Holonomy groups

in Riemannian geometry

Lecture 7

Exceptional holonomy groups

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Groups $Spin(3)$, $Spin(4)$, and $Sp(1)$

Recall: For $n \geq 3$, $\mathrm{Spin}\,(n)$ is a connected simply connected group fitting into the short exact sequence

$$0 \to \{\pm 1\} \to \mathrm{Spin}\,(n) \to SO(n) \to 0,$$

In other words, $SO(n) \cong \mathrm{Spin}\,(n)/\pm 1$.

The group $Sp(1) = \{q \in \mathbb{H} \mid q\bar{q} = 1\}$ acts on $\mathrm{Im}\,\mathbb{H}$: $\quad q \cdot x = qx\bar{q}$. Hence, we have the short exact sequence

$$0 \to \{\pm 1\} \to Sp(1) \to SO(3) \to 0,$$

which establishes the isomorphism $\mathrm{Spin}\,(3) \cong Sp(1) \cong SU(2)$.

Consider also the action of $Sp_+(1) \times Sp_-(1)$ on $\mathbb{H}$: $(q_+, q_-) \cdot x = q_+ x \bar{q}_-$. This leads to the short exact sequence

$$0 \to \{\pm 1\} \to Sp_+(1) \times Sp_-(1) \to SO(4) \to 0.$$

Hence, $\mathrm{Spin}\,(4) \cong Sp_+(1) \times Sp_-(1)$.

The group G_2

Put $V = \operatorname{Im}\mathbb{H}_x \oplus \mathbb{H}_y \cong \mathbb{R}^7$, which is considered as oriented Euclidean vector space. $SO(4)$ acts on V:

$$[q_+, q_-] \cdot (x, y) = (q_- x \bar{q}_-, q_+ y \bar{q}_-).$$

Write

$$\begin{aligned}\tfrac{1}{2}\, d\bar{y} \wedge dy =& \omega_1 i + \omega_2 j + \omega_3 k \\ =& (dy_0 \wedge dy_1 - dy_2 \wedge dy_3) i + (dy_0 \wedge dy_2 + dy_1 \wedge dy_3) j + \\ &+ (dy_0 \wedge dy_3 - dy_1 \wedge dy_2) k.\end{aligned}$$

Notice that $(\omega_1, \omega_2, \omega_3)$ is the standard basis of $\Lambda^2_-(\mathbb{R}^4)^*$. Put

$$\begin{aligned}\varphi &= vol_x - \tfrac{1}{2}\operatorname{Re}(dx \wedge dy \wedge d\bar{y}) \\ &= dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge \omega_1 + dx_2 \wedge \omega_2 + dx_3 \wedge \omega_3.\end{aligned}$$

Def. The stabilizer of φ in $GL_7(\mathbb{R})$ is called G_2.

$$\varphi = vol_x - \tfrac{1}{2}\operatorname{Re}(dx \wedge dy \wedge d\bar{y}).$$

Observe the following:

- $L^*_{[q_+,q_-]} d\bar{y} \wedge dy = q_- d\bar{y} \wedge dy\, \bar{q}_- \;\Rightarrow\; \operatorname{Re}(dx \wedge dy \wedge d\bar{y})$ is $SO(4)$-invariant $\Rightarrow$ $SO(4) \subset G_2$.
- Write $V = (\mathbb{R} \oplus \mathbb{C}_z) \oplus \mathbb{C}^2_{w_1,w_2}$, $(x_0, z, w_1, w_2) \mapsto x_0 i + z j + \bar{w}_1 + w_2 j$. Then

  $$\begin{aligned}\varphi = & \tfrac{1}{2} dx_0 \wedge \operatorname{Im}(dz \wedge d\bar{z} + dw_1 \wedge d\bar{w}_1 + dw_2 \wedge d\bar{w}_2) \\ & + \operatorname{Re}(dz \wedge dw_1 \wedge dw_2)\end{aligned}$$

  Hence, $G_2 \supset SU(3)$.
- $SO(4) \subset G_2$, $SU(3) \subset G_2$ $\Rightarrow G_2 \cap SO(7)$ acts transitively on S^6.

- For $Q : V \to \Lambda^7 V$, $Q(v) = (i_v\varphi)^2 \wedge \varphi$ we have $Q(e_1) = \|e_1\|^2 vol_7 \Rightarrow Q(v) = \|v\|^2 vol_7$ for all $v \in V$.
- $g \in G_2 \Rightarrow g^*Q(gv) = Q(v) \Rightarrow (\det g) \cdot \|gv\|^2 = \|v\|^2$ $\Rightarrow \det g = 1$, i.e. $G_2 \subset SO(7)$
- $\{g \in G_2 \mid ge_1 = e_1\} \cong SU(3)$. Hence, we have that topologically G_2 is the fibre bundle

$$\begin{array}{ccc} SU(3) & \hookrightarrow & G_2 \\ & & \downarrow \\ & & S^6 \end{array}$$

In particular, $\dim G = 14$; G is connected and simply connected.

- $\Lambda^3 V^* \supset GL_7(\mathbb{R}) \cdot \varphi \cong GL_7(\mathbb{R})/G_2$ has dimension $35 = \dim \Lambda^3 V^*$. Hence, $GL_7(\mathbb{R}) \cdot \varphi$ is an open set in $\Lambda^3 V^*$.

Fact. G_2 is the automorphism group of octonions, i.e.

$$\{g \in GL_8(\mathbb{R}) \mid g(ab) = g(a) \cdot g(b)\} \cong G_2.$$

Some representation theory of G_2

Consider $V \cong \mathbb{R}^7$ as a G_2-representation via the embedding $G_2 \subset SO(7)$. Then V is irreducible.

Further $\Lambda^2 V^*$ contains the following G_2-invariant subspaces

- $\Lambda^2_{14} V^* \cong \mathfrak{g}_2$
- $\Lambda^2_7 V^* = \{i_v\varphi \mid v \in V\} \cong V$

which are irreducible. By dimension counting,

$$\Lambda^2 V^* \cong \Lambda^2_{14} V^* \oplus \Lambda^2_7 V^*.$$

Rem. The subspaces Λ^2_7 and Λ^2_{14} can be described equivalently as follows:

$$\Lambda^2_7 = \{\alpha \mid *(\varphi \wedge \alpha) = 2\alpha\}$$
$$\Lambda^2_{14} = \{\alpha \mid *(\varphi \wedge \alpha) = -\alpha\}$$

To decompose $\Lambda^3 V^*$, consider

$$\gamma\colon \mathrm{End}(V) \cong V \otimes V \mapsto \Lambda^3 V^*, \qquad \gamma(a) = a^*\varphi.$$

Then $\mathrm{Ker}\,\gamma = \mathfrak{g}_2$. Since $\dim \mathrm{Im}\,\gamma = 7 \times 7 - \dim \mathrm{Ker}\,\gamma = 35$ $= \dim \Lambda^3 V^*$, γ is surjective. Hence,

$$\Lambda^3 V^* \cong S^2 V^* \oplus \Lambda^2_7 V^* \cong \mathbb{R} \oplus S^2_0 V^* \oplus V^*$$

and $S^2_0 V^*$ is irreducible. We summarize,

Lem.

$$\Lambda^2 V^* \cong \mathfrak{g}_2 \oplus V,$$
$$\Lambda^3 V^* \cong \mathbb{R} \oplus V \oplus S^2_0 V^*$$

G_2 as a structure group

A G_2-structure on M^7 is determined by a 3-form φ, which is pointwise linearly equivalent to the 3–form $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$. In particular, φ determines a Riemannian metric g_φ and an orientation.

The following Lemma is auxiliary and will be proved in the next lecture.

Lem. *Denote by $\sigma : \mathbb{R}^n \otimes \Lambda^k(\mathbb{R}^n)^* \to \Lambda^{k-1}(\mathbb{R}^n)^*$ the contraction map.Then, for any Riemannian mfld M, the map*

$$\Gamma(\Lambda^k T^*M) \xrightarrow{\nabla^{LC}} \Gamma(T^*M \otimes \Lambda^k T^*M) \xrightarrow{-\sigma} \Gamma(\Lambda^{k-1} T^*M)$$

coincides with $d^* : \Omega^k \to \Omega^{k-1}$.

Theorem

φ is parallel wrt the Levi-Vita connection of g_φ iff $d\varphi = 0 = d(*_\varphi\varphi)$.

Proof. Recall that the intrinsic torsion of the G_2–structure can be identified with $\nabla\varphi$. In particular, $\nabla\varphi$ takes values in $V^* \otimes \mathfrak{g}_2^\perp \cong V^* \otimes V \cong (S_0^2 V^* \oplus \mathbb{R}) \oplus (\mathfrak{g}_2 \oplus V)$. Observe that $d\varphi$ and $d(*\varphi)$ can be obtained from $\nabla\varphi$ by means of the algebraic maps

$$V^* \otimes V \hookrightarrow V^* \otimes \Lambda^3 V^* \longrightarrow \Lambda^4 V^* \cong \Lambda^3 V^* \cong \mathbb{R} \oplus V \oplus S_0^2 V^*.$$
$$V^* \otimes V \hookrightarrow V^* \otimes \Lambda^3 V^* \mapsto \Lambda^2 V^* \cong \mathfrak{g}_2 \oplus V.$$

One can show that both maps are surjective. Comparing components of target spaces with the components of

$$V^* \otimes V \cong S_0^2 V^* \oplus \mathbb{R} \oplus \mathfrak{g}_2 \oplus V$$

we obtain that $\nabla\varphi = 0 \iff d\varphi = 0 = d(*\varphi)$. □

Curvature of a G_2–manifold

Let $c : S^2\mathfrak{g}_2 \to S^2 V^*$ be the Ricci contraction. Denote $F = \operatorname{Ker} c$. This is an irreducible G_2-representation of dimension 77.

Recall that $\mathcal{R}^{G_2} \cong \operatorname{Ker} b \cap S^2\mathfrak{g}_2$, where

$$b : S^2(\Lambda^2 V^*) \to \Lambda^4 V^*$$

is the Bianchi map. Notice that

$$S^2\mathfrak{g}_2 \cong F \oplus S_0^2 V^* \oplus \mathbb{R},$$
$$\Lambda^4 V^* \cong \Lambda^3 V^* \cong V \oplus S_0^2 V^* \oplus \mathbb{R}$$

The Bianchi map is injective on $S_0^2 V^* \oplus \mathbb{R}$. Hence $\mathcal{R}^{G_2} \cong F$. We summarize

Prop. *$\mathcal{R}^{G_2} \cong F$. A 7-mfld with holonomy in G_2 is Ricci-flat.*

The group $Spin(7)$

Put $U = \mathbb{H}_x \oplus \mathbb{H}_y$. Let $Sp_0(1) \times Sp_+(1) \times Sp_-(1)$ act on U via

$$(q_0, q_+, q_-) \cdot (x, y) = (q_0 x \bar{q}_-, q_+ y \bar{q}_-).$$

Define the Cayley 4-form $\Omega_0 \in \Omega^4(V)$ by

$$\Omega_0 = vol_x + \omega_x^1 \wedge \omega_y^1 + \omega_x^2 \wedge \omega_y^2 + \omega_x^3 \wedge \omega_y^3 + vol_y =$$
$$= vol_x - \mathrm{Re}(d\bar{x} \wedge dx \wedge d\bar{y} \wedge dy) + vol_y.$$

Denote by K the stabilizer of Ω_0 in $GL_8(\mathbb{R})$. The following facts are obtained in a similar fashion as for the group G_2:

- $\Omega_0 = dx_0 \wedge \varphi_0 + *_4\varphi_0 \quad \Longrightarrow \quad G_2 = K \cap SO(7)$
- $SU(4) \subset K$
- $K \subset SO(8)$
- K is a compact, connected and simply connected Lie group of dimension 21 acting transitively on S^7

- Consider U as a G_2-representation. Then $U \cong \mathbb{R} \oplus V \;\Rightarrow\; \Lambda^2 U \cong \Lambda^2 V \oplus V \cong \mathfrak{g}_2 \oplus V \oplus V$. By dimension counting, $\mathfrak{K} \cong \mathfrak{g}_2 \oplus V$. Hence,

$$\Lambda^2 U \cong \mathfrak{K} \oplus \mathfrak{K}^\perp \quad \text{with} \quad \dim \mathfrak{K}^\perp = 7.$$

- Obviously, $-\mathbf{1}_U \in K$ acts trivially on $\Lambda^2 U$. One can show that the map

$$K / \pm 1 \to SO(\mathfrak{K}^\perp)$$

is an isomorphism. Hence,

$$K \cong Spin(7).$$

Rem. Unlike in the G_2 case, the orbit of Ω_0 in $\Lambda^4(\mathbb{R}^8)^*$ is not open.

$Spin(7)$ as a structure group

A $Spin(7)$-structure on M^8 is determined by $\Omega \in \Omega^4(M)$, which is pointwise linearly equivalent to the Cayley form.

Theorem

Ω is parallel wrt the Levi-Civita connection of g_Ω iff $d\Omega = 0$.

Proof. [Salamon, Prop. 12.4]. □

Prop. *$\mathcal{R}^{Spin(7)} \cong W$, where W is an irreducible $Spin(7)$-representation of dimension 168. In particular, an 8-mfld with holonomy in $Spin(7)$ is Ricci-flat.*

Proof. [Salamon, Cor. 12.6]. □

Examples

Ex.

- Since $SU(3) \subset G_2$, for any Z with $\mathrm{Hol}(Z) \subset SU(3)$, $M = Z \times \mathbb{R}$ can be considered as G_2-mfld
- First local examples were constructed by Bryant in 1987.

Theorem (Bryant-Salamon)

Let M be a positive self-dual Einstein four–manifold. Then there exists a metric with holonomy in G_2 on the total space of $\Lambda^2_- T^*M$.

Sketch of the proof. Let $P \to M$ be the principal $SO(4)$-bundle. Since $\mathfrak{so}(4) = \mathfrak{so}_+(3) \oplus \mathfrak{so}_-(3)$ we can decompose the Levi-Vita connection: $\tau = \tau_+ + \tau_-$. Further, since $Sp(1) \cong Spin(3)$ we have

$$\mathfrak{so}(3) = \mathfrak{spin}(3) \cong \mathfrak{sp}(1) = \mathrm{Im}\,\mathbb{H}.$$

Hence, $\tau_\pm \in \Omega^1(P; \mathrm{Im}\,\mathbb{H})$. Similarly, the canonical 1-form θ can be thought of as an element of $\Omega^1(P; \mathbb{H})$.

Consider the action of $SO(4) = Sp_+(1) \times Sp_-(1)/\pm 1$ on $P \times \operatorname{Im}\mathbb{H}_x$

$$[q_+, q_-] \cdot (p, x) = (p \cdot [q_+, q_-], q_- x \bar{q}_-).$$

Clearly, $P \times \operatorname{Im}\mathbb{H}/SO(4) \cong \Lambda^2_- T^* M$.

Put $\alpha = dx + \tau_- x - x\tau_- \in \Omega^1(P \times \operatorname{Im}\mathbb{H}, \operatorname{Im}\mathbb{H})$. It is easy to check that the following forms are $SO(4)$-equivariant:

$$\begin{aligned}
\gamma_1 =& \alpha_1 \wedge \alpha_2 \wedge \alpha_3, \\
\gamma_2 =& -\operatorname{Re}(\alpha \wedge \bar{\theta} \wedge \theta) = \alpha_1 \wedge \omega_1 + \alpha_2 \wedge \omega_2 + \alpha_3 \wedge \omega_3, \\
\varepsilon_1 =& \tfrac{1}{6}\operatorname{Re}(\bar{\theta} \wedge \theta \wedge \bar{\theta} \wedge \theta) = \pi^* vol_M, \\
\varepsilon_2 =& \tfrac{1}{4}\operatorname{Re}(\alpha \wedge \alpha \wedge \bar{\theta} \wedge \theta) = \\
=& \alpha_2 \wedge \alpha_3 \wedge \omega_1 + \alpha_3 \wedge \alpha_1 \wedge \omega_2 + \alpha_1 \wedge \alpha_2 \wedge \omega_3.
\end{aligned}$$

Moreover, for any functions $f = f(|x|^2)$, $h = h(|x|^2)$ without zeros the symmetric tensor

$$g = f^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + h^2(\theta_0^2 + \theta_1^2 + \theta_2^2 + \theta_4^2)$$

determines a metric on $\Lambda^2_- T^* M$. Then

$$\varphi = f^3\gamma_1 + fh^2\gamma_2$$

determines a G_2-structure on $\Lambda^2_- T^* M$. We have also

$$*\varphi = h^4\varepsilon_1 - f^2h^2\varepsilon_2.$$

With the help of the fact that M is positive, self-dual, and Einstein, equations $d\varphi = 0 = d*\varphi$ essentially imply that

$$f(r) = (1+r)^{-1/4} \qquad h(r) = \sqrt{2\varkappa}(1+r)^{1/4}.$$

Here $\varkappa = (sc.curv.)/12 > 0$. □

Rem. Hitchin showed that the only complete self-dual Einstein 4-mflds with positive sc. curvature are S^4 and $\mathbb{C}P^2$ with their standard metrics. For these 4-mflds the holonomy of the Bryant-Salamon metric *equals* G_2.

Using similar technique, Bryant and Salamon prove the following.

Theorem

Let M^3 be S^3 or its quotient by a finite group. Then there exists an explicite metric with holonomy G_2 on $M \times \mathbb{R}^4$ (total space of the spinor bundle).

Consider S^4 as $\mathbb{HP}^1$. Let $\mathbb{S}$ denote the tautological quaternionic line bundle (the spinor bundle).

Theorem

The total space of $\mathbb{S}$ carries an explicit metric with holonomy $Spin(7)$.

Calabi metric revisited

Recall: If S^1 acts on $\mathbb{C}^4 \cong \mathbb{H}^2$ via

$$\lambda \cdot (z_1, z_2, w_1, w_2) = (\lambda z_1, \lambda z_2, \bar{\lambda} w_1, \bar{\lambda} w_2),$$

then the hyperKähler moment map is given by

$$\mu = -(|z_1|^2 + |z_2|^2 - |w_1|^2 - |w_2|^2)i - 2k(z_1 w_1 + z_2 w_2).$$

In particular, the induced metric on $\mu^{-1}(i)/S^1 \cong T^*\mathbb{C}P^1$ has holonomy $Sp(1) \cong SU(2)$.

Want to study asymptotic properties of the Calabi metric. First consider

$$\left.\begin{array}{c}\mu = 0\\ z \neq 0\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c}(w_1, w_2) = a(z_2, -z_1)\\ |a| = 1\end{array}\right.$$

Hence, the map $\mathbb{C}^2 \to \mathbb{C}^4$

$$(t_1, t_2) \mapsto (t_1, t_2, t_2, -t_1)$$

induces a diffeomorphism $\mathbb{C}^2/\pm 1 \cong \mu^{-1}(0)/S^1$ (away from the singular pt). It is easy to see that in fact this is an isometry.

Observe also that we have a commutative diagram

$$\begin{array}{ccc}\mu^{-1}(-i) & \hookrightarrow & \mu_c^{-1}(0)\\ \downarrow /S^1 & & \downarrow /\mathbb{C}^*\\ T^*\mathbb{P}^1 & \xrightarrow{\chi} & \mathbb{C}^2/\pm 1\end{array}$$

where the map χ is induced by the inclusion in the top row. Moreover, χ is holomorphic and

$$\chi^{-1}(z) = \left\{\begin{array}{ll} pt, & z \neq 0\\ \mathbb{P}^1, & z = 0\end{array}\right.$$

i.e. χ is a resolution of singularity.

Prop. *Let g denote the Calabi metric on $T^*\mathbb{CP}^1$. Then*

$$\chi^* g = g_{flat} + O(r^{-4}),$$

where r is the radial function on $\mathbb{C}^2/\pm 1$.

A metric with asymptotics as in the Prop. above is called ALE (asymptotically locally Euclidean).

The fact that the leading term is g_{flat} follows from the following observation. Denote by $M_\rho = \mu^{-1}(-i\rho)/S^1$, where $\rho \in \mathbb{R}$. Clearly, M_ρ is diffeomorphic to $T^*\mathbb{C}P^1$ for any ρ. As $\rho \to 0$, the metric g_ρ tends to the flat metric on $M_0 \cong \mathbb{C}^2/\pm 1$ (away from the singularity).

A sketch of the construction of a compact G_2-mfld

Consider $\mathbb{T}^7$ with its flat G_2-structure (g_0, φ_0). The group $\mathbb{Z}_2^3$ acts on $\mathbb{T}^7$ via

$$\alpha(x_1, \dots, x_7) = (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7)$$
$$\beta(x_1, \dots, x_7) = (x_1, -x_2, -x_3, x_4, x_5, \frac{1}{2} - x_6, -x_7)$$
$$\gamma(x_1, \dots, x_7) = (-x_1, x_2, -x_3, x_4, \frac{1}{2} - x_5, x_6, \frac{1}{2} - x_7)$$

Lem. *The singular set S of $\mathbb{T}^7/\mathbb{Z}_2^3$ consists of 12 disjoint $\mathbb{T}^3$ with singularities modelled on $\mathbb{T}^3 \times \mathbb{C}^2/\pm 1$.*

Since $T^*\mathbb{P}^1$ is asymptotic to flat $\mathbb{C}^2/\pm 1$, we can cut out a small neihbourhood of each connected component of S and replace it with $\mathbb{T}^3 \times T^*\mathbb{P}^1$. The metric on the resulting mfld, as well as a G_2-structure, is obtained by glueing the flat metric on $\mathbb{T}^7$ to the product (non-flat) metric on $\mathbb{T}^3 \times T^*\mathbb{P}^1$. The 3-form φ is not parallel, but can be chosen so that $d\varphi = 0$ and $d * \varphi$ is small.

Then Joyce proves that such (g, φ) can be deformed into a metric with holonomy G_2.

Examples of compact $Spin(7)$-mflds can be constructed in a similar manner.