Aufgabe 1 (5 Punkte)

Es sei $f: I \to \mathbb{R}$ reell analytisch, $I \subseteq \mathbb{R}$ ein offenes Intervall. Zeigen Sie:

1. Es gibt ein größtes offenes Intervall $I \subseteq J \subseteq \mathbb{R}$, auf das sich f analytisch fortsetzen lässt.
2. Finden sie ein Beispiel, in dem sich f über dieses größte Intervall hinaus glatt fortsetzen lässt.

Zur Erinnerung: Dass ein Element p einer partiell geordneten Menge $(P, \leq)$ *größt* ist, bedeutet dass $q \leq p$ für alle $q \in P$ gilt und nicht nur, dass es kein $q \in P$ gibt mit $p < q$ (solche Elemente heißen maximal). Insbesondere zeigen die komplexen Wurzelfunktionen, dass das komplexe Analog der ersten Aussage nicht stimmt.

Lösungsskizze. We first note that by the real version of the) identity theorem, the extension of f to any interval J containing I is necessarily unique, since I clearly has an accumulation point; this can in fact also be deduced from the complex version since by 6.19 any real analytic function on an open interval extends to a open neighbourhood $U \subseteq \mathbb{C}$ of J, whence the complex version applies. We will denote by f_J this extension when it exists.

Let now $\mathcal{J}$ be the set of open intervals J which contain I and onto which f admits an analytic continuation. Then, $\cup_{J\in\mathcal{J}} J$ is again open and connected as every two intervals in $\mathcal{J}$ must intersect at least on I; in particular, $\cup_{J\in\mathcal{J}} J$ is an open interval.

To answer the first question, we now claim that f admits an analytic continuation to the whole of $\cup_{J\in\mathcal{J}} J$, i.e. that all the different possible extensions glue to a global one. This is then certainly the largest such interval by construction, since by definition it extends every other. More preciely, we claim that choosing for each $t \in \cup_{J\in\mathcal{J}} J$ a $J \in \mathcal{J}$ with $x \in J$ and prescribing a function $\cup_{J\in\mathcal{J}} J \longrightarrow \mathbb{C}$ by stipulating $x \mapsto f_J(x)$ is well-defined (clearly it is then analytic). For this we need to check that any two intervals $J_1, J_2 \in \mathcal{J}$ the associated analytic extensions agree on $J_1 \cap J_0$. But $J_0 \cap J_1$ is still an interval containing I and both f_{J_0} and f_{J_1} extend f. Thus $(f_{J_0})_{|J_0\cap J_1} = (f_{J_1})_{|} J_0 \cap J_1$ by the uniqueness assertion in the first paragraph.

For the second part, recall that

$$f: \mathbb{R}_{>0} \longrightarrow \mathbb{R}, \quad x \longmapsto e^{-\frac{1}{x^2}}$$

can be extended by 0 to non-positive reals to a smooth function $\mathbb{R} \to \mathbb{R}$. And since $\lim_{t\searrow 0} f^{(k)}(t) = 0$ for all $k \in \mathbb{N}$, all smooth extensions g of f to some $(-\epsilon, \infty)$ must have $g^{(k)}(0) = 0$, and thus none of them can be analytic at 0, since $\mathrm{T}_0 g = 0$, but f does not vanish in on any positive real number, no matter how close to 0.

Hence $\mathbb{R}_{>0}$ is the largest domain of analytic definition for f, but it admits a (in fact many) smooth continuation to the whole of $\mathbb{R}$. □

Aufgabe 2 (5 Punkte)

1. Zeigen Sie, dass mit $T \subseteq \mathbb{R}^n$ auch $\overline{T} \subseteq \mathbb{R}^n$ zusammenhängend ist.
2. Finden Sie eine wegzusammenhängende Menge $T \subseteq \mathbb{R}^n$, derart dass $\overline{T}$ nicht wegzusammenhängend ist.

Finden Sie auch ein Beispiel eines offenen solchen T?

Lösungsskizze. Let $T \subset \mathbb{R}^n$ be connected. Suppose $\overline{T} \subseteq U \cup V$ with $U \cap V \cap \overline{T} = \emptyset$. Clearly, this implies $T \subseteq U \cup V$ with $U \cap V \cap T = \emptyset$ and thus by connectedness of T we find $T \subseteq U$ or $T \subseteq V$. Let's assume the former. Then $T \cap V = \emptyset$ and thus $T \subseteq \mathbb{R}^n \setminus V$. But here the right hand side is closed, so it follows that also $\overline{T} \subseteq \mathbb{R}^n \setminus V$ and thus $\overline{T} \cap V = \emptyset$ and thus $\overline{T} \subseteq U$ as desired.

For the second part, consider $\Gamma = \{(x, \sin(\frac{1}{x})) \mid x \in \mathbb{R}_{>0}\}$, the graph of $x \mapsto \sin(\frac{1}{x})$ on $\mathbb{R}_{>0}$. This is a path-connected subset of $\mathbb{R}^2 = \mathbb{C}$ because $\sin(\frac{1}{x}))$ is continuous. But its closure $\overline{\Gamma}$ is precisely in $\{(0, y) \mid y \in [-1, 1]\} \cup \Gamma$ and this is not path-connected: The key reason is that $\sin(\frac{1}{x})$ sends any interval $]0, \varepsilon[$ to the whole of $[-1, 1]$, so as any path on Γ approaches the vertical axis its second coordinate has to oscillate between 1 and -1 and can thus never attain a limit there.

Indeed, pick a continuous path $\gamma: [0, 1] \to \overline{\Gamma}$ starting in the vertical line segment, i.e. $\mathrm{Re}\gamma(0) = 0$. We claim that then $\mathrm{Re}\gamma(t) = 0$ for all $t \in [0, 1]$. For the set of t for which this is true is clearly closed and non-empty so be the connectedness of $[0, 1]$ the claim follows once we show this set is also open. So let $t \in [0, 1]$ have $\mathrm{Re}\gamma(t) = 0$. By continuity of γ there is a $\delta > 0$ such $|\mathrm{Im}\gamma(t) - \mathrm{Im}\gamma(s)| < \frac{1}{2}$ for all $s \in (t-\delta, t+\delta)$. But if say $\mathrm{Re}\gamma(t') > 0$ for some such $t < t' < t+\delta$, then by the intermediate value theorem $\mathrm{Re}\gamma\colon [t,t'] \to \mathbb{R}$ attains every value of $[0, \mathrm{Re}\gamma(t')]$. But by construction γ maps the interval $[t,t']$ into

$$\{z \in \overline{\Gamma} \mid \mathrm{Im}\gamma(t) - \frac{1}{2} < \mathrm{Im}(z) < \mathrm{Im}\gamma(t) + \frac{1}{2}\}$$

But not all point of $[0, \mathrm{Re}\gamma(t')]$ even occur as real parts of points in this subset: It can certainly only contain points with one of $\mathrm{Im}(z) = 1$ or $\mathrm{Im}(z) = -1$, and these are the only ones in $\overline{\Gamma}$ with $\mathrm{Re}(z) = \frac{2}{(4k+1)\pi}$ and $\mathrm{Re}(z) = \frac{2}{(4k+3)\pi}$, respectively. Since the interval $[0, \mathrm{Re}\gamma(t')]$ must, however, contain both types of real parts the moment $\mathrm{Re}(\gamma(t')) > 0$ we have reached a contradiction. The case $t - \delta < t' < t$ is dealt with entirely analogously by considering the interval $[t', t]$.

Finally, note that Γ is not open, but we can tweak it to be open rather easily: consider

$$\Delta = \{(x,y) \mid x \in \mathbb{R}_{>0} \text{ and } |y - \sin(\frac{1}{x})| < x\}$$

This is some sort of shrinking tubular neighbourhood of $\sin(\frac{1}{x})$, which is clearly open and

$$\overline{\Delta} = \{(0,y) \mid y \in [-1,1]\} \cup \{(x,y) \mid x \in \mathbb{R}_{>0} \text{ and } |y - \sin(\frac{1}{x})| \leq x\}.$$

The argument above applies essentially verbatim. The only points $z \in \overline{\Delta}$ with real part $\frac{2}{(4k+1)\pi}$ and $\frac{2}{(4k+3)\pi}$ have imaginary part in $\left[1 - \frac{(4k+1)\pi}{2}, 1 + \frac{(4k+1)\pi}{2}\right]$ and $\left[1 - \frac{(4k+3)\pi}{2}, 1 + \frac{(4k+3)\pi}{2}\right]$, respectively. But for k large enough, these intervals are at distance > 1 so the image of $\mathrm{Im}\gamma$ on $(t-\delta, t+\delta)$ can intersectat most one of them, and thus $\mathrm{Re}\gamma$ must miss the other kind. □

Aufgabe 3 (5 Punkte)

Zeigen Sie: Ist $G \subsetneq S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ eine unendliche Untergruppe (bezüglich der Multiplikation), so ist G dicht in S^1 (also $\overline{G} = S^1$).

Zur Erinnerung: Jede unendliche Teilmenge einer abgeschlossenen, beschränkten Teilmenge $K \subset \mathbb{R}^n$ hat einen Häufungspunkt.

Lösungsskizze. Since G is not finite, it has an accumulation point by the hint, i.e. there is $z \in G$ and a sequence $x\colon \mathbb{N} \to \in G \setminus \{z\}$ which converges to z. Multiplying this sequence with z^{-1}, we obtain a sequence $y\colon \mathbb{N} \to G \setminus \{1\}$ converging to 1.

Now fix $w = \mathrm{cis}(\psi) \in S^1$ and let $\varepsilon > 0$ be arbitrarily small. Then we find an $n \in \mathbb{N}$ with $|1 - y_n| < \epsilon$ and by making n larger we can also assure y_n lies in the right half space, i.e. $y_n = \mathrm{cis}(\varphi_n)$ with $-\frac{\pi}{2} < \varphi_n < \frac{\pi}{2}$. This latter condition assures that, when we pick the unique $k \in \mathbb{Z}$ with $k\varphi_n < \psi < (k+1)\varphi_n$, then w lies on the circle segment connecting y_n^k and y_n^{k+1}, whose angle is less than π (i.e. the short one, not the long one) and we clearly have $|y_n^{k+1} - y_n^k| = |y_n^k| \cdot |y_n - 1| < \epsilon$. But generally, every point c on the short circle segment between any two points a, b on the unit circle that are not antipodal, satisfies $|c - a| \leq |b - a|$ (and also $|c - b| < |a - b|$; draw a picture!). In the present case, taking $a = y_n^k$, $b = y_n^{k+1}$ and $c = w$, this yields $|w - y_n^k| \leq \epsilon$ and thus the claim since $y_n^k \in G$.

To see this general fact, we can (by symmetry) clearly assume that $a = 1$ and that $b = \mathrm{cis}(\theta)$ with $0 < \theta < \pi$, to spare ourselves some notation. Then we want to show that $|\mathrm{cis}(\theta') - 1| < |\mathrm{cis}(\theta) - 1|$ for all $0 < \theta' < \theta$. But we have

$$|\mathrm{cis}(\vartheta) - 1|^2 = (\cos(\vartheta) - 1)^2 + \sin(\vartheta)^2 = \cos(\vartheta)^2 - 2\cos(\vartheta) + 1 + \sin(\vartheta)^2 = 2 - 2\cos(\vartheta)$$

and cos is a monotone decreasing function on $[0, \pi]$, so the above is a monotone increasing function on that interval, which gives the claim. □

Aufgabe 4 (5 Punkte)

Geben Sie eine komplexe Zahl $z \in \mathbb{C}$ mit $\sin(z) = 2$ an.

Angeben heißt hier, durch eine explizite Formel beschreiben (die natürlich keinen Arcussinus verwenden sollte).

Lösungsskizze. In the lecture we saw that $\sin(z) = 2$ is equivalent to $\exp(iz)$ being a root of $T^2 - 4iT - 1$. By completing the square these roots are $(2+\sqrt{3})i$ and $(2-\sqrt{3})i$. We thus need to determine the logarithms of these. But in polar coordinates they simply are $(2+\sqrt{3}) \cdot \operatorname{cis}(\pi/2)$ and $(2-\sqrt{3}) \cdot \operatorname{cis}(\pi/2)$ and thus their logarithms are given by $\ln(2+\sqrt{3}) + \frac{\pi}{2}i + 2\pi k i$ and $\ln(2-\sqrt{3}) + \frac{\pi}{2}i + 2\pi k i$ with $k \in \mathbb{Z}$. Dividing these by i (i.e. multiplying them by $-i$) we thus find that z can be any one of

$$\frac{(4k+1)\pi}{2} - \ln(2+\sqrt{3})i \quad \text{or} \quad \frac{(4k+1)\pi}{2} - \ln(2-\sqrt{3})i$$

as a solution to the exercise. □