Holonomy groups

in Riemannian geometry

Lecture 6

December 1, 2011

Some results from the previous lecture

Prop. *The first Chern class* $c_1(M)$ *is represented by* $\frac{1}{2\pi}\rho$*, where* ρ *is the Ricci form.*

Cor. *The curvature tensor of the canonical line bundle* $K_M = \Lambda^{m,0}T^*M = \Lambda^m(T^*M)^{1,0}$ *equals* $i\rho$*.*

Theorem

Let M^{2m} *be a Kähler mfld. Then* $\mathrm{Hol}^0(M) \subset SU(m)$ *iff* $Ric \equiv 0$*.*

Theorem

$\mathrm{Hol}(M) \subset SU(M)$ *iff* M *admits a parallel* $(m, 0)$*–form.*

Calabi-Yau and Kähler-Einstein metrics

Let (M, I) be be a closed connected complex mfld.

Def. A Kähler metric g is said to be Kähler-Einstein if it is Einstein, i.e. if there exists a constant λ such that

$$\rho = \lambda\omega. \tag{1}$$

Rem.

(i) $\lambda\colon M \to \mathbb{R}$ in (1) $\implies$ $\lambda = const.$

(ii) (1) $\iff$ $R(\omega) = \lambda\omega.$

Def. A class $c \in H^2(M; \mathbb{R})$ is said to be

- positive, if $\exists \beta \in c \cap \Omega^{1,1}$ s.t. $\beta(\cdot, I\cdot) > 0$;
- negative, if $\exists \beta \in c \cap \Omega^{1,1}$ s.t. $\beta(\cdot, I\cdot) < 0$.

Main Theorems

Theorem (Calabi-Yau)

Let $\rho' \in 2\pi\, c_1(M)$ be a closed real $(1,1)$-form. Then there exists a unique Kähler metric g' on M with Kähler form ω' cohomologous to ω and with Ricci form ρ'.

Cor. *If $c_1(M) = 0$, then M has a unique Ricci-flat Kähler metric g' with $[\omega'] = [\omega]$.*

Theorem (Aubin-Calabi-Yau)

Assume $c_1(M) < 0$. Then, up to a scaling constant, M has a unique Kähler-Einstein metric (with negative Einstein constant) .

On the proof of Calabi-Yau and Aubin-Calabi-Yau theorems

Let $\Omega \in \Omega^{m,0}(U)$, where $U \subset M$ is open. Write

$$\nabla\Omega = \psi \otimes \Omega,$$

where ψ is a local connection form of $\Lambda^{m,0}T^*M$.

Observe: $\Omega \in \Omega^{m,0} \;\Rightarrow\; \partial\Omega = 0 \;\Rightarrow\; \bar\partial\Omega = d\Omega = \psi \wedge \Omega$. By definition, Ω is holomorphic, if $\bar\partial\Omega = 0$. Since Ω is a complex volume form,

$$\bar\partial\Omega = 0 \quad\Longleftrightarrow\quad \psi^{0,1} \wedge \Omega = 0 \quad\Longleftrightarrow\quad \psi \in \Omega^{1,0}.$$

We have

$$\begin{aligned} d(\log\|\Omega\|^2) &= \frac{1}{\|\Omega\|^2}\, d\langle\Omega,\Omega\rangle \\ &= \frac{1}{\|\Omega\|^2}\,(\psi\|\Omega\|^2 + \bar\psi\|\Omega\|^2) \\ &= \psi + \bar\psi. \end{aligned}$$

Ω is holomorphic $\Longrightarrow \quad \psi = (d(\log\|\Omega\|^2))^{1,0} = \partial(\log\|\Omega\|^2)$.

Hence, the curvature of $\Lambda^{m,0}T^*M$ is represented by $d\psi = \bar\partial\partial\log\|\Omega\|^2$. In particular, $d\psi$ is purely imaginary $(1,1)$-form. Hence,

$$\boxed{\rho = i\, d\psi = -i\,\partial\bar\partial\log\|\Omega\|^2.}$$

Further, observe that

$$*\Omega = a \cdot \bar{\Omega},$$

where $a \in \mathbb{C}^*$. Hence, $a \cdot m!\,\Omega \wedge \bar{\Omega} = \|\Omega\|^2 \omega^m$. If g' is another Kähler metric s.t. $[\omega'] = [\omega]$, then

$$(\omega')^m = e^f \cdot \omega^m$$

for some $f : M \to \mathbb{R}$. Therefore,

$$\|\Omega\|^2_{g'} = e^{-f} \|\Omega\|^2_g \quad \Longrightarrow \rho' = \rho - i\partial\bar{\partial} f.$$

Vice versa, by the $\partial\bar{\partial}$-Lemma, for any real closed $(1,1)$-form ρ' cohomologous to ρ, there exists $f \colon M \to \mathbb{R}$ s.t.

$$\rho' - \rho = -i\,\partial\bar{\partial} f.$$

Moreover, f is unique up to an additive constant. Similarly,

$$\omega' - \omega = i\,\partial\bar{\partial}\varphi, \qquad \varphi \colon M \to \mathbb{R}.$$

Thus, in the setting of the CY thm, we are looking for φ s.t.

(i) $(\omega + i\,\partial\bar{\partial}\varphi)^m = e^f \cdot \omega^m,$ $\quad (*)$

(ii) $\omega + i\,\partial\bar{\partial}\varphi > 0,$

where f is a fixed function.

Claim. $(i) \;\Rightarrow\; (ii)$

Proof. [Ballmann. Lectures on Kähler mflds, p.90]. □

Rem. For Kähler mflds, eqn $Ric(g) = 0$ is therefore equivalent to $(*)$. Notice that

- $(*)$ is an eqn for a *function* rather than for a metric tensor,
- $(*)$ is highly nonlinear (nonlinear in derivatives of the highest order).

Claim. The Kähler-Einstein condition (under the setup of Aubin-Calabi-Yau thm) is equivalent to the eqn

$$(\omega + i\,\partial\bar{\partial}\varphi)^m = e^{f - \lambda\varphi} \cdot \omega^m,$$

where ω is a suitably chosen Kähler metric on M.

Proof. [see Ballmann, p.91 for details]. □

Idea of the proof of the Calabi-Yau thm

Uniqueness: Let φ_1, φ_2 be solutions of the eqn

$$(\omega + i\,\partial\bar\partial\varphi)^m = e^{F(p,\varphi)}\omega^m.$$

It can be shown that

$$\frac{1}{m}\int |\mathrm{grad}(\varphi_1 - \varphi_2)|^2_{g_1}\omega_1^m + \\ + \int (\varphi_1 - \varphi_2)(e^{F(p,\varphi_1)} - (e^{F(p,\varphi_2)})\omega^m \le 0.$$

Hence, uniqueness follows from the (weak) monotonicity of F in φ (for each fixed $p \in M$).

Existence (by the continuity method): Consider the eqn

$$(\omega + i\,\partial\bar\partial\varphi)^m = e^{tf}\omega^m,$$

where $t \in [0,1]$ is a parameter. Denote by $\mathcal{T}$ the set of those t, for which there exists a solution. Then $\mathcal{T} \ni 0$, hence $\mathcal{T} \neq \emptyset$. Moreover, $\mathcal{T}$ is open and closed. Hence, $1 \in \mathcal{T}$.

Examples of Calabi–Yau manifolds

A compact (simply connected) Riemannian mfld with $\mathrm{Hol}(M,g) \subset SU(m)$ is called *Calabi-Yau*. If $\pi_1(M) = \{1\}$ this is equivalent to $c_1(M) = 0$.

Ex.

1) Let M be a degree d hypersurface in $\mathbb{C}P^N$. From the adjunction formula we have

$$K_M = \big(K_{\mathbb{CP}^N} \otimes \mathcal{O}(d)\big)\big|_M \cong \mathcal{O}(-N-1+d)\big|_M.$$

Therefore, $c_1(K_M) = 0 \;\Leftrightarrow\; d = N+1$. Hence, the Fermat quartic $M = \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{CP}^3$ admits a metric with holonomy $SU(2)$.

2) Let M be a complete intersection: $M = M_{d_1} \cap \cdots \cap M_{d_k} \subset \mathbb{C}P^N$. Then $c_1(M) = 0 \;\Leftrightarrow\; d_1 + \cdots + d_k = N+1$.

A non-compact example: Calabi metric

Theorem (Calabi)

Let M be Kähler–Einstein with positive sc. curvature. Then there exists a metric on the total space of K_M with $\mathrm{Hol}^0 \subset SU(m+1)$.

Proof. Let $P \to M$ be the $U(m)$-structure. Since $\mathfrak{u}(m) \cong \mathfrak{su}(m) \oplus i\mathbb{R}$, the Levi-Civita connection on P decomposes: $\varphi_{LC} = \varphi_0 + \psi i$. Observe that ψi is essentially the connection of K_M. It follows that M is KE iff $d\psi = \lambda \pi^* \omega$, where $\pi : P \to M$.
Consider $\beta = dz + z\psi i \in \Omega^1(P \times \mathbb{C}; \mathbb{C})$, where z is a coordinate on $\mathbb{C}$. Put $\rho = |z|^2 = z\bar{z}$. With the help of

$$d\beta = (\beta \wedge \psi + \lambda z \pi^* \omega) i, \qquad d\rho = dz \cdot \bar{z} + z d\bar{z} = \beta \cdot \bar{z} + z\beta,$$

one easily shows that the 2-form

$$\tilde{\omega} = u\pi^*\omega - \frac{1}{\lambda} u' \cdot i\beta \wedge \bar{\beta}$$

is closed, where $u = u(\rho)$.

Proof of the Calabi theorem (continued)

Moreover, $\tilde{\omega} = u\pi^*\omega - \frac{1}{\lambda} u' \cdot i\beta \wedge \bar{\beta}$ is $U(m)$-invariant and basic and therefore descends to a $(1,1)$-form $\tilde{\omega}$ on $(P \times \mathbb{C})/U(m) = K$. If both u and u' are positive, $\tilde{\omega}$ is also positive.
Recall that each $p \in P$ is a unitary basis of $T_{\pi(p)}M$, i.e. $p = (p_1, \dots, p_m)$. Then $\Omega = p_1^* \wedge \dots \wedge p_m^*$ is a global complex m-form on P. Consider

$$\tilde{\Omega} = \beta \wedge \Omega.$$

Just like $\tilde{\omega}$, $\tilde{\Omega}$ descends to an $(m+1, 0)$-form on K. Then $\tilde{\Omega}$ is parallel iff $\|\tilde{\Omega}\| = const \Rightarrow u^m u' = \lambda(m+1) \Rightarrow$ $u(\rho) = (\lambda\rho + l)^{\frac{1}{m+1}}$. Hence we obtain an explicit metric on K with $\mathrm{Hol}^0 \subset SU(m+1)$, namely

$$g = u(p)\pi_K^* g_M \oplus u'(\rho) \mathrm{Re}(\beta \otimes \bar{\beta}). \qquad \square$$

Rem. If the scalar curvature of M is negative, the Calabi metric is defined on a neighbourhood of the zero section only.

HyperKähler manifolds

A quaternionic vector space is a real vector space V equipped with a triple (I_1, I_2, I_3) of endomorphisms s.t.

$$I_r^2 = -1, \quad I_1 I_2 = I_3 = -I_2 I_1.$$

In other words, V is an $\mathbb{H}$-module.

V is *quaternion-Hermitian*, if V is equipped with an Euclidean scalar product, which is Hermitian wrt each complex structure I_r. Denote $\omega_r(\cdot,\cdot) = \langle I_r \cdot, \cdot\rangle$, $\omega = \omega_1 i + \omega_2 j + \omega_3 k$.

Ex. $V = \mathbb{H}^m$, $I_1(h) = h\bar{i}$, $I_2(h) = h\bar{j}$, $I_3(h) = h\bar{k}$, $\langle h_1, h_2\rangle = \mathrm{Re}(\bar{h}_1 h_2)$. Then $\omega(h_1, h_2) = \mathrm{Im}(\bar{h}_1 h_2)$

Put $h = \langle\cdot,\cdot\rangle + i\omega_1$ and $\omega_c = \omega_2 + \omega_3 i$. Then h is an Hermitian scalar product and ω_c is a complex symplectic form. Hence,

$$\begin{aligned} Sp(m) &= \{A \in O(\mathbb{H}^n) \mid AI_r = I_r A, \quad r = 1,2,3\} \\ &= O(4n) \cap GL_n(\mathbb{H}) \\ &= U(2n) \cap Sp(2n; \mathbb{C}). \end{aligned}$$

Assume M^{4m} is endowed with with an $Sp(m)$–structure. In other words, M is a Riemannian mfld equipped with a triple (I_1, I_2, I_3) of almost complex structures s.t. the metric is Hermitian wrt each I_r.

Alternatively, M can be seen as an almost Hermitian mfld equipped with a complex symplectic form $\omega_c \in \Omega^{2,0}(M)$.

M is called *hyperKähler*, if $\mathrm{Hol(M)} \subset Sp(m)$. This is equivalent to one of the following conditions:

(i) $\nabla I_1 = \nabla I_2 = \nabla I_3 = 0$;

(ii) $\nabla \omega_1 = \nabla \omega_2 = \nabla \omega_3 = 0$;

(iii) g is Kähler wrt each complex structure I_r.

Prop. *For an almost hyperKähler manifold the following holds:*

$$\nabla\omega_1 = \nabla\omega_2 = \nabla\omega_3 = 0 \iff d\omega_1 = d\omega_2 = d\omega_3 = 0.$$

Proof. Need to show that each almost complex structure is integrable. Observe: $v \in \mathfrak{X}^{1,0}_{I_1}(M) \Leftrightarrow \imath_v\omega_2 = i\,\imath_v\omega_3$. Indeed,

$$i_v\omega_2 = g(I_2 v, \cdot) = g(I_3 I_1 v, \cdot) = \omega_3(I_1 v, \cdot).$$

Then $\imath_v\omega_2 = i\imath_v\omega_3 \Leftrightarrow I_1 v = iv$.
Assume now $v, w \in \mathfrak{X}^{1,0}_{I_1}(M)$. Then

$$\begin{aligned}
\imath_{[v,w]}\omega_2 &= \mathcal{L}_v(\imath_w\omega_2) - \imath_w(\mathcal{L}_v\omega_2) \\
&= \mathcal{L}_v(\imath_w\omega_2) - \imath_w(\imath_v\omega_2) && \text{(Cartan)} \\
&= \mathcal{L}_v(i\imath_w\omega_3) - \imath_w(i\imath_w\omega_3) \\
&= i\,\imath_{[v,w]}\omega_3.
\end{aligned}$$

□

Examples of hyperKähler manifolds

Ex.

(i) We have an exceptional isomorphism $Sp(1) \cong SU(2)$, since $\omega_c \in \lambda^{2,0}\mathbb{C}^2$ is a complex volume form. Hence, if $\dim_{\mathbb{R}} M = 4$

Calabi-Yau ≡ hyperKähler

Hence, there is a hK metric on the Fermat quartic.

(ii) Similar methods as in the proof of the fact that for KE M the total space of K_M has a Ricci-flat metric, also give that the total space of $T^*\mathbb{C}P^m$ has a complete metric with holonomy $Sp(m)$ for any m (this fact is also due to Calabi).

Let M^{4m} be a *compact* Kähler with a complex sympl. form ω_c. Then ω_c^m trivializes K_M and hence there exists a Ricci-flat Kähler metric on M.

Observe that any closed $(p, 0)$–form on closed Ricci–flat Kähler mfld must be parallel. This follows from the fact that the Weitzenböck formula for $(p, 0)$–forms involves Ricci–curvature only.

Hence, with respect to the new Ricci–flat metric $\nabla\omega_c = 0$. Thus if M is compact Kähler

$$\text{hyperKähler} \equiv \text{complex symplectic}$$

This is used to show that there are compact 8-mflds with holonomy $Sp(2)$ by blowing-up the diagonal in $M_4 \times M_4$ and quotening by the involution. Further generalization of this yields compact mflds with holonomy $Sp(m)$.

HyperKähler reduction

Let M be a hK mfld and assume G acts on M preserving hK structure. Then for any $\xi \in \mathfrak{g}$

$$0 = \mathcal{L}_{K_\xi}\omega_r = \imath_{K_\xi} d\omega_r + d\imath_{K_\xi}\omega_r = 0 + d\imath_{K_\xi}\omega_r,$$

where K_ξ is the Killing v.f.
Assume there exists $\mu_r(\xi) : M \to \mathbb{R}$ s.t. $i_{K_\xi}\omega_r = d\mu_r(\xi)$.
Construct a G-equivariant map

$$\mu = \mu_1 i + \mu_2 j + \mu_3 k : \quad M \to \mathfrak{g}^* \otimes \operatorname{Im}\mathbb{H},$$

which is called the hK moment map.

Theorem

If $M /\!\!/\!\!/_\tau G = \mu^{-1}(\tau)/G$ is a mfld, where $\tau \in \mathfrak{g}^*$ is central, then it is hyperKähler (with respect to the induces metric).

Proof. For $m \in \mu^{-1}(\tau)$ put $\mathcal{K}_m = \{K_\xi(m) \mid \xi \in \mathfrak{g}\}$. Since $d\mu_r(\xi) = g(I_r K_\xi, \cdot)$, the orthogonal complement to

$$\mathcal{K}_m \oplus I_1\mathcal{K}_m \oplus I_2\mathcal{K}_m \oplus I_3\mathcal{K}_m$$

can be identified with $T_{[m]}(M /\!\!/\!\!/_\tau G)$. Hence $M /\!\!/\!\!/_\tau G$ is almost hyperKähler. The corresponding 2-forms are closed, hence $M /\!\!/\!\!/_\tau G$ is hyperKähler. □

Further examples of hyperKähler manifolds

Ex.

1) S^1 acts on $\mathbb{H}^{n+1}$ by multiplication on the left. The moment map is

$$\mu(x) = -\sum_{p=1}^{n+1} \bar{x}_p i x_p = i\sum_{p=1}^{n+1}(|w_p|^2 - |z_p|^2) - 2k\sum_{p=1}^{n+1} z_p w_p,$$

where $x_p = z_p + jw_p,\ z_p, w_p \in \mathbb{C}$. Clearly,

$$\mathbb{H}^{n+1} /\!\!/\!\!/ S^1 = \mu^{-1}(-i)/S^1 \cong$$

$$\cong \{(z_p, w_p) \in \mathbb{C}^{2n+2} | \sum_{p=1}^{n+1} z_p w_p = 0,\ (z_1, \dots, z_{n+1}) \neq 0\}/\mathbb{C}^*$$

$$\cong T^*\mathbb{C}P^n.$$

Hence, the total space of $T^*\mathbb{CP}^n$ is hK and the metric obtained via the hK reduction coincides with the Calabi metric.

Ex.

2) $T^*Gr_p(\mathbb{C}^{p+q})$ is hK. This is also obtained as a hK reduction: $T^*Gr_p(\mathbb{C}^{p+q}) \cong \mathbb{H}^{p(p+q)} /\!/\!/ U(p)$.

3) Let X^4 be a hK mfld. Pick a G-bundle $P \to X$. Then the space $\mathcal{A}(P)$ inherits a hK structure. The action of the gauge gp $\mathcal{G} = AutP$ preserves this hK structure and the moment map is

$$\begin{aligned} \mu : A \longmapsto F_A^+ \in \Omega^2_+(X; \operatorname{ad} P) &\cong \\ \cong \Gamma(\operatorname{ad} P) \otimes \operatorname{Im} \mathbb{H} &\cong \\ \cong \operatorname{Lie}(\mathcal{G})^* \otimes \operatorname{Im} \mathbb{H}. \end{aligned}$$

Hence, the moduli space of asd instantons

$$\mu^{-1}(0)/\mathcal{G} \cong \{A \mid F_A^+ = 0\}/\mathcal{G}$$

is hyperKähler.

Quaternion-Kähler manifolds

Consider the action of $Sp(n) \times Sp(1)$ on $\mathbb{H}^n$:

$$(A, q) \cdot x = Ax\bar{q}.$$

Obviously, $(-1, -1)$ acts trivially and we define

$$Sp(n)Sp(1) = Sp(n) \times Sp(1)/\pm 1 \subset SO(4n).$$

Consider $\Lambda^1 = \mathbb{R}^{4n}$ as $Sp(n)Sp(1)$-representation. Then

$$\Lambda^1_{\mathbb{C}} \cong E \otimes_{\mathbb{C}} W,$$

where E denotes the complex tautological representation of $Sp(n) \subset SU(2n)$ of dimension $2n$ and W denotes the two dimensional complex representation of $Sp(1) \cong SU(2)$. Explicitly,

$$v \longmapsto v^{1,0} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + I_2 v^{0,1} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then
$$\begin{aligned}\mathfrak{so}(4n) &\cong \Lambda^2(\mathbb{R}^{4n})^* \cong \Lambda^2[E\otimes W]_r \\ &\cong [S^2E\otimes\Lambda^2W]_r \oplus [\Lambda^2E\otimes S^2W]_r \\ &\cong \mathfrak{sp}(n)\oplus[\Lambda^2E\otimes W_2]_r \\ &\cong \mathfrak{sp}(n)\oplus\mathfrak{sp}(1)\oplus[\Lambda^2_0E\otimes W_2]_r.\end{aligned}$$

Here: $W_p = S^pW$ is the irreducible $(p+1)$-dimensional $Sp(1)$-representation. In particular, $W_1 = W$, $W_2 = \mathfrak{sp}(1)_{\mathbb{C}}$. Consider the 4-form

$$\Omega_0 = \omega_1\wedge\omega_1 + \omega_2\wedge\omega_2 + \omega_3\wedge\omega_3 \quad \in \Lambda^4(\mathbb{R}^{4n})^*,$$

which is $Sp(n)Sp(1)$-invariant.

Lem. *For $n \geq 2$, the subgp of $GL_{4n}(\mathbb{R})$ preserving Ω_0 is equal to $Sp(n)Sp(1)$.*

Proof. [Salamon. Lemma 9.1] □

Rem. Hence, the 4-form Ω_0 determines the Euclidean scalar product.

An $Sp(n)Sp(1)$-structure on M^{4n}, $n \geq 2$ can be described by $\Omega \in \Omega^4(M)$, which is linearly equivalent to Ω_0 at each pt. Then M is quaternion-Kähler, i.e. $\mathrm{Hol}(M) \subset Sp(n)Sp(1)$, iff $\nabla\Omega = 0$. In particular, $d\Omega = 0$.

Theorem (Swann)

If $\dim M \geq 12$, then $\nabla\Omega = 0 \;\Leftrightarrow\; d\Omega = 0$.

In contrast to hK mflds, qK mflds do not have global almost complex structures but rather are endowed with rank 3 subbundle of $End(TM)$ admitting *local* trivialization (I_1, I_2, I_3) satisfying quaternionic relations. This is apparent from the decomposition

$$\mathfrak{so}(4n) \cong \mathfrak{sp}(n)\oplus\mathfrak{sp}(1)\oplus[\Lambda^2_0E\otimes W_2]_r.$$

Prop. *The spaces of algebraic curvature tensors for qK and hK mflds are given respectively by*

$$\mathcal{R}^{Sp(n)Sp(1)} \cong [S^4E]_r \oplus \mathbb{R},$$
$$\mathcal{R}^{Sp(n)} \cong [S^4E]_r.$$

Proof. Similar to the corresponding proof for Kähler mflds. For details see [Salamon. Prop. 9.3]. □

Cor. *Any qK mfld is Einstein, and its Ricci tensor vanishes iff it is locally hK, i.e.* $\mathrm{Hol}^0 \subset Sp(n)$.

Ex. $\mathbb{H}P^n = \mathbb{H}^{n+1} \backslash \{0\} / \mathbb{H}^* \cong \frac{Sp(n+1)}{Sp(n) \times Sp(1)}$ is a symmetric qK mfld. All qK symmetric spaces were classified by Woff.

CY and KE mflds HyperKähler mflds Quaternion–Kahler mflds

Theorem (Swann)

Let M^{4n} be a positive qK mfld with the corresponding $Sp(n)Sp(1)$-structure P. Then the total space of the bundle $\mathcal{U}(M) = P \times_{Sp(n)Sp(1)} \mathbb{H}^* / \pm 1$ carries a hK metric.

The construction of this hK metric is similar to the construction of the Calabi metrics (Ricci–flat on K_M and hK on $T^*\mathbb{CP}^n$).