Holonomy groups

in Riemannian geometry

Lecture 2

October 27, 2011

LIE GROUPS VECTOR BUNDLES PRINCIPAL BUNDLES CONNECTIONS ON G–BUNDLES HOLONOMY TORSION

Smooth manifold comes equipped with a collection of charts $(U_\alpha, \varphi_\alpha)$, where $\{U_\alpha\}$ is an open covering and the maps $\varphi_\beta \circ \varphi_\alpha^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$ are smooth.

A Lie group G is a group which has a structure of a smooth mfld such that the structure maps, i.e. $m\colon G \times G \to G$, $\cdot^{-1}\colon G \to G$, are smooth.

$\mathfrak{g} := T_e G$ is a *Lie algebra*, i.e. a vector space endowed with a map $[\cdot,\cdot]\colon \Lambda^2\mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity:

$$\big[\xi,[\eta,\zeta]\big] + \big[\eta,[\zeta,\xi]\big] + \big[\zeta,[\xi,\eta]\big] = 0.$$

Ex.

G	$GL_n(\mathbb{R})$	$GL_n(\mathbb{C})$	$SO(n)$	$U(n)$
$\mathfrak{g}$	$\operatorname{End}\mathbb{R}^n$	$\operatorname{End}\mathbb{C}^n$	$\{A^t = -A\}$	$\{\bar{A}^t = -A\}$

Identification: $\mathfrak{g} \cong \{\text{left-invariant vector fields on } G\}$

- $\xi_1, \ldots, \xi_n$ a basis of $\mathfrak{g}$
- $\omega_1, \ldots, \omega_n$ dual basis

$\omega := \sum \omega_i \otimes \xi_i \in \Omega^1(G; \mathfrak{g})$ canonical 1-form with values in $\mathfrak{g}$, which satisfies the Maurer–Cartan equation

$$d\omega + \frac{1}{2}[\omega \wedge \omega] = \sum_i d\omega_i \otimes \xi_i + \frac{1}{2}\sum_{i,j} \omega_i \wedge \omega_j \otimes [\xi_i, \xi_j] = 0.$$

Vector bundles

A vector bundle E over M satisfies:

- E is a manifold endowed with a submersion $\pi \colon E \to M$
- $\forall m \in M \ E_m := \pi^{-1}(m)$ has the structure of a vector space
- $\forall m \in M \quad \exists U \ni m$ s.t. $\quad \pi^{-1}(U) \cong U \times E_m$

$\Gamma(E) = \{s \colon M \to E \mid \pi \circ s = id_M\}$ space of sections of E

Ex.

E	$\Gamma(E)$	
TM	$\mathfrak{X}(M)$	vector fields
$\Lambda^k T^*M$	$\Omega^k(M)$	differential k-forms
$T^p_q(M) := \bigotimes^p TM \otimes \bigotimes^q T^*M$	?	tensors of type (p, q)

de Rham complex

Exterior derivative $d\colon \Omega^k \to \Omega^{k+1}$ is the unique map with the properties:

- df is the differential of f for $f \in \Omega^0(M) = C^\infty(M)$
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$, if $\alpha \in \Omega^p$
- $d^2 = 0$

Thus, we have the de Rham complex:

$$0 \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^n \to 0, \qquad n = \dim M.$$

Betti numbers:

$$b_k = \dim H^k(M; \mathbb{R}) = \dim \frac{\operatorname{Ker} d\colon \Omega^k \to \Omega^{k+1}}{\operatorname{im} d\colon \Omega^{k-1} \to \Omega^k}.$$

Lie bracket of vector fields

A vector field can be viewed as an $\mathbb{R}$–linear derivation of the algebra $C^\infty(M)$. Then $\mathfrak{X}(M)$ is a Lie algebra:

$$[v, w] \cdot f = v \cdot (w \cdot f) - w \cdot (v \cdot f).$$

The exterior derivative and the Lie bracket are related by

$$2d\omega(v, w) = v \cdot \omega(w) - w \cdot \omega(v) - \omega([v, w])$$

Rem. "2" is optional in the above formula.

Lie derivative

For $v \in \mathfrak{X}(M)$ let φ_t be the corresponding 1-parameter (semi)group of diffeomorphisms of M, i.e.

$$\frac{d}{dt}\varphi_t(m) = v(\varphi_t(m)), \qquad \varphi_0 = id_M.$$

The *Lie derivative* of a tensor S is defined by

$$\mathcal{L}_v S = \frac{d}{dt}\Big|_{t=0} \varphi_t^* S$$

In particular, this means:

$$\mathcal{L}_v f(m) = \frac{d}{dt}\Big|_{t=0} f(\varphi_t(m)) = df_m\big(v(m)\big), \qquad \text{if } f \in C^\infty(M),$$

$$\mathcal{L}_v w(m) = \frac{d}{dt}\Big|_{t=0} \big(d\varphi_t\big)_m^{-1} w(\varphi_t(m)), \qquad \text{if } w \in \mathfrak{X}(M)$$

Properties of the Lie derivative

- $\mathcal{L}_v(S \otimes T) = (\mathcal{L}_v S) \otimes T + S \otimes (\mathcal{L}_v T)$
- $\mathcal{L}_v w = [v, w]$ for $w \in \mathfrak{X}(M)$
- $[\mathcal{L}_v, \mathcal{L}_w] = \mathcal{L}_{[v,w]}$
- Cartan formula

$$\boxed{\mathcal{L}_v \omega = \imath_v d\omega + d(\imath_v \omega)} \qquad \text{where } \omega \in \Omega(M).$$

- $[\mathcal{L}_v, d] = 0$ on $\Omega(M)$

Connections on vector bundles

Def. *A connection* on E is a linear map
$\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ satisfying the Leibnitz rule:

$$\nabla(fs) = df \otimes s + f\nabla s, \qquad \forall f \in C^\infty(M) \quad \text{and } \forall s \in \Gamma(E)$$

For $v \in \mathfrak{X}(M)$ we write

$$\nabla_v s = v \cdot \nabla s, \qquad \text{where " $\cdot$ "is a contraction.}$$

Then

$$\nabla_{\alpha v}(\beta s) = \alpha \nabla_v(\beta s) = \alpha(v \cdot \beta)\nabla_v s + \alpha\beta\nabla_v s.$$

Curvature

Prop. *For $v, w \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$ the expression*

$$\nabla_v(\nabla_w s) - \nabla_w(\nabla_v s) - \nabla_{[v,w]} s$$

is $C^\infty(M)$–linear in v, w, and s.

Def. The unique section $R = R(\nabla)$ of $\Lambda^2 T^*M \otimes \mathrm{End}(E)$ satisfying

$$R(\nabla)(v \wedge w \otimes s) = \nabla_v(\nabla_w s) - \nabla_w(\nabla_v s) - \nabla_{[v,w]} s$$

is called the *curvature* of the connection ∇.

Choose local coordinates $(x_1, \dots, x_n)$ on M

$v_i := \frac{\partial}{\partial x_i} \qquad \Rightarrow \qquad [v_i, v_j] = 0$

Then $R(v_i, v_j)s = \nabla_v(\nabla_w s) - \nabla_w(\nabla_v s)$

Think of $\nabla_{v_i} s$ as "partial derivative" of s

Curvature measures how much "partial derivatives" of sections of E fail to commute.

Twisted differential forms

Denote $\Omega^k(E) := \Gamma(\Lambda^k T^* M \otimes E)$

Then $\nabla \colon \Omega^0(E) \to \Omega^1(E)$ extends uniquely to $d^\nabla \colon \Omega^k(E) \to \Omega^{k+1}(E)$ via the rule

$$d^\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s$$

We obtain the sequence

$$\Omega^0(E) \xrightarrow{\nabla = d^\nabla} \Omega^1(E) \xrightarrow{d^\nabla} \Omega^2(E) \xrightarrow{d^\nabla} \dots \xrightarrow{d^\nabla} \Omega^n(E) \qquad (1)$$

Then $$\boxed{(d^\nabla \circ d^\nabla)\sigma = R(\nabla) \cdot \sigma}$$

Curvature measures the extend to which sequence (1) fails to be a complex.

Principal bundles

Let G be a Lie group

A principal bundle P over M satisfies:

- P is a manifold endowed with a submersion $\pi\colon P \to M$
- G acts on P on the right and $\pi(p \cdot g) = \pi(p)$
- $\forall m \in M$ the group G acts freely and transitively on $P_m := \pi^{-1}(m)$. Hence $P_m \cong G$
- Local triviality: $\forall m \in M \quad \exists U \ni m$ s.t. $\quad \pi^{-1}(U) \cong U \times G$

Example: Frame bundle

Let $E \to M$ be a vector bundle. A *frame* at a point m is a linear isomorphism $p\colon \mathbb{R}^k \to E_m$.

$$Fr(E) := \bigcup_{m,p} \{(m,p) \mid p \text{ is a frame at } m\}$$

(i) $GL(k;\mathbb{R}) = Aut(\mathbb{R}^k)$ acts freely and transitively on $Fr_m(E)$:

$$p \cdot g = p \circ g.$$

(ii) A *moving frame* on $U \subset M$ is a set $\{s_1, \ldots, s_k\}$ of pointwise linearly independent sections of E over U. This gives rise to a section s of $Fr(E)$ over U:

$$s(m)x = \sum x_i s_i(m), \qquad x \in \mathbb{R}^k.$$

By *(i)* this defines a trivialization of $Fr(E)$ over U.

Frame bundle: variations

If in addition E is

- *oriented*, i.e. $\Lambda^{\text{top}}E$ is trivial, $Fr^+(E)$ is a principal $GL^+(k;\mathbb{R})$–bundle
- *Euclidean* Fr_O is a principal $O(k)$–bundle
- *Hermitian* Fr_U is a principal $U(k)$–bundle
- *quaternion–Hermitian* is a principal $Sp(k)$–bundle
- ......

Def. Let G be a subgroup of $GL(n;\mathbb{R})$, $n = \dim M$. A *G–structure* on M is a principal G–subbundle of $Fr_M = Fr(TM)$.

- orientation $\Leftrightarrow$ $GL^+(n;\mathbb{R})$–structure
- Riemannian metric $\Leftrightarrow$ $O(n)$–structure
- ......

Associated bundle

$P \to M$ principal G–bundle
V G–representation, i.e. a homomorphism $\rho\colon G \to GL(V)$ is given

$$P \times_G V := \big(P \times V\big)/G, \qquad \text{action: } (p, v) \cdot g = (pg, \rho(g^{-1})v)$$

is called the *bundle associated to P with fibre V*.

Ex. For $P = Fr_M$, $G = GL(n;\mathbb{R})$, and $E = P \times_G V$ we have

- $E = TM$ for $V = \mathbb{R}^n$ (tautological representation)
- $E = T^*M$ for $V = \big(\mathbb{R}^n\big)^*$
- $E = \Lambda^k T^*M$ for $V = \Lambda^k\big(\mathbb{R}^n\big)^*$

Sections of associated bundles correspond to equivariant maps:

$$\big\{f\colon P \to V \mid f(pg) = \rho(g^{-1})f(p)\big\} \equiv \Gamma(E)$$
$$f \mapsto s_f, \qquad s_f(m) = [p, f(p)], \quad p \in P_m$$

Connection as horizontal distribution

For $\xi \in \mathfrak{g}$ the Killing vector at $p \in P$ is given by

$$K_\xi(p) := \frac{d}{dt}\Big|_{t=0} (p \cdot \exp t\xi)$$

$\mathcal{V}_p = \{K_\xi(p) \mid \xi \in \mathfrak{g}\} \cong \mathfrak{g}$ is called *vertical space* at p

Def. A connection on P is a subbundle $\mathcal{H}$ of TP satisfying

(i) $\mathcal{H}$ is G–invariant, i.e. $\mathcal{H}_{pg} = (R_g)_* \mathcal{H}_p$

(ii) $TP = \mathcal{V} \oplus \mathcal{H}$

$\mathcal{H}$ is called a *horizontal* bundle.

Connection as a 1–form

Given a connection on P, define $\omega \in \Omega^1(P; \mathfrak{g})$ as follows

$$T_pP \to \mathcal{V}_p \cong \mathfrak{g}$$

ω is called the *connection form* and satisfies:

(a) $\omega(K_\xi) = \xi$

(b) $R_g^* \omega = ad_{g^{-1}}\, \omega$, where ad denotes the adjoint representation

Prop. *Every $\omega \in \Omega^1(P; \mathfrak{g})$ satisfying (a) and (b) defines a connection via*

$$\mathcal{H} = \operatorname{Ker} \omega.$$

Horizontal lift

$\mathrm{Ker}(\pi_*)_p = \mathcal{V}_p$. Hence $(\pi_*)_p \colon \mathcal{H}_p \to T_{\pi(p)}M$ is an isomorphism. In particular, $\mathcal{H} \cong \pi^*TM$. Hence, we have

Prop. *For any $w \in \mathfrak{X}(M)$ there exists $\tilde{w} \in \mathfrak{X}(P)$ s.t.*

(i) $\tilde{w}$ *is G–invariant and horizontal*

(ii) $(\pi_*)_p\tilde{w} = w(\pi(p))$

Vice versa, if $\tilde{w} \in \mathfrak{X}(P)$ is G–invariant and horizontal, then $\exists!\ w \in \mathfrak{X}(M)$ s.t. $\pi_*\tilde{w} = w$.

Invariant and equivariant forms

$\tilde{\alpha} \in \Omega^k(P)$ is called *basic* if $\imath_v\tilde{\alpha} = 0$ for any vertical vector field v.

Then $\forall \alpha \in \Omega^k(M)$ the form $\tilde{\alpha} = \pi^*\alpha$ is G–invariant and basic. On the other hand, any G–invariant and basic k–form $\tilde{\alpha}$ on P induces a k–form on M. **Notice:** no connection required here.

V is a representation of G
$\tilde{\alpha} \in \Omega^k(P; V)$ is G–equivariant if $R_g^*\tilde{\alpha} = \rho(g^{-1})\tilde{\alpha}$.

Ex. Connection 1-form is an equivariant form for $V = \mathfrak{g}$.

For basic and equivariant forms we have the identification

$$\Omega^k_{G,bas}(P, V) \cong \Omega^k(M; E), \qquad \pi^*\alpha \leftarrow\!\shortmid \alpha$$

Curvature tensor

Prop. *Let ω be a connection form. The 2–form $\tilde{F}_\omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$ is basic and G–equivariant, i.e. $R_g^* \tilde{F} = ad_{g^{-1}} \tilde{F}$.*

Cor. *Denote $ad\, P := P \times_{G,ad} \mathfrak{g}$. Then there exists $F \in \Omega^2(M; ad\, P)$ s.t. $\pi^* F = \tilde{F}$.*

The 2-form F is called the *curvature form* of the connection ω. The defining equation for F is often written as

$$d\omega = -\frac{1}{2}[\omega \wedge \omega] + F$$

and is called the *structural equation*.

Covariant differentiation

$P \to M$ G–bundle, $\rho\colon G \to GL(V)$, $E := P \times_G V$,
$f\colon P \to V$ equivariant map, i.e. section of E.

Def. $\nabla f = d^h f = df\big|_{\mathcal{H}}$ is called the covariant derivative of f.

Rem. Denote $\tau = d\rho_e\colon \mathfrak{g} \to \mathfrak{gl}(V) = \mathrm{End} V$. Then for a vertical vector $K_\xi(p)$ we have: $df(K_\xi(p)) = -\tau(\xi) f(p)$, that is all information about df is contained in $d^h f$.

Prop.

$$\nabla f = df + \omega \cdot f$$

Here "$\cdot$" means the action of $\mathfrak{g}$ on V via the map τ.

Prop. *$\nabla f \in \Omega^1(P;V)$ is G–equivariant and basic form.*

Thus ∇f can be interpreted as an element of $\Omega^1(M;E)$ and we have a diagram

$$\begin{array}{ccc} Map^G(P;V) & \xrightarrow{\ \nabla\ } & \Omega^1_{G,bas}(P;V) \\ \Vert & & \Vert \\ \Gamma(E) & \xrightarrow{\ \nabla^E\ } & \Omega^1(M;E) \end{array} \qquad \begin{array}{ccc} f & \longmapsto & \nabla f \\ \downarrow & & \downarrow \\ s_f & \longmapsto & \nabla^E s_f \end{array}$$

Prop. *∇^E is a connection on E.*

Bianchi identity

ω connection on P, F curvature
$ad\,P$ has an induced connection ∇

Theorem (Bianchi identity)

$$\boxed{d^\nabla F = 0}$$

Proof. For $\tilde\varphi \in \Omega^k(P;\mathfrak{g})$ denote $D\tilde\varphi = d\tilde\varphi + [\omega \wedge \tilde\varphi]$

Step 1. For any $\varphi \in \Omega^k(M; ad\,P)$ we have $\widetilde{d^\nabla\varphi} = D\tilde\varphi$.

Can assume $\varphi = s \cdot \varphi_0$, where $\varphi_0 \in \Omega^k(M)$ and

$$\Gamma(ad\,P) \ni s \leftrightsquigarrow f \in Map^G(P;\mathfrak{g}).$$

Then

$$\begin{aligned} \widetilde{d^\nabla\varphi} &= \widetilde{\nabla s} \wedge \tilde\varphi_0 + \tilde s \cdot d\tilde\varphi_0 \\ &= (df + [\omega, f]) \wedge \tilde\varphi_0 + f\, d\tilde\varphi_0 \\ &= d(f\tilde\varphi_0) + [\omega \wedge f\tilde\varphi_0] \\ &= D\varphi \end{aligned}$$

Proof of the Bianchi identity (continued)

Step 2. $D\tilde F = 0$, where $\tilde F = d\omega + \frac{1}{2}[\omega \wedge \omega]$.

$$\begin{aligned} d\tilde F &= \tfrac{1}{2}\big([d\omega \wedge \omega] - [\omega \wedge d\omega]\big) \\ &= [d\omega \wedge \omega] \\ &= [\tilde F \wedge \omega] - \tfrac{1}{2}[[\omega \wedge \omega] \wedge \omega] \end{aligned}$$

$$\text{Jacobi identity} \implies [[\omega \wedge \omega] \wedge \omega] = 0$$

Thus, $D\tilde F = 0 \iff d^\nabla F = 0.$ □

Horizontal lift of a curve

$\gamma\colon [0,1] \to M$ (piecewise) smooth curve, $p_0 \in P_{\gamma(0)}$.

Prop. *[KN, Prop. II.3.1] For any γ there exists a unique horizontal lift of γ through p_0, i.e. a curve $\Gamma\colon [0,1] \to P$ with the following properties:*

(i) $\dot\Gamma(t) \in \mathcal{H}_{\Gamma(t)}$ *for any $t \in [0,1]$ ("Γ is horizontal")*

(ii) $\Gamma(0) = p_0$

(iii) $\pi \circ \Gamma = \gamma$

Sketch of the proof. Let Γ_0 be an arbitrary lift of γ, $\Gamma_0(0) = p_0$. Then $\Gamma = \Gamma_0 \cdot g$ for some curve $g\colon [0,1] \to G$. Hence,

$$\dot\Gamma = \dot\Gamma_0 \cdot g + \Gamma_0 \cdot \dot g \quad \implies \quad \omega(\dot\Gamma) = ad_{g^{-1}}\omega(\dot\Gamma_0) + g^{-1}\dot g.$$

Then there exists a unique curve g, $g(0) = e$, such that
$g^{-1}\dot g + ad_{g^{-1}}\omega(\dot\Gamma_0) = 0 \quad \iff \quad \omega(\dot\Gamma) = 0.$ □

Parallel transport

$\gamma\colon [0,1] \to M,\ \gamma(0) = m,\ \gamma(1) = n$
Parallel transport $\Pi_\gamma\colon P_m \to P_n$ is defined by

$$\Pi_\gamma(p) = \Gamma(1),$$

where Γ is the horizontal lift of γ satisfying $\Gamma(0) = p$.

Prop.

(i) Π_γ *commutes with the action of* G *for any curve* γ

(ii) Π_γ *is bijective*

(iii) $\Pi_{\gamma_1 * \gamma_2} = \Pi_{\gamma_1} \circ \Pi_{\gamma_2}, \quad \Pi_{\gamma^{-1}} = \Pi_\gamma^{-1}$

Holonomy group

Denote $\Omega_m := \{$piecewise smooth loops in M based at $m\}$

$$Hol_p(\omega) := \{g \in G \mid \exists \gamma \in \Omega_m \text{ s.t. } \Pi_\gamma(p) = pg\}$$

Prop.

(i) Hol_p *is a Lie group*

(ii) $Hol_{pg} = Ad_{g^{-1}}(Hol_p)$

Proof. Group structure follows from *(iii)* of the previous Prop.
For the structure of Lie group see [Kobayashi–Nomizu, Thm 4.2].
Statement *(ii)* follows from the observation

$$\Gamma \text{ is horizontal} \implies R_g \circ \Gamma \text{ is also horizontal.}$$

□

Reduction of connections

Let $H \subset G$ be a Lie subgroup and $Q \subset P$ be a principal H–bundle ("structure group reduces to H").

Def. A connection $\mathcal{H}$ on P reduces to Q if $\mathcal{H}_q \subset T_qQ \quad \forall q \in Q$.

Prop. *A connection reduces to Q $\iff$ $\imath^*\omega$ takes values in $\mathfrak{h}$, where $\imath\colon Q \hookrightarrow P$.*

Proof. ($\Rightarrow$):

$$\begin{array}{ccc} T_qQ \cong \mathcal{H}_q \oplus \mathfrak{h} & \xrightarrow{(0,\, id)} & \mathfrak{h} \\ \big\downarrow & & \big\downarrow \\ T_qP & \xrightarrow{\ \omega\ } & \mathfrak{g} \end{array}$$

($\Leftarrow$): $\imath^*\omega$ is a connection on Q, hence $TQ = \mathcal{H}^Q \oplus \mathfrak{h}$. Since $\mathcal{H}^Q \subset \mathcal{H}^P$ and $\operatorname{rk}\mathcal{H}^P = \dim M = \operatorname{rk}\mathcal{H}^Q$, we obtain $\mathcal{H}^Q = \mathcal{H}^P$. □

Lie groups | Vector bundles | Principal bundles | Connections on G–bundles | Holonomy | Torsion

Reduction theorem

For $p_0 \in P$ define the *holonomy bundle* through p_0 as follows:

$Q(p_0) := \{p \in P \mid \exists$ a horizontal curve Γ s.t. $\Gamma(0) = p_0,\ \Gamma(1) = p\}$.

Theorem ("Reduction theorem")

Put $H = Hol_{p_0}(P,\omega)$. Then the following holds:

(i) Q is a principal H–bundle

(ii) connection ω reduces to Q

Proof. *(i):* $p \in Q,\ g \in H \ \Rightarrow\ pg \in Q$ (by the def of H).

Exercise: Show that $Hol_p(\omega) = H \quad \forall p \in Q$.

From the def of Q follows, that H acts transitively on fibres.

Local triviality: Use parallel transport over coordinate chart U wrt segments to obtain a local section of Q (see [KN, Thm II.7.1] for details).

(ii): Follows immediately from the def of Q. □

Parallel transport and covariant derivative

Let $\Gamma\colon [0,1] \to P$ be a horizontal lift of γ
$\Gamma_E(t) := [\Gamma(t), v], \quad v \in V, \; E = P \times_G V$
$\Gamma_E\colon [0,1] \to E$ is called the horizontal lift of γ to E
$\Pi_t\colon E_{\gamma(t)} \to E_m$ parallel transport in E, $m = \gamma(0)$

Lem. $\nabla_w s = \lim_{t\to 0} \frac{1}{t}\Big(\Pi_t s\big(\gamma(t)\big) - s(m)\Big)$, *where* $w = \dot\gamma(0)$.

Proof. Let $s \leftrightsquigarrow f$, i.e. $[p, f(p)] = s(\pi(p))$. First observe that

$$\Pi^E_\gamma [p, v] = [\Pi_\gamma p, v].$$

Since $\big[\Gamma(t), f(\Gamma(t))\big] = s(\gamma(t))$, we obtain

$$\Pi_t s = \big[p, f(\Gamma(t))\big].$$

⇓ to be continued ⇓

Lie groups Vector bundles Principal bundles Connections on G–bundles Holonomy Torsion

Lem. $\nabla_w s = \lim_{t\to 0} \frac{1}{t}\Big(\Pi_t s\big(\gamma(t)\big) - s(m)\Big)$, *where* $w = \dot\gamma(0)$.

Proof. Let $s \leftrightsquigarrow f$, i.e. $[p, f(p)] = s(\pi(p))$. First observe that

$$\Pi^E_\gamma [p, v] = [\Pi_\gamma p, v].$$

Since $\big[\Gamma(t), f(\Gamma(t))\big] = s(\gamma(t))$, we obtain

$$\Pi_t s = \big[p, f(\Gamma(t))\big].$$

Then

$$\begin{aligned}
\nabla_w s &= [p, df(\tilde w)] \\
&= \big[p, \tfrac{d}{dt}\big|_{t=0} f \circ \Gamma(t)\big] \\
&= \lim_{t\to 0} \tfrac{1}{t}\Big(\big[p, f(\Gamma(t))\big] - [p, f(p)]\Big) \\
&= \lim_{t\to 0} \tfrac{1}{t}\Big(\Pi_t s\big(\gamma(t)\big) - s(m)\Big).
\end{aligned}$$

□

Rem. Let $w \in \mathfrak{X}(M)$. If $s \;\leftrightsquigarrow\; f$, then $\nabla_w s \;\leftrightsquigarrow\; df(\tilde{w})$.

Lem. *Let $s \in \Gamma(E)$, $s_0 = s(m)$. Assume $\nabla s = 0$. Then for any loop γ based at m we have $\Pi^E_\gamma s_0 = s_0$.*

Proof. Let Γ be a horizontal lift of γ. Then $f \circ \Gamma = const$. Hence $\Pi_t s(\gamma(t)) = [p, f \circ \Gamma]$ does not depend on t. □

V is a G–representation, $H = Stab_\eta$, where $\eta \in V$.
$Q \subset P$ is a principal H–subbundle
The constant function $q \mapsto \eta$ can be extended to an equivariant function η on P

Theorem

ω reduces to Q $\iff$ $\nabla^E \eta = 0$.

Proof. ($\Rightarrow$): $\forall q \in Q$ $d\eta\big|_{\mathcal{H}_q} = 0$, since η is constant on Q and $\mathcal{H} \subset TQ$.

($\Leftarrow$): For any $q \in Q$ we have

$$[q, \eta] = \Pi^E_\gamma [q, \eta] = \left[\Pi_\gamma q, \eta\right] = [qg, \eta] = [q, \rho(g^{-1})\eta].$$

Hence $Hol_q(\omega) \subset H$. Then the holonomy bundle through q is contained in Q. Therefore, ω reduces to Q. □

Ambrose–Singer theorem

Theorem (Ambrose–Singer)

Let Q be the holonomy bundle through p_0, $\tilde{F} \in \Omega^2(P;\mathfrak{g})$ curvature of ω. Then

$$\mathfrak{hol}_{p_0} = \operatorname{span}\big\{\tilde{F}_q(w_1,w_2) \mid q \in Q,\ w_1,w_2 \in \mathcal{H}_q\big\}.$$

Sketch of the proof. Can assume $Q = P$. Denote

$$\mathfrak{g}' = \operatorname{span}\big\{\tilde{F}_q(w_1,w_2) \mid q \in Q,\ w_1,w_2 \in \mathcal{H}_q\big\} \subset \mathfrak{g}.$$

Further, $S_p := \mathcal{H}_p \oplus \{K_\xi(p) \mid \xi \in \mathfrak{g}'\}$. Then the distribution S is integrable. If $P_0 \ni p_0$ is a maximal integral submanifold, then $P_0 = P$, since each horizontal curve must lie in P_0. Then $\dim \mathfrak{g} = \dim P - \dim M = \dim P_0 - \dim M = \dim \mathfrak{g}'$. Hence $\mathfrak{g} = \mathfrak{g}'$. □

From now on $P = Fr(M)$ is the principal $G = GL_n(\mathbb{R})$–bundle of linear frames

Def. A *canonical 1–form* $\theta \in \Omega^1(P;\mathbb{R}^n)$ is given by

$$\theta(v) = p^{-1}(d\pi(v)), \qquad v \in T_pP.$$

Rem. θ is defined for bundles of linear frames only.

θ is G–equivariant in the following sense: $R_g^*\theta = g^{-1}\theta$. Indeed, for any $v \in T_pP$ we have

$$R_g^*\theta(v) = (pg)^{-1}\big(d\pi(R_g v)\big) = g^{-1}p^{-1}\big(d\pi(v)\big) = g^{-1}\theta(v).$$

Torsion

ω is a connection on $Fr(M)$. In particular, ω is $\mathfrak{gl}_n(\mathbb{R})$–valued. Thus, we have induced connections on TM, T^*M, $\Lambda^k T^*M \dots$

Def. $\Theta = d\theta + \frac{1}{2}[\omega, \theta] \in \Omega^2(Fr(M); \mathbb{R}^n)$ is called the *torsion form* of ω.

Rem. $[\omega, \theta](v, w) = \omega(v)\theta(w) - \omega(w)\theta(v)$.

Prop. *Θ is horizontal and equivariant. Hence there exists $T \in \Omega^2(M; TM)$ s.t. $2\Theta = \pi^* T$.*

T can be viewed as a skew–symmetric linear map $TM \otimes TM \to TM$ and is called the *torsion tensor*.

Lie groups Vector bundles Principal bundles Connections on G–bundles Holonomy Torsion

Theorem

For $v, w \in \mathfrak{X}(M)$ we have

$$\boxed{T(v, w) = \nabla_v w - \nabla_w v - [v, w]}$$

Proof. Represent v, w by equivariant functions f_v, $f_w \colon Fr \to \mathbb{R}^n$. Then $\nabla_v w$ is represented by $df_w(\tilde{v})$.

For the bundle of frames, $f_w = \theta(\tilde{w})$. Hence $\nabla_v w = p\big(\tilde{v} \cdot \theta(\tilde{w})\big)$. Therefore we obtain

$$\begin{aligned} T(v, w) &= p\big(2\Theta(\tilde{v}, \tilde{w})\big) \\ &= p\big(\tilde{v} \cdot \theta(\tilde{w}) - \tilde{w} \cdot \theta(\tilde{v}) - \theta([\tilde{v}, \tilde{w}])\big) \\ &= \nabla_v w - \nabla_w v - [v, w]. \end{aligned}$$

The last equality follows from $[\tilde{v}, \tilde{w}]^h = \widetilde{[v, w]}$ (*exercise*). $\square$

Denote
$\Gamma(T^*M) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M) \xrightarrow{\mathrm{Alt}} \Omega^2(M), \quad \alpha \mapsto \mathrm{Alt}(\nabla\alpha).$

Theorem

$$\mathrm{Alt}(\nabla\alpha) = d\alpha - \alpha \circ T$$

In particular, for torsion–free connections $\mathrm{Alt}(\nabla\alpha) = d\alpha$.

Proof. This follows from the previous Thm with the help of the formulae $v \cdot \alpha(w) = \nabla_v(\alpha(w)) = (\nabla_v\alpha)(w) + \alpha(\nabla_v w)$. □