*Choose any 3 of the following problems and send me your solutions no later than **Feb 2, 2012.***

Problem 1. Let R be an algebraic Riemannian curvature tensor, i.e. R belongs to $\Lambda^2\mathbb{R}^n \otimes \Lambda^2\mathbb{R}^n$ and satisfies the algebraic identity. Prove that $R \in S^2(\Lambda^2\mathbb{R}^n)$.

Problem 2. Prove that there exists a complex manifold, which does not admit a Kähler metric.

Problem 3. Prove that $T^*Gr_p(\mathbb{C}^{p+q})$ admits a hyperKähler metric.

Problem 4. Let $\Omega \in \Lambda^4(\mathbb{R}^8)^*$ be the Cayley form. Denote by U the 8–dimensional $Spin(7)$–representation, given by the embedding $Spin(7) = Stab_\Omega \subset SO(8)$. Prove that there exists the decomposition $\Lambda^2 U = \Lambda^2_+ \oplus \Lambda^2_-$, where Λ^2_+ and Λ^2_- are eigenspaces of the linear map

$$T\colon \Lambda^2 \to \Lambda^2, \quad \omega \mapsto -*(\omega \wedge \Omega)$$

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Problem 5. Identify $\mathbb{R}^7$ with the space of imaginary octonions $\mathrm{Im}\,\mathbb{O}$. Define an almost complex structure J on $S^6 \subset \mathrm{Im}\,\mathbb{O}$ as follows: J_u is the restriction of the right multiplication by u to $u^\perp$. Prove that J is not integrable.

Problem 6. Let M^4 be an oriented Riemannian 4–manifold, which is self–dual. Denote by $P \to M$ the principal $SO(4)$–bundle of orthonormal oriented frames. Let $\varphi_- \in \Omega^1(P; \mathrm{Im}\,\mathbb{H})$ be the component of the Levi–Civita connection. Prove that

$$d\varphi_- + \varphi_- \wedge \varphi_- = \varkappa \bar{\theta} \wedge \theta,$$

where $\theta \in \Omega^1(P; \mathbb{H})$ is the canonical 1–form and $\varkappa$ is proportional to the scalar curvature.