Holonomy groups

in Riemannian geometry

Lecture 1

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Parallel translation in $\mathbb{R}^n$

- $\gamma\colon [0,1] \to \mathbb{R}^n$ arbitrary (smooth) curve
- $v\colon [0,1] \to \mathbb{R}^n$ vector field along γ

Then v is *parallel*, if $\dot{v} = \frac{dv}{dt} \equiv 0$.

Parallel translation on curved spaces

- $S \subset \mathbb{R}^n$ oriented hypersurface (e.g. $S^2 \subset \mathbb{R}^3$)
- n unit normal vector along S
- $\gamma\colon [0,1] \to S$ curve
- $v\colon [0,1] \to \mathbb{R}^n$ vector field along γ s.t.

$$v(t) \in T_{\gamma(t)}S \quad \Leftrightarrow \quad \big\langle v(t), n(\gamma(t))\big\rangle = 0 \qquad \forall t \qquad (1)$$

(1) $\Rightarrow$ v can not be constant in t. The eqn $\dot{v} = 0$ is replaced by

$$\mathrm{proj}_{TS}\dot{v} = 0 \quad \Leftrightarrow \quad \dot{v} - \langle \dot{v}, n(\gamma)\rangle n(\gamma) = 0.$$

Differentiating (1) we obtain a first order ODE for *parallel* v:

$$\boxed{\dot{v} + \langle v, \tfrac{d}{dt}n(\gamma)\rangle n(\gamma) = 0}$$

Parallel transport

Parallel transport is a linear isomorphism

$$P_\gamma\colon T_{\gamma(0)}S \to T_{\gamma(1)}S, \qquad v_0 \mapsto v(1)$$

where v is the solution of the problem

$$\dot{v} + \langle v, \tfrac{d}{dt}n(\gamma)\rangle n(\gamma) = 0, \qquad v(0) = v_0.$$

P_γ is an isometry, since

$$v, w \ \text{ are parallel } \Rightarrow \langle v(t), w(t)\rangle \ \text{ is constant in } t$$

Holonomy group

- $s \in S$ basepoint
- $Hol_s := \{P_\gamma \mid \gamma(0) = s = \gamma(1)\} \subset SO(T_sS)$ *based holonomy group*
- $Hol_{s'}$ is conjugated to Hol_s ("Holonomy group does not depend on the choice of the basepoint")
- Holonomy group is intrinsic to S, i.e. depends on the Riemannian metric on S but not on the embedding $S \subset \mathbb{R}^n$
- Ex: $Hol(S^2) = SO(2)$

Properties:

- ◇ definition generalises to any Riemannian manifold (M, g)
- ◇ encodes both local and global features of the metric
- ◇ "knows" about additional structures compatible with metric

Classification of holonomy groups

Berger's list, 1955

Assume M is a simply–connected irreducible nonsymmetric Riemannian mfld of dimension n. Then $Hol(M)$ is one of the following:

	Holonomy	Geometry	Extra structure
•	$SO(n)$		
•	$U(n/2)$	Kähler	complex
•	$SU(n/2)$	Calabi–Yau	complex + hol. vol.
•	$Sp(n/4)$	hyperKähler	quaternionic
•	$Sp(1)Sp(n/4)$	quaternionic Kähler	"twisted" quaternionic
•	G_2 (n=7)	exceptional	"octonionic"
•	$Spin(7)$ (n=8)	exceptional	"octonionic"

Plan

- General theory (torsion, Levi–Civita connection, Riemannian curvature, holonomy)
- Proof of Berger's theorem (Olmos 2005)
- Properties of manifolds with non–generic holonomies (some constructions, examples, curvature tensors. . .)