathcal{K}$ as $\operatorname{supp} k$ is is specialization closed in $\sigma\mathcal{K}$. □

The next lemma is the key to our theorem on the relative telescope conjecture for good actions. Before stating and proving it we recall from [7] Proposition 2.9 that the space $\operatorname{Spc}\mathcal{T}^c$ is T_0; given points $x, y \in \operatorname{Spc}\mathcal{T}^c$ we have $x = y$ if and only if $\mathcal{V}(x) = \mathcal{V}(y)$. In fact $\operatorname{Spc}\mathcal{T}^c$ is spectral in the sense of Hochster [39] so every irreducible closed subset has a unique generic point.

Lemma 2.4.11. *Suppose the support of any compact object of $\mathcal{K}$ is a specialization closed subset of $\sigma\mathcal{K}$ and that for each irreducible closed subset $\mathcal{V} \subseteq \operatorname{Spc}\mathcal{T}^c$ there exists a compact object of $\mathcal{K}$ whose support is precisely $\mathcal{V} \cap \sigma\mathcal{K}$. If x and y are distinct points of $\sigma\mathcal{K}$ with $y \in \mathcal{V}(x)$ then*

$$\langle \Gamma_{y'}\mathcal{K} \mid y' \in (\mathcal{V}(x) \cap \mathcal{U}(y)) \setminus \{x\}\rangle_{\mathrm{loc}} \nsubseteq \Gamma_x\mathcal{K}^{\perp}$$

where $\mathcal{U}(y) = \{y' \in \operatorname{Spc}\mathcal{T}^c \mid y \in \mathcal{V}(y')\}$ is the complement of $\mathcal{Z}(y)$.

Proof. By hypothesis there is a compact object k of $\mathcal{K}$ satisfying

$$\operatorname{supp} k = \mathcal{V}(x) \cap \sigma\mathcal{K}.$$

The object $L_{\mathcal{Z}(y)}k$ is compact in $L_{\mathcal{Z}(y)}\mathcal{K}$ and has support

$$\operatorname{supp} L_{\mathcal{Z}(y)}k = \operatorname{supp} k \cap (\operatorname{Spc}\mathcal{T}^c \setminus \mathcal{Z}(y)) \cap \sigma\mathcal{K} = \mathcal{V}(x) \cap \mathcal{U}(y) \cap \sigma\mathcal{K}$$

by Proposition 2.2.20.

Suppose for a contradiction that

$$\langle \Gamma_{y'}\mathcal{K} \mid y' \in (\mathcal{V}(x) \cap \mathcal{U}(y)) \setminus \{x\}\rangle_{\mathrm{loc}} \subseteq \Gamma_x\mathcal{K}^{\perp}.$$

Consider the localization triangle for $L_{\mathcal{Z}(y)}k$

$$\Gamma_{\mathcal{Z}(x)}L_{\mathcal{Z}(y)}k \longrightarrow L_{\mathcal{Z}(y)}k \longrightarrow L_{\mathcal{Z}(x)}L_{\mathcal{Z}(y)}k \longrightarrow \Sigma\Gamma_{\mathcal{Z}(x)}L_{\mathcal{Z}(y)}k.$$

We have, via Proposition 2.2.20,

$$\operatorname{supp} L_{\mathcal{Z}(x)}L_{\mathcal{Z}(y)}k = \mathcal{U}(x) \cap \mathcal{V}(x) \cap \mathcal{U}(y) \cap \sigma\mathcal{K} = \{x\}$$

and

$$\operatorname{supp} \Sigma\Gamma_{\mathcal{Z}(x)}L_{\mathcal{Z}(y)}k = \mathcal{Z}(x) \cap \mathcal{V}(x) \cap \mathcal{U}(y) \cap \sigma\mathcal{K} = (\mathcal{V}(x) \cap \mathcal{U}(y) \cap \sigma\mathcal{K}) \setminus \{x\}.$$

So, as the local-to-global principle holds, the morphism $L_{\mathcal{Z}(x)}L_{\mathcal{Z}(y)}k \longrightarrow \Sigma\Gamma_{\mathcal{Z}(x)}L_{\mathcal{Z}(y)}k$ must be zero by our orthogonality assumption. This forces the triangle to split giving

$$L_{\mathcal{Z}(y)}k \cong L_{\mathcal{Z}(x)}L_{\mathcal{Z}(y)}k \oplus \Gamma_{\mathcal{Z}(x)}L_{\mathcal{Z}(y)}k.$$

As $L_{\mathcal{Z}(y)}k$ is compact in $L_{\mathcal{Z}(y)}\mathcal{K}$ it follows that $L_{\mathcal{Z}(x)}L_{\mathcal{Z}(y)}k$ must also be compact. But we have already seen that the support of $L_{\mathcal{Z}(x)}L_{\mathcal{Z}(y)}k$ is $\{x\}$ which is not specialization closed in $\mathcal{U}(y)\cap\sigma\mathcal{K}$. This yields a contradiction as by Lemma 2.4.10 the compact objects in $L_{\mathcal{Z}(y)}\mathcal{K}$ have specialization closed support in $\mathcal{U}(y)\cap\sigma\mathcal{K}$. □

Lemma 2.4.12. *Let $\mathcal{S}$ be a smashing $\mathcal{T}$-submodule of $\mathcal{K}$. Then*

$$\sigma\mathcal{S} \cup \sigma\mathcal{S}^\perp = \sigma\mathcal{K} \quad \textit{and} \quad \sigma\mathcal{S} \cap \sigma\mathcal{S}^\perp = \varnothing.$$

Proof. Suppose x is a point of $\sigma\mathcal{K}$ satisfying $x \in \sigma\mathcal{S} \cap \sigma\mathcal{S}^\perp$. Then as we have assumed σ and τ are inverse bijections and $\mathcal{S}^\perp$ is a localizing submodule by Lemma 2.4.3 we would have

$$\Gamma_x\mathcal{K} \subseteq \mathcal{S} \cap \mathcal{S}^\perp = 0.$$

This contradicts $x \in \sigma\mathcal{K}$ as x is a point of $\sigma\mathcal{K}$ if and only if $\Gamma_x\mathcal{K} \neq 0$.

We now show that every point of $\sigma\mathcal{K}$ lies in either $\sigma\mathcal{S}$ or $\sigma\mathcal{S}^\perp$. Let x be a point of $\sigma\mathcal{K}$ and suppose $x \notin \sigma\mathcal{S}^\perp$. In particular $\Gamma_x\mathcal{K} \nsubseteq \mathcal{S}^\perp$ so there is an object X of $\Gamma_x\mathcal{K}$ with $\Gamma_\mathcal{S}X \neq 0$ where $\Gamma_\mathcal{S}$ is the acyclization functor associated to $\mathcal{S}$. Consider the localization triangle for X associated to $\mathcal{S}$

$$\Gamma_\mathcal{S}X \longrightarrow X \longrightarrow L_\mathcal{S}X \longrightarrow \Sigma\Gamma_\mathcal{S}X.$$

Applying Γ_x we get another triangle

$$\Gamma_x\Gamma_\mathcal{S}X \longrightarrow \Gamma_xX \longrightarrow \Gamma_xL_\mathcal{S}X \longrightarrow \Sigma\Gamma_x\Gamma_\mathcal{S}X.$$

Since $x \notin \sigma\mathcal{S}^\perp$ we have $\Gamma_xL_\mathcal{S}X \cong 0$. Hence

$$0 \neq X \cong \Gamma_xX \cong \Gamma_x\Gamma_\mathcal{S}X$$

so $\Gamma_x\mathcal{S}$ is not the zero subcategory and $x \in \sigma\mathcal{S}$.

□

Lemma 2.4.13. *Suppose the support of any compact object of $\mathcal{K}$ is a specialization closed subset of $\sigma\mathcal{K}$ and that for each irreducible closed subset $\mathcal{V}$ in* $\operatorname{Spc}\mathcal{T}^c$ *there exists a compact object of $\mathcal{K}$ whose support is precisely $\mathcal{V} \cap \sigma\mathcal{K}$. Let $\mathcal{S} \subseteq \mathcal{K}$ be a smashing $\mathcal{T}$-submodule. Then the subset $\sigma\mathcal{S}$ is specialization closed in $\sigma\mathcal{K}$.*

Proof. We prove the lemma by contradiction. Let x be a point of $\sigma\mathcal{S}$ and suppose y is a point of $\mathcal{V}(x) \cap \sigma\mathcal{K}$ which does not lie in $\sigma\mathcal{S}$. Then by the last lemma we must have $y \in \sigma\mathcal{S}^\perp$. We have assumed $\operatorname{Spc}\mathcal{T}^c$ is noetherian so there exists a point x' of $\sigma\mathcal{S} \cap \mathcal{U}(y)$ which is maximal with respect to specialization. We thus have

$$((\mathcal{V}(x') \cap \mathcal{U}(y)) \setminus \{x'\}) \cap \sigma\mathcal{S} = \varnothing$$

by virtue of the maximality of x'. From the previous lemma we deduce that every point of $((\mathcal{V}(x') \cap \mathcal{U}(y)) \setminus \{x'\})$ lies in $\sigma\mathcal{S}^{\perp}$. As σ and τ are inverse there are containments

$$\Gamma_{x'}\mathcal{K} \subseteq \mathcal{S} \quad \text{and} \quad \langle \Gamma_{y'}\mathcal{K} \mid y' \in (\mathcal{V}(x') \cap \mathcal{U}(y)) \setminus \{x'\}\rangle_* \subseteq \mathcal{S}^{\perp}$$

the first as $x \in \sigma\mathcal{S}$ and the second by what we have just shown. Taking orthogonals in the first containment and combining we deduce that

$$\langle \Gamma_{y'}\mathcal{K} \mid y' \in (\mathcal{V}(x') \cap \mathcal{U}(y)) \setminus \{x'\}\rangle_* \subseteq \mathcal{S}^{\perp} \subseteq \Gamma_{x'}\mathcal{K}^{\perp}$$

contradicting Lemma 2.4.11 and completing the proof.

□

Theorem 2.4.14. *Suppose the hypotheses of 2.4.6 hold, the support of any compact object of $\mathcal{K}$ is a specialization closed subset of $\sigma\mathcal{K}$ and that for each irreducible closed subset $\mathcal{V}$ of $\operatorname{Spc}\mathcal{T}^c$ there exists a compact object whose support is precisely $\mathcal{V} \cap \sigma\mathcal{K}$. Then the relative telescope conjecture holds for $\mathcal{K}$ i.e., every smashing $\mathcal{T}$-submodule of $\mathcal{K}$ is generated, as a localizing subcategory, by compact objects of $\mathcal{K}$.*

Proof. Let $\mathcal{S}$ be a smashing submodule of $\mathcal{K}$. Recall from Lemma 2.4.9 that there is an equality

$$\mathcal{T}_W * \mathcal{K} = \mathcal{S} \tag{2.3}$$

for any $W \subseteq \operatorname{Spc}\mathcal{T}^c$ whose intersection with $\sigma\mathcal{K}$ is $\sigma\mathcal{S}$. By the lemma we have just proved the subset $\sigma\mathcal{S}$ is specialization closed in $\sigma\mathcal{K}$ so we can find a specialization closed subset W of $\operatorname{Spc}\mathcal{T}^c$ with $W \cap \sigma\mathcal{K} = \sigma\mathcal{S}$. As W is specialization closed in $\operatorname{Spc}\mathcal{T}^c$ the tensor ideal $\mathcal{T}_W$ is generated by objects of $\mathcal{T}^c$. It then follows from the equality (2.3) that $\mathcal{S}$ is generated by objects of $\mathcal{K}^c$ - this last statement is the content of Proposition 2.2.11. □

2.5 Working Locally

We now show that the support theory we have developed is compatible with passing to quasi-compact open subsets of the spectrum; in particular, certain properties can be checked locally on an open cover.

Let $\mathcal{T}$ be a rigidly-compactly generated tensor triangulated category such that $\operatorname{Spc}\mathcal{T}^c$ is noetherian. We recall that, as $\operatorname{Spc}\mathcal{T}^c$ is noetherian, every open subset is quasi-compact. Let U be an open subset with closed complement Z. There is an associated smashing localization sequence

$$\Gamma_Z\mathcal{T} = \mathcal{T}_Z \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{T} \underset{p_*}{\overset{p^*}{\rightleftarrows}} L_Z\mathcal{T} = \mathcal{T}(U)$$

where we have introduced the notation $\mathcal{T}(U)$ for the category on the right; we feel that this is worthwhile as when working locally it is better to keep open subsets in mind rather than their closed complements. Both $\mathcal{T}_Z$ and $\mathcal{T}(U)$ are tensor ideals and we recall that by definition

$$i_*i^! = \Gamma_Z\mathbf{1} \otimes (-) \quad \text{and} \quad p_*p^* = L_Z\mathbf{1} \otimes (-).$$

By Thomason's localization theorem (see for example [60] Theorem 2.1) the subcategory of compact objects of $\mathcal{T}(U)$ is the idempotent completion of $\mathcal{T}^c/\mathcal{T}^c_Z$ i.e., it is precisely the subcategory $\mathcal{T}^c(U)$ of Balmer. By [8] Proposition 2.15 the category $\mathcal{T}^c(U)$ is a rigid tensor category and so $\mathcal{T}(U)$ is a rigidly-compactly generated tensor triangulated category. We also wish to remind the reader that $\operatorname{Spc}\mathcal{T}^c(U)$ is naturally isomorphic to U by [10] Proposition 1.11. The quotient functor p^* is monoidal and we will denote by $\mathbf{1}_U$ the tensor unit $p^*\mathbf{1}$ of $\mathcal{T}(U)$.

We will use the notation introduced above throughout this section and it will be understood that U carries the subspace topology. The category $\mathcal{T}(U)$ acts on itself giving rise to a support theory; in order to avoid confusion we will include $\mathbf{1}_U$ in the notation for acyclization, localization, and support functors this gives rise to, $\mathcal{T}(U)$ in the notation for the associated subcategories, and write the support as $\operatorname{supp}_{\mathcal{T}(U)}$.

Let us now recall that p^* behaves nicely with respect to tensor idempotents in $\mathcal{T}$.

Lemma 2.5.1. *Let $\mathcal{V} \subseteq \operatorname{Spc}\mathcal{T}^c$ be specialization closed. Then*

$$p^*\Gamma_{\mathcal{V}}\mathbf{1} \cong \Gamma_{\mathcal{V}\cap U}\mathbf{1}_U \quad \textit{and} \quad p^*L_{\mathcal{V}}\mathbf{1} \cong L_{\mathcal{V}\cap U}\mathbf{1}_U.$$

Proof. This is just a different way of stating [11] Corollary 6.5. □

We next show the projection formula holds in this generality.

Lemma 2.5.2. *Suppose $X \in \mathcal{T}$ and $Y \in \mathcal{T}(U)$. Then there is an isomorphism*

$$X \otimes p_*Y \cong p_*(p^*X \otimes Y).$$

Proof. As Y is in $\mathcal{T}(U)$ we have $p^*p_*Y \cong Y$ and hence

$$p_*Y \cong p_*p^*p_*Y \cong L_Z\mathbf{1} \otimes p_*Y.$$

From this we see

$$\begin{aligned} \Gamma_Z\mathbf{1} \otimes X \otimes p_*Y &\cong X \otimes \Gamma_Z\mathbf{1} \otimes p_*Y \\ &\cong X \otimes \Gamma_Z\mathbf{1} \otimes L_Z\mathbf{1} \otimes p_*Y \\ &\cong 0 \end{aligned}$$

showing $X \otimes p_*Y$ is in the image of p_*. Using this we deduce that

$$\begin{aligned} p_*(p^*X \otimes Y) &\cong p_*(p^*X \otimes p^*p_*Y) \\ &\cong p_*p^*(X \otimes p_*Y) \\ &\cong L_Z\mathbf{1} \otimes X \otimes p_*Y \\ &\cong X \otimes p_*Y \end{aligned}$$

which is the claimed isomorphism. □

It follows easily from these facts that one can work locally when considering the subcategories $\Gamma_x\mathcal{T}$ for $x \in \operatorname{Spc}\mathcal{T}^c$.

Proposition 2.5.3. *For all $x \in U$ there is an isomorphism*

$$p_*\Gamma_x\mathbf{1}_U \cong \Gamma_x\mathbf{1}.$$

Proof. To see this is the case just note there are isomorphisms

$$\begin{aligned} p_*\Gamma_x\mathbf{1}_U &\cong p_*(\Gamma_{\mathcal{V}(x)\cap U}\mathbf{1}_U \otimes L_{\mathcal{Z}(x)\cap U}\mathbf{1}_U) \\ &\cong p_*(p^*\Gamma_{\mathcal{V}(x)}\mathbf{1} \otimes p^*L_{\mathcal{Z}(x)}\mathbf{1}) \\ &\cong p_*p^*(\Gamma_{\mathcal{V}(x)}\mathbf{1} \otimes L_{\mathcal{Z}(x)}\mathbf{1}) \\ &\cong L_Z\Gamma_x\mathbf{1} \\ &\cong \Gamma_x\mathbf{1} \end{aligned}$$

where we have used Lemma 2.5.1 for the second isomorphism and the fact that $\Gamma_x\mathbf{1} \in L_Z\mathcal{T} = \mathcal{T}(U)$ for the final isomorphism. □

Proposition 2.5.4. *For all $x \in U$ the functor p_* induces an equivalence*

$$\Gamma_x\mathcal{T} \underset{p_*}{\overset{p^*}{\rightleftarrows}} \Gamma_x\mathcal{T}(U)\ .$$

Proof. The essential image of p^* restricted to $\Gamma_x\mathcal{T}$ is $\Gamma_x\mathcal{T}(U)$ as we have isomorphisms

$$\begin{aligned} p^*(\Gamma_x\mathbf{1}\otimes A) &\cong p^*\Gamma_x\mathbf{1}\otimes p^*A \\ &\cong p^*p_*\Gamma_x\mathbf{1}_U\otimes p^*A \\ &\cong \Gamma_x\mathbf{1}_U\otimes p^*A \end{aligned}$$

where A is any object of $\mathcal{T}$ and we have used the Proposition we have just proved for the second isomorphism.

If X is in $\mathcal{T}$ we have, using the projection formula,

$$p_*(\Gamma_x\mathbf{1}_U\otimes p^*X)\cong p_*\Gamma_x\mathbf{1}_U\otimes X\cong \Gamma_x\mathbf{1}\otimes X$$

showing the essential image of p_* restricted to $\Gamma_x\mathcal{T}(U)$ is $\Gamma_x\mathcal{T}$.

Finally, as p_* is fully faithful we have $p^*p_*\cong \mathrm{id}_{\mathcal{T}(U)}$ and $p_*p^*\cong \mathrm{id}_{\mathrm{im}\,p_*}$. From what we have just shown it is clear that this equivalence restricts to give the equivalence in the statement of the proposition. □

Let us now fix some action of $\mathcal{T}$ on a compactly generated triangulated category $\mathcal{K}$ and consider the relative version. For $U\subseteq \operatorname{Spc}\mathcal{T}^c$ as above we have a smashing localization sequence

$$\Gamma_Z\mathcal{K} \underset{j^!}{\overset{j_*}{\rightleftarrows}} \mathcal{K} \underset{q_*}{\overset{q^*}{\rightleftarrows}} L_Z\mathcal{K}=\mathcal{K}(U)$$

by Lemma 2.2.6 and Corollary 2.2.13, where

$$j_*j^! = \Gamma_Z\mathbf{1}*(-) \quad \text{and} \quad q_*q^* = L_Z\mathbf{1}*(-).$$

Our first observation is that $\mathcal{T}(U)$ acts on $\mathcal{K}(U)$ in a way which is compatible with the quotient functors.

Proposition 2.5.5. *There is an action of $\mathcal{T}(U)$ on $\mathcal{K}(U)$ defined by commutativity of the diagram*

$$\begin{array}{ccc} \mathcal{T}\times\mathcal{K} & \xrightarrow{p^*\times q^*} & \mathcal{T}(U)\times\mathcal{K}(U) \\ {\scriptstyle *}\downarrow & & \downarrow{\scriptstyle *_U} \\ \mathcal{K} & \xrightarrow[q^*]{} & \mathcal{K}(U). \end{array}$$

Proof. As in the diagram we define the action of $\mathcal{T}(U)$ on $\mathcal{K}(U)$ by setting, for $X \in \mathcal{T}$ and $A \in \mathcal{K}$,

$$p^*X *_U q^*A = q^*(X * A)$$

and similarly for morphisms. This is well defined as if $X' \in \mathcal{T}$, $A' \in \mathcal{K}$ with $p^*X \cong p^*X'$ and $q^*A \cong q^*A'$ then

$$\begin{aligned} q_*(p^*X *_U q^*A) &= q_*q^*(X * A) \\ &= L_Z(X * A) \\ &\cong L_Z X * L_Z A \\ &\cong L_Z X' * L_Z A' \\ &\cong q_*(p^*X' *_U q^*A') \end{aligned}$$

which implies $p^*X *_U q^*A \cong p^*X' *_U q^*A'$.

The associator and unitor are defined by the diagrams

$$\begin{array}{ccc} (p^*X \otimes p^*Y) *_U q^*A & \xrightarrow[\sim]{a_U} & p^*X *_U (p^*Y *_U q^*A) \\ \| & & \| \\ q^*((X \otimes Y) * A) & \xrightarrow[q^*a]{\sim} & q^*(X * (Y * A)) \end{array}$$

and

$$\begin{array}{ccc} \mathbf{1}_U *_U q^*A & \xrightarrow[\sim]{l_U} & q^*A \\ \| & & \| \\ q^*(\mathbf{1} * A) & \xrightarrow[q^*l]{\sim} & q^*A \end{array}$$

respectively for $X, Y \in \mathcal{T}$ and $A \in \mathcal{K}$. It is easily verified that $*_U$ fulfils the necessary conditions to be an action. □

We next prove the relative analogue of Proposition 2.5.4:

Proposition 2.5.6. *For $x \in U$ there is an equivalence*

$$\Gamma_x\mathcal{K} \underset{q_*}{\overset{q^*}{\rightleftarrows}} \Gamma_x\mathcal{K}(U)\ .$$

Proof. The category $\Gamma_x\mathcal{K}$ is contained in $q_*\mathcal{K}(U)$ so q^* is fully faithful when restricted to $\Gamma_x\mathcal{K}$. It just remains to note that for $A \in \mathcal{K}$

$$q^*(\Gamma_x\mathbf{1} * A) = p^*\Gamma_x\mathbf{1} *_U q^*A \cong \Gamma_x\mathbf{1}_U *_U q^*A$$

so that $q^*\Gamma_x\mathcal{K} = \Gamma_x\mathcal{K}(U)$. □

Remark 2.5.7. In particular, the last proposition implies that from an open cover $\operatorname{Spc}\mathcal{T}^c = \cup_{i=1}^n U_i$ we get an open cover

$$\sigma\mathcal{K} = \cup_{i=1}^n \sigma\mathcal{K}(U_i).$$

Now let us fix some cover $\operatorname{Spc}\mathcal{T}^c = \cup_{i=1}^n U_i$ by open subsets and denote the projections from $\mathcal{K}$ to $\mathcal{K}(U_i)$ by q_i^*. We will prove two results showing that one can deduce information about $\mathcal{K}$ from the corresponding statements for the $\mathcal{K}(U_i)$. First let us show that compact objects having (specialization) closed support is local in this sense.

Lemma 2.5.8. *Suppose that for all $1 \leq i \leq n$ and $a \in \mathcal{K}(U_i)$ compact the subset $\operatorname{supp}_{\mathcal{T}(U_i)} a$ is (specialization) closed in U_i. Then for all $b \in \mathcal{K}^c$ the subset $\operatorname{supp} b$ is (specialization) closed in $\operatorname{Spc}\mathcal{T}^c$.*

Proof. Let b be compact in $\mathcal{K}$. Then

$$\begin{aligned} \operatorname{supp} b &= \cup_{i=1}^n (\operatorname{supp} b \cap U_i) \\ &= \cup_{i=1}^n \{x \in U_i \mid \Gamma_x \mathbf{1}_U *_U q_i^* b \neq 0\} \\ &= \cup_{i=1}^n \operatorname{supp}_{\mathcal{T}(U_i)} q_i^* b \end{aligned}$$

as we have

$$\Gamma_x \mathbf{1}_U *_U q_i^* b = q_i^*(\Gamma_x b) \neq 0$$

if and only if x is in $\operatorname{supp} b \cap U_i$. Now q_i^* sends compacts to compacts as the associated localization is smashing, so by hypothesis each $\operatorname{supp}_{\mathcal{T}(U_i)} q_i^* b$ is (specialization) closed in U_i. Thus $\operatorname{supp} b$ is (specialization) closed in $\operatorname{Spc}\mathcal{T}^c$. □

Remark 2.5.9. It is worth noting from the proof that for any $A \in \mathcal{K}$ there is an equality

$$\operatorname{supp} A = \cup_{i=1}^n \operatorname{supp}_{\mathcal{T}(U_i)} q_i^* A.$$

Finally we show it is also possible to check that $\sigma\mathcal{K}$ classifies localizing $\mathcal{T}$-submodules locally. It is easily seen that, provided $\mathcal{T}$ satisfies the local-to-global principle, a bijection between subsets of $\sigma\mathcal{K}$ and the collection of localizing submodules of $\mathcal{K}$ is equivalent to each of the $\Gamma_x\mathcal{K}$ being minimal in the following sense (cf. [17] Section 4 and our Lemma 2.3.3):

Definition 2.5.10. We say a localizing submodule $\mathcal{L} \subseteq \mathcal{K}$ is *minimal* if it has no proper and non-trivial localizing submodules.

By Proposition 2.3.4 we have that σ is left inverse to τ. To see τ is an inverse to σ one just needs to note that if the $\Gamma_x\mathcal{K}$ are minimal then the local-to-global principle completely determines any localizing submodule in terms of its support. In fact the converse is also true: such a bijection is easily seen to imply that the $\Gamma_x\mathcal{K}$ are minimal. Thus the following theorem should not come as a surprise.

Theorem 2.5.11. *Suppose $\mathcal{T}$ has a model and that for $i = 1, \ldots, n$ the action of $\mathcal{T}(U_i)$ on $\mathcal{K}(U_i)$ yields bijections*

$$\left\{ \textit{subsets of}\, \sigma\mathcal{K}(U_i) \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \textit{localizing submodules of}\, \mathcal{K}(U_i) \right\}.$$

Then σ and τ give a bijection

$$\left\{ \textit{subsets of}\, \sigma\mathcal{K} \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \textit{localizing submodules of}\, \mathcal{K} \right\}.$$

Proof. By the discussion before the theorem it is sufficient to check that $\Gamma_x\mathcal{K}$ is minimal for each $x \in \sigma\mathcal{K}$. But for any such x there exists an i such that $x \in U_i$ and by Proposition 2.5.6 the subcategory $\Gamma_x\mathcal{K}$ is equivalent to $\Gamma_x\mathcal{K}(U_i)$. This latter category is a minimal $\mathcal{T}(U_i)$-submodule by hypothesis and by the diagram of Proposition 2.5.5 this implies it is also minimal with respect to the action of $\mathcal{T}$. ∎

Chapter 3

The Singularity Category of a Ring

We begin by introducing, for a ring R, an infinite completion $S(R)$, as in Krause's [48], of the usual singularity category $D_{\mathrm{Sg}}(R)$ ([26], [63]). This completion has a natural action of the unbounded derived category of R. We can thus bring the machinery we have developed to bear on the problem of determining the structure of the lattice of localizing subcategories of $S(R)$. We obtain a complete classification for rings which are locally hypersurfaces extending work of Takahashi [68] and removing the hypothesis that R be essentially of finite type over a field from the hypersurface case of a result reported by Iyengar [42].

3.1 Preliminaries

Throughout R denotes a commutative noetherian ring with unit.

Given a ring R we set

$$D_{\mathrm{Sg}}(R) = D^b(R\text{-mod})/D^{\mathrm{perf}}(R).$$

This category, usually called the singularity category, provides a measure of the singularities of the scheme $\operatorname{Spec} R$. Throughout we will prefer to work with an infinite completion of $D_{\mathrm{Sg}}(R)$ and will reserve the term singularity category for this larger category:

Theorem 3.1.1 ([48] Theorem 1.1)**.** *Let R be a ring.*

(1) *There is a recollement*

$$S(R\text{-Mod}) \underset{I_\rho}{\overset{I_\lambda}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{I}}}}} K(\operatorname{Inj} R) \underset{Q_\rho}{\overset{Q_\lambda}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{Q}}}}} D(R\text{-Mod})$$

where each functor is right adjoint to the one above it. We call $S(R\text{-Mod}) = K_{\mathrm{ac}}(\operatorname{Inj} R)$*, the homotopy category of acyclic complexes of injective* R*-modules, the* singularity category *of* R*.*

(2) *The triangulated category* $K(\operatorname{Inj} R)$ *is compactly generated, and* Q *induces an equivalence*

$$K(\operatorname{Inj} R)^{\mathrm{c}} \longrightarrow D^b(R\text{-mod}).$$

(3) *The sequence*

$$D(R\text{-Mod}) \xrightarrow{Q_\lambda} K(\operatorname{Inj} R) \xrightarrow{I_\lambda} S(R\text{-Mod})$$

is a localization sequence. Therefore $S(R\text{-Mod})$ *is compactly generated, and* $I_\lambda \circ Q_\rho$ *induces (up to direct factors) an equivalence*

$$D_{\mathrm{Sg}}(R) \longrightarrow S(R\text{-Mod})^{\mathrm{c}}.$$

Notation 3.1.2. As in the theorem we call $S(R\text{-Mod})$ the *singularity category* of R and we shall denote it by $S(R)$. By (3) of the theorem $S(R)$ contains $D_{\mathrm{Sg}}(R)$. The closure under summands of $D_{\mathrm{Sg}}(R)$ in $S(R)$ is $S(R)^c$, the thick subcategory of compact objects, so it is reasonable to call $S(R)$ the singularity category.

Remark 3.1.3. Krause's theorem is more general than the version we state here. In particular, it covers the case of quasi-coherent sheaves on a noetherian separated scheme which we shall treat in Chapter 5.

Before continuing let us briefly remind the reader of Matlis' classification of indecomposable injective R-modules [51]. Given an R-module M we denote by $E(M)$ (or $E_R(M)$ if the ring is not clear from the context) the injective envelope of M.

Theorem 3.1.4. *Given any prime ideal* $\mathfrak{p} \in \operatorname{Spec} R$ *the injective module* $E(R/\mathfrak{p})$ *is indecomposable and every indecomposable injective* R*-module has this form for a unique prime ideal.*

Let I *be an injective* R*-module. Then* I *decomposes as a direct sum of indecomposable injective* R*-modules. This decomposition is unique in the sense that for each prime ideal* $\mathfrak{p}$ *the cardinality of the summands isomorphic to* $E(R/\mathfrak{p})$ *depends only upon* E *and* $\mathfrak{p}$*.*

3.2 An Action of $D(R)$

We prove that the unbounded derived category of R, denoted here by $D(R)$, acts on the singularity category $S(R)$.

First let us recall the following result originally proved for the homotopy category of spectra in [50].

Theorem 3.2.1. *Suppose $\mathcal{T}$ is a compactly generated triangulated category and H is a coproduct preserving homological functor on $\mathcal{T}$ i.e., H is a functor to an abelian category taking coproducts to coproducts and triangles to long exact sequences. Then the full subcategory*

$$\ker(H) = \{X \in \mathcal{T} \mid H(\Sigma^i X) = 0 \; \forall \, i \in \mathbb{Z}\}$$

is strictly localizing i.e., it is a localizing subcategory of $\mathcal{T}$ and its inclusion admits a right adjoint.

Proof. Margolis' original proof carries over to the case of any compactly generated triangulated category; see for example [46] Theorem 7.5.1 which generalizes this even further. □

Let us consider $E = \coprod_\lambda E_\lambda$ where E_λ runs through a set of representatives for the isomorphism classes of compact objects in $S(R)$. Denote by $K(\mathrm{Flat}\, R)$ the homotopy category of complexes of flat R-modules. We define a homological functor $H\colon K(\mathrm{Flat}\, R) \longrightarrow \mathrm{Ab}$ by setting for X in $K(\mathrm{Flat}\, R)$

$$H(X) = H^0(X \otimes_R E).$$

This is a coproduct preserving homological functor since we are merely composing the exact coproduct preserving functor $(-) \otimes_R E$ with the coproduct preserving homological functor H^0 (where we work inside of $K(R)$).

We recall some facts from [58] in the following definition.

Definition 3.2.2. A complex X in $K(\mathrm{Flat}\, R)$ is *pure acyclic* if it is exact and has flat syzygies. Such complexes form a triangulated subcategory of $K(\mathrm{Flat}\, R)$ which we denote by $K_{\mathrm{pac}}(\mathrm{Flat}\, R)$ and we say that a morphism with pure acyclic mapping cone is a *pure quasi-isomorphism*.

We also wish to remind the reader that since R is noetherian the tensor product of a complex of flats with a complex of injectives is again a complex of injectives and that tensoring a pure acyclic complex of flats with a complex of injectives yields a contractible complex (this second fact is [62] Corollary 9.7 (i)). Thus the category of pure acyclic complexes $K_{\mathrm{pac}}(\mathrm{Flat}\, R)$ is contained in the kernel of H.

Definition 3.2.3. With notation as above we denote by $A_\otimes(\mathrm{Inj}\, R)$ the quotient $\ker(H)/K_{\mathrm{pac}}(\mathrm{Flat}\, R)$. As we are about to show this is the category of complexes of flat modules which act on $S(R)$ in a way that is not automatically trivial.

Lemma 3.2.4. *An object X of $K(\mathrm{Flat}\, R)$ lies in $\ker(H)$ if and only if the exact functor*

$$X \otimes_R (-) \colon K(\mathrm{Inj}\, R) \longrightarrow K(\mathrm{Inj}\, R)$$

restricts to

$$X \otimes_R (-) \colon S(R) \longrightarrow S(R).$$

In particular, $A_\otimes(\mathrm{Inj}\, R)$ consists of the pure quasi-isomorphism classes of objects which act on $S(R)$.

Proof. The object X is in $\ker(H)$ if and only if $X \otimes_R E$ is acyclic so it is sufficient to show that $X \otimes_R E$ is acyclic if and only if $X \otimes_R (-)$ preserves acyclicity of complexes of injectives. The if part of this statement is trivial.

So suppose $X \otimes_R E$ is acyclic. Since $X \otimes_R (-)$ preserves coproducts in $K(\mathrm{Inj}\, R)$ and acyclicity is preserved by extensions, suspension, and coproducts we deduce that $X \otimes_R (-)$ preserves acyclicity of complexes in the localizing subcategory $\langle E \rangle_{\mathrm{loc}}$ of $K(\mathrm{Inj}\, R)$. But this is precisely $S(R)$ since $\langle E \rangle_{\mathrm{loc}}$ contains a compact generating set: it is a localizing subcategory and hence closed under splitting idempotents so contains all compact objects of the compactly generated category $S(R)$. □

Restricting H to $N(\mathrm{Flat}\, R) = K(\mathrm{Flat}\, R)/K_{\mathrm{pac}}(\mathrm{Flat}\, R)$ we obtain by Margolis' theorem an adjoint pair

$$A_\otimes(\mathrm{Inj}\, R) \rightleftarrows N(\mathrm{Flat}\, R).$$

In particular $A_\otimes(\mathrm{Inj}\, R)$ is well generated (this can be deduced from the statement of the version of Margolis' result in [46] 7.5.1).

We will restrict our attention to studying the action of a full subcategory of $A_\otimes(\mathrm{Inj}\, R)$, namely $D(R)$. Of course we first need to show that $D(R)$ has a fully faithful embedding into $A_\otimes(\mathrm{Inj}\, R)$ so this last comment makes sense. Before checking this let us remind the reader of the notion of K-flatness.

Definition 3.2.5. We say that a complex of flat R-modules F is *K-flat* provided $F \otimes_R (-)$ sends quasi-isomorphisms to quasi-isomorphisms (or equivalently if $F \otimes_R X$ is an exact complex for any exact complex of R-modules X).

Lemma 3.2.6. *There is a fully faithful, exact, coproduct preserving functor $D(R) \longrightarrow A_{\otimes}(\operatorname{Inj} R)$.*

Proof. There is, by Theorem 5.5 of [57], a fully faithful exact coproduct preserving functor $D(R) \longrightarrow N(\operatorname{Flat} R)$ given by taking K-flat resolutions and inducing an equivalence

$$D(R) \cong {}^{\perp}N_{\mathrm{ac}}(\operatorname{Flat} R).$$

This functor factors via $A_{\otimes}(\operatorname{Inj} R)$ since K-flat complexes send acyclics to acyclics under the tensor product. □

Proposition 3.2.7. *The embedding $D(R) \longrightarrow A_{\otimes}(\operatorname{Inj} R)$ defines an action of $D(R)$ on $S(R)$*

$$D(R) \times S(R) \xrightarrow{\odot} S(R)$$

in the sense of Definition 2.1.1.

Proof. Since one can view this as taking place inside $K(R)$ biexactness follows from the good properties of the tensor product and taking K-flat resolutions (which are not unique in $K(R)$ but are in $N(\operatorname{Flat} R)$ and so the choice of K-flat resolution in $K(R)$ does not matter once one tensors with something in $S(R)$). For A, B in $D(R)$ and X in $S(R)$ we have natural isomorphisms

$$(A \otimes^{L} B) \odot X \cong A \odot (B \odot X)$$

since one obtains a K-flat resolution for $A \otimes^{L} B$ by tensoring K-flat resolutions for A and B and taking K-flat resolutions is functorial modulo pure acyclics. Taking K-flat resolutions preserves coproducts as in the statement of the lemma so that $(-) \odot (-)$ is coproduct preserving in both variables. The stalk complex R concentrated in degree 0 is already K-flat and gives the unit for the action. The associativity and unit conditions then follow from those of the tensor product of complexes. □

Remark 3.2.8. Recall that every complex in $K^{-}(\operatorname{Flat} R)$, the homotopy category of bounded above complexes of flat R-modules, is K-flat. Thus when acting by the subcategory $K^{-}(\operatorname{Flat} R)$ there is an equality $\odot = \otimes_R$.

3.3 First Properties of the Action

We begin by recalling some facts about the tensor triangulated category $(D(R), \otimes, R)$ (we use $\otimes$ to denote the left derived tensor product). The category $D(R)$ is rigidly-compactly generated: $\{\Sigma^i R \mid i \in \mathbb{Z}\}$ is a generating set, so the tensor unit is not only compact it also generates, and the compact objects $D^{\mathrm{perf}}(R)$ are a rigid tensor subcategory. The spectrum of the compact objects $\operatorname{Spc} D^{\mathrm{perf}}(R)$ is canonically isomorphic to $\operatorname{Spec} R$ (by [59]) and we will identify these spaces. It is well known that $D(R)$ has a model so Theorem 2.3.9 applies to $D(R)$. The support gives a complete classification of the localizing subcategories of $D(R)$ by [59], with the specialization closed subsets of $\operatorname{Spec} R$ corresponding to subcategories generated by objects of $D^{\mathrm{perf}}(R)$.

Notation 3.3.1. We follow the conventions of Chapter 2 and denote, for $\mathcal{V} \subseteq \operatorname{Spec} R$ specialization closed, the associated Rickard idempotents by $\Gamma_{\mathcal{V}} R$ and $L_{\mathcal{V}} R$. For an object X of $S(R)$ we often write $\Gamma_{\mathcal{V}} X$ for $\Gamma_{\mathcal{V}} R \odot X$ and we denote the associated subcategory of $S(R)$ by $\Gamma_{\mathcal{V}} S(R)$. For a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ we denote by $\Gamma_{\mathfrak{p}} R$ the associated object $\Gamma_{\mathcal{V}(\mathfrak{p})} R \otimes L_{\mathcal{Z}(\mathfrak{p})} R$.

The support assignment $\operatorname{supp}_{(D(R),\odot)}$ taking values in $\operatorname{Spec} R$ will simply be denoted by supp.

It is possible to give an explicit description of the Rickard idempotents associated to certain specialization closed subsets of $\operatorname{Spec} R$. First we fix notation for the relevant complexes of R-modules and specialization closed subsets of $\operatorname{Spec} R$.

Definition 3.3.2. Given an element $f \in R$ we define the *stable Koszul complex* $K_\infty(f)$ to be the complex concentrated in degrees 0 and 1

$$\cdots \longrightarrow 0 \longrightarrow R \longrightarrow R_f \longrightarrow 0 \longrightarrow \cdots$$

where the only non-zero morphism is the canonical map to the localization. Given a sequence of elements $\mathbf{f} = \{f_1, \ldots, f_n\}$ of R we set

$$K_\infty(\mathbf{f}) = K_\infty(f_1) \otimes \cdots \otimes K_\infty(f_n).$$

We define the *Ĉech complex* of $\mathbf{f}$ to be the suspension of the kernel of the canonical morphism $K(\mathbf{f}) \longrightarrow R$. This is a degreewise split epimorphism and so we get a triangle in $K(A)$

$$K_\infty(\mathbf{f}) \longrightarrow R \longrightarrow \check{C}(\mathbf{f}) \longrightarrow \Sigma K_\infty(\mathbf{f}).$$

Explicitly we have

$$\check{C}(\mathbf{f})^t = \bigoplus_{i_0<\cdots<i_t} R_{f_{i_0}\cdots f_{i_t}}$$

for $0 \leq t \leq n-1$ and $K_\infty(\mathbf{f})$ is degreewise the same complex desuspended and with R in degree 0. For an ideal I of R we define $K(I)$ and $\check{C}(I)$ by choosing a set of generators for I; the complex obtained is independent of the choice of generators up to quasi-isomorphism in $D(R)$ and hence up to pure quasi-isomorphism in $A_\otimes(\operatorname{Inj} R)$. We note that these complexes are K-flat.

Notation 3.3.3. We fix the notation we will use for the subsets of $\operatorname{Spec} R$ of interest to us. Let

$$\mathcal{Z}(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}$$

and denote its complement by

$$\mathcal{U}(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \subseteq \mathfrak{p}\}.$$

The other main collection of specialization closed subsets we will be interested in are the usual closed subsets associated to primes, namely

$$\mathcal{V}(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{p} \subseteq \mathfrak{q}\}.$$

In several cases there are explicit descriptions of the Rickard idempotents corresponding to these specialization closed subsets.

Proposition 3.3.4. *For an ideal $I \subseteq R$ and $\mathfrak{p} \in \operatorname{Spec} R$ a prime ideal there are natural isomorphisms in $D(R)$:*

(1) $\Gamma_{\mathcal{V}(I)} R \cong K_\infty(I)$;

(2) $L_{\mathcal{V}(I)} R \cong \check{C}(I)$;

(3) $L_{\mathcal{Z}(\mathfrak{p})} R \cong R_\mathfrak{p}$.

In particular the objects $\Gamma_\mathfrak{p} R = \Gamma_{\mathcal{V}(\mathfrak{p})} R \otimes L_{\mathcal{Z}(\mathfrak{p})} R$ giving rise to supports on $D(R)$ and $S(R)$ are naturally isomorphic to $K_\infty(\mathfrak{p}) \otimes R_\mathfrak{p}$.

Proof. Statements (1) and (2) are special cases of [34] Lemma 5.8. For the third statement simply observe that the full subcategory of complexes with homological support in $\mathcal{U}(\mathfrak{p})$ is the essential image of the inclusion of $D(R_\mathfrak{p})$. ☐

We are now in a position to obtain, very cheaply, some first results about the singularity category and the action of $D(R)$ on it. We first observe that all localizing subcategories of $S(R)$ are $D(R)$-submodules.

Lemma 3.3.5. *Every localizing subcategory $\mathcal{L}$ of $S(R)$ is stable under the action of $D(R)$.*

Proof. As $D(R)$ is generated by the tensor unit Lemma 2.1.15 applies. □

We already observed that Theorem 2.3.9 applies to $D(R)$. This has the following consequence for $S(R)$:

Proposition 3.3.6. *Given an object X of $S(R)$ there is an isomorphism $X \cong 0$ if and only if $\operatorname{supp} X = \varnothing$. We also have for each object X of $S(R)$ an equality*

$$\langle X \rangle_{\mathrm{loc}} = \langle \Gamma_{\mathfrak{p}} X \mid \mathfrak{p} \in \operatorname{supp} X \rangle_{\mathrm{loc}}.$$

Proof. This is an immediate consequence of Lemma 3.3.5 and Theorem 2.3.9. □

Using the explicit description in Proposition 3.3.4 of certain Rickard idempotents in $D(R)$ we are able to give representatives for the objects resulting from their action on objects of $S(R)$.

Proposition 3.3.7. *For each object X of $S(R)$ and ideal $I \subseteq R$ the complex $\Gamma_{\mathcal{V}(I)} X$ is homotopic to a complex whose degree i piece is the summand of X^i consisting of those indecomposable injectives corresponding to primes in $\mathcal{V}(I)$.*

Proof. Let us fix an ideal I and choose generators $I = (f_1, \ldots, f_n)$. By Proposition 3.3.4 we have

$$\Gamma_{\mathcal{V}(I)} X \cong K_\infty(f_1, \ldots, f_n) \odot X \cong K_\infty(f_n) \odot (K_\infty(f_{n-1}) \odot \cdots (K_\infty(f_1) \odot X) \cdots).$$

We can thus reduce to the case that $I = (f)$. By Proposition 3.3.4 again we have

$$L_{\mathcal{V}((f))} X \cong \check{C}(f) \otimes_R X \cong R_f \otimes_R X$$

where the last isomorphism uses the explicit description of the Ĉech complex given in Definition 3.3.2. The canonical map $X \longrightarrow R_f \otimes_R X$ is a degreewise split epimorphism in the category of chain complexes which fits into the localization triangle

$$K_\infty(f) \otimes_R X \longrightarrow X \longrightarrow R_f \otimes_R X \longrightarrow \Sigma K_\infty(f) \otimes_R X$$

in $S(R)$. So up to homotopy $K_\infty(f) \odot X$ is the kernel of this split epimorphism. The kernel in each degree is precisely the summand consisting of those indecomposable injectives corresponding to primes in $\mathcal{V}(I)$ which proves the claim. □

3.4 Subsets versus Subcategories

Recall from Definition 2.2.22 that the action of $D(R)$ on $S(R)$ gives rise to order preserving assignments

$$\left\{ \text{subsets of } \operatorname{Spec} R \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \text{localizing subcategories of } S(R) \right\}$$

where for a localizing subcategory $\mathcal{L}$ we set

$$\sigma(\mathcal{L}) = \operatorname{supp} \mathcal{L} = \{\mathfrak{p} \in \operatorname{Spec} R \mid \Gamma_\mathfrak{p}\mathcal{L} \neq 0\}$$

and

$$\tau(W) = \{A \in S(R) \mid \operatorname{supp} A \subseteq W\}.$$

Here we have used Lemma 3.3.5 to replace submodules by localizing subcategories and the fact that R is noetherian so that there are no complications with invisible points. As $D(R)$ satisfies the local-to-global principle one can say a little more.

Proposition 3.4.1. *Given a subset $W \subseteq \operatorname{Spec} R$ there is an equality of subcategories*

$$\tau(W) = \langle \Gamma_\mathfrak{p} S(R) \mid \mathfrak{p} \in W \rangle_{\mathrm{loc}}.$$

Proof. This is just a restatement of Lemma 2.3.3. □

We next note that, as one would expect, $S(R)$ is supported on the singular locus $\operatorname{Sing} R$ of $\operatorname{Spec} R$.

Lemma 3.4.2. *There is a containment $\sigma S(R) \subseteq \operatorname{Sing} R$.*

Proof. If $\mathfrak{p} \in \operatorname{Spec} R$ is a regular point then $S(R_\mathfrak{p}) = 0$. Thus for any object X of $S(R)$

$$\Gamma_\mathfrak{p} X \cong R_\mathfrak{p} \otimes_R (\Gamma_{\mathcal{V}(\mathfrak{p})} R \odot X) \cong 0$$

as it is an acyclic complex of injective $R_\mathfrak{p}$-modules. □

It is clear that $D(R)$ also acts, by K-flat resolutions, on itself and on $K(\operatorname{Inj} R)$. It will be convenient for us to show that these actions are compatible with each other and the action on $S(R)$ in an appropriate sense. We write $\otimes$ for the action of $D(R)$ on itself and $\odot$ for the action of $D(R)$ on $K(\operatorname{Inj} R)$ which extends the action on $S(R)$.

Proposition 3.4.3. *These actions of $D(R)$ are compatible with the localization sequence*

$$\xymatrix{ D(R) \ar[r]^{Q_\lambda} & K(\operatorname{Inj} R) \ar@/^/[l]^{Q} \ar[r]^{I_\lambda} & S(R) \ar@/^/[l]^{I} }$$

in the sense that, up to natural isomorphism, the action commutes with each of the functors in the diagram. Explicitly, for $J \in K(\operatorname{Inj} R)$, $X \in S(R)$, and $E, F \in D(R)$ we have isomorphisms

$$Q(E \odot J) \cong E \otimes QJ \quad , \quad Q_\lambda(E \otimes F) \cong E \odot Q_\lambda F$$

$$I(E \odot X) \cong E \odot IX \quad , \quad I_\lambda(E \odot J) \cong E \odot I_\lambda J.$$

Proof. It is obvious that the inclusion I is compatible with the action of $D(R)$. As $D(R)$ acts on $K(\operatorname{Inj} R)$ via K-flat resolutions the action commutes with Q; an object J of $K(\operatorname{Inj} R)$ is quasi-isomorphic to QJ so for $E \in D(R)$ the object $E \odot J$ computes the left derived tensor product.

To treat the other two functors let J be an object of $K(\operatorname{Inj} R)$ and consider the localization triangle

$$Q_\lambda QJ \longrightarrow J \longrightarrow II_\lambda J \longrightarrow \Sigma Q_\lambda QJ.$$

As $D(R)$ is generated by the tensor unit R every localizing subcategory of $K(\operatorname{Inj} R)$ is stable under the action by Lemma 2.1.15. Thus for $E \in D(R)$ we get a triangle,

$$E \odot Q_\lambda QJ \longrightarrow E \odot J \longrightarrow E \odot II_\lambda J \longrightarrow \Sigma E \odot Q_\lambda QJ,$$

where $E \odot Q_\lambda QJ \in Q_\lambda D(R)$ and $E \odot II_\lambda J$ is acyclic. Hence this triangle must be uniquely isomorphic to the localization triangle for $E \odot J$ giving

$$E \odot Q_\lambda QJ \cong Q_\lambda Q(E \odot J) \qquad \text{and} \qquad E \odot II_\lambda J \cong II_\lambda(E \odot J).$$

We already know that the action commutes with Q and I so the remaining two commutativity relations follow immediately. □

We can also say something about compatibility with the right adjoint Q_ρ of Q.

Lemma 3.4.4. *Suppose E and F are objects of $D(R)$ such that E has a bounded flat resolution and F has a bounded below injective resolution. Then*

$$E \odot Q_\rho F \cong Q_\rho(E \otimes F).$$

Proof. Let $\tilde{E}$ be a bounded flat resolution of E and recall that, by virtue of being bounded, $\tilde{E}$ is K-flat. In [48] Krause identifies Q_ρ with taking K-injective resolutions, where the K-injectives are the objects of the colocalizing subcategory of $K(\operatorname{Inj} R)$ generated by the bounded below complexes of injectives (such resolutions exist, see for example [18]). Thus $Q_\rho F$ is a K-injective resolution of F and so, as bounded below complexes of injectives are K-injective, we may assume it is bounded below as it is homotopic to the bounded below resolution we have required of F. We have $Q_\rho F \cong F$ in $D(R)$ so there are isomorphisms in the derived category

$$E \otimes F \cong E \otimes Q_\rho F \cong \tilde{E} \otimes_R Q_\rho F.$$

Hence in $K(\operatorname{Inj} R)$ we have isomorphisms

$$\begin{aligned} Q_\rho(E \otimes F) &\cong Q_\rho(E \otimes Q_\rho F) \\ &\cong Q_\rho(\tilde{E} \otimes_R Q_\rho F) \\ &\cong \tilde{E} \otimes_R Q_\rho F \\ &\cong E \odot Q_\rho F \end{aligned}$$

where the penultimate isomorphism is a consequence of the fact that $\tilde{E} \otimes_R Q_\rho F$ is, by the assumption that $\tilde{E}$ is bounded and $Q_\rho F$ is bounded below, a bounded below complex of injectives and hence K-injective.

□

Before proceeding let us record the following easy observation for later use.

Lemma 3.4.5. *The diagram*

$$\begin{array}{ccc} D^+(R) & \xrightarrow{I_\lambda Q_\rho} & S(R) \\ \downarrow & & \downarrow \\ D^+(R_\mathfrak{p}) & \xrightarrow[I_\lambda Q_\rho]{} & S(R_\mathfrak{p}), \end{array}$$

where the vertical functors are localization at $\mathfrak{p} \in \operatorname{Spec} R$, commutes.

Proof. The square

$$\begin{array}{ccc} D^+(R) & \xrightarrow{Q_\rho} & K(\operatorname{Inj} R) \\ \downarrow & & \downarrow \\ D^+(R_\mathfrak{p}) & \xrightarrow[Q_\rho]{} & K(\operatorname{Inj} R_\mathfrak{p}), \end{array}$$

is commutative by Lemma 3.4.4.

To complete the proof it is sufficient to show that the square

$$\begin{array}{ccc} K(\operatorname{Inj} R) & \xrightarrow{I_\lambda} & S(R) \\ \downarrow & & \downarrow \\ K(\operatorname{Inj} R_\mathfrak{p}) & \xrightarrow[I_\lambda]{} & S(R_\mathfrak{p}), \end{array}$$

also commutes. This follows by observing that the square

$$\begin{array}{ccc} K(\operatorname{Inj} R) & \xleftarrow{I} & S(R) \\ \uparrow & & \uparrow \\ K(\operatorname{Inj} R_\mathfrak{p}) & \xleftarrow[I]{} & S(R_\mathfrak{p}), \end{array}$$

commutes and taking left adjoints, where we are using the fact that tensoring and restricting scalars along $R \longrightarrow R_\mathfrak{p}$ are both exact and preserve injectives so give rise to an adjoint pair or functors between the relevant homotopy categories of injectives and singularity categories. □

Given these compatibilities it is not hard to see that $\sigma S(R)$ is precisely the singular locus.

Proposition 3.4.6. *For any $\mathfrak{p} \in \operatorname{Sing} R$ the object $\Gamma_\mathfrak{p} I_\lambda Q_\rho k(\mathfrak{p})$ is non-zero in $S(R)$. Thus $\Gamma_\mathfrak{p} S(R)$ is non-trivial for all such $\mathfrak{p}$ yielding the equality $\sigma S(R) = \operatorname{Sing} R$.*

Proof. Let $\mathfrak{p}$ be a singular point of $\operatorname{Spec} R$. Applying the last lemma we may check that $I_\lambda Q_\rho k(\mathfrak{p})$ is non-zero over $R_\mathfrak{p}$. By [52] Section 18 Theorem 41 one has

$$\operatorname{pd}_{R_\mathfrak{p}} k(\mathfrak{p}) = \operatorname{gl.dim} R_\mathfrak{p} = \infty$$

so $k(\mathfrak{p})$ is finitely generated over $R_\mathfrak{p}$ but not perfect. Theorem 3.1.1 then tells us that $I_\lambda Q_\rho k(\mathfrak{p})$ is not zero in $S(R_\mathfrak{p})$.

We now show $I_\lambda Q_\rho k(\mathfrak{p})$ lies in $\Gamma_\mathfrak{p} S(R)$. By Proposition 3.4.3 there is an isomorphism

$$\Gamma_\mathfrak{p} R \odot I_\lambda Q_\rho k(\mathfrak{p}) \cong I_\lambda(\Gamma_\mathfrak{p} R \odot Q_\rho k(\mathfrak{p})). \tag{3.1}$$

As $\Gamma_\mathfrak{p} R \cong K_\infty(\mathfrak{p}) \otimes R_\mathfrak{p}$ (by Proposition 3.3.4) is a bounded K-flat complex and the injective resolution of $k(\mathfrak{p})$ is certainly bounded below we can apply Lemma 3.4.4. This gives us isomorphisms

$$I_\lambda(\Gamma_\mathfrak{p} R \odot Q_\rho k(\mathfrak{p})) \cong I_\lambda Q_\rho(\Gamma_\mathfrak{p} R \otimes k(\mathfrak{p})) \cong I_\lambda Q_\rho k(\mathfrak{p}).$$

Combining these with (3.1) shows that, up to homotopy, $\Gamma_\mathfrak{p} R \odot (-)$ fixes the non-zero object $I_\lambda Q_\rho k(\mathfrak{p})$ proving that $\Gamma_\mathfrak{p} S(R)$ is non-zero. □

It is thus natural to restrict the support and the assignments σ and τ to subsets of the singular locus. This result then implies that the assignment τ taking a subset of $\operatorname{Sing} R$ to a localizing subcategory of $S(R)$ is injective with left inverse σ.

Corollary 3.4.7. *For every $W \subseteq \operatorname{Sing} R$ we have $\sigma\tau(W) = W$. In particular, τ is injective when restricted to subsets of the singular locus.*

Proof. By Proposition 2.3.4 the assignment τ is injective when restricted to $\sigma S(R) = \operatorname{Sing} R$ and we have

$$\sigma\tau(W) = W \cap \operatorname{Sing} R = W$$

which proves the corollary. □

We now prove some results concerning generators for the subcategories produced via the action of $D(R)$. This will allow us to describe the image of τ as the localizing subcategories which contain certain objects.

The next lemma is an easy modification of an argument of Krause in [48]. We give the details, including those straight from Krause's proof, as it is clearer to present them along with the modifications than to just indicate what else needs to be checked.

Lemma 3.4.8. *Let $\mathcal{V}$ be a specialization closed subset of* $\operatorname{Sing} R$. *The set of objects*

$$\{\Sigma^i I_\lambda Q_\rho R/\mathfrak{p} \mid \mathfrak{p} \in \mathcal{V}, i \in \mathbb{Z}\}$$

is a generating set for $\Gamma_\mathcal{V} S(R)$ consisting of objects which are compact in $S(R)$.

Proof. Let X be a non-zero object of $\Gamma_\mathcal{V} S(R)$. In particular X is a complex of injectives satisfying $H^n X = 0$ for all $n \in \mathbb{Z}$. As X is not nullhomotopic we can choose n such that $Z^n X$ is not injective. Consider the beginning of an augmented minimal injective resolution of $Z^n X$

$$0 \longrightarrow Z^n X \longrightarrow E^0(Z^n X) \longrightarrow E^1(Z^n X).$$

Note that for $\mathfrak{q} \notin \mathcal{V}$ the object $R_\mathfrak{q} \otimes \Gamma_\mathcal{V} R$ is zero in $D(R)$ as the cohomology of $\Gamma_\mathcal{V} R$ is supported in $\mathcal{V}$ by definition. Thus for $\mathfrak{q} \notin \mathcal{V}$ the complex $X_\mathfrak{q}$ is nullhomotopic by virtue of being in the essential image of $\Gamma_\mathcal{V} R \odot (-)$. So $Z^n X_\mathfrak{q}$ is injective as a nullhomotopic complex is split exact. Since, for modules, localization at a prime sends minimal injective resolutions to minimal injective resolutions (see for example [53] Section 18) for any such $\mathfrak{q}$ it holds that $E^1(Z^nX)_\mathfrak{q} = 0$. So writing

$$E^1(Z^nX) \cong \bigoplus_i E(R/\mathfrak{p}_i)$$

we have $\mathfrak{p} \in \mathcal{V}$ for each distinct $\mathfrak{p}$ occurring in the direct sum as otherwise it would not vanish when localized (see for example [13] Lemma 2.1). Now fix some $\mathfrak{p}$ such that $E(R/\mathfrak{p})$ occurs in $E^1(Z^nX)$. By [33] Theorem 9.2.4, as the injective envelope of $\mathfrak{p}$ occurs in $E^1(Z^nX)$, we have

$$0 \neq \dim_{k(\mathfrak{p})} \operatorname{Ext}^1(k(\mathfrak{p}), Z^nX_\mathfrak{p}) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^1(R/\mathfrak{p}, Z^nX)_\mathfrak{p}.$$

In particular $\operatorname{Ext}^1(R/\mathfrak{p}, Z^nX)$ is non-zero. Using [48] Lemma 2.1 and the adjunction between I and I_λ there are isomorphisms

$$\begin{aligned}\operatorname{Ext}^1(R/\mathfrak{p}, Z^nX) &\cong \operatorname{Hom}_{K(R\text{-Mod})}(R/\mathfrak{p}, \Sigma^{n+1}IX)\\ &\cong \operatorname{Hom}_{K(\operatorname{Inj} R)}(Q_\rho R/\mathfrak{p}, \Sigma^{n+1}IX)\\ &\cong \operatorname{Hom}_{S(R)}(\Sigma^{-n-1}I_\lambda Q_\rho R/\mathfrak{p}, X).\end{aligned}$$

Thus the set in question is certainly generating and it consists of compact objects by Theorem 3.1.1 (3). □

Lemma 3.4.9. *The object $I_\lambda Q_\rho k(\mathfrak{p})$ generates $\Gamma_\mathfrak{p} S(R)$ for every $\mathfrak{p} \in \operatorname{Sing} R$ i.e,*

$$\Gamma_\mathfrak{p} S(R) = \langle I_\lambda Q_\rho k(\mathfrak{p})\rangle_{\text{loc}}.$$

Proof. By Lemma 3.4.8 we have an equality

$$\Gamma_{\mathcal{V}(\mathfrak{p})} S(R) = \langle I_\lambda Q_\rho R/\mathfrak{q} \mid \mathfrak{q} \in \mathcal{V}(\mathfrak{p})\rangle_{\text{loc}}.$$

Noticing that

$$\Gamma_\mathfrak{p} S(R) = L_{\mathcal{Z}(\mathfrak{p})}\Gamma_{\mathcal{V}(\mathfrak{p})} S(R) = \langle L_{\mathcal{Z}(\mathfrak{p})} R\rangle_{\text{loc}} \odot \Gamma_{\mathcal{V}(\mathfrak{p})} S(R)$$

we thus get, by Lemma 2.1.14, equalities

$$\begin{aligned}\langle L_{\mathcal{Z}(\mathfrak{p})} R\rangle_{\text{loc}} \odot \Gamma_{\mathcal{V}(\mathfrak{p})} S(R) &= \langle R_\mathfrak{p}\rangle_{\text{loc}} \odot \langle I_\lambda Q_\rho R/\mathfrak{q} \mid \mathfrak{q} \in \mathcal{V}(\mathfrak{p})\rangle_{\text{loc}}\\ &= \langle R_\mathfrak{p} \odot I_\lambda Q_\rho R/\mathfrak{q} \mid \mathfrak{q} \in \mathcal{V}(\mathfrak{p})\rangle_{\text{loc}}\end{aligned}$$

where we have used Proposition 3.3.4 to identify $L_{\mathcal{Z}(\mathfrak{p})} R$ with $R_\mathfrak{p}$. Hence, using Proposition 3.4.3 and Lemma 3.4.4 to move the action by $R_\mathfrak{p}$ past $I_\lambda Q_\rho$, we obtain equalities

$$\Gamma_\mathfrak{p} S(R) = \langle I_\lambda Q_\rho (R_\mathfrak{p} \otimes R/\mathfrak{q}) \mid \mathfrak{q} \in \mathcal{V}(\mathfrak{p})\rangle_{\text{loc}} = \langle I_\lambda Q_\rho k(\mathfrak{p})\rangle_{\text{loc}}$$

completing the proof. □

Next we consider the behaviour of σ and τ with respect to the collection of subcategories of $S(R)$ generated by objects of $S(R)^c$. These assignments are sensitive to such subcategories. Indeed we already saw in Proposition 2.2.11 that the subcategory τ associates to a specialization closed subset of $\operatorname{Sing} R$ is generated by compact objects of $S(R)$. Now let us demonstrate that, when R is Gorenstein, the support of any compact object of $S(R)$ is closed. We begin by showing that it is sufficient to consider the images under $I_\lambda Q_\rho$ of finitely generated R-modules when considering the supports of compact objects.

Lemma 3.4.10. *Let x be a compact object of $S(R)$. Then there exists a finitely generated R-module M and integer i such that $x \oplus \Sigma x$ is isomorphic to $\Sigma^i I_\lambda Q_\rho M$. In particular there is an equality*

$$\operatorname{supp} x = \operatorname{supp} I_\lambda Q_\rho M.$$

Proof. By Theorem 3.1.1 $I_\lambda Q_\rho$ induces an equivalence up to summands between $D_{\mathrm{Sg}}(R)$ and $S(R)^c$ so $x \oplus \Sigma x$ is in the image of $I_\lambda Q_\rho$ by [61] Corollary 4.5.12. By the argument of [63] Lemma 1.11 every object of $D_{\mathrm{Sg}}(R)$ is, up to suspension, the image of a finitely generated R-module so we can find a finitely generated M as claimed.

Since $\operatorname{supp} x = \operatorname{supp}(x \oplus \Sigma x)$ by Proposition 2.2.20 and, by the same Proposition, suspending doesn't change the support the last statement now follows. □

Lemma 3.4.11. *If x is an object of $S(R)^c$ then the set*

$$\{\mathfrak{p} \in \operatorname{Sing} R \mid L_{\mathcal{Z}(\mathfrak{p})} R \odot x \neq 0\}$$

is closed in $\operatorname{Spec} R$.

Proof. Clearly we may, by applying Lemma 3.4.10, suppose x is $I_\lambda Q_\rho M$ where M is a finitely generated R-module. By the compatibility conditions of 3.4.3 and 3.4.4 we have an isomorphism

$$L_{\mathcal{Z}(\mathfrak{p})} R \odot I_\lambda Q_\rho M \cong I_\lambda Q_\rho M_\mathfrak{p}$$

where $L_{\mathcal{Z}(\mathfrak{p})} R \cong R_\mathfrak{p}$ by Proposition 3.3.4.

By considering the diagram of Lemma 3.4.5 and noting that the module $M_\mathfrak{p}$ is finitely generated over $R_\mathfrak{p}$ we see the object $I_\lambda Q_\rho M_\mathfrak{p}$ is zero precisely when $M_\mathfrak{p}$ has finite projective dimension.

Thus

$$\{\mathfrak{p} \in \operatorname{Sing} R \mid L_{\mathcal{Z}(\mathfrak{p})} R \odot I_\lambda Q_\rho M \neq 0\} = \{\mathfrak{p} \in \operatorname{Sing} R \mid \operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} = \infty\}$$

and this latter set is closed as M is finitely generated. □

Lemma 3.4.12. *Let R be Gorenstein and let x be a compact object of $S(R)$. Then $\mathfrak{p} \in \operatorname{Sing} R$ is in the support of x if and only if $L_{\mathcal{Z}(\mathfrak{p})}R \odot x$ is not zero.*

Proof. One direction is easy: if $\mathfrak{p} \in \operatorname{supp} x$ then

$$\Gamma_{\mathfrak{p}}R \odot x \cong \Gamma_{\mathcal{V}(\mathfrak{p})}R \odot L_{\mathcal{Z}(\mathfrak{p})}R \odot x \neq 0$$

so $L_{\mathcal{Z}(\mathfrak{p})}R \odot x$ is certainly not zero.

Now let us prove the converse. By Lemma 3.4.10 it is sufficient to prove the result for $I_\lambda Q_\rho M$ where M is a finitely generated R-module. So suppose M is a finitely generated R-module of infinite projective dimension such that the projection of $R_{\mathfrak{p}} \otimes_R M = M_{\mathfrak{p}}$ to $S(R_{\mathfrak{p}})^c$ is not zero, where this projection is $L_{\mathcal{Z}(\mathfrak{p})}I_\lambda Q_\rho M$ by the compatibility conditions of 3.4.3 and 3.4.4. In particular $M_{\mathfrak{p}}$ also has infinite projective dimension.

As $R_{\mathfrak{p}}$ is Gorenstein of finite Krull dimension $M_{\mathfrak{p}}$ has, as an $R_{\mathfrak{p}}$-module, a Gorenstein injective envelope $G(M_{\mathfrak{p}})$ by [33] Theorem 11.3.2 which fits into an exact sequence

$$0 \longrightarrow M_{\mathfrak{p}} \longrightarrow G(M_{\mathfrak{p}}) \longrightarrow L \longrightarrow 0$$

where L has finite injective dimension (details about Gorenstein injectives and Gorenstein injective envelopes can be found in Section 4.1.1). So for i sufficiently large (i.e., exceeding the dimension of $R_{\mathfrak{p}}$) we have isomorphisms

$$\begin{aligned}
&\operatorname{Hom}_{S(R)}(I_\lambda Q_\rho R/\mathfrak{p}, \Sigma^i L_{\mathcal{Z}(\mathfrak{p})} I_\lambda Q_\rho M) \\
\cong\ &\operatorname{Hom}_{S(R_{\mathfrak{p}})}(I_\lambda Q_\rho k(\mathfrak{p}), \Sigma^i I_\lambda Q_\rho M_{\mathfrak{p}}) \\
\cong\ &\operatorname{Hom}_{S(R_{\mathfrak{p}})}(I_\lambda Q_\rho k(\mathfrak{p}), \Sigma^i I_\lambda Q_\rho G(M_{\mathfrak{p}})) \\
\cong\ &\operatorname{Ext}^i_{R_{\mathfrak{p}}}(k(\mathfrak{p}), G(M_{\mathfrak{p}})) \\
\cong\ &\operatorname{Ext}^i_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}})
\end{aligned}$$

where the first isomorphism is by adjunction, the second by the identification of a complete injective resolution for $M_{\mathfrak{p}}$ with the defining complex of $G(M_{\mathfrak{p}})$ (see Proposition 4.1.10 and Corollary 4.1.11, cf. [48] Section 7), the third by [48] Proposition 7.10, and the last isomorphism by the finiteness of the injective dimension of L.

From [33] Proposition 9.2.13 we learn that for $\mathfrak{q} \subseteq \mathfrak{q}'$ distinct primes with no prime ideal between them that $\mu_j(\mathfrak{q}, M_{\mathfrak{p}}) \neq 0$ implies that $\mu_{j+1}(\mathfrak{q}', M_{\mathfrak{p}}) \neq 0$ where

$$\mu_j(\mathfrak{q}, M) = \dim_{k(\mathfrak{q})} \operatorname{Ext}^j_{R_{\mathfrak{q}}}(k(\mathfrak{q}), M_{\mathfrak{q}})$$

are the Bass invariants. As $M_\mathfrak{p}$ is not perfect infinitely many of the Bass invariants are non-zero and so in particular, as $\mathfrak{p}$ is the maximal ideal of $R_\mathfrak{p}$, there are infinitely many non-zero $\mu_j(\mathfrak{p}, M_\mathfrak{p})$. Thus, taking i larger if necessary, we get that

$$0 \neq \operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \cong \operatorname{Hom}_{S(R)}(I_\lambda Q_\rho R/\mathfrak{p}, \Sigma^i L_{\mathcal{Z}(\mathfrak{p})} I_\lambda Q_\rho M).$$

Hence $\Gamma_{\mathcal{V}(\mathfrak{p})} L_{\mathcal{Z}(\mathfrak{p})} I_\lambda Q_\rho M \neq 0$ as by Lemma 3.4.8 the object $I_\lambda Q_\rho R/\mathfrak{p}$ is one of the generators for $\Gamma_{\mathcal{V}(\mathfrak{p})} S(R)$. It follows that $\mathfrak{p} \in \operatorname{supp} I_\lambda Q_\rho M$ as desired. □

Proposition 3.4.13. *Let R be Gorenstein. If x is a compact object of $S(R)$ then $\operatorname{supp} x$ is a closed subset of $\operatorname{Sing} R$.*

Proof. By the last lemma

$$\operatorname{supp} x = \{\mathfrak{p} \in \operatorname{Sing} R \mid L_{\mathcal{Z}(\mathfrak{p})} x \neq 0\}$$

which is closed by Lemma 3.4.11.

□

Remark 3.4.14. If $\mathfrak{p} \in \operatorname{Sing} R$ the proof of Lemma 3.4.12 gives the equality

$$\operatorname{supp} I_\lambda Q_\rho R/\mathfrak{p} = \mathcal{V}(\mathfrak{p}).$$

Indeed, as $R_\mathfrak{p}$ is not regular the residue field $k(\mathfrak{p})$ must have an infinite free resolution over $R_\mathfrak{p}$ so if $(R/\mathfrak{p})_\mathfrak{q}$ had finite projective dimension over R for $\mathfrak{q} \in \mathcal{V}(\mathfrak{p})$ one could localize to find a finite resolution for $k(\mathfrak{p})$ giving a contradication. Thus

$$\operatorname{supp} I_\lambda Q_\rho R/\mathfrak{p} = \{\mathfrak{q} \in \operatorname{Sing} R \mid L_{\mathcal{Z}(\mathfrak{q})} I_\lambda Q_\rho R/\mathfrak{p} \neq 0\}$$

which is precisely $\mathcal{V}(\mathfrak{p})$.

It follows from this proposition and the compatibility of supports with extensions, coproducts, and suspensions (Proposition 2.2.20) that, provided R is Gorenstein, for any localizing subcategory $\mathcal{L} \subseteq S(R)$ generated by objects of $S(R)^c$ the subset $\sigma\mathcal{L} \subseteq \operatorname{Sing} R$ is specialization closed. As mentioned above τ sends specialization closed subsets to localizing subcategories of $S(R)$ generated by objects compact in $S(R)$ so τ and σ restrict, i.e.:

Proposition 3.4.15. *The assignments σ and τ restrict to well-defined functions*

$$\left\{ \begin{array}{c} \textit{specialization closed} \\ \textit{subsets of } \operatorname{Sing} R \end{array} \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{localizing subcategories of } S(R) \\ \textit{generated by objects of } S(R)^c \end{array} \right\}.$$

We are now ready to state and prove our first classification theorem for subcategories of the singularity category (cf. Theorem 7.5 [68]).

Theorem 3.4.16. *Let R be a commutative Gorenstein ring. Then there are order preserving bijections*

$$\left\{ \textit{subsets of } \operatorname{Sing} R \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{localizing subcategories } \mathcal{L} \textit{ of } S(R) \\ \textit{containing } I_\lambda Q_\rho k(\mathfrak{p}) \textit{ for } \mathfrak{p} \in \sigma(\mathcal{L}) \end{array} \right\}$$

and

$$\left\{ \begin{array}{c} \textit{specialization closed} \\ \textit{subsets of } \operatorname{Sing} R \end{array} \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{subcategories } \mathcal{L} \textit{ of } S(R) \\ \textit{generated by objects of } S(R)^c \\ \textit{and containing } I_\lambda Q_\rho k(\mathfrak{p}) \textit{ for } \mathfrak{p} \in \sigma(\mathcal{L}) \end{array} \right\}.$$

This second being equivalent to the bijection

$$\left\{ \begin{array}{c} \textit{specialization closed} \\ \textit{subsets of } \operatorname{Sing} R \end{array} \right\} \rightleftarrows \left\{ \begin{array}{c} \textit{thick subcategories } \mathcal{L} \textit{ of } D_{\mathrm{Sg}}(R) \\ \textit{such that } \mathcal{L}_\mathfrak{p} \subseteq D_{\mathrm{Sg}}(R_\mathfrak{p}) \textit{ contains } k(\mathfrak{p}) \\ \textit{(up to summands)} \end{array} \right\}.$$

Proof. We proved in Corollary 3.4.7 that $\sigma\tau(W) = W$ for every subset W of $\operatorname{Sing} R$. If $\mathcal{L}$ is a localizing subcategory then by the local-to-global principle Theorem 2.3.9

$$\mathcal{L} = \langle \Gamma_\mathfrak{p}\mathcal{L} \mid \mathfrak{p} \in \sigma(\mathcal{L}) \rangle_{\mathrm{loc}}.$$

Given that $I_\lambda Q_\rho k(\mathfrak{p})$ lies in $\mathcal{L}$ for each $\mathfrak{p} \in \sigma(\mathcal{L})$ we must have $\Gamma_\mathfrak{p}\mathcal{L} = \Gamma_\mathfrak{p} S(R)$ by Lemma 3.4.9. Thus $\mathcal{L} = \tau\sigma(\mathcal{L})$ by Proposition 3.4.1.

The restricted assignments of the second claim make sense by Proposition 3.4.15 and it is a bijection by the same argument we have just used above.

The last bijection is a consequence of the second one together with Krause's result Theorem 3.1.1 (3) which identifies $S(R)^c$, up to summands, with the singularity category $D_{\mathrm{Sg}}(R)$. □

Chapter 4

Hypersurface Rings

The aim of this chapter is to strengthen Theorem 3.4.16 in the case where R is locally a hypersurface. We prove that for such rings subsets of the singular locus completely classify localizing subcategories of $S(R)$. This allows us to deduce that the telescope conjecture holds for $S(R)$.

4.1 Preliminary Material on Commutative Algebra

We give here a brief summary of some commutative algebra definitions and results. Specifically the theory of Gorenstein homological algebra and local complete intersections. Our main reference for Gorenstein homological algebra is [33], particularly Chapters 10 and 11.

4.1.1 Gorenstein Homological Algebra

Let us denote by R a noetherian ring.

Definition 4.1.1. An R-module G is *Gorenstein injective* if there exists an exact sequence

$$E = \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

of injective R-modules such that $\mathrm{Hom}(I, E)$ is exact for every injective module I and $G = \ker(E^0 \longrightarrow E^1) = Z^0E$ is the zeroth syzygy of E. We say that the complex E is *totally acyclic* and call it a *complete resolution* of G.

Remark 4.1.2. Of course one can extend this definition to other abelian categories with enough injectives. For instance there is a notion of Gorenstein injective sheaves on a scheme.

Definition 4.1.3. We denote by $K_{\mathrm{tac}}(\mathrm{Inj}\, R)$ the homotopy category of totally acyclic complexes of injective R-modules. It is a full triangulated subcategory of $S(R) = K_{\mathrm{ac}}(\mathrm{Inj}\, R)$ and the two coincide when R is Gorenstein by [48] Proposition 7.13.

Let us recall the following two results concerning Gorenstein injective modules.

Proposition 4.1.4 ([33] Proposition 10.1.2)**.** *Let G be a Gorenstein injective R-module. Then*

$$\mathrm{Ext}^i(I, G) = 0$$

for all $i > 0$ and injective modules I and either $\mathrm{id}_R\, G = 0$ *or* $\mathrm{id}_R\, G = \infty$.

Proposition 4.1.5 ([33] Corollary 11.2.2)**.** *Let R be a Gorenstein ring of finite Krull dimension and G an R-module. The following are equivalent:*

(1) *G is Gorenstein injective;*

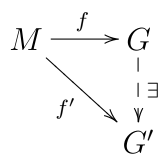
(2) $\mathrm{Ext}^i(L, G) = 0$ *for all R-modules L with* $\mathrm{pd}_R\, L < \infty$ *and all $i \geq 1$;*

(3) $\mathrm{Ext}^1(L, G) = 0$ *for all R-modules L with* $\mathrm{pd}_R\, L < \infty$*;*

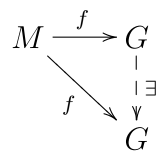
(4) $\mathrm{Ext}^i(I, G) = 0$ *for all injective R-modules I and $i \geq 1$.*

We now define envelopes with respect to a class of R-modules.

Definition 4.1.6. Let R be a ring and fix some class $\mathcal{G}$ of R-modules. A $\mathcal{G}$-*preenvelope* of an R-module M is a pair (G, f) where G is a module in $\mathcal{G}$ and $f\colon M \longrightarrow G$ is a morphism such that for any $G' \in \mathcal{G}$ and $f' \in \mathrm{Hom}(M, G')$ there exists a morphism making the triangle



commute. We say that a preenvelope (G, f) is a $\mathcal{G}$-*envelope* of M if, when we consider the diagram



every choice of morphism $G \longrightarrow G$ making the triangle commute is an automorphism of G. When speaking of envelopes we shall omit the morphism from the notation and refer to G as the $\mathcal{G}$-envelope of M.

It is clear that $\mathcal{G}$-envelopes, when they exist, are unique up to isomorphism. In the case $\mathcal{G} = \operatorname{Inj} R$, the class of injective R-modules, we see that the $\operatorname{Inj} R$-envelope of an R-module is precisely its injective envelope. In the case $\mathcal{G} = \operatorname{GInj} R$, the class of Gorenstein injective R-modules, we get the notion of *Gorenstein injective envelope*. As every injective R-module is Gorenstein injective it can be shown that whenever (G, f) is a Gorenstein injective envelope of M the morphism f is injective.

Notation 4.1.7. For an R-module M we shall denote its Gorenstein injective envelope, if it exists, by $G_R(M)$.

In many cases Gorenstein injective envelopes exist and have certain nice properties.

Theorem 4.1.8 ([33] 11.3.2, 11.3.3)**.** *If R is Gorenstein of Krull dimension n, then every R-module M admits a Gorenstein injective envelope $M \longrightarrow G$ such that if*

$$0 \longrightarrow M \longrightarrow G \longrightarrow L \longrightarrow 0$$

is exact then $\operatorname{id}_R L \leq n - 1$ whenever $n \geq 1$. Furthermore, $\operatorname{id}_R M < \infty$ if and only if $M \longrightarrow G$ is an injective envelope.

Proposition 4.1.9 ([33] 10.4.25, 11.3.9)**.** *Let R be a Gorenstein ring of finite Krull dimension. Then the class of Gorenstein injective R-modules is closed under small coproducts and summands and if $M_i \longrightarrow G_i$ is a Gorenstein injective envelope of the R-module M_i for each $i \in I$, then*

$$\oplus_i M_i \longrightarrow \oplus_i G_i$$

is a Gorenstein injective envelope.

Following the notation above we denote by $\operatorname{GInj} R$ the category of Gorenstein injective R-modules. It is a Frobenius category i.e., it is an exact category with enough projectives and enough injectives and the projective and injective objects coincide. The exact structure comes from taking the exact sequences to be those exact sequences of R-modules whose terms are Gorenstein injective. It is Frobenius as every injective R-module is Gorenstein injective and by Proposition 4.1.4 injective R-modules are also projective in $\operatorname{GInj} R$.

The *stable category* of $\operatorname{GInj} R$ denoted $\underline{\operatorname{GInj}}R$ is the category whose objects are those of $\operatorname{GInj} R$ and whose hom-sets are

$$\begin{aligned}\underline{\operatorname{Hom}}(G,H) &:= \operatorname{Hom}_{\underline{\operatorname{GInj}}R}(G,H)\\ &= \operatorname{Hom}_R(G,H)/\{f \mid f \text{ factors via an injective module}\}.\end{aligned}$$

This category is triangulated with suspension functor given by taking syzygies of complete resolutions and triangles coming from short exact sequences (see for example [37] Chapter 1 for details).

The following result shows that we can study part of the singularity category $S(R)$ by working with Gorenstein injective R-modules.

Proposition 4.1.10. *For a noetherian ring R there is an equivalence*

$$K_{\mathrm{tac}}(\operatorname{Inj} R) \underset{c}{\overset{Z^0}{\rightleftarrows}} \operatorname{GInj} \underline{R}$$

where Z^0 takes the zeroth syzygy of a complex of injectives and c sends a Gorenstein injective R-module to a complete resolution.

Proof. The result is standard so we only sketch the proof. The functor c is well defined since complete resolutions are unique up to homotopy equivalence, exist by definition for every Gorenstein injective module, and morphisms of modules lift uniquely up to homotopy to morphisms of complete resolutions. The zeroth syzygy of any totally acyclic complex of injectives is by definition Gorenstein injective. It is clear that, up to injectives, the Gorenstein injective R-module obtained by applying Z^0 to a totally acyclic complex of injectives only depends on its homotopy class so that Z^0 is well defined.

It is easy to check that the requisite composites are naturally isomorphic to the corresponding identity functors. □

For R Gorenstein every acyclic complex of injectives is totally acyclic by [48] Proposition 7.13 so there is an equivalence between $S(R)$ and $\underline{\operatorname{GInj}}R$. We thus obtain an action of $D(R)$ on $\underline{\operatorname{GInj}}R$ via this equivalence and we use the same notation to denote this action.

Corollary 4.1.11. *Let R be a Gorenstein ring and M an R-module. There is an isomorphism in $\underline{\operatorname{GInj}}R$*

$$Z^0 I_\lambda Q_\rho M \cong G_R(M).$$

Proof. By Theorem 4.1.8 there is a short exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow G_R(M) \longrightarrow L \longrightarrow 0$$

where L has finite projective dimension. Considering this as a triangle in $D(R)$ we obtain a triangle in $\underline{\mathrm{GInj}}R$

$$Z^0 I_\lambda Q_\rho M \longrightarrow Z^0 I_\lambda Q_\rho G_R(M) \longrightarrow 0 \longrightarrow \Sigma Z^0 I_\lambda Q_\rho M$$

where L is sent to zero as it is a perfect complex when viewed as an object of $D(R)$. By [48] Corollary 7.14 the object $Z^0 I_\lambda Q_\rho G_R(M)$ is naturally isomorphic to the image of $G_R(M)$ in $\underline{\mathrm{GInj}}R$ under the canonical projection which proves the claim. □

4.1.2 Complete Intersections and Complexity

We now give a brief summary of certain invariants related to the growth of minimal free resolutions over local rings and their relation to local complete intersection rings. First of all let us recall the definition of the rings which will be of most interest to us.

Definition 4.1.12. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. We say R is a *complete intersection* if its $\mathfrak{m}$-adic completion $\hat{R}$ can be written as the quotient of a regular ring by a regular sequence. The minimal length of the regular sequence in such a presentation of $\hat{R}$ is the *codimension* of R.

A not necessarily local ring R is a locally complete intersection if $R_\mathfrak{p}$ is a complete intersection for each $\mathfrak{p} \in \operatorname{Spec} R$.

If R is a complete intersection of codimension 1 we say that it is a *hypersurface*. Similarly if R is a complete intersection of codimension 1 when localized at each prime of $\operatorname{Spec} R$ we say that R is *locally a hypersurface*.

Remark 4.1.13. Rings satisfying the conditions of the above definition are sometimes called *abstract* complete intersections to differentiate them from those local rings which are quotients of regular rings by regular sequences without the need to complete. We use the term complete intersection as in the definition above and when we need to impose that R itself is a quotient of a regular ring by a regular sequence it will be stated explicitly.

Remark 4.1.14. The property of being a complete intersection is stable under localization. Furthermore, the property of being a complete intersection is intrinsic (cf. Theorem 4.1.15) and if $(R, \mathfrak{m}, k)$ is a complete intersection then any presentation of $\hat{R}$ as a quotient of a regular local ring has kernel generated by a regular sequence.

Let $(R, \mathfrak{m}, k)$ be a local ring. Given a finitely generated R-module M we denote by $\beta_i(M)$ the ith *Betti number* of M

$$\beta_i(M) = \dim_k \operatorname{Tor}_i(M, k) = \dim_k \operatorname{Ext}^i(M, k).$$

The asymptotic behaviour of the Betti numbers is expressed by the *complexity* of M. For M in R-mod the *complexity* of M, $\operatorname{cx}(M)$, is defined to be

$$\operatorname{cx}(M) = \inf\{c \in \mathbb{N} \mid \text{there exists } a \in \mathbb{R} \text{ such that } \beta_n(M) \leq an^{c-1} \text{ for } n \gg 0\}.$$

By a result of Gulliksen ([36] Theorem 2.3) the complexity of the residue field k detects whether or not R is a complete intersection. In fact one can say slightly more:

Theorem 4.1.15 ([1] Theorem 3)**.** *The following conditions on a local ring $(R, \mathfrak{m}, k)$ are equivalent:*

(1) *R is a complete intersection of codimension c;*

(2) $\operatorname{cx} k = c$*;*

(3) *for every M a finitely generated R-module $\operatorname{cx} M \leq c$ and some module has complexity c.*

Finally we recall how the complexity of a module changes under certain changes of the base ring. This result together with further properties can be found in [4] as Proposition 5.2.

Proposition 4.1.16. *For a local ring $(R, \mathfrak{m}, k)$ and a finitely generated R-module M the following hold.*

(1) *Let R' be another local ring and $R \longrightarrow R'$ a local flat morphism, then*

$$\operatorname{cx}_R M = \operatorname{cx}_{R'} M \otimes_R R'.$$

(2) *Let $Q \longrightarrow R$ be a surjective local map of local rings whose kernel is generated by a Q-regular sequence of length c. Then*

$$\operatorname{cx}_Q M \leq \operatorname{cx}_R M \leq \operatorname{cx}_Q M + c.$$

4.2 The Classification Theorem for Hypersurfaces

Throughout this section $(R, \mathfrak{m}, k)$ is a local Gorenstein ring unless otherwise specified. We consider the relationship between the categories $S(R)$ and $S(R/(x))$ for x a regular element. Our results allow us to classify the localizing subcategories of $S(R)$ in the case that R is a hypersurface ring.

By the classification result we have already proved in Theorem 3.4.16 together with the fact that every localizing subcategory is closed under the action of $D(R)$ (Lemma 3.3.5) it is sufficient to consider subcategories of $\Gamma_{\mathfrak{p}} S(R)$. As in the discussion before Theorem 2.5.11 a bijection between subsets of the singular locus and the collection of localizing subcategories is equivalent to each of the $\Gamma_{\mathfrak{p}} S(R)$ being minimal in the sense of Definition 2.5.10

Remark 4.2.1. Note that we can reduce to the case of local rings when studying minimality. Indeed, suppose R is a noetherian ring and $\mathfrak{p} \in \operatorname{Spec} R$. Then since $\Gamma_{\mathfrak{p}} R \otimes L_{\mathcal{Z}(\mathfrak{p})} R \cong \Gamma_{\mathfrak{p}} R$ we can study $\Gamma_{\mathfrak{p}} S(R)$ in $S(R_{\mathfrak{p}}) \subseteq S(R)$, the essential image of $L_{\mathcal{Z}(\mathfrak{p})} R \odot (-) \cong R_{\mathfrak{p}} \otimes_R (-)$.

We now prove several lemmas leading to a key proposition. The first two of these lemmas are well known.

Lemma 4.2.2. *Let x be a regular element of R. Then the quotient ring $R/(x)$ is also Gorenstein.*

Proof. By the second injective change of rings theorem (see for example [44] Theorem 205) we have

$$\operatorname{id}_{R/(x)} R/(x) \leq \operatorname{id}_R R - 1 = \dim R - 1.$$

Thus $R/(x)$ has finite self-injective dimension so is Gorenstein as claimed. □

Lemma 4.2.3. *Let G be a Gorenstein injective R-module and $x \in R$ an R-regular element. Then G is x-divisible i.e., multiplication by x is surjective on G.*

Proof. As x is regular we get a projective resolution of the R-module $R/(x)$

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/(x) \longrightarrow 0.$$

Recall from Proposition 4.1.5 that Gorenstein injective modules are right Ext^i-orthogonal to the modules of finite projective dimension for $i \geq 1$. So applying $\operatorname{Hom}_R(-, G)$ to the above short exact sequence yields an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(R/(x), G) \longrightarrow G \xrightarrow{x} G \longrightarrow 0$$

which proves the claim. □

Notation 4.2.4. We will consider $D(R)$ to act on $\underline{\mathrm{GInj}}R$ via the equivalence of Proposition 4.1.10. Thus by $\Gamma_{\mathfrak{m}}G$ for $G \in \underline{\mathrm{GInj}}R$ we mean the class represented by the Gorenstein injective $Z^0\Gamma_{\mathfrak{m}}c(G)$.

Lemma 4.2.5. *Let G be a Gorenstein injective R-module such that $\Gamma_{\mathfrak{m}}G \neq 0$ in the stable category. Then for all $i \geq 1$*

$$\mathrm{Ext}^i(k, G) \neq 0.$$

Proof. For $i \geq 1$ there are isomorphisms

$$\begin{aligned}\mathrm{Ext}^i(k, G) &\cong \mathrm{Hom}(I_\lambda Q_\rho k, \Sigma^i I_\lambda Q_\rho G)\\ &\cong \mathrm{Hom}(I_\lambda Q_\rho k, \Sigma^i \Gamma_{\mathfrak{m}} I_\lambda Q_\rho G)\\ &\cong \mathrm{Ext}^i(k, \Gamma_{\mathfrak{m}}G)\end{aligned}$$

where the first and last isomorphisms are via [48] Proposition 7.10, together with Lemma 3.4.4 for the last isomorphism, and the middle one is by adjunction and the fact that as R is local there is an equality $\Gamma_{\mathfrak{m}}R = \Gamma_{\mathcal{V}(\mathfrak{m})}R$ of tensor idempotents in $D(R)$. By Proposition 3.3.7 the minimal complete resolution of $\Gamma_{\mathfrak{m}}G$ consists solely of copies of $E(k)$. Hence there is a representative for $\Gamma_{\mathfrak{m}}G$ which is $\mathfrak{m}$-torsion and, as it represents a non-zero object in the stable category, of infinite injective dimension. So using the isomorphisms above we see that the Ext's are nonvanishing as claimed: their dimensions give the cardinalities of the summands of $E(k)$ in each degree of a minimal injective resolution for our representative of $\Gamma_{\mathfrak{m}}G$ ([33] 9.2.4). □

Lemma 4.2.6. *Let G be a Gorenstein injective R-module such that $\Gamma_{\mathfrak{m}}G \neq 0$ in the stable category, $x \in R$ a regular element, and denote by M the R-module* $\mathrm{Hom}_R(R/(x), G)$. *Then* $\mathrm{id}_R\, M = \infty = \mathrm{pd}_R\, M$.

Proof. Recall from the proof of Lemma 4.2.3 that M fits into the short exact sequence

$$0 \longrightarrow M \longrightarrow G \xrightarrow{x} G \longrightarrow 0.$$

Applying $\mathrm{Hom}_R(k, -)$ gives a long exact sequence

$$0 \longrightarrow \mathrm{Hom}(k, M) \longrightarrow \mathrm{Hom}(k, G) \longrightarrow \mathrm{Hom}(k, G) \longrightarrow \mathrm{Ext}^1(k, M) \longrightarrow \cdots$$

where for $i \geq 0$ the maps

$$\mathrm{Ext}^i(k, G) \longrightarrow \mathrm{Ext}^i(k, G)$$

are multiplication by x (see for example [71] 3.3.6) and hence are all 0 as ([71] Corollary 3.3.7) the $\mathrm{Ext}^i(k,G)$ are k-vector spaces. Thus for $i \geq 0$ the morphisms

$$\mathrm{Ext}^i(k,M) \longrightarrow \mathrm{Ext}^i(k,G)$$

are surjective. By the last lemma the groups $\mathrm{Ext}^i(k,G)$ are non-zero for $i \geq 1$. Thus $\mathrm{Ext}^i(k,M) \neq 0$ for $i \geq 1$ so M necessarily has infinite injective dimension. Since R is Gorenstein M must also have infinite projective dimension (see for example [33] Theorem 9.1.10). □

Proposition 4.2.7. *Let G be an object of $\Gamma_{\mathfrak{m}}\underline{\mathrm{GInj}}R$ and suppose x is a regular element of R. Then $\langle G\rangle_{\mathrm{loc}}$ contains the image of a non-injective Gorenstein injective envelope of an $R/(x)$-module.*

Proof. By Lemmas 4.2.6 and 4.2.3 there is a short exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow G \xrightarrow{x} G \longrightarrow 0$$

with M an $R/(x)$-module of infinite projective dimension over R. Applying $Z^0 I_\lambda Q_\rho$ gives a triangle in $\langle G\rangle_{\mathrm{loc}}$

$$G_R(M) \longrightarrow G \longrightarrow G \longrightarrow \Sigma G_R(M)$$

where we use Corollary 4.1.11 to identify $Z^0 I_\lambda Q_\rho M$ with the class of its Gorenstein injective envelope. The module $G_R(M)$ is not injective by Theorem 4.1.8 as $\mathrm{pd}_R M = \infty$ (i.e., it is not in the kernel of $I_\lambda Q_\rho$), which completes the proof. □

Suppose R is an artinian local hypersurface. Then necessarily R is, up to isomorphism, of the form $S/(x^n)$ for a discrete valuation ring S with x a uniformiser. In particular, R is an artinian principal ideal ring so by [41] Theorem 2 every R-module is a direct sum of cyclic R-modules. Using this fact we show that $\underline{\mathrm{GInj}}R$ is minimal when R is an artinian local hypersurface. This provides the base case for our inductive argument that the maps of Theorem 3.4.16 are bijections for any hypersurface ring without the requirement that the categories in question contain certain objects.

Lemma 4.2.8. *Suppose R is an artinian local hypersurface. Then the category $\underline{\mathrm{GInj}}R = \Gamma_{\mathfrak{m}}\underline{\mathrm{GInj}}R$ is minimal.*

Proof. Since R is 0-Gorenstein there is an equality $\underline{R\text{-Mod}} = \underline{\mathrm{GInj}}R$, where $\underline{R\text{-Mod}}$ is the stable category of the Frobenius category R-Mod, as every R-module is Gorenstein injective by [33] Proposition 11.2.5 (4).

As remarked above we have an isomorphism $R \cong S/(x^n)$ where S is a discrete valuation ring and x is a uniformiser. We also recalled that every R-module is a coproduct of cyclic R-modules; this remains true in the stable category $\underline{R\text{-Mod}}$. Since the subcategory of compacts in $\underline{R\text{-Mod}}$ is precisely $\underline{R\text{-mod}}$ every object is thus a coproduct of compact objects (so in particular $\underline{R\text{-Mod}}$ is a pure-semisimple triangulated category cf. [12] Corollary 12.26). It follows that every localizing subcategory of $\underline{R\text{-Mod}}$ is generated by objects of $\underline{R\text{-mod}}$.

We deduce minimality from the existence of Auslander-Reiten sequences. For each $1 \leq i \leq n-1$ there is an Auslander-Reiten sequence

$$0 \longrightarrow R/(x^i) \xrightarrow{f} R/(x^{i-1}) \oplus R/(x^{i+1}) \xrightarrow{g} R/(x^i) \longrightarrow 0$$

where, using $\overline{(-)}$ to denote residue classes,

$$f(\overline{a}) = \begin{pmatrix} \overline{a} \\ \overline{ax} \end{pmatrix} \quad \text{and} \quad g\begin{pmatrix} \overline{a} \\ \overline{b} \end{pmatrix} = \overline{ax - b}.$$

So the smallest thick subcategory containing any non-zero compact object is all of $\underline{R\text{-mod}}$: every object is up to isomorphism a coproduct of the classes of cyclic modules and the Auslander-Reiten sequences show that any cyclic module which is non-zero in the stable category is a generator. Thus, as we observed above that every localizing subcategory is generated by objects of $\underline{R\text{-mod}}$, we see there are no non-trivial localizing subcategories except for all of $\underline{R\text{-Mod}}$ so it is minimal as claimed. □

We next need a result that is essentially contained in [48] Section 6 but which we reformulate in a way which is more convenient for our purposes. To prove this lemma we need a very straightforward result which we include for the sake of completeness.

Lemma 4.2.9. *Suppose R and S are local rings and $\pi \colon R \longrightarrow S$ is a surjection with kernel generated by an R-regular sequence. Then the functor*

$$\pi_* \colon S\text{-Mod} \longrightarrow R\text{-Mod}$$

sends modules of finite projective dimension to modules of finite projective dimension.

Proof. We prove the result by induction on the length of the R-sequence $\{x_i\}_{i=1}^n$ generating the kernel of π. Suppose first that $S = R/(x)$ for an R-regular element

x. The first change of rings theorem ([71] Theorem 4.3.3) gives for any S-module M of finite projective dimension

$$\mathrm{pd}_R M = \mathrm{pd}_S M + 1$$

so π_* sends modules of finite S-projective dimension to modules of finite R-projective dimension as claimed.

Suppose the result holds for R-sequences of length strictly less than n and let $S = R/(x_1, \ldots, x_n)$ where the sequence $\{x_1, \ldots, x_n\}$ is R-regular. Then, as above, we deduce from the first change of rings theorem that for an S-module M of finite S-projective dimension

$$\mathrm{pd}_{R/(x_1,\ldots,x_{n-1})} M = \mathrm{pd}_S M + 1.$$

Thus using the natural factorization of the functor π_* via $R/(x_1, \ldots, x_{n-1})$-Mod we are done by the inductive hypothesis. □

Lemma 4.2.10. *Suppose $\pi\colon R \longrightarrow S$ is a surjective map of Gorenstein local rings with kernel generated by an R-regular sequence. Then there is an induced coproduct preserving exact functor*

$$\underline{\pi_*}\colon \underline{\mathrm{GInj}}S \longrightarrow \underline{\mathrm{GInj}}R$$

which sends an object of $\underline{\mathrm{GInj}}S$ to its GInj *R-envelope.*

Proof. Let us denote by ν the composite

$$\nu\colon\ D(S) \xrightarrow{\pi_*} D(R) \xrightarrow{I_\lambda Q_\rho} S(R) \xrightarrow{Z^0} \underline{\mathrm{GInj}}R\,.$$

Recall from [48] Corollary 5.5 and Example 5.6 that the composite

$$\mu\colon\ R\text{-Mod} \longrightarrow D(R) \xrightarrow{I_\lambda Q_\rho} S(R)$$

where the functor R-Mod $\longrightarrow D(R)$ is the canonical inclusion, preserves all coproducts and annihilates all modules of finite projective dimension. Thus by Lemma 4.2.9 the equal composites

$$\begin{array}{ccccc} \mathrm{GInj}\, S & \longrightarrow & D(S) & \xrightarrow{\nu} & \underline{\mathrm{GInj}}R \\ & \searrow^{\pi_*} & & \nearrow_{Z^0\mu} & \\ & & R\text{-Mod} & & \end{array}$$

must factor via the stable category $\underline{\mathrm{GInj}}S$. Indeed, as R is Gorenstein S is also Gorenstein by Lemma 4.2.2. Thus injective S-modules have finite S-projective dimension and π_* sends them to modules of finite R-projective dimension by the last lemma. In particular S-injectives are killed by both composites. We get a commutative diagram

$$\begin{array}{ccccc} \mathrm{GInj}\, S & \longrightarrow & D(S) & \xrightarrow{\nu} & \underline{\mathrm{GInj}}R \\ & \searrow^{p} & & \nearrow_{\underline{\pi_*}} & \\ & & \underline{\mathrm{GInj}}S. & & \end{array}$$

The functors π_*, p, and $Z^0\mu$ are all coproduct preserving: we have already noted that μ preserves coproducts, Z^0 is the equivalence of Proposition 4.1.10, and it is easily seen that the projection p also preserves coproducts (the concerned reader may consult [48] Corollary 7.14). As p is essentially surjective we see that $\underline{\pi_*}$ also preserves coproducts. Indeed, the top composite is equal to $Z^0\mu\pi_*$ which preserves coproducts and any coproduct of objects in $\underline{\mathrm{GInj}}S$ is the image under p of a coproduct of S-modules. Exactness follows similarly by noting that the top composite sends short exact sequences to triangles as $\mathrm{GInj}\, S$ is an exact subcategory of S-Mod and π_* is exact.

The explicit description of $\underline{\pi_*}$ is clear from the construction: by the commutativity of the diagram $\underline{\pi_*}$ sends the image of an object M of $\mathrm{GInj}\, S$ to $Z^0 I_\lambda Q_\rho \pi_* M$ which is precisely its Gorenstein injective envelope as an R-module by Corollary 4.1.11. □

Remark 4.2.11. Given an S-module M we see from the above that, letting $G_S(M)$ and $G_R(\pi_* M)$ denote its Gorenstein injective envelopes over S and R respectively, there are isomorphisms in the stable category

$$\underline{\pi_*} G_S(M) \cong G_R(\pi_* G_S(M)) \cong G_R(\pi_* M).$$

The first isomorphism is a consequence of the last lemma. The second isomorphism follows from Theorem 4.1.8 which provides us, after an application of π_*, with a short exact sequence

$$0 \longrightarrow \pi_* M \longrightarrow \pi_* G_S(M) \longrightarrow \pi_* L \longrightarrow 0$$

where $\pi_* L$ has finite projective and injective dimension. Thus the R-Gorenstein injective envelopes of $\pi_* M$ and $\pi_* G_S(M)$ agree in $\underline{\mathrm{GInj}}R$ which gives the second isomorphism.

We are now ready to prove the theorem which gives us a complete classification of the localizing subcategories of $S(R)$ when R is a hypersurface.

Theorem 4.2.12. *If $(R, \mathfrak{m}, k)$ is a hypersurface then $\Gamma_{\mathfrak{m}}\underline{\mathrm{GInj}}R$ is minimal.*

Proof. We prove the theorem by induction on the dimension of R. In the case $\dim R = 0$ then R is an artinian hypersurface and $\underline{\mathrm{GInj}}R$ is minimal by Lemma 4.2.8.

So suppose the theorem holds for hypersurfaces of dimension strictly less than n and let $\dim R = n \geq 1$. Then as $\operatorname{depth} R = n \geq 1$ the maximal ideal $\mathfrak{m}$ is not contained in any of the associated primes or $\mathfrak{m}^2$ so we can choose, by prime avoidance, a regular element x not lying in $\mathfrak{m}^2$. The quotient $R/(x)$ is again a hypersurface, for example one can see this by noting that the second deviations agree $\varepsilon_2(R) = \varepsilon_2(R/(x)) = 1$ and the higher deviations vanish (see [2] section 7 for details).

Let us denote the projection $R \longrightarrow R/(x)$ by π. By Proposition 4.2.7 for every $0 \neq G \in \Gamma_{\mathfrak{m}}\underline{\mathrm{GInj}}R$ the subcategory $\langle G \rangle_{\mathrm{loc}}$ contains a non-zero object in the image of the functor $\underline{\pi_*}$ of Lemma 4.2.10. The ring $R/(x)$ has dimension $n-1$ so by the inductive hypothesis the category $\Gamma_{\mathfrak{m}/(x)}\underline{\mathrm{GInj}}R/(x)$ is minimal.

The functor $\underline{\pi_*}$ is exact and coproduct preserving by Lemma 4.2.10 so as $\langle G \rangle_{\mathrm{loc}}$ contains one object in the image of $\underline{\pi_*}$ it must contain the whole image by minimality of $\Gamma_{\mathfrak{m}/(x)}\underline{\mathrm{GInj}}R/(x)$. In particular $\langle G \rangle_{\mathrm{loc}}$ contains $G_R(k) \cong \underline{\pi_*}G_{R/(x)}(k)$. This object generates $\Gamma_{\mathfrak{m}}\underline{\mathrm{GInj}}R$ by Z^0 applied to the statement of Lemma 3.4.9. Hence $\langle G \rangle_{\mathrm{loc}} = \Gamma_{\mathfrak{m}}\underline{\mathrm{GInj}}R$ so $\Gamma_{\mathfrak{m}}\underline{\mathrm{GInj}}R$ is minimal as claimed. □

Using the other techniques we have developed this is enough to give a classification of the localizing subcategories of $S(R)$ for R a not necessarily local ring which is locally a hypersurface.

Theorem 4.2.13. *If R is a noetherian ring which is locally a hypersurface then there is an order preserving bijection*

$$\left\{ \textit{subsets of } \operatorname{Sing} R \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \textit{localizing subcategories of } S(R) \right\}$$

given by the assignments of Theorem 3.4.16. This restricts to the equivalent order preserving bijections

$$\left\{ \begin{array}{c} \textit{specialization closed} \\ \textit{subsets of } \operatorname{Sing} R \end{array} \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{localizing subcategories of } S(R) \\ \textit{generated by objects of } S(R)^c \end{array} \right\}$$

and

$$\left\{ \begin{array}{c} \textit{specialization closed} \\ \textit{subsets of } \operatorname{Sing} R \end{array} \right\} \rightleftarrows \left\{ \textit{thick subcategories of } D_{\mathrm{Sg}}(R) \right\}.$$

Proof. By Theorem 3.4.16 it is sufficient to check every localizing subcategory $\mathcal{L}$ contains $I_\lambda Q_\rho k(\mathfrak{p})$ for each $\mathfrak{p} \in \operatorname{Sing} R$ such that $\Gamma_\mathfrak{p}\mathcal{L} \neq 0$. As there are equivalences

$$\Gamma_\mathfrak{p} S(R) \cong \Gamma_\mathfrak{p} S(R_\mathfrak{p})$$

each of the subcategories $\Gamma_\mathfrak{p} S(R)$ is minimal by Theorem 4.2.12 as each $R_\mathfrak{p}$ is a local hypersurface. Hence if $\Gamma_\mathfrak{p}\mathcal{L} \neq 0$ for a localizing subcategory $\mathcal{L}$ we must have $\Gamma_\mathfrak{p} S(R) = \Gamma_\mathfrak{p}\mathcal{L} \subseteq \mathcal{L}$ where the containment is a consequence of the closure of localizing subcategories under the action of $D(R)$ (Lemma 3.3.5). In particular the generator $I_\lambda Q_\rho k(\mathfrak{p})$ of $\Gamma_\mathfrak{p} S(R)$ is an object of $\mathcal{L}$. Thus the image of the injection τ, namely those localizing subcategories $\mathcal{L}$ containing $I_\lambda Q_\rho k(\mathfrak{p})$ for each $\mathfrak{p} \in \operatorname{Sing} R$ such that $\Gamma_\mathfrak{p}\mathcal{L} \neq 0$, is in fact the set of all localizing subcategories. This proves the first bijection.

As in Theorem 3.4.16 the second bijection is a consequence of the first and Proposition 3.4.13 which states that compact objects have closed supports so σ of a compactly generated subcategory is specialization closed. The third bijection is equivalent to the second as by Theorem 3.1.1 (3) there is an equivalence up to summands $D_{\mathrm{Sg}}(R) \cong S(R)^c$ so our restatement is a consequence of Thomason's localization theorem ([60] Theorem 2.1). □

Remark 4.2.14. Our result implies Takahashi's Theorem 7.6 of [68].

We can use this theorem to give a proof of the telescope conjecture for $S(R)$ when R is locally a hypersurface.

Theorem 4.2.15. *If R is locally a hypersurface then the singularity category $S(R)$ satisfies the telescope conjecture i.e., every smashing subcategory of $S(R)$ is generated by objects of $S(R)^c$.*

Proof. As every localizing subcategory of $S(R)$ is a $D(R)$-submodule by Lemma 3.3.5 and the $D(R)$ action classifies the localizing subcategories of $S(R)$ by Theorem 4.2.13 the relative telescope conjecture (Definition 2.4.1) for this action agrees with the usual telescope conjecture. Thus it is sufficient to verify that the conditions of Theorem 2.4.14 hold.

The local-to-global principle holds for the action as Theorem 2.3.9 applies to $D(R)$. The support of every compact object of $S(R)$ is specialization closed by Proposition 3.4.13 and for every irreducible closed subset $\mathcal{V}(\mathfrak{p}) \subseteq \operatorname{Sing} R$ the object $I_\lambda Q_\rho R/\mathfrak{p}$ has support $\mathcal{V}(\mathfrak{p})$ by Remark 3.4.14.

Thus the theorem applies and every smashing subcategory of $S(R)$ is compactly generated. □

Chapter 5

The Singularity Category of a Scheme

We now present global versions of our results for affine schemes. Using these results we give a complete classification of the localizing subcategories of the singularity category for local complete intersection rings and certain complete intersection schemes over a base field. Throughout (unless explicitly mentioned otherwise) we will denote by X a separated noetherian scheme. We will use the following notation

$$D(X) := D(\mathrm{QCoh}\, X), \quad S(X) := K_{\mathrm{ac}}(\mathrm{Inj}\, X) \quad \text{and} \quad K(X) := K(\mathrm{QCoh}\, X)$$

where $\mathrm{QCoh}\, X$ is the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules and $\mathrm{Inj}\, X$ is the category of injective quasi-coherent sheaves of $\mathcal{O}_X$-modules.

Remark. We have defined $\mathrm{Inj}\, X$ to be the category of injective objects in $\mathrm{QCoh}\, X$, but we could just as well have taken it to be the category of those injective objects in the category of all $\mathcal{O}_X$-modules which are quasi-coherent. This fact can be found as Lemma 2.1.3 of [28]. We thus feel free to speak either of quasi-coherent injective $\mathcal{O}_X$-modules or injective quasi-coherent $\mathcal{O}_X$-modules as they are the same thing when X is (locally) noetherian which is the only case we consider.

We begin this section by showing that, as in the affine case, $D(X)$ acts on $S(X)$. We can then apply the machinery developed in Section 2.3 together with local arguments on X to globalise most of the results we have proved in the affine case.

5.1 An Action of $D(X)$

We recall that the machinery of [48], [58], and [57] works perfectly well in the generality that X is a noetherian separated scheme (in fact the machinery in each of these papers works in greater generality than we will use - the noetherian separated case is the intersection of what is known with our interests). Let us prove there is an action

$$D(X) \times S(X) \xrightarrow{\odot} S(X)$$

as in the affine case.

Recall from [48] Corollary 5.4 that $S(X)$ is a compactly generated triangulated category. Consider $E = \coprod_\lambda E_\lambda$ where E_λ runs through a set of representatives for the isomorphism classes of compact objects in $S(X)$. We define a homological functor $H\colon K(\mathrm{Flat}\, X) \longrightarrow \mathrm{Ab}$ by setting, for F an object of $K(\mathrm{Flat}\, X)$,

$$H(F) = H^0(F \otimes_{\mathcal{O}_X} E)$$

where the tensor product is taken in $K(X)$. This is a coproduct preserving homological functor since we are merely composing the exact coproduct preserving functor $(-) \otimes_{\mathcal{O}_X} E$ with the coproduct preserving homological functor H^0.

We again remind the reader of the notion of pure acyclicity. In [58] a complex F in $K(\mathrm{Flat}\, X)$ is defined to be *pure acyclic* if it is exact and has flat syzygies. Such complexes form a triangulated subcategory of $K(\mathrm{Flat}\, X)$ which we denote by $K_{\mathrm{pac}}(\mathrm{Flat}\, X)$ and we say that a morphism with pure acyclic mapping cone is a *pure quasi-isomorphism.* We recall that when X is noetherian the tensor product of a complex of flats with a complex of injectives is again a complex of injectives. As in the affine case tensoring a pure acyclic complex of flats with an injective complex yields a contractible complex. This can be checked locally using [62] Corollary 9.7, see for example [57] Lemma 8.2. In particular every pure acyclic complex lies in the kernel of H.

Definition 5.1.1. With notation as above we denote by $A_\otimes(\mathrm{Inj}\, X)$ the quotient $\ker(H)/K_{\mathrm{pac}}(\mathrm{Flat}\, X)$.

Lemma 5.1.2. *An object F of $K(\mathrm{Flat}\, X)$ lies in $\ker(H)$ if and only if the exact functor*

$$F \otimes_{\mathcal{O}_X} (-)\colon K(\mathrm{Inj}\, X) \longrightarrow K(\mathrm{Inj}\, X)$$

restricts to

$$F \otimes_{\mathcal{O}_X} (-)\colon S(X) \longrightarrow S(X).$$

In particular, $A_{\otimes}(\operatorname{Inj} X)$ *consists of the pure quasi-isomorphism classes of objects which act on* $S(X)$*.*

Proof. The proof is essentially the same as the one given for Lemma 3.2.4; the point is that E generates $S(X)$.

□

Remark 5.1.3. Restricting H to $N(\operatorname{Flat} X) = K(\operatorname{Flat} X)/K_{\mathrm{pac}}(\operatorname{Flat} X)$ we obtain by Theorem 3.2.1 an adjoint pair

$$A_{\otimes}(\operatorname{Inj} X) \rightleftarrows N(\operatorname{Flat} X).$$

Lemma 5.1.4. *There is a fully faithful, exact, coproduct preserving functor* $D(X) \longrightarrow A_{\otimes}(\operatorname{Inj} X)$*.*

Proof. There is, by the proof of Theorem 5.5 of [57], a fully faithful, exact, coproduct preserving functor $D(X) \longrightarrow N(\operatorname{Flat} X)$ given by taking K-flat resolutions and inducing an equivalence

$$D(X) \cong {}^{\perp}N_{\mathrm{ac}}(\operatorname{Flat} X).$$

This functor given by taking resolutions factors via $A_{\otimes}(\operatorname{Inj} X)$ since K-flat complexes send acyclics to acyclics under the tensor product. □

Remark 5.1.5. Taking K-flat resolutions and then tensoring gives an action

$$(-) \odot (-) \colon D(X) \times S(X) \longrightarrow S(X)$$

by an argument which is the same (*mutatis mutandis*) as the one given in Proposition 3.2.7: K-flat resolutions are well behaved with respect to the tensor product so the necessary compatibilities follow from those of the tensor product of complexes.

The tensor triangulated category $(D(X), \otimes, \mathcal{O}_X)$ is rigidly-compactly generated. Thus we can apply all of the machinery we have developed for actions of rigidly-compactly generated triangulated categories. In particular, recalling from [69] that $\operatorname{Spc} D(X)^c = \operatorname{Spc} D^{\mathrm{perf}}(X) \cong X$, we can associate to every specialization closed subset $\mathcal{V}$ of X a localization sequence of submodules

$$\Gamma_{\mathcal{V}} S(X) \rightleftarrows S(X) \rightleftarrows L_{\mathcal{V}} S(X)$$

where $\Gamma_{\mathcal{V}} S(X)$ is generated by objects compact in $S(R)$, by Corollary 2.2.13 and Lemma 2.2.6. Since X is noetherian we get for every $x \in X$ objects $\Gamma_x \mathcal{O}_X$ which allow us to define supports on $S(X)$ with values in X. We also wish to note that by Lemma 2.2.8 the action restricts to the level of compact objects

$$D^{\mathrm{perf}}(X) \times S(X)^{\mathrm{c}} \xrightarrow{\odot} S(X)^{\mathrm{c}}.$$

Finally as the category $D(X)$ has a model Theorem 2.3.9 applies.

5.2 Subsets of X Versus Subcategories of $S(X)$

We are now in a position to demonstrate that what we have proved in the affine case extends in a straightforward way to noetherian separated schemes via the machinery of Section 2.5. As in Definition 2.2.22 we have assignments

$$\left\{ \text{ subsets of } X \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \text{ localizing } D(X)\text{-submodules of } S(X) \right\}$$

where for a localizing submodule $\mathcal{L}$ we set

$$\sigma(\mathcal{L}) = \operatorname{supp} \mathcal{L} = \{x \in X \mid \Gamma_x \mathcal{L} \neq 0\}$$

and for a subset W of X

$$\tau(W) = \{A \in S(X) \mid \operatorname{supp} A \subseteq W\}.$$

In this section, unless stated otherwise, submodules are localizing submodules. In order to apply our formalism to the situation of $D(X)$ acting on $S(X)$ we first need to understand what the effect of restricting to an open subset of X is.

Before continuing let us remind the reader of some of the notation of Chapter 2. Given a specialization closed subset $\mathcal{V} \subseteq X$ we denote by $D_{\mathcal{V}}(X)$ the smashing subcategory generated by those compact objects whose support, in the sense of [7], lies in $\mathcal{V}$. We recall that the corresponding localization sequence gives rise to the tensor idempotents $\Gamma_{\mathcal{V}}\mathcal{O}_X$ and $L_{\mathcal{V}}\mathcal{O}_X$. For a closed subset Z of X with complement U we denote the quotient $D(X)/D_Z(X)$ by either $L_Z D(X)$ or $D(X)(U)$, as in Section 2.5. The action of $D(X)$ on $S(X)$ gives rise to an action of $D(X)(U)$ on $S(X)(U) = L_Z S(X)$ as in Proposition 2.5.5.

Lemma 5.2.1. *Let $U \subseteq X$ be an open set with complement $Z = X \setminus U$, and let $f \colon U \longrightarrow X$ be the inclusion. If E is an object of $D(X)$ then the map $E \longrightarrow \mathbf{R}f_*f^*E$ agrees with the localization map $E \longrightarrow L_Z E$. In particular, $D(X)(U)$ is precisely $D(U)$.*

Proof. By definition the smashing subcategory $D_Z(X)$ giving rise to L_Z is the localizing subcategory generated by the compact objects whose support is contained in Z. The kernel of f^* is the localizing subcategory generated by those compact objects whose homological support is contained in Z by [66]. As these two notions of support coincide for compact objects of $D(X)$ (see for example [7] Corollary 5.6) the lemma follows immediately. □

We recall from [48] Theorems 1.5 and 6.6 that for $f\colon U \longrightarrow X$ an open immersion we obtain an adjoint pair of functors

$$S(X) \underset{f_*}{\overset{f^*}{\rightleftarrows}} S(U).$$

These functors are easily seen to be, using the classification of injective quasi-coherent sheaves on a locally noetherian scheme (see for example [28] Lemma 2.1.5), just the usual pullback and pushforward of complexes.

Lemma 5.2.2. *With notation as in Lemma 5.2.1 suppose $U \subseteq X$ is an open affine and let A be an object of $S(X)$. Then the natural map $A \longrightarrow f_*f^*A$ agrees with $A \longrightarrow L_ZA$. In particular, $S(X)(U)$ is canonically identified with $S(U)$.*

Proof. Since $f\colon U \longrightarrow X$ is an affine morphism we have that $f_*\colon D(U) \longrightarrow D(X)$ is exact and $\mathbf{R}f_* = f_*$. The map $A \longrightarrow L_ZA$ is, by definition, obtained by taking the morphism $\mathcal{O}_X \longrightarrow L_Z\mathcal{O}_X$ in $D(X)$ and tensoring with $A \in S(X)$. By Lemma 5.2.1 the map $\mathcal{O}_X \longrightarrow L_Z\mathcal{O}_X$ is just $\mathcal{O}_X \longrightarrow f_*\mathcal{O}_U$, which is a map of K-flat complexes. Thus the map $A \longrightarrow L_ZA$ is

$$A \longrightarrow f_*\mathcal{O}_U \otimes_{\mathcal{O}_X} A \cong f_*(\mathcal{O}_U \otimes_{\mathcal{O}_U} f^*A) \cong f_*f^*A,$$

where the first isomorphism is by the projection formula, completing the proof. □

Now we are in business: we know that for an open affine $U \cong \operatorname{Spec} R$ in X the construction of Section 2.5 gives us $D(R)$ acting on $S(R)$. It just remains to verify that this is the action we expect.

Lemma 5.2.3. *Suppose U is an open subscheme of X with inclusion $f\colon U \longrightarrow X$. Then the diagram*

$$\begin{array}{ccc} D(X) \times S(X) & \xrightarrow{f^* \times f^*} & D(U) \times S(U) \\ \downarrow{\scriptstyle \odot} & & \downarrow{\scriptstyle \odot} \\ S(X) & \xrightarrow{f^*} & S(U) \end{array}$$

commutes up to natural isomorphism.

Proof. By virtue of being an open immersion f^* sends K-flat complexes to K-flat complexes and commutes with taking K-flat resolutions. Thus, as f^* commutes with tensor products up to natural isomorphism, resolving by a K-flat, tensoring, and then pulling back agrees with pulling back, resolving and then tensoring (up to natural isomorphism). So the square is commutative as claimed. □

This is the diagram of Proposition 2.5.5, so it follows that the action $\odot_U$ of said proposition is precisely our old friend $\odot$. Thus we can use the machinery we have developed to obtain a classification of the localizing $D(X)$-submodules of $S(X)$ when X is locally a hypersurface.

Lemma 5.2.4. *There is an equality $\sigma S(X) = \operatorname{Sing} X$ i.e., for any $x \in X$ the subcategory $\Gamma_x S(X)$ is non-trivial if and only if $x \in \operatorname{Sing} X$.*

Proof. Let $\cup_{i=1}^n U_i$ be an open affine cover of X. By Remark 2.5.7 the subset $\sigma S(X)$ is the union of the $\sigma S(U_i)$. Thus it is sufficient to note that $x \in U_i$ lies in $\operatorname{Sing} X$ if and only if it lies in $\operatorname{Sing} U_i$ and invoke Proposition 3.4.6 which tells us that $\sigma S(U_i) = \operatorname{Sing} U_i$. □

Proposition 5.2.5. *If X is a Gorenstein separated scheme then every compact object of $S(X)$ has closed support.*

Proof. We proved that for any open affine U_i the compact objects of $S(U_i)$ have closed support in Proposition 3.4.13. The result then follows by covering X by open affines and applying Lemma 2.5.8. □

Remark 5.2.6. It follows that the support of any triangulated subcategory generated by compact objects of $S(X)$ is a specialization closed subset of $\operatorname{Sing} X$.

We are now ready to state our first theorem concerning the singularity categories of schemes with hypersurface singularities.

Theorem 5.2.7. *Suppose X is a noetherian separated scheme with only hypersurface singularities. Then there is an order preserving bijection*

$$\left\{ \textit{subsets of } \operatorname{Sing} X \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \textit{localizing submodules of } S(X) \right\}$$

given by the assignments discussed before Lemma 5.2.3. This restricts to the equivalent bijections

$$\left\{ \begin{array}{c} \textit{specialization closed} \\ \textit{subsets of } \operatorname{Sing} X \end{array} \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{submodules of } S(X) \textit{ generated} \\ \textit{by objects of } S(X)^c \end{array} \right\}$$

and

$$\left\{ \begin{array}{c} \textit{specialization closed} \\ \textit{subsets of } \operatorname{Sing} X \end{array} \right\} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \left\{ \textit{thick } D^{\mathrm{perf}}(X)\textit{-submodules of } D_{\mathrm{Sg}}(X) \right\}.$$

Proof. The first bijection is an application of Theorem 4.2.13 and Theorem 2.5.11 to an open affine cover of X together with the observation of Lemma 5.2.4 that $\sigma S(X) = \operatorname{Sing} X$. To see that the first bijection restricts to the second recall from Proposition 5.2.5 that compact objects of $S(X)$ have specialization closed support. The statement now follows immediately from what we have already proved and using [48] Theorem 1.1 (this is the general form of Theorem 3.1.1, which as we remarked also applies *mutatis mutandis* to noetherian separated schemes) it is easily deduced that the second and third bijections are equivalent. □

It is natural to ask when one can strengthen this result to a complete classification of the localizing subcategories of $S(X)$. We now prove that if X is a hyperplane section of a regular scheme then every localizing subcategory of $S(X)$ is closed under the action of $D(X)$. This gives a complete description of the lattice of localizing subcategories of $S(X)$ for such schemes.

Let T be a regular separated noetherian scheme of finite Krull dimension and let L be an ample line bundle on T. Suppose $t \in H^0(T, L)$ is a section giving rise to an exact sequence

$$0 \longrightarrow L^{-1} \xrightarrow{t^\vee} \mathcal{O}_T \longrightarrow \mathcal{O}_X \longrightarrow 0$$

which defines a hypersurface $X \xrightarrow{i} T$. The scheme X is a noetherian separated scheme with hypersurface singularities so our theorem applies to classify localizing $D(X)$-submodules of $S(X)$. The key observation in strengthening this result is the following easy computation.

Lemma 5.2.8. *Let $F \in D(X)$ be a quasi-coherent sheaf concentrated in degree zero. There is an isomorphism in $S(X)$*

$$I_\lambda Q_\rho(F \otimes i^* L^{-1}) \cong \Sigma^{-2} I_\lambda Q_\rho F.$$

Proof. By the way we have defined X the coherent $\mathcal{O}_T$-module $\mathcal{O}_X$ comes with a flat resolution

$$0 \longrightarrow L^{-1} \xrightarrow{t^\vee} \mathcal{O}_T \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Thus the complex $\mathbf{L}i^* i_* F$ has two non-zero cohomology groups namely

$$H^0(\mathbf{L}i^* i_* F) \cong F \quad \text{and} \quad H^{-1}(\mathbf{L}i^* i_* F) \cong F \otimes_{\mathcal{O}_X} i^* L^{-1}.$$

As the scheme T is regular of finite Krull dimension the object i_*F of $D(T)$ is locally isomorphic to a bounded complex of projectives. Hence $\mathbf{L}i^*i_*F$ is also locally isomorphic to a bounded complex of projectives. In particular, since being the zero object is local in $S(X)$ by Lemma 5.2.2 and the local-to-global principle, we have $I_\lambda Q_\rho \mathbf{L}i^*i_*F \cong 0$. The standard t-structure on $D(X)$ gives a triangle

$$\Sigma F \otimes_{\mathcal{O}_X} i^*L^{-1} \longrightarrow \mathbf{L}i^*i_*F \longrightarrow F \longrightarrow \Sigma^2 F \otimes_{\mathcal{O}_X} i^*L^{-1}.$$

Thus applying $I_\lambda Q_\rho$ to this triangle yields an isomorphism

$$I_\lambda Q_\rho F \cong I_\lambda Q_\rho \Sigma^2 F \otimes_{\mathcal{O}_X} i^*L^{-1}$$

in $S(X)$ i.e., $I_\lambda Q_\rho(F \otimes_{\mathcal{O}_X} i^*L^{-1}) \cong \Sigma^{-2} I_\lambda Q_\rho F$. □

Let us write i^*L^n for the tensor product of n copies of i^*L. By Proposition 3.4.3 and Lemma 3.4.4 twisting by i^*L^n and applying $I_\lambda Q_\rho$ to a sheaf F commute up to natural isomorphism. We thus have isomorphisms

$$i^*L^n \odot I_\lambda Q_\rho F \cong I_\lambda Q_\rho(F \otimes_{\mathcal{O}_X} i^*L^n) \cong \Sigma^{2n} I_\lambda Q_\rho F$$

in $S(X)$.

Corollary 5.2.9. *Let X be as above. Then there are order preserving bijections*

$$\left\{ \textit{subsets of } \operatorname{Sing} X \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \textit{localizing subcategories of } S(X) \right\}$$

and

$$\left\{ \begin{array}{c} \textit{specialization closed} \\ \textit{subsets of } \operatorname{Sing} X \end{array} \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{localizing subcategories} \\ \textit{of } S(X) \textit{ generated by} \\ \textit{objects of } S(X)^c \end{array} \right\}.$$

Proof. As X is a locally complete intersection in the regular scheme T it is certainly Gorenstein. In particular it has a dualising complex so by [58] (Proposition 6.1 and Theorem 4.31) every complex in $S(X)$ is totally acyclic. Thus [48] Proposition 7.13 applies telling us that every object of $S(X)$ is the image, under $I_\lambda Q_\rho$, of a Gorenstein injective sheaf on X.

Let $\mathcal{L} \subseteq S(X)$ be a localizing subcategory and suppose A is an object of $\mathcal{L}$. Then there exists a Gorenstein injective sheaf G such that $A \cong I_\lambda Q_\rho G$ by the discussion above. There are isomorphisms

$$\begin{aligned} \Sigma^m i^*L^n \odot A \cong \Sigma^m i^*L^n \odot I_\lambda Q_\rho G &\cong \Sigma^m I_\lambda Q_\rho(G \otimes i^*L^n) \\ &\cong \Sigma^{m+2n} I_\lambda Q_\rho G \\ &\cong \Sigma^{m+2n} A \end{aligned}$$

where we can interchange the action of i^*L^n and $I_\lambda Q_\rho$ as in the discussion before the proposition.

As L is ample on T the line bundle i^*L is ample on X so the set of objects

$$\{\Sigma^m i^* L^n \mid m, n \in \mathbb{Z}\}$$

is a compact generating set for $D(X)$, see for example 1.10 of [60]. We have just seen $\mathcal{L}$ is stable under the action of each of the generators. Thus the full subcategory of $D(X)$ consisting of objects whose action sends $\mathcal{L}$ to itself is localizing, as $\mathcal{L}$ is localizing, and contains a generating set so must be all of $D(X)$. This proves $\mathcal{L}$ is a submodule as claimed. □

Remark 5.2.10. The action of i^*L can be viewed in the context of the degree 2 periodicity operator of Gulliksen [35] (see also [32] and [5]). As i^*L is invertible in $D(X)$ one can consider, as in [9], the graded commutative ring

$$E^*_{i^*L} = \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}(\mathcal{O}_X, i^*L^j)$$

with multiplication defined by sending $(\mathcal{O}_X \longrightarrow i^*L^j, \mathcal{O}_X \longrightarrow i^*L^k)$ to the composite

$$\begin{array}{ccccc} \mathcal{O}_X & \longrightarrow & & \longrightarrow & i^*L^{j+k} \\ \downarrow & & & & \uparrow \wr \\ i^*L^j & \xrightarrow{\sim} & i^*L^j \otimes \mathcal{O}_X & \longrightarrow & i^*L^j \otimes i^*L^k. \end{array}$$

In analogy with Lemma 2.1.7 the degree j elements of the ring $E^*_{i^*L}$ act on $S(X)$ by natural transformations $\mathrm{id}_{S(X)} \longrightarrow i^*L^j \otimes (-)$. In particular, in the above situation Lemma 5.2.8 implies that $E^*_{i^*L}$ acts via the even part of the central ring $\mathcal{Z}(S(X))$ (see Definition 2.1.5).

To end the section we show that our relative version of the telescope conjecture (Definition 2.4.1) holds for the action of $D(X)$ on $S(X)$ when X is any separated noetherian scheme with hypersurface singularities.

Lemma 5.2.11. *Let X be a Gorenstein separated scheme. For any irreducible closed subset $\mathcal{V} \subseteq \operatorname{Sing} X$ there exists a compact object of $S(X)^c$ whose support is precisely $\mathcal{V}$, namely $I_\lambda Q_\rho \mathcal{O}_\mathcal{V}$ where $\mathcal{O}_\mathcal{V}$ is the structure sheaf associated to the reduced induced structure on $\mathcal{V}$.*

Proof. Let $\mathcal{V}$ be an irreducible closed subset of $\operatorname{Sing} X$ as in the statement. We have claimed the object $I_\lambda Q_\rho \mathcal{O}_\mathcal{V}$ of $S(X)^c$ has the desired support. To see this let

X be covered by open affine subschemes $\{U_i\}_{i=1}^n$ where $U_i \cong \operatorname{Spec} R_i$. The restriction $\mathcal{O}_{\mathcal{V}_i}$ of $\mathcal{O}_{\mathcal{V}}$ to U_i is the sheaf associated to $R/\mathfrak{p}_i$ where $\mathcal{V}(\mathfrak{p}_i) = \mathcal{V}_i = \mathcal{V} \cap U_i$. By Remark 2.5.9

$$\begin{aligned}\operatorname{supp} I_\lambda Q_\rho \mathcal{O}_{\mathcal{V}} &= \bigcup_{i=1}^n \operatorname{supp} I_\lambda Q_\rho \mathcal{O}_{\mathcal{V}_i} \\ &= \bigcup_{i=1}^n \operatorname{supp} I_\lambda Q_\rho R/\mathfrak{p}_i \\ &= \bigcup_{i=1}^n \mathcal{V}_i \\ &= \mathcal{V}\end{aligned}$$

where the second last equality comes from Remark 3.4.14. □

Theorem 5.2.12. *Let X be a noetherian separated scheme with hypersurface singularities. Then the action of $D(X)$ on the singularity category $S(X)$ satisfies the relative telescope conjecture i.e., every smashing $D(X)$-submodule of $S(X)$ is generated by objects of $S(X)^c$. In particular, if X is a hypersurface defined by a section of an ample line bundle on some ambient regular separated noetherian scheme T as above then $S(X)$ satisfies the usual telescope conjecture.*

Proof. The result is an application of Theorem 2.4.14. We have seen in Theorem 5.2.7 that $D(X)$-submodules are classified by $\operatorname{Sing} X$ via the assignments σ and τ. By Proposition 5.2.5 compact objects of $S(X)$ have specialization closed support. Finally, we have proved in the last lemma that every irreducible closed subset of $\operatorname{Sing} X$ can be realised as the support of a compact object.

Thus the conditions of Theorem 2.4.14 are met for the action of $D(X)$ on $S(X)$ and it follows that the relative telescope conjecture holds. In the case Corollary 5.2.9 applies this reduces to the usual telescope conjecture. □

5.3 A General Classification Theorem

We are now ready to prove a version of Theorem 5.2.7 valid in higher codimension. Our strategy is to reduce to the hypersurface case so we may deduce the result from what we have already proved. Let us begin by fixing some terminology and notation for the setup we will be considering following Section 2 of [64].

Throughout this section by a complete intersection ring we mean a ring R such that there is a regular ring Q and a surjection $Q \longrightarrow R$ with kernel generated by a regular sequence. A locally complete intersection scheme X is a closed subscheme of a regular scheme such that the corresponding sheaf of ideals is locally generated by a regular sequence. All schemes considered from this point onward are over some fixed base field and are assumed to have enough locally free sheaves. Let T be a separated regular noetherian scheme of finite Krull dimension and $\mathcal{E}$ a vector bundle on T of rank c. For a section $t \in H^0(T, \mathcal{E})$ we denote by $Z(t)$ the *zero scheme* of t. We recall that $Z(t)$ can be defined globally by the exact sequence

$$\mathcal{E}^\vee \xrightarrow{t^\vee} \mathcal{O}_T \longrightarrow \mathcal{O}_{Z(t)} \longrightarrow 0.$$

It can also be defined locally by taking a cover $X = \cup_i U_i$ trivializing $\mathcal{E}$ via $f_i \colon \mathcal{E}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}^{\oplus c}$ and defining an ideal sheaf $\mathscr{I}(s)$ by

$$\mathscr{I}(t)|_{U_i} = (f_i(t)_1, \ldots, f_i(t)_c).$$

We say that the section t is *regular* if the ideal sheaf $\mathscr{I}(t)$ is locally generated by a regular sequence. Thus the zero scheme $Z(t)$ of a regular section t is a locally complete intersection in T of codimension c. In our situation t is regular if and only if $\operatorname{codim} Z(t) = \operatorname{rk} \mathcal{E} = c$ (cf. [52] 16.B).

Let T and $\mathcal{E}$ be as above and let $t \in H^0(T, \mathcal{E})$ be a regular section with zero scheme X. Denote by $\mathcal{N}_{X/T}$ the normal bundle of X in T. There are projective bundles $\mathbb{P}(\mathcal{E}^\vee) = T'$ and $\mathbb{P}(\mathcal{N}_{X/T}^\vee) = Z$ with projections which we denote q and p respectively. Associated to these projective bundles are canonical line bundles $\mathcal{O}_{\mathcal{E}}(1)$ and $\mathcal{O}_{\mathcal{N}}(1)$ together with canonical surjections

$$q^*\mathcal{E} \longrightarrow \mathcal{O}_{\mathcal{E}}(1) \quad \text{and} \quad p^*\mathcal{N}_{X/S} \longrightarrow \mathcal{O}_{\mathcal{N}}(1).$$

The section t induces a section $t' \in H^0(T', \mathcal{O}_{\mathcal{E}}(1))$ and we denote its divisor of zeroes by Y. The natural closed immersion $Z \longrightarrow T'$ factors via Y. To summarize there is a commutative diagram

$$\begin{array}{ccccc} Z = \mathbb{P}(\mathcal{N}_{X/T}^\vee) & \xrightarrow{i} & Y & \xrightarrow{u} & \mathbb{P}(\mathcal{E}^\vee) = T' \\ \downarrow{\scriptstyle p} & & & \searrow{\scriptstyle \pi} & \downarrow{\scriptstyle q} \\ X & & \xrightarrow[j]{} & & T. \end{array} \tag{5.1}$$

This gives rise to functors $Si_* \colon S(Z) \longrightarrow S(Y)$ and $Sp^* \colon S(X) \longrightarrow S(Z)$ by [48] Theorem 1.5 and Theorem 6.6 respectively. Orlov proves the following theorem in Section 2 of [64]:

Theorem 5.3.1. *Let T, T', X, and Y be as above. Then the functor*

$$\Phi_Z := i_* p^* \colon D^b(\operatorname{Coh} X) \longrightarrow D^b(\operatorname{Coh} Y)$$

induces an equivalence of triangulated categories

$$\overline{\Phi}_Z \colon D_{\mathrm{Sg}}(X) \longrightarrow D_{\mathrm{Sg}}(Y).$$

Remark 5.3.2. Recently Dima Arinkin has used Orlov's theorem to define a notion of support for objects of $D_{\mathrm{Sg}}(X)$. His definition agrees with the one obtained by allowing $D^{\mathrm{perf}}(Y)$ to act on $D_{\mathrm{Sg}}(X)$ via the above equivalence (using a notion of support as in Balmer's [7]) and the one we will construct below.

We wish to show this equivalence extends to the infinite completions $S(X)$ and $S(Y)$; it is natural to ask if the theorem extends and considering the larger categories allows us to bring the machinery we have developed to bear. In order to show the equivalence extends we demonstrate that it is compatible with the functor Si_*Sp^*, induced by i and p as in Section 6 of [48], via $I_\lambda Q_\rho$. General nonsense about triangulated categories then implies Si_*Sp^* must also be an equivalence.

Notation 5.3.3. We will frequently be concerned below with commuting diagrams involving the functors of the general version of Theorem 3.1.1 ([48] Theorem 1.1) for pairs of schemes. As in [48] we will tend not to clutter the notation by indicating which scheme the various functors correspond to as it is always identifiable from the context.

Lemma 5.3.4. *Let $i \colon Z \longrightarrow Y$ be a regular closed immersion i.e., the ideal sheaf on Y defining Z is locally generated by a regular sequence, where Z and Y are noetherian separated schemes. Then the functor*

$$\hat{R}i_* \colon K(\operatorname{Inj} Z) \longrightarrow K(\operatorname{Inj} Y)$$

of [48] Theorem 1.4 has a coproduct preserving right adjoint $K(i^!)$ and sends compact objects to compact objects.

Proof. Since i is a closed immersion we have an adjunction at the level of categories of quasi-coherent sheaves

$$\operatorname{QCoh} Z \underset{i^!}{\overset{i_*}{\rightleftarrows}} \operatorname{QCoh} Y.$$

The right adjoint $i^!$ sends injectives to injectives as i_* is exact.

These functors give an adjunction

$$K(\mathrm{QCoh}\, Z) \underset{K(i^!)}{\overset{K(i_*)}{\rightleftarrows}} K(\mathrm{QCoh}\, Y)$$

and $K(i^!)$ restricts to a functor from $K(\mathrm{Inj}\, Y) \longrightarrow K(\mathrm{Inj}\, Z)$. We claim that this restricted functor is the right adjoint of $\hat{R}i_*$. Recall that $\hat{R}i_*$ is defined by the composite

$$K(\mathrm{Inj}\, Z) \xrightarrow{J} K(\mathrm{QCoh}\, Z) \xrightarrow{K(i_*)} K(\mathrm{QCoh}\, Y) \xrightarrow{J_\lambda} K(\mathrm{Inj}\, Y)$$

where J is the inclusion and J_λ is left adjoint to the corresponding inclusion for Y. For $A \in K(\mathrm{Inj}\, Z)$ and $B \in K(\mathrm{Inj}\, Y)$ there are isomorphisms

$$\begin{aligned} \mathrm{Hom}(\hat{R}i_* A, B) &= \mathrm{Hom}(J_\lambda K(i_*) JA, B) \\ &\cong \mathrm{Hom}(JA, K(i^!) JB) \\ &\cong \mathrm{Hom}(JA, JK(i^!) B) \\ &\cong \mathrm{Hom}(A, K(i^!) B) \end{aligned}$$

the first equality by definition, the third isomorphism $JK(i^!) \cong K(i^!)J$ as $K(i^!)$ sends complexes of injectives to complexes of injectives, and the fourth isomorphism as J is fully faithful. This proves that the right adjoint to $\hat{R}i_*$ is induced by $K(i^!)$ as claimed.

To complete the proof note that $i^!$ preserves coproducts. The functor $K(i^!)$ and hence the right adjoint of $\hat{R}i_*$ are thus also coproduct preserving. It now follows from [60] Theorem 5.1 that $\hat{R}i_*$ sends compact objects to compact objects. □

Thus from [48], namely the first diagram of Theorem 6.1 and Remark 3.8, we deduce, whenever i is a regular closed immersion, a commutative square

$$\begin{array}{ccc} D^b(\mathrm{Coh}\, Z) & \xrightarrow[\sim]{Q_\rho} & K^c(\mathrm{Inj}\, Z) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle \hat{R}i_*} \\ D^b(\mathrm{Coh}\, Y) & \xrightarrow[Q_\rho]{\sim} & K^c(\mathrm{Inj}\, Y). \end{array} \tag{5.2}$$

Lemma 5.3.5. *Let Z and Y be Gorenstein separated schemes and suppose $i\colon Z \longrightarrow Y$ is a regular closed immersion. Then the functor $K(i^!)$ sends acyclic complexes of injectives to acyclic complexes of injectives.*

Proof. As i is a regular closed immersion i_* sends perfect complexes to perfect complexes. Thus $\mathbf{R}i^!\colon D(Y) \longrightarrow D(Z)$ preserves coproducts by [60] Theorem 5.1 so is isomorphic to $\mathbf{L}i^*(-) \otimes \mathbf{R}i^!\mathcal{O}_Y$ by *ibid.* Theorem 5.4. The scheme Y is Gorenstein so $\mathbf{R}i^!\mathcal{O}_Y$ is a dualizing complex on Z. As Z is also Gorenstein the dualizing complex $\mathbf{R}i^!\mathcal{O}_Y$ is (at least on each connected component) a suspension of an invertible sheaf. Thus we can choose $n \in \mathbb{Z}$ so that $H^j(\mathbf{R}i^!F) = 0$ for every quasi-coherent sheaf F on Y and $j > n$ as $\mathbf{L}i^*(F)$ is always bounded above.

If A is an acyclic complex of injectives then the truncation

$$0 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow A^2 \longrightarrow \cdots$$

is an injective resolution of $B = \ker(A^0 \longrightarrow A^1)$. Thus applying $K(i^!)$ to this truncation computes $\mathbf{R}i^!B$ so the resulting complex is acyclic above degree n. By taking suspensions we deduce that $K(i^!)A$ is in fact acyclic everywhere and we have already noted that $i^!$ preserves injectivity as it has an exact left adjoint. $\square$

Remark 5.3.6. As the notation in the last two lemmas indicates they apply to the situation we are interested in, namely the one given at the start of the section: the morphism $i\colon Z \longrightarrow Y$ is a regular closed immersion. Let us indicate why this is the case. Pick some open affine subscheme $\operatorname{Spec} Q$ of T, with preimage in X isomorphic to $\operatorname{Spec} R$, on which $\mathcal{E}$ is trivial and such that the kernel of $Q \longrightarrow R$ is generated by the regular sequence $\{q_1, \dots, q_c\}$. We get a diagram of open subschemes of the diagram (5.1)

$$\begin{array}{ccccc} \mathbb{P}^{c-1}_R & \xrightarrow{i} & Y' & \xrightarrow{u} & \mathbb{P}^{c-1}_Q \\ {\scriptstyle p}\downarrow & & & \searrow^{\pi} & \downarrow{\scriptstyle q} \\ \operatorname{Spec} R & & \xrightarrow[j]{} & & \operatorname{Spec} Q. \end{array}$$

The hypersurface Y' is defined by the section $t' = \sum_{i=1}^{c} q_i x_i$ of $\mathcal{O}_{\mathbb{P}^{c-1}_Q}(1)$, where the x_i are a basis for the global sections of $\mathcal{O}_{\mathbb{P}^{c-1}_Q}$. Let z be a point in the cth standard open affine $\mathbb{A}^{c-1}_R$ in $\mathbb{P}^{c-1}_R$ (we choose this open affine for ease of notation, little changes if z lies in another standard open affine) and consider the local maps of local rings

$$\mathcal{O}_{T',ui(z)} \xrightarrow{\alpha} \mathcal{O}_{Y,i(z)} \xrightarrow{\beta} \mathcal{O}_{Z,z}.$$

We wish to show that $\ker\beta$ is generated by a regular sequence. Note that both α and $\beta\alpha$ have kernels generated by regular sequences: the kernel of α is generated by the image of $s = q_1x_1 + \cdots + q_{c-1}x_{c-1} + q_c$ in $\mathcal{O}_{T',ui(z)}$ and the kernel of $\beta\alpha$ is generated by the image of the regular sequence $\{q_1, \dots, q_c\}$.

It is clear that the image of $\{q_1, \ldots, q_{c-1}, s\}$ is a regular sequence in $\mathcal{O}_{T',ui(z)}$ and as this ring is local and noetherian we may permute the order of the elements in this sequence and it remains regular by [53] Theorem 16.3. Thus $\{s, q_1, \ldots, q_{c-1}\}$ is a regular sequence in $\mathcal{O}_{T',ui(z)}$. It follows that the image of $\{q_1, \ldots, q_{c-1}\}$ is a regular sequence in $\mathcal{O}_{Y,i(z)}$ and it generates the kernel of β. Thus i is a regular closed immersion as claimed.

So we have an adjoint pair of functors

$$K(\operatorname{Inj} Z) \underset{K(i^!)}{\overset{\hat{R}i_*}{\rightleftarrows}} K(\operatorname{Inj} Y)$$

which both send acyclic complexes to acyclic complexes: $\hat{R}i_*$ by Theorem 1.5 of [48] and $K(i^!)$ by Lemma 5.3.5. Thus they restrict to an adjoint pair

$$S(Z) \underset{Si^!}{\overset{Si_*}{\rightleftarrows}} S(Y).$$

So we have a commutative square

$$\begin{array}{ccc} S(Y) & \xrightarrow{I} & K(\operatorname{Inj} Y) \\ {\scriptstyle Si^!}\downarrow & & \downarrow{\scriptstyle K(i^!)} \\ S(Z) & \xrightarrow[I]{} & K(\operatorname{Inj} Z). \end{array}$$

Taking left adjoints of the functors in this last square we get another commutative diagram

$$\begin{array}{ccc} K(\operatorname{Inj} Z) & \xrightarrow{I_\lambda} & S(Z) \\ {\scriptstyle \hat{R}i_*}\downarrow & & \downarrow{\scriptstyle Si_*} \\ K(\operatorname{Inj} Y) & \xrightarrow[I_\lambda]{} & S(Y). \end{array}$$

By Lemma 5.3.4 the composite $I_\lambda \hat{R}i_*$ sends compact objects to compact objects. As I_λ sends compacts to compacts and is essentially surjective, up to summands, on compacts we see that Si_* must preserve compacts too. So restricting this square to compact objects and juxtaposing with the square (5.2) we get a commutative diagram

$$\begin{array}{ccccc} D^b(\operatorname{Coh} Z) & \xrightarrow[\sim]{Q_\rho} & K^c(\operatorname{Inj} Z) & \xrightarrow{I_\lambda} & S^c(Z) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle \hat{R}i_*} & & \downarrow{\scriptstyle Si_*} \\ D^b(\operatorname{Coh} Z) & \xrightarrow[Q_\rho]{\sim} & K^c(\operatorname{Inj} Y) & \xrightarrow[I_\lambda]{} & S^c(Y). \end{array}$$

In particular, the functor $\bar{i}_*\colon D_{\mathrm{Sg}}(Z) \longrightarrow D_{\mathrm{Sg}}(Y)$ induced by i is compatible with Si_* under the embeddings of $D_{\mathrm{Sg}}(Z)$ and $D_{\mathrm{Sg}}(Y)$ as the compact objects in $S(Z)$ and $S(Y)$.

Proposition 5.3.7. *There is an equivalence of triangulated categories*

$$Si_*Sp^*\colon S(X) \longrightarrow S(Y)$$

which when restricted to compact objects is Orlov's equivalence.

Proof. We have just seen that the square

$$\begin{array}{ccc} D_{\mathrm{Sg}}(Z) & \longrightarrow & S(Z) \\ {\scriptstyle \bar{i}_*}\downarrow & & \downarrow{\scriptstyle Si_*} \\ D_{\mathrm{Sg}}(Y) & \longrightarrow & S(Y) \end{array}$$

commutes. By [48] Theorem 6.6 the square

$$\begin{array}{ccc} D_{\mathrm{Sg}}(X) & \longrightarrow & S(X) \\ {\scriptstyle \bar{p}^*}\downarrow & & \downarrow{\scriptstyle Sp^*} \\ D_{\mathrm{Sg}}(Z) & \longrightarrow & S(Z) \end{array}$$

commutes. Putting this second square on top of the first the equivalence $\overline{\Phi}_Z$ fits into a commutative diagram

$$\begin{array}{ccc} D_{\mathrm{Sg}}(X) & \longrightarrow & S(X) \\ {\scriptstyle \overline{\Phi}_Z}\downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle Si_*Sp^*} \\ D_{\mathrm{Sg}}(Y) & \longrightarrow & S(Y). \end{array}$$

Hence Si_*Sp^* is a coproduct preserving exact functor between compactly generated triangulated categories inducing an equivalence on compact objects. It follows from abstract nonsense that it must be an equivalence. □

We have thus reduced the problem of understanding $S(X)$ to that of understanding $S(Y)$. The scheme Y is locally a hypersurface as it is a locally complete intersection in the regular scheme T' and has codimension 1. Theorem 5.2.7 thus applies and we have the following theorem, where we use the notation introduced at the beginning of the section.

Theorem 5.3.8. *The category $D(Y)$ acts on $S(X)$ via the equivalence $S(X) \cong S(Y)$ giving order preserving bijections*

$$\left\{ \textit{subsets of } \operatorname{Sing} Y \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \textit{localizing } D(Y)\textit{-submodules of } S(X) \right\}$$

and

$$\left\{ \begin{array}{c} \textit{specialization closed} \\ \textit{subsets of } \operatorname{Sing} Y \end{array} \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{localizing } D(Y)\textit{-submodules} \\ \textit{of } S(X) \textit{ generated by} \\ \textit{objects of } S(X)^c \end{array} \right\}.$$

Furthermore if the line bundle $\mathcal{O}_{\mathcal{E}}(1)$ is ample, for example if S is affine, then every localizing subcategory of $S(X)$ is a $D(Y)$-submodule so one obtains a complete classification of the localizing subcategories of $S(X)$.

Proof. Let us denote the equivalence $S(X) \xrightarrow{\sim} S(Y)$ by Ψ. We define an action of $D(Y)$ on $S(X)$ by setting, for $E \in D(Y)$ and $A \in S(X)$

$$E \square A = \Psi^{-1}(E \odot \Psi A).$$

It is easily checked that this is in fact an action.

The equivalence Ψ sends localizing subcategories (generated by objects of $S(X)^c$) to localizing subcategories (generated by objects of $S(Y)^c$). A localizing subcategory $\mathcal{L} \subseteq S(X)$ is a $D(Y)$-submodule if and only if for every $E \in D(Y)$

$$E \square \mathcal{L} = \Psi^{-1}(E \odot \Psi \mathcal{L}) \subseteq \mathcal{L}$$

if and only if $E \odot \Psi\mathcal{L} \subseteq \Psi\mathcal{L}$. In other words $\mathcal{L}$ is a $D(Y)$-submodule if and only if $\Psi\mathcal{L}$ is a $D(Y)$-submodule. Thus the theorem follows from Theorem 5.2.7 as Y is locally a hypersurface.

The last statement is a consequence of Corollary 5.2.9. □

Corollary 5.3.9. *The relative telescope conjecture holds for the action of $D(Y)$ on $S(X)$. In particular if $\mathcal{O}_{\mathcal{E}}(1)$ is ample then the usual telescope conjecture holds for $S(X)$.*

Proof. This is immediate from the corresponding statements for the action of $D(Y)$ on $S(Y)$ given in Theorem 5.2.12. □

5.3.1 Embedding Independence

To prove Theorem 5.3.8 we have relied on the choice of some ambient scheme T, vector bundle $\mathcal{E}$, and a regular section t of $\mathcal{E}$. Thus it is not clear that the support theory one produces, via the hypersurface Y associated to this data, is independent of the choices we have made. We now show this is in fact the case: the choices one makes do not matter as far as the support theory is concerned.

The setup will be exactly the same as previously, except we will have two possibly different regular noetherian separated schemes of finite Krull dimension T_1 and T_2 each carrying a vector bundle $\mathcal{E}_i$ with a regular section t_i for $i = 1, 2$ such that

$$Z(t_1) \cong X \cong Z(t_2).$$

Thus there are, by Proposition 5.3.7, two equivalences

$$\Psi_1 \colon S(X) \longrightarrow S(Y_1) \quad \text{and} \quad \Psi_2 \colon S(X) \longrightarrow S(Y_2)$$

giving rise to a third equivalence $S(Y_1) \xrightarrow{\sim} S(Y_2)$ which we shall denote by Θ.

We first treat the case in which both $\mathcal{O}_{\mathcal{E}_1}(1)$ and $\mathcal{O}_{\mathcal{E}_2}(1)$ are ample.

Lemma 5.3.10. *Suppose $\mathcal{O}_{\mathcal{E}_i}(1)$ is ample for $i = 1, 2$. Then there is a homeomorphism*

$$\theta \colon \operatorname{Sing} Y_1 \longrightarrow \operatorname{Sing} Y_2$$

such that for any A in $S(Y_1)$ we have

$$\theta \operatorname{supp} A = \operatorname{supp} \Theta A.$$

In particular the two support theories for $S(X)$ obtained via the actions of $D(Y_1)$ and $D(Y_2)$ coincide up to this homeomorphism.

Proof. We first define θ and show it is a bijection. Let y be a point of $\operatorname{Sing} Y_1$. By Theorem 5.3.8 the subcategory $\Gamma_y S(Y_1)$ is a minimal localizing subcategory. Thus its essential image $\Theta\Gamma_y S(Y_1)$ is a minimal localizing subcategory of $S(Y_2)$. So by Corollary 5.2.9 the subcategory $\Theta\Gamma_y S(Y_1)$ is necessarily of the form $\Gamma_{\theta(y)} S(Y_2)$. This defines a function $\theta \colon \operatorname{Sing} Y_1 \longrightarrow \operatorname{Sing} Y_2$ which is a bijection as Θ is an equivalence.

Let us now show that θ is compatible with supports. If A is an object of $S(Y_1)$ then by Corollary 5.2.9 and Theorem 2.3.9 we have

$$\langle A \rangle_{\mathrm{loc}} = \langle \Gamma_y S(Y_1) \mid y \in \operatorname{supp} A \rangle_{\mathrm{loc}}.$$

Applying Θ gives two sets of equalities, namely

$$\Theta\langle A\rangle_{\mathrm{loc}} = \langle \Theta A\rangle_{\mathrm{loc}} = \langle \Gamma_w S(Y_2) \mid w \in \operatorname{supp}\Theta A\rangle_{\mathrm{loc}}$$

and

$$\begin{aligned}\Theta\langle A\rangle_{\mathrm{loc}} &= \Theta\langle \Gamma_y S(Y_1) \mid y \in \operatorname{supp} A\rangle_{\mathrm{loc}} \\ &= \langle \Gamma_{\theta(y)} S(Y_2) \mid y \in \operatorname{supp} A\rangle_{\mathrm{loc}}.\end{aligned}$$

We thus obtain $\theta \operatorname{supp} A = \operatorname{supp}\Theta A$ which shows that θ respects the support.

Finally, let us show that θ is a homeomorphism. Let $\mathcal{V}$ be a closed subset of $\operatorname{Sing} Y_1$. Then it follows from Lemma 5.2.11 that there exists a compact object c in $S(Y_1)$ whose support is $\mathcal{V}$. Hence

$$\theta\mathcal{V} = \theta \operatorname{supp} c = \operatorname{supp}\Theta c$$

is closed by Proposition 5.2.5 as Θ is an equivalence and so preserves compactness. The whole argument works just as well reversing the roles of Y_1 and Y_2 so θ^{-1} is also closed and thus θ is a homeomorphism. □

By working locally we are now able to extend this to arbitrary X admitting a suitable embedding.

Proposition 5.3.11. *Suppose we are given regular noetherian separated schemes of finite Krull dimension T_1 and T_2 each carrying a vector bundle $\mathcal{E}_i$ with a regular section t_i for $i = 1, 2$ such that*

$$Z(t_1) \cong X \cong Z(t_2).$$

Then there is a homeomorphism $\theta\colon \operatorname{Sing} Y_1 \longrightarrow \operatorname{Sing} Y_2$ satisfying

$$\theta \operatorname{supp} A = \operatorname{supp}\Theta A$$

for any A in $S(Y_1)$. In particular the two support theories for $S(X)$ obtained via the actions of $D(Y_1)$ and $D(Y_2)$ coincide up to this homeomorphism.

Proof. Let $\{W_1^j\}_{j=1}^n$ and $\{W_2^k\}_{k=1}^m$ be open affine covers of T_1 and T_2. Denote by $\{U_1^j\}_{j=1}^n$ and $\{U_2^k\}_{k=1}^m$ the two open affine covers of X obtained by restriction. For any of the open affines W_i^l we may consider $\mathcal{E}_i|_{W_i^l}$ and $t_i|_{W_i^l}$; the zero scheme of $t_i|_{W_i^l}$ is precisely the open subscheme U_i^l so each of the opens in the two covers satisfies the set up for Proposition 5.3.7 to apply. We denote by Y_i^l the associated hypersurface. Furthermore, as W_i^l is affine the canonical line bundle on $\mathbb{P}(\mathcal{E}_i|_{W_i^l}^\vee)$ is ample so Lemma 5.3.10 applies.

Now fix one of the $U_1^j \subseteq X$ and cover it by the open affines $U_{12}^{jk} = U_1^j \cap U_2^k$ for $k = 1, \ldots, m$. There are diagrams

$$\begin{array}{ccc} & & S(U_1^j) \xrightarrow[\sim]{\Psi_1^j} S(Y_1^j) \\ & \nearrow & \\ S(U_{12}^{jk}) & & \\ & \searrow & \\ & & S(U_2^k) \xrightarrow[\sim]{\Psi_2^k} S(Y_2^k) \end{array}$$

where the equivalences are the restrictions of Ψ_1 and Ψ_2 and the diagonal maps are inclusions. We thus get an equivalence

$$\Theta^{jk} \colon \Psi_1^j S(U_{12}^{jk}) \longrightarrow \Psi_2^k S(U_{12}^{jk})$$

restricting Θ, and so as in Lemma 5.3.10 a support preserving homeomorphism

$$\theta^{jk} \colon \operatorname{Sing} Y_1^{jk} \longrightarrow \operatorname{Sing} Y_2^{jk}$$

where Y_1^{jk} is the subset corresponding to $\Psi_1^j S(U_{12}^{jk})$ and Y_2^{jk} corresponds to $\Psi_2^k S(U_{12}^{jk})$.

We have produced support preserving homeomorphisms θ^{jk} for each $j = 1, \ldots, n$ and $k = 1, \ldots, m$ and the Y_i^{jk} cover the singular locus of Y_i for $i = 1, 2$. It just remains to note that these glue to the desired homeomorphism $\operatorname{Sing} Y_1 \longrightarrow \operatorname{Sing} Y_2$; the required compatibility on overlaps is immediate as the θ^{jk} are defined via restrictions of Θ. □

5.3.2 Local Complete Intersection Rings

Let us now restrict our attention to the case of local complete intersection rings over some fixed base field. Theorem 5.3.8 applies in this case and we will explicitly describe the singular locus of the associated hypersurface Y; this can be done working with any choice of embedding as the associated support theory is invariant by the last subsection.

Suppose $(R, \mathfrak{m}, k)$ is a local complete intersection of codimension c i.e., R is the quotient of a regular local ring Q by an ideal generated by a regular sequence and

$$\operatorname{cx}_R k = \dim_k \mathfrak{m}/\mathfrak{m}^2 - \dim R = c.$$

Replacing Q by a quotient if necessary we may assume that the kernel of $Q \longrightarrow R$ is generated by a regular sequence of length precisely c. To see this is the case suppose the kernel is generated by a regular sequence $\{q_1, \ldots, q_r\}$ with $r > c$. Then by considering the effect on the embedding dimension and the dimension of successive quotients by the q_i we see that $r - c$ of the q_i must lie in $\mathfrak{n} \setminus \mathfrak{n}^2$ where $\mathfrak{n}$ is the maximal ideal of Q. By [53] Theorem 16.3 any permutation of the q_i is again a regular sequence so we may rearrange to first take the quotient by the $r - c$ of the q_i not in $\mathfrak{n}^2$. This quotient is again regular, surjects onto R and this surjection has kernel generated by a regular sequence of length c.

Set $X = \operatorname{Spec} R$, $T = \operatorname{Spec} Q$, $\mathcal{E} = \mathcal{O}_T^{\oplus c}$, and $t = (q_1, \ldots, q_c)$ where the q_i are a regular sequence generating the kernel of $Q \longrightarrow R$. Let Y be the hypersurface defined by the section $\Sigma_{i=1}^c q_i x_i$ of $\mathcal{O}_{\mathbb{P}_Q^{c-1}}(1)$ where the x_i are a basis for the free Q-module $H^0(\mathbb{P}_Q^{c-1}, \mathcal{O}_{\mathbb{P}_Q^{c-1}}(1))$. In summary we are concerned with the following commutative diagram

$$\begin{array}{ccccc} \mathbb{P}_R^{c-1} & \xrightarrow{i} & Y & \xrightarrow{u} & \mathbb{P}_Q^{c-1} \\ \downarrow{\scriptstyle p} & & & \searrow^{\pi} & \downarrow{\scriptstyle q} \\ X & & \xrightarrow[j]{} & & T. \end{array}$$

Let us first make the following trivial observation about the singular locus of $\mathbb{P}_R^{c-1}$.

Lemma 5.3.12. *There is an equality*

$$\operatorname{Sing} \mathbb{P}_R^{c-1} = p^{-1} \operatorname{Sing} R.$$

Now we show that the singular locus of Y can not be any bigger than the singular locus of $\mathbb{P}_R^{c-1}$.

Lemma 5.3.13. *The singular locus of Y, $\operatorname{Sing} Y$, is contained in $i(\operatorname{Sing} \mathbb{P}_R^{c-1})$.*

Proof. We first show the singular locus of Y is contained in the image of i. The image of i is precisely $Y \cap q^{-1}X$, so we want to show that away from $q^{-1}X$ the scheme Y is regular. Let $\mathfrak{p} \in T \setminus X$, so the section $t = (q_1, \ldots, q_c)$ is not zero at $k(\mathfrak{p})$. Thus in a neighbourhood of any point of $q^{-1}(\mathfrak{p})$ the section defining $Y \cap q^{-1}(\mathfrak{p})$ is just a linear polynomial with invertible coefficients and so Y is regular along its intersection with $q^{-1}(\mathfrak{p})$. Thus $\operatorname{Sing} Y \subseteq i(\mathbb{P}_R^{c-1})$ as claimed.

Next let us prove that $\operatorname{Sing} Y$ is in fact contained in $i(\operatorname{Sing} \mathbb{P}_R^{c-1})$. Given $z \in \mathbb{P}_R^{c-1}$ such that $i(z) \in \operatorname{Sing} Y$ we need to show $z \in \operatorname{Sing} \mathbb{P}_R^{c-1}$. By Remark 5.3.6 the surjection

$$\mathcal{O}_{Y,i(z)} \longrightarrow \mathcal{O}_{\mathbb{P}_R^{c-1},z}$$

has kernel generated by a regular sequence. Thus Proposition 4.1.16 applies yielding

$$\operatorname{cx}_{\mathcal{O}_{\mathbb{P}^{c-1}_R,z}} k(z) \geq \operatorname{cx}_{\mathcal{O}_{Y,i(z)}} i_*k(z) = \operatorname{cx}_{\mathcal{O}_{Y,i(z)}} k(i(z)) > 0$$

where we have also used Theorem 4.1.15, so $z \in \operatorname{Sing} \mathbb{P}^{c-1}_R$. □

In fact the part of the singular locus of Y corresponding to $\mathfrak{m}$ can not be any smaller than $p^{-1}(\mathfrak{m})$ either.

Lemma 5.3.14. *Every point in $ip^{-1}(\mathfrak{m})$ is contained in* $\operatorname{Sing} Y$.

Proof. By Lemma 5.3.12 every point in $p^{-1}(\mathfrak{m})$ is singular in $\mathbb{P}^{c-1}_R$. Consider for $z \in p^{-1}(\mathfrak{m})$ the local maps

$$\mathcal{O}_{\mathbb{P}^{c-1}_Q,ui(z)} \xrightarrow{\alpha} \mathcal{O}_{Y,i(z)} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^{c-1}_R,z}$$

where the kernel of each of these morphisms and the composite is generated by a regular sequence (see Remark 5.3.6). We have assumed $Q \longrightarrow R$ minimal i.e., the elements q_i occuring in the regular sequence generating the kernel all lie in $\mathfrak{n}^2$ where $\mathfrak{n}$ is the maximal ideal of Q. Thus as z lies over $\mathfrak{m}$ the image of each q_i is in the square of the maximal ideal of $\mathcal{O}_{\mathbb{P}^{c-1}_Q,ui(z)}$.

By passing to a standard open affine in $\mathbb{P}^{c-1}_Q$ containing $ui(z)$ (and reordering the q_i if necessary) we see that the morphism α has kernel generated by the image of $\sum_{i=1}^{c-1} q_i x_i + q_c$ where the x_i are now coordinates on $\mathbb{A}^{c-1}_Q$. As the image of each q_i is in the square of the maximal ideal of $\mathcal{O}_{\mathbb{P}^{c-1}_Q,ui(z)}$ the element $\sum_{i=1}^{c-1} q_i x_i + q_c$ defining $\mathcal{O}_{Y,i(z)}$ must also lie in the square of the maximal ideal. Hence $i(z)$ lies in $\operatorname{Sing} Y$. □

It follows from this that $\operatorname{supp}_{(D(Y),\square)} \Gamma_{\mathfrak{m}} S(R) = \mathbb{P}^{c-1}_k$. By Lemma 3.4.9 the object $I_\lambda Q_\rho k$ generates $\Gamma_{\mathfrak{m}} S(R)$. Thus its image under $\bar{i}_*\bar{p}^*$, which is precisely $I_\lambda Q_\rho$ of the structure sheaf of $ip^{-1}(\mathfrak{m})$ with the reduced induced scheme structure, generates $Si_*Sp^*\Gamma_{\mathfrak{m}} S(R)$. By Lemma 5.2.11 this generating object has support, with respect to the $D(Y)$ action on $S(Y)$, precisely $ip^{-1}(\mathfrak{m})$. Thus, identifying the topological spaces $\mathbb{P}^{c-1}_k$ and $ip^{-1}(\mathfrak{m})$, we see $\Gamma_{\mathfrak{m}} S(R)$ has the claimed support.

We now show the singular locus of Y is composed completely of such projective pieces with dimensions corresponding to the complexities of the residue fields of the points in $\operatorname{Sing} R$.

Proposition 5.3.15. *As a set the singular locus of Y is*

$$\operatorname{Sing} Y \cong \coprod_{\mathfrak{p} \in \operatorname{Sing} R} \mathbb{P}^{c_\mathfrak{p}-1}_{k(\mathfrak{p})}$$

where $c_\mathfrak{p} = \operatorname{cx}_{R_\mathfrak{p}} k(\mathfrak{p})$ is the codimension of $R_\mathfrak{p}$.

Proof. We can write $\operatorname{Sing} Y$, using the classification of Theorem 5.3.8, as

$$\operatorname{Sing} Y \cong \coprod_{\mathfrak{p} \in \operatorname{Sing} R} \operatorname{supp}_{(D(Y),\square)} \Gamma_{\mathfrak{p}} S(R).$$

Again using the classification theorem and the independence results of the previous subsection we may compute the $D(Y)$-support of $\Gamma_{\mathfrak{p}} S(R) = \Gamma_{\mathfrak{p}} S(R_{\mathfrak{p}})$ over $R_{\mathfrak{p}}$. By the discussion before the proposition this is precisely $\mathbb{P}^{c_{\mathfrak{p}}-1}_{k(\mathfrak{p})}$. □

This gives the following refined version of Theorem 5.3.8 for local complete intersection rings.

Corollary 5.3.16. *Suppose $(R, \mathfrak{m}, k)$ is a local complete intersection of finite type over a field. Then there are order preserving bijections*

$$\left\{ \begin{array}{c} \textit{subsets of} \\ \coprod_{\mathfrak{p} \in \operatorname{Sing} R} \mathbb{P}^{c_{\mathfrak{p}}-1}_{k(\mathfrak{p})} \end{array} \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \textit{localizing subcategories of } S(R) \right\}$$

and

$$\left\{ \begin{array}{c} \textit{specialization closed} \\ \textit{subsets of } \operatorname{Sing} Y \end{array} \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{localizing subcategories} \\ \textit{of } S(R) \textit{ generated by} \\ \textit{objects of } S(R)^c \end{array} \right\}.$$

Furthermore the telescope conjecture holds for $S(R)$.

Proof. We apply Theorem 5.3.8 setting $X = \operatorname{Spec} R$, $S = \operatorname{Spec} Q$, $\mathcal{E} = \mathcal{O}_S^{\oplus c}$, and $s = (q_1, \ldots, q_c)$ where the q_i are a regular sequence generating the kernel of $Q \longrightarrow R$. The line bundle $\mathcal{O}_{\mathcal{E}}(1)$ is ample on $\mathbb{P}^{c-1}_Q$ so we obtain a complete classification of the localizing subcategories of $S(R)$ in terms of $\operatorname{Sing} Y$. By Proposition 5.3.15 the singular locus of Y is, as a set, precisely the given disjoint union of projective spaces. The final statement is Corollary 5.3.9. □

Remark 5.3.17. A similar result has been announced by Iyengar [42] for locally complete intersection rings essentially of finite type over a field.

Remark 5.3.18. The support theory obtained here may be compared to results of Avramov and Buchweitz [3]. They consider supports in the cone over the piece of $\operatorname{Sing} Y$ corresponding to the closed point of $\operatorname{Spec} R$ after changing base to the algebraic closure of k. As in their work our support theory has consequences for cohomological vanishing which will be pursued in further work.

Bibliography

[1] L. L. Avramov. Modules with extremal resolutions. *Math. Res. Lett.*, 3(3):319–328, 1996.

[2] L. L. Avramov. Infinite free resolutions. In *Six lectures on commutative algebra*, Mod. Birkhäuser Class., pages 1–118. Birkhäuser Verlag, Basel, 2010.

[3] L. L. Avramov and R.-O. Buchweitz. Support varieties and cohomology over complete intersections. *Invent. Math.*, 142(2):285–318, 2000.

[4] L. L. Avramov, V. N. Gasharov, and I. V. Peeva. Complete intersection dimension. *Inst. Hautes Études Sci. Publ. Math.*, (86):67–114 (1998), 1997.

[5] L. L. Avramov and L.-C. Sun. Cohomology operators defined by a deformation. *J. Algebra*, 204(2):684–710, 1998.

[6] J. Ayoub. The n-motivic t-structure for $n = 0, 1$ and 2. *Adv. Math.*, (to appear).

[7] P. Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005.

[8] P. Balmer. Supports and filtrations in algebraic geometry and modular representation theory. *Amer. J. Math.*, 129(5):1227–1250, 2007.

[9] P. Balmer. Spectra, spectra, spectra - Tensor triangular spectra versus Zariski spectra of endomorphism rings. *Algebraic and Geometric Topology*, 10(3):1521–1563, 2010.

[10] P. Balmer and G. Favi. Gluing techniques in triangular geometry. *Q. J. Math.*, 58(4):415–441, 2007.

[11] P. Balmer and G. Favi. Generalized Rickard idempotents and the telescope conjecture. *http://www.math.ucla.edu/ balmer/research/Pubfile/Idemp.pdf*, 2009.

[12] A. Beligiannis. Relative homological algebra and purity in triangulated categories. *J. Algebra*, 227(1):268–361, 2000.

[13] D. Benson, S. B. Iyengar, and H. Krause. Local cohomology and support for triangulated categories. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(4):573–619, 2008.

[14] D. J. Benson, J. F. Carlson, and J. Rickard. Complexity and varieties for infinitely generated modules. *Math. Proc. Cambridge Philos. Soc.*, 118(2):223–243, 1995.

[15] D. J. Benson, J. F. Carlson, and J. Rickard. Complexity and varieties for infinitely generated modules. II. *Math. Proc. Cambridge Philos. Soc.*, 120(4):597–615, 1996.

[16] D. J. Benson, S. B. Iyengar, and H. Krause. Stratifying modular representations of finite groups. *arXiv:0810.1339*.

[17] D. J. Benson, S. B. Iyengar, and H. Krause. Stratifying triangulated categories. *arXiv:0910.0642*.

[18] M. Bökstedt and A. Neeman. Homotopy limits in triangulated categories. *Compositio Math.*, 86(2):209–234, 1993.

[19] A. Bondal and D. Orlov. Reconstruction of a variety from the derived category and groups of autoequivalences. *Compositio Math.*, 125(3):327–344, 2001.

[20] A. I. Bondal. Representations of associative algebras and coherent sheaves. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(1):25–44, 1989.

[21] A. I. Bondal and D. Orlov. Semiorthogonal decompositions of algebraic varieties. *arXiv:9506012*, 1995.

[22] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[23] A. K. Bousfield. Correction to: "The Boolean algebra of spectra" [Comment. Math. Helv. **54** (1979), no. 3, 368–377;]. *Comment. Math. Helv.*, 58(4):599–600, 1983.

[24] T. Bridgeland. Flops and derived categories. *Invent. Math.*, 147(3):613–632, 2002.

[25] A. B. Buan, H. Krause, and Ø. Solberg. Support varieties: an ideal approach. *Homology, Homotopy Appl.*, 9(1):45–74, 2007.

[26] R.-O. Buchweitz. Maximal Cohen-Macaulay modules and Tate cohomology over Gorenstein rings. *unpublished*, 1987.

[27] J. F. Carlson, P. W. Donovan, and W. W. Wheeler. Complexity and quotient categories for group algebras. *J. Pure Appl. Algebra*, 93(2):147–167, 1994.

[28] B. Conrad. *Grothendieck duality and base change*, volume 1750 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.

[29] I. Dell'Ambrogio. Tensor triangular geometry and KK-theory. *arXiv:1001.2637*, 2010.

[30] E. S. Devinatz, M. J. Hopkins, and J. H. Smith. Nilpotence and stable homotopy theory. I. *Ann. of Math. (2)*, 128(2):207–241, 1988.

[31] A. Dold and D. Puppe. Homologie nicht-additiver Funktoren. Anwendungen. *Ann. Inst. Fourier Grenoble*, 11:201–312, 1961.

[32] D. Eisenbud. Homological algebra on a complete intersection, with an application to group representations. *Trans. Amer. Math. Soc.*, 260(1):35–64, 1980.

[33] E. E. Enochs and O. M. G. Jenda. *Relative homological algebra*, volume 30 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 2000.

[34] J. P. C. Greenlees. Tate cohomology in axiomatic stable homotopy theory. In *Cohomological methods in homotopy theory (Bellaterra, 1998)*, volume 196 of *Progr. Math.*, pages 149–176. Birkhäuser, Basel, 2001.

[35] T. H. Gulliksen. A change of ring theorem with applications to Poincaré series and intersection multiplicity. *Math. Scand.*, 34:167–183, 1974.

[36] T. H. Gulliksen. On the deviations of a local ring. *Math. Scand.*, 47(1):5–20, 1980.

[37] D. Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*, volume 119 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.

[38] R. Hartshorne. *Residues and duality.* Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin, 1966.

[39] M. Hochster. Prime ideal structure in commutative rings. *Trans. Amer. Math. Soc.*, 142:43–60, 1969.

[40] M. Hovey, J. H. Palmieri, and N. P. Strickland. Axiomatic stable homotopy theory. *Mem. Amer. Math. Soc.*, 128(610):x+114, 1997.

[41] B. Huisgen-Zimmermann. Purity, algebraic compactness, direct sum decompositions, and representation type. In *Infinite length modules (Bielefeld, 1998)*, Trends Math., pages 331–367. Birkhäuser, Basel, 2000.

[42] S. Iyengar. Stratifying derived categories associated to finite groups and commutative rings. *Kyoto RIMS Workshop on Algebraic Triangulated Categories and Related Topics*, http://www.mi.s.osakafu-u.ac.jp/ kiriko/seminar/09JulRIMS/lectures/iyengar.pdf.

[43] S. Iyengar and H. Krause. Acyclicity versus total acyclicity for complexes over Noetherian rings. *Doc. Math.*, 11:207–240 (electronic), 2006.

[44] I. Kaplansky. *Commutative rings.* The University of Chicago Press, Chicago, Ill.-London, revised edition, 1974.

[45] B. Keller. A remark on the generalized smashing conjecture. *Manuscripta Math.*, 84(2):193–198, 1994.

[46] H. Krause. Localization theory for triangulated categories. *arXiv:0806.1324*.

[47] H. Krause. Smashing subcategories and the telescope conjecture—an algebraic approach. *Invent. Math.*, 139(1):99–133, 2000.

[48] H. Krause. The stable derived category of a Noetherian scheme. *Compos. Math.*, 141(5):1128–1162, 2005.

[49] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[50] H. R. Margolis. *Spectra and the Steenrod algebra*, volume 29 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1983. Modules over the Steenrod algebra and the stable homotopy category.

[51] E. Matlis. Injective modules over Noetherian rings. *Pacific J. Math.*, 8:511–528, 1958.

[52] H. Matsumura. *Commutative algebra*, volume 56 of *Mathematics Lecture Note Series*. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., second edition, 1980.

[53] H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[54] C. Mazza, V. Voevodsky, and C. Weibel. *Lecture Notes on Motivic Cohomology*. American Mathematical Society, Providence, RI, 2006.

[55] Z. Mebkhout. Sur le problème de Hilbert-Riemann. In *Complex analysis, microlocal calculus and relativistic quantum theory (Proc. Internat. Colloq., Centre Phys., Les Houches, 1979)*, volume 126 of *Lecture Notes in Phys.*, pages 90–110. Springer, Berlin, 1980.

[56] H. Miller. Finite localizations. *Bol. Soc. Mat. Mexicana (2)*, 37(1-2):383–389, 1992. Papers in honor of José Adem (Spanish).

[57] D. Murfet. The mock homotopy category of projectives and Grothendieck duality. *Ph.D. thesis*, Available from: http://www.therisingsea.org/thesis.pdf, 2007.

[58] D. Murfet and S. Salarian. Totally acyclic complexes over noetherian schemes. *arXiv:0902:3013*.

[59] A. Neeman. The chromatic tower for $D(R)$. *Topology*, 31(3):519–532, 1992. With an appendix by Marcel Bökstedt.

[60] A. Neeman. The Grothendieck duality theorem via Bousfield's techniques and Brown representability. *J. Amer. Math. Soc.*, 9(1):205–236, 1996.

[61] A. Neeman. *Triangulated categories*, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001.

[62] A. Neeman. The homotopy category of flat modules, and Grothendieck duality. *Invent. Math.*, 174(2):255–308, 2008.

[63] D. O. Orlov. Triangulated categories of singularities and D-branes in Landau-Ginzburg models. *Tr. Mat. Inst. Steklova*, 246(Algebr. Geom. Metody, Svyazi i Prilozh.):240–262, 2004.

[64] D. O. Orlov. Triangulated categories of singularities, and equivalences between Landau-Ginzburg models. *Mat. Sb.*, 197(12):117–132, 2006.

[65] D. C. Ravenel. Localization with respect to certain periodic homology theories. *Amer. J. Math.*, 106(2):351–414, 1984.

[66] R. Rouquier. Dimensions of triangulated categories. *J. K-Theory*, 1(2):193–256, 2008.

[67] G. Stevenson. An extension of Dwyer's and Palmieri's proof of Ohkawa's theorem on Bousfield classes. *In preparation*, 2010.

[68] R. Takahashi. Classifying thick subcategories of the stable category of Cohen-Macaulay modules. *Adv. Math.*, 225(4):2076–2116, 2010.

[69] R. W. Thomason. The classification of triangulated subcategories. *Compositio Math.*, 105(1):1–27, 1997.

[70] J.-L. Verdier. Des catégories dérivées des catégories abéliennes. *Astérisque*, (239):xii+253 pp. (1997), 1996. With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.

[71] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.