Holonomy groups

in Riemannian geometry

Lecture 3

November 3, 2011

Recap of the previous lecture

$$Fr(M) := \bigcup_{m,p} \{(m,p) \mid p\colon \mathbb{R}^n \xrightarrow{\cong} T_mM\}$$ frame bundle;

$$\theta(v) = p^{-1}(d\pi(v)),\ v \in T_pFr(M)$$ canonical 1–form

$$\Theta = d\theta + \tfrac{1}{2}[\omega, \theta] \in \Omega^2(Fr(M); \mathbb{R}^n),$$ torsion form

$$\exists T \in \Omega^2(M; TM), \text{s.t.} \qquad 2\Theta = \pi^*T,$$ torsion tensor

$$T(v,w) = \nabla_v w - \nabla_w v - [v,w], \quad v, w \in \mathfrak{X}(M)$$

$$\mathrm{Alt}(\nabla\alpha) = d\alpha - \alpha \circ T, \quad \alpha \in \Omega^1(M)$$

Curvature tensor

For $P = Fr(M)$ we have $ad\,P = \mathrm{End}(TM)$. Then the curvature can be viewed as a skew–symmetric map

$$TM \otimes TM \to \mathrm{End}(TM), \qquad (v, w) \mapsto R(v, w).$$

R is called the *curvature tensor*.

Theorem (KN, Thm. II.5.1)

For $v, w, x \in \mathfrak{X}(M)$ we have

$$R(v, w)x = [\nabla_v, \nabla_w]x - \nabla_{[v,w]}x.$$

Theorem

For any G–bundle P the space $\mathcal{A}(P)$ of all connections is an affine space modelled on $\Omega^1(M; ad\,P)$.

Proof. Pick an arbitrary connection ω on P. Then for any $\omega' \in \mathcal{A}(P)$, the 1-form $\xi = \omega - \omega'$ is basic and ad–equivariant. Vice versa, for any basic and equivariant 1–form ξ, the form $\omega' = \omega - \xi$ is a connection. Hence, the statement of the thm. □

Assume $G \subset GL_n(\mathbb{R})$ and therefore $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R}) \cong (\mathbb{R}^n)^* \otimes \mathbb{R}^n$.
$Fr(M) \supset P$ is a G–bundle, $\omega, \omega' \in \mathcal{A}(P)$, $\xi = \omega - \omega'$.
For any $p \in P$, the map $\theta_p \colon \mathcal{H}_p \to \mathbb{R}^n$ is an isomorphism.
Therefore we can write

$$\xi_p \in (\mathbb{R}^n)^* \otimes \mathfrak{g}, \qquad T_p \colon \Lambda^2\mathbb{R}^n \cong \Lambda^2\mathcal{H}_p \xrightarrow{\Theta_p} \mathbb{R}^n.$$

Then

$$\Theta' - \Theta = \tfrac{1}{2}[\xi, \theta] \quad \Longleftrightarrow \quad \big(T_p' - T_p\big)x \wedge y = \tfrac{1}{2}\big(\xi_p(x)y - \xi_p(y)x\big).$$

Consider the G–equivariant homomorphism

$$\delta \colon (\mathbb{R}^n)^* \otimes \mathfrak{g} \hookrightarrow (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \otimes \mathbb{R}^n \longrightarrow \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n.$$

Then, $T' - T = \delta\xi$.

Prop. *P has a torsion–free connection if and only if $T_p \in \operatorname{Im}\delta$ for all $p \in P$.*

(M, g) Riemannian manifold (by default, M **is oriented**)
$Fr(M) \supset P$ is the $G = SO(n)$–bundle of orthonormal oriented frames

We have the commutative diagram of $SO(n)$–representations:

$$\begin{array}{ccc} \mathfrak{so}(n) & \hookrightarrow & \mathfrak{gl}_n(\mathbb{R}) = \operatorname{End}\mathbb{R}^n \\ \cong\downarrow & & \downarrow\cong \\ \Lambda^2\mathbb{R}^n & \hookrightarrow & \mathbb{R}^n \otimes \mathbb{R}^n \end{array} \qquad (\mathbb{R}^n)^* \cong \mathbb{R}^n.$$

Prop. *The map $\delta_{\mathfrak{so}(n)} \colon \mathbb{R}^n \otimes \Lambda^2\mathbb{R}^n \to \Lambda^2\mathbb{R}^n \otimes \mathbb{R}^n$ is an isomorphism.*

Proof. For $a = \sum a_{ijk} e_i \otimes e_j \wedge e_k$ we have (*exercise*):

$$\delta a = \frac{1}{2}\sum (a_{ijk} - a_{jik}) e_i \wedge e_j \otimes e_k.$$

Hence, if $a \in \operatorname{Ker}\delta$, then $a_{ijk} = a_{jik} = -a_{jki} = -a_{kji} = a_{kij} = a_{ikj} = -a_{ijk} \quad \Longrightarrow \quad a = 0.$ □

The Levi–Civita connection

Theorem ("Fundamental theorem of Riemannian geometry")

Any $SO(n)$–subbundle of $Fr(M)$ admits a unique torsion–free connection.

Theorem ("Fundamental theorem", reformulation)

For any Riemannian metric g there exists a unique torsion–free connection on $Fr(M)$ such that $\nabla g = 0$.

The unique connection in the "Fundamental thm" is called the *Levi–Civita* (or *Riemannian*) connection. The corresponding curvature tensor is called *Riemannian curvature tensor*.

For any $p \in P$ we have

$$R_p\colon \Lambda^2\mathbb{R}^n \cong \Lambda^2\mathcal{H}_p \longrightarrow \mathfrak{so}(n) \cong \Lambda^2\mathbb{R}^n.$$

Theorem ("algebraic Bianchi identity")

$R_p(x,y)z + R_p(y,z)x + R_p(z,x)y = 0$ *for all* $x, y, z \in \mathbb{R}^n$.

Proof. $d\theta + \frac{1}{2}[\omega, \theta] = \Theta = 0 \ \Rightarrow [d\omega, \theta] - [\omega, d\theta] = 0$. This implies the *first Bianchi identity*:

$$\begin{aligned} [R,\theta] &= [d\omega,\theta] + \tfrac{1}{2}\big[[\omega\wedge\omega],\theta\big] \\ &= [\omega, d\theta] + \tfrac{1}{2}\big[[\omega\wedge\omega],\theta\big] \\ &= -\tfrac{1}{2}\big[\omega,[\omega,\theta]\big] + \tfrac{1}{2}\big[[\omega\wedge\omega],\theta\big] \\ &= 0. \end{aligned}$$

$[R,\theta](px, py, pz) = 0 \iff$ algebraic Bianchi identity. □

Cor. $\langle R_p(x,y)z, t\rangle = \langle R_p(z,t)x, y\rangle$, *i.e.* $R_p \in S^2(\Lambda^2\mathbb{R}^n)$.

Proof. *Exercise.* □

Observation: If $V = V_1 \oplus V_2$ as G–representation, then $E = E_1 \oplus E_2$, where $E_i := P \times_G V_i$.

Determine irreducible components of the $SO(n)$–representation

$$\mathfrak{R} = \{R \in \Lambda^2(\mathbb{R}^n)^* \otimes \mathfrak{so}(n) \mid R \text{ satisfies alg. Bianchi id.}\}.$$

We can decompose

$$\mathrm{End}\,\mathbb{R}^n = \mathfrak{so}(n) \oplus \mathrm{Sym}\,\mathbb{R}^n = \mathfrak{so}(n) \oplus \mathrm{Sym}_0\,\mathbb{R}^n \oplus \mathbb{R},$$

where $\mathrm{Sym}_0\,\mathbb{R}^n = \mathrm{Ker}(\mathrm{tr}\colon \mathrm{Sym}\,\mathbb{R}^n \to \mathbb{R})$. In other words,

$$\mathbb{R}^n \otimes \mathbb{R}^n \cong \Lambda^2\mathbb{R}^n \oplus S^2_0\,\mathbb{R}^n \oplus \mathbb{R}. \qquad (1)$$

Prop. (1) *is decomposition into irreducible components if* $n \neq 4$. *For* $n = 4$ *we have in addition* $\Lambda^2\mathbb{R}^4 = \Lambda^2_+\mathbb{R}^4 \oplus \Lambda^2_-\mathbb{R}^4$.

Here: $*\colon \Lambda^m\mathbb{R}^{2m} \to \Lambda^m\mathbb{R}^{2m}$ is the Hodge operator, $*^2 = id$
$\Lambda^m_\pm\mathbb{R}^{2m}$ are eigenspaces corresponding to $\lambda = \pm 1$.

Think of $\bigotimes^4 \mathbb{R}^n$ as the space of quadrilinear forms on $(\mathbb{R}^n)^*$. Consider the map

$$b(R)(\alpha,\beta,\gamma,\delta) = \tfrac{1}{3}\Big(R(\alpha,\beta,\gamma,\delta) + R(\beta,\gamma,\alpha,\delta) + R(\gamma,\alpha,\beta,\delta)\Big)$$

(cyclic permutation in the first 3 variables; *Bianchi map*). Then

- b is $SO(n)$–invariant
- $b^2 = b$
- $b\colon S^2(\Lambda^2\mathbb{R}^n) \to S^2(\Lambda^2\mathbb{R}^n)$

Hence, we have

$$S^2(\Lambda^2\mathbb{R}^n) = \mathrm{Ker}\,b \oplus \mathrm{Im}\,b = \mathfrak{R} \oplus \Lambda^4\mathbb{R}^n.$$

The *Ricci contraction* is the $SO(n)$–equivariant map

$$c\colon S^2(\Lambda^2\mathbb{R}^n) \to S^2\mathbb{R}^n, \qquad c(R)(x,y) = \operatorname{tr} R(x,\cdot,y,\cdot)$$

The *Kulkarni–Nomizu product* of $h, k \in S^2\mathbb{R}^n$ is the 4–tensor $h \owedge k$ given by

$$h \owedge k(\alpha,\beta,\gamma,\delta) = h(\alpha,\gamma)k(\beta,\delta) + h(\beta,\delta)k(\alpha,\gamma) - h(\alpha,\delta)k(\beta,\gamma) - h(\beta,\gamma)k(\alpha,\delta).$$

Prop.

- $h \owedge k = k \owedge h$;
- $h \owedge k \in \operatorname{Ker} b = \mathfrak{R}$;
- $q \owedge q = 2\, id_{\Lambda^2\mathbb{R}^n}$, *where* q =*standard scalar product on* $\mathbb{R}^n$.

Lem. *If* $n \geq 3$, *the map* $q \owedge \cdot\colon S^2\mathbb{R}^n \to \mathfrak{R}$ *is injective and its adjoint is the restriction of the Ricci contraction* $c\colon \mathfrak{R} \to S^2\mathbb{R}^n$.

Components of the Riemannian curvature tensor

Theorem

We have the following decomposition:

$$\mathfrak{R} \cong \mathbb{R} \oplus S^2_0\mathbb{R}^n \oplus \mathcal{W},$$

where $\mathcal{W} = \operatorname{Ker} c \cap \operatorname{Ker} b$. *If* $n \geq 5$, *each component is irreducible.*

Explicitly:

- $\frac{1}{n}\operatorname{tr} c(R) + c(R)_0$ are the components of R in $\mathbb{R} \oplus S^2_0\mathbb{R}^n$;
- the inclusions of the first two spaces are given by

$$\mathbb{R} \ni 1 \mapsto q \owedge q, \qquad S^2_0\mathbb{R}^n \ni h \mapsto q \owedge h. \tag{2}$$

Def. For the Riemannian curvature tensor R we define:

- $Ric(R) = c(R)$ *Ricci curvature*;
- $s = \operatorname{tr} c(R)$ *scalar curvature*, Ric_0 *traceless Ricci curvature*;
- $W(R) \in \operatorname{Ker} c \cap \operatorname{Ker} b$ *Weyl tensor*.

From (2) follows that $R = \lambda\, q \owedge q + \mu\, Ric_0 \owedge q + W$. The coefficients λ, μ can be determined from the equality $c(q \owedge h) = (n-2)h + (\mathrm{tr}\, h)q$. Hence, we obtain

$$\boxed{R = \frac{s}{2n(n-1)}\, q \owedge q + \frac{1}{n-2}\, Ric_0 \owedge q + W.}$$

Observe: Ric is a symmetric quadratic form on the tangent bundle.

Def. A Riemannian mfld (M, g) is called *Einstein*, if there exists $\lambda \in \mathbb{R}$ such that

$$Ric(g) = \lambda g.$$

Local expressions

Choose local coordinates $(x_1, \ldots, x_n)$ on M and write:

$$\nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}, \qquad g_{ij} = g\big(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}\big), \quad (g^{ij}) = (g_{ij})^{-1}$$

Local functions Γ_{ij}^k are called *Chistoffel symbols*.

Theorem ([KN, Prop. III.7.6 + Cor. IV.2.4])

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} \big(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij} \big),$$

$$T\big(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}\big) = \sum_k (\Gamma_{ij}^k - \Gamma_{ji}^k) \tfrac{\partial}{\partial x_k},$$

$$R\big(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}\big) \tfrac{\partial}{\partial x_k} = \sum_l R_{ijk}^l \tfrac{\partial}{\partial x_l},$$

$$R_{ijk}^l = \big(\partial_j \Gamma_{ki}^l - \partial_k \Gamma_{ji}^l\big) + \sum_m \big(\Gamma_{ki}^m \Gamma_{jm}^l - \Gamma_{ji}^m \Gamma_{km}^l\big)$$

Low dimensions

$n = 2$. The curvature tensor is determined by the scalar curvature:

$$S^2(\Lambda^2\mathbb{R}^2) = \mathbb{R}q \owedge q, \qquad R = \frac{s}{4} q \owedge q.$$

Notice: Einstein $\Leftrightarrow$ constant sc. curvature

$n = 3$. The curvature tensor is determined by the Ricci curvature:

$$S^2(\Lambda^2\mathbb{R}^3) = \mathbb{R}q \owedge q \oplus S^2_0(\mathbb{R}^3) \owedge q, \qquad R = \frac{s}{12} q \owedge q + Ric_0 \owedge q.$$

$n = 4$. Recall: $\Lambda^2\mathbb{R}^4 = \Lambda^2_+ \oplus \Lambda^2_-$. Then

$$S^2_0(\mathbb{R}^4) \cong \Lambda^2_+ \otimes \Lambda^2_-, \qquad \mathcal{W} \cong S^2_0(\Lambda^2_+) \oplus S^2_0(\Lambda^2_-).$$

Hence, the Weyl tensor splits: $W = W^+ + W^-$, $W^\pm \in S^2_0(\Lambda^2_\pm)$.
If we consider R as a linear symmetric map of $\Lambda^2\mathbb{R}^4 = \Lambda^2_+ \oplus \Lambda^2_-$, we have

$$R = \left(\begin{array}{c|c} W^+ + \frac{s}{12} id & Ric_0 \\ \hline Ric_0^* & W^- + \frac{s}{12} id \end{array}\right)$$

Two Riemannian metrics g and g' are *conformally equivalent* if $g' = e^\varphi g$ for some $\varphi \in C^\infty(M)$. The class $[g]$ is called the conformal class of g.

conformal class $\iff$ $CO(n) = O(n) \times \mathbb{R}_+$–structure on M

Prop. *The Weyl tensor is conformally invariant.*

Proof. $g' \sim g$; ω', ω corresponding LC connections, $\omega' = \omega + \xi$.
Recall: $0 = T' - T = \delta\xi$, where
$\delta\colon (\mathbb{R}^n)^* \otimes \mathfrak{co}(n) \to \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$, $\mathfrak{co}(n) = \mathfrak{so}(n) \oplus \mathbb{R}$. Since $\delta\colon (\mathbb{R}^n)^* \otimes \mathfrak{so}(n) \to \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ is an isomorphism, we have $\xi \in \operatorname{Ker}\delta \cong (\mathbb{R}^n)^*$. Then

$$\begin{aligned} \tilde F' - \tilde F &= d\omega' - d\omega + \tfrac{1}{2}[\omega' \wedge \omega'] - \tfrac{1}{2}[\omega \wedge \omega] \\ &= d\xi + [\omega \wedge \xi] + \tfrac{1}{2}[\xi \wedge \xi] \\ &= \nabla\xi + \tfrac{1}{2}[\xi \wedge \xi]. \end{aligned}$$

Hence, $R' - R$ takes values in $(\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^*$ and thus belongs to $\mathbb{R} \oplus S^2_0(\mathbb{R}^n)$. □

Geodesics

Def. A curve $\gamma\colon \mathbb{R} \to M$ is called *geodesic* if $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$ for all t, i.e. if the vector field $\dot\gamma$ is parallel along γ.

Choose local coordinates $(x_1, \cdots, x_n)$ and write $\gamma:\ x_i = x_i(t)$.

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0 \quad\Longleftrightarrow\quad \frac{d^2 x_i}{dt^2} + \sum_{j,k} \Gamma^i_{jk}\dot x_i \dot x_j = 0, \quad i = 1, \dots, n.$$

Cor. *For any $m \in M$ and any $\mathrm{v} \in T_m M$ there exists a unique geodesic γ such that $\gamma(0) = m$ and $\dot\gamma(0) = \mathrm{v}$.*

Rem. γ is not necessarily defined on the whole real line.

Def. (M, g) is called *complete*, if each geodesic is defined on the whole $\mathbb{R}$.

Def (Exponential map). For $m \in M$ we define

$$\exp\colon T_m M \to M \qquad \exp(t\mathrm{v}) = \gamma_{\mathrm{v}}(t).$$

Rem. In general, $\exp$ is defined on $B_\varepsilon(0)$ only.

Since $\exp_* = \mathrm{id}$ at m, $\exp$ is a diffeomorphism between some neighbourhoods of $0 \in T_m M$ and $m \in M$.

Def (Normal coordinates). The map

$$M \xrightarrow{\exp^{-1}} T_m M \xrightarrow{p} \mathbb{R}^n, \qquad p \text{ is an isometry},$$

defined in a neighbourhood of m is called *normal coordinate system*.

Theorem (Gauss Lemma)

$$g_{\exp_m(\mathrm{v})}\big((\exp_m)_*\mathrm{v}, (\exp_m)_*\mathrm{v}\big) = g_m(\mathrm{v}, \mathrm{v}), \qquad \textit{for all } \mathrm{v} \in T_mM.$$

Recall: A solution to the equation

$$\ddot{J} + R(J, \dot{\gamma}_{\mathrm{v}})\dot{\gamma}_{\mathrm{v}} = 0, \qquad J \in \Gamma(\gamma_{\mathrm{v}}^* TM)$$

is called a *Jacobi vector field* along γ. If J_{v} is the unique Jacobi vector field satisfying $J_{\mathrm{v}}(0) = m,\ \dot{J}_{\mathrm{v}}(0) = \mathrm{v}$, then

$$(exp_m)_*\mathrm{v} = J_{\mathrm{v}}(1).$$

Def. $\mathrm{Hol}_p^0 = \big\{g \mid \Pi_\gamma(p) = pg,\ \gamma$ is contractible$\big\} \subset \mathrm{Hol}_p$ is called the *restricted holonomy group* at $p \in P$.

Hol_p^0 is the identity component of Hol_p.

Consider $\mathbb{R}^n$ as an $H = \mathrm{Hol}_p$–representation and write

$$\mathbb{R}^n = V_0 \oplus V_1 \oplus \cdots \oplus V_k. \tag{3}$$

Here V_0 is a trivial representation (may be 0), all $V_i,\ i \geq 1$, are irreducible. All V_i are pairwise orthogonal.

Prop. *Under* (3), $H^0 = \mathrm{Hol}_p^0$ *is isomorphic to a product*

$$\{e\} \times H_1 \times \cdots \times H_k.$$

Prop. *Under* (3), $H^0 = \mathrm{Hol}_p^0$ *is isomorphic to a product*

$$\{e\} \times H_1 \times \cdots \times H_k.$$

Proof. Let P be the holonomy bundle through $p \in Fr(M)$. Then, $\forall q \in P$ and $\forall x, y \in \mathbb{R}^n$ we have $R_q(x,y) \in \mathfrak{h}$. Hence

$$R_q(x,y)(V_i) \subset V_i.$$

Write $x = \sum x_i,\ y = \sum y_i$ with $x_i, y_i \in V_i$. Then

$$\langle R(x,y)u, v\rangle = \langle R(u,v)x, y\rangle = \sum_i \langle R(u,v)x_i, y_i\rangle$$
$$= \sum_i \langle R(x_i, y_i)u, v\rangle,$$

i.e. $R(x,y) = \sum_i R(x_i, y_i)$. By the Ambrose–Singer thm,

$$\mathfrak{h} = 0 \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_k, \qquad \text{with } \mathfrak{h}_i \subset \mathrm{End}\, V_i.$$

This implies the statement of the Proposition. □

Torsion Levi–Civita con–n Decomposable metrics Symmetric spaces Berger Thm

Prop. *Under* (3), M *is locally isomorphic to a Riemannian product*

$$M_0 \times M_1 \times \cdots \times M_k, \qquad \textit{where } M_0 \textit{ is flat.}$$

Proof. Denote $E_i := P \times_H V_i$, where P is the holonomy bundle. Then $TM = \bigoplus_i E_i$. Each distribution E_i is integrable:

$$v, w \in \Gamma(E_i) \Rightarrow \ \nabla_v w \in \Gamma(E_i) \ \Rightarrow \ [v,w] = \nabla_v w - \nabla_w v - 0 \in \Gamma(E_i).$$

From the Frobenius thm, in a neigbhd of m we may choose coordinates

$$x_1^1, \dots x_1^{r_1}; \dots; x_k^1, \dots x_k^{r_k}$$

s.t. $\frac{\partial}{\partial x_i^j}$ is belongs to E_i. If $v = \frac{\partial}{\partial x_i^j},\ w = \frac{\partial}{\partial x_s^t},\ i \neq s$, then $\nabla_v w = \nabla_w v$ belongs to $E_s \cap E_i = 0$. Hence,

$$\frac{\partial}{\partial x_s^t} g\Big(\frac{\partial}{\partial x_i^{j_1}}, \frac{\partial}{\partial x_i^{j_2}}\Big) = g\big(\nabla_w v_i^{j_1}, v_i^{j_2}\big) + g\big(v_i^{j_1}, \nabla_w v_i^{j_2}\big) = 0$$

provided $s \neq i$. Hence, the restriction of g to E_i depends on x_i^j only. □

Def. Under the circumstances of the previous Proposition, M is called *locally reducible*. M is called *locally irreducible* if the holonomy representation is irreducible.

Cor. *M is locally irreducible iff M is locally a Riemannian product.*

Theorem (de Rham decomposition theorem)

Let M be connected, simply connected, and complete. If the holonomy representation is reducible, then M is isometric to a Riemannian product.

Proof. [KN, Thm. IV.6.1] □

Symmetric spaces

Def. (M, g) is called *symmetric* if $\forall m \in M$ $\exists$ an isometry $s = s_m$ with the following properties:

$$s(m) = m, \qquad (s_*)_m = -\mathrm{id} \quad \text{on } T_mM.$$

Prop. *Let M be symmetric. Then*

(i) s_m *is a local geodesic symmetry, i.e.* $s_m(\exp_m(\mathrm{v})) = \exp_m(-\mathrm{v})$ *whenever* $\exp_m$ *is defined on* $\pm\mathrm{v}$*;*

(ii) (M, g) *is complete;*

(iii) $s_m^2 = \mathrm{id}_M$.

Proof. *(i):* s_m is isometry $\Rightarrow$ $s_m(\exp_m(\mathrm{v})) = \exp_m(s_*\mathrm{v}) = \exp_m(-\mathrm{v})$. *(ii):* If $\gamma\colon (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = m$ is a geodesic, then $s_m(\gamma(t)) = \gamma(-t)$ $\Rightarrow s_{\gamma(\tau/2)}(\gamma(t)) = \gamma(\tau - t) \Rightarrow s_{\gamma(\tau/2)} \circ s_m(\gamma(t)) = \gamma(\tau + t)$ whenever $\tau/2, t, \tau + t \in (-\varepsilon, \varepsilon)$. Since $s_{\gamma(\tau/2)} \circ s_m$ is globally defined, γ extends to $(0, +\infty)$. □

Prop. *A Riemannian symmetric space M is homogeneous, i.e. the group of isometries acts transitively on M.*

Proof. If γ is a geodesic, then $\gamma(t_1)$ is mapped to $\gamma(t_2)$ by s_m with $m = \gamma(\frac{t_1+t_2}{2})$.
For any $(p, q) \in M \times M$ there exists a sequence of geodesic segments put end to end which joins p and q (in fact, there is a single geodesic). Then the composition of reflections in the corresponding middle points maps p to q. □

Rem. In fact, we have shown, that the identity component G of the isometry group acts transitively.

Pick $m \in M$ and denote $K = Stab_m \subset G$. Then $M \cong G/K$.
Observe, that G is endowed with the involution

$$\sigma: G \to G, \qquad f \mapsto s_m \circ f \circ s_m$$

Theorem ([Helgason. Diff geom and symm spaces, IV.4])

- *(i)* *Let G be a connected Lie group with an involution σ and a left invariant metric which is also right–invariant under $\hat{K} = \{\sigma(g) = g\}$. Let K be a closed subgroup of G s.t. $\hat{K}^0 \subset K \subset \hat{K}$. Then $M = G/K$ is a symmetric space with its induced metric.*
- *(ii)* *Every symmetric space arises as in (i).*
- *(iii)* *We have the Cartan decomposition: $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ with*

  $$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}.$$

  Moreover, $T_m M \cong \mathfrak{m}$.
- *(iv)* $\mathrm{Hol}_m \subset K$.

Rem. Holonomy groups of Riemannian symmetric spaces were classified by Cartan (see [Besse. Einstein mflds, 7.H, 10.K])

Theorem

For a Riemannian mfld M the following conditions are equivalent:

(i) $\nabla R = 0$;

(ii) *the local geodesic symmetry s_m is an isometry for any $m \in M$.*

Def. (M, g) is called *locally symmetric*, if *(i)* $\Leftrightarrow$ *(ii)* holds.

Proof. *(ii)$\Rightarrow$(i):*
s_m isometry $\Rightarrow$ s_m preserves ∇R. On the other hand, since ∇R is of order 5, we must have $s_m^*(\nabla R)_m = -(\nabla R)_m$. Hence, $(\nabla R)_m = 0\ \forall m$.

$\nabla R = 0 \;\Rightarrow\; s_m$ is isometry:

$\gamma = \gamma_{\mathrm{w}}$ geodesic through m, $(e_1, \ldots, e_n)$ orthonormal frame of T_mM. Define $E_i \in \Gamma(\gamma^*TM) : \nabla_{\dot\gamma} E_i = 0,\ E_i(0) = e_i$.

$\nabla R = 0 \;\Rightarrow\; R(E_i, \dot\gamma)\dot\gamma$ is parallel along $\gamma \Rightarrow$
$R(E_i, \dot\gamma)\dot\gamma = \sum_j r_{ij} E_j$ with $r_{ij} = \langle R(E_i, \dot\gamma)\dot\gamma, E_j\rangle$, which is constant in t.

Write $J_{\mathrm{v}}(t) = \sum a^i_{\mathrm{v}}(t) E_i(t)$. Then a_{v} satisfies ODE with <u>constant coefficients</u> $\ddot a_{\mathrm{v}} + r a_{\mathrm{v}} = 0$.

Similarly, for $\bar\gamma = \gamma_{-\mathrm{w}}$ put $\bar E_i :\ \nabla_{\dot{\bar\gamma}} \bar E_i = 0,\ \bar E_i(0) = -e_i$;
$\bar J_{\mathrm{v}} = \sum \bar a^i_{\mathrm{v}} \bar E_i$. Then $\ddot{\bar a}_{\mathrm{v}} + r\bar a_{\mathrm{v}} = 0$ (with the same matrix r!).
Moreover, $\bar a_{\mathrm{v}}(0) = 0 = a_{\mathrm{v}}(0)$ and $\dot{\bar a}_{\mathrm{v}}(0) = \dot a_{\mathrm{v}}(0)$. Hence $\bar J_{\mathrm{v}}(1) = J_{\mathrm{v}}(1)$. Then

$$\begin{aligned}\langle J_{\mathrm{v}}(1), J_{\mathrm{v}}(1)\rangle = \langle \mathrm{v}, \mathrm{v}\rangle &= \langle \bar J_{\mathrm{v}}(1), \bar J_{\mathrm{v}}(1)\rangle \\ &= \langle (s_m)_* J_{\mathrm{v}}(1), (s_m)_* J_{\mathrm{v}}(1)\rangle.\end{aligned}$$

□

Berger theorem revisited

Theorem (Berger thm)

Assume M is a simply–connected irreducible not locally symmetric Riemannian mfld of dimension n. Then Hol is one of the following:

	Holonomy	Geometry	Extra structure
•	$SO(n)$		
•	$U(n/2)$	*Kähler*	*complex*
•	$SU(n/2)$	*Calabi–Yau*	*complex + hol. vol.*
•	$Sp(n/4)$	*hyperKähler*	*quaternionic*
•	$Sp(1)Sp(n/4)$	*quaternionic Kähler*	*"twisted" quaternionic*
•	G_2 *(n=7)*	*exceptional*	*"octonionic"*
•	$Spin(7)$ *(n=8)*	*exceptional*	*"octonionic"*

Comments to the Berger theorem

- The assumption $\pi_1(M) = 0$ could be dropped by restricting attention to Hol^0.
- M is locally symmetric $\Rightarrow$ M is locally isometric to a symmetric space. Holonomies of simply connected symmetric spaces are known.
- Irreducibility could be dropped by taking all possible products of the entries of the Berger list.
- In the theorem, Hol is not just an abstract group, but rather a subgroup of $SO(n)$, or, equivalently, comes together with an irreducible n–dimensional representation.

Ex. For instance,

$$SO(m) = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & A \end{array} \right) \right\} \subset SO(2m)$$

is never a holonomy representation of an irreducible manifold (in fact, this is never a holonomy representation of any Riemannian manifold).