Universität Bielefeld Winter 2025
Complex Analysis — Solutions to Sheet 7

Exercise 1

Throughout, we will say that t_i, ε_i, f_i are the *datum* of an analytic extension of some F along w if they satisfy the axioms of the introduction.

1. The function is clearly well-defined, since by assumption the f_i's coincide on successive intersections of the $D(t_i, \varepsilon_i)$. Moreover, note that F_w is also continuous as a composite of continuous functions on each ball.

We now check it is independent of the choice of t_i, ε_i and f_i's. For this, take two such choices (denoting the second choices by a letter further in the alphabet) and denote F_w and G_w the resulting functions. Consider:

$$\Gamma = \{t \in [0;1] \mid \forall s \in [0;t], F_w(s) = G_w(s)\}$$

Then Γ is non-empty by the first assumption: there exists a small ball around x where both f_0 and g_0 coincide with the original F, hence $[0, \min(\varepsilon_0, \zeta_0)[\subset \Gamma$.

To conclude, it suffices to check that Γ is both open and closed in $[0, 1]$. The closure is clear by continuity. Now pick $t \in \Gamma$. Then, there must be i such that $t \in D(w(t_i), \varepsilon_i)$ and j such that $t \in D(w(u_j), \zeta_j)$. By construction, f_i and g_j are two holomorphic functions defined on $D(w(u_j), \zeta_j) \cap D(w(t_i), \varepsilon_i)$ and they are equal on a subset which contains an accumulation point (the part of the path w up to t). Therefore, the coincide on the whole intersection.

Since this intersection is again open, it follows that there is a small δ such that $|t - s| < \delta$ implies $f_i(w(s)) = g_j(w(s))$ hence Γ is open.

2. Write P for the space of paths $[0, 1] \to U$ which are homotopic relative endpoints to a fixed w; here P is endowed the the topology induced by the $\|\cdot\|_\infty$ norm. Note that P is connected; in fact even better, it is path-connected as every $\overline{w} \in P$ admits a path to w precisely given by the homotopy between w and $\overline{w}$ (P is precisely the path-component of w in the space of all paths).

Write Γ for the subspace of P of paths $\overline{w}$ such that $F_{\overline{w}}(1) = F_w(1)$. Γ is non-empty because $w \in \Gamma$. We check that Γ is both open and closed to conclude.

Fix a finite set of t_i, ε_i, f_i such that they define an analytic continuation of F along v. There is a $r > 0$ such that $|w(t) - z| < r$ implies that z is in some $B(w(t_i), \varepsilon_i)$, i.e. a tube around w of radius r is contained in the union of the balls; for instance, pick r to be the minimum of the radius of balls inside all of the possible non-empty 2-by-2 intersections of balls and then divide it by 2. Now suppose $\|v - v'\|_\infty < r$ for some v'; then, necessarily the same collection of t_i, ε_i, f_i also provides a analytic extension along the path v'; in particular,

$$F_v(1) = f_n(v(1)) = f_n(v'(1)) = F_{v'}(1)$$

hence Γ is open.

But note that actually, what we have shown is that the association $v \mapsto F_v$ is continuous (without even assumptions on the endpoint) since all the f_i are also continuous.

Therefore, using that evaluation at 1 is also continuous, we get that Γ is also closed, which concludes.

3. By the simply connected assumption, every path in U is homotopic to any other path with the same start and endpoints. By the above, this means that $F_w(1)$ only depends of $w(1)$.

In particular, we claim that the following defines a function $f : U \to \mathbb{C}$: if $z \in U$, we let $f(z)$ be the value of $F_w(1)$ for any path $w : [0,1] \to U$ with $w(0) = x$ and $w(1) = z$. This is well-defined, and we have to check it is holomorphic in U, with $T_x f = F$.

The second part is actually straightforward: by the first question, for any path (so say for instance the constant path equal to x), there is a small ball $D(x, \varepsilon_0)$, centred around x such that $T_x f_0 = F$. But now, if we consider the radial path from x to some $z \in D(x, \varepsilon_0)$, we find that f_0, t_0 and ε_0 are a valid datum of analytic continuation along this path so that by question 1, $f(z) = f_0(z)$. In particular, $T_x f = F$.

In fact, the above line of reasoning holds more generally in the neighborhood of any point: fix $z \in U$ and any path w with $w(0) = x$, $w(1) = z$ and pick f_i, t_i, ε_i the datum of an analytic continuation along w. Then, we claim that $f(t) = f_n(t)$ for any $t \in D(z, \varepsilon_n)$: indeed, consider the concatenated path where we first do w and then the radial path from z to t. Then, the same collection of f_i, t_i, ε_i defines an analytic continuation along this new path and therefore $f(t) = f_n(t)$.

We have shown that in a neighborhood of any point of U, f coincides with an analytic function, therefore it is analytic itself.

Exercise 2

Any analytic continuation G of g is necessarily of the form

$$G(z) := \frac{1}{\widetilde{G}(z) - 2\pi i}$$

where $\widetilde{G}(z)$ is a logarithm on U, i.e. a primitive of $\frac{1}{z}$, since $\tilde{G}(z) := \frac{1}{G(z)} + 2\pi i$ is analytic and coincides with the standard ln in a neighborhood of 1. If there was an analytic continuation of g along say $\gamma : t \mapsto \exp(2\pi i t)$, then there would be a well-defined logarithm on an open domain of $\mathbb{C}$ containing a circle. We have already seen this to be impossible.

Note however that $D_g = \mathbb{C}^\times$, by simply taking a logarithm defined on a different slit domain but with still $\ln(1) = 0$, and writing the exact same formula.

Exercise 3

We recall that $\tan(z) = \frac{\sin(z)}{\cos(z)}$ and $\operatorname{cotan}(z) = \frac{\cos(z)}{\sin(z)}$.

We first deal with the fourth integral. The function tan is holomorphic on an open that contains γ by the quotient rule, hence

$$\int_\gamma \tan(z) = 0$$

We now turn to the second integral, which can be dealt in similar fashion. Indeed, note that $\frac{\sin(z)}{z}$ admits a holomorphic continuation on all of $\mathbb{C}$, by virtue of both Corollary II.3.11 and $\sin(z) \to 0$ when $z \to 0$. Hence, the integral

$$\int_\gamma \frac{\sin(z)}{z} = 0$$

is null, as the path is closed.

The next integrals are non-zero, so we have to be a little be more clever: for the first integral, since cos is holomorphic on $\mathbb{C}$, we have

$$\int_\gamma \frac{\cos(z)}{z} = 2\pi i \cos(0) = 2\pi i$$

by Cauchy's integral formula, see Proposition II.3.1 of the lecture notes.

Finally, note that $z \operatorname{cotan}(z)$ is holomorphic at 0 by using the above result on the sinus cardinalis, with value at 0 $\cos(0) = 1$. Hence, the third integral evaluates to:

$$\int_\gamma \operatorname{cotan}(z) = \int_\gamma \frac{z \operatorname{cotan}(z)}{z} = 2\pi i$$

by Cauchy's integral formula again.

Exercice 4

Write $P = \sum_{k=0}^n a_k X^k$ with $n \geqslant 1$, then for $z \neq 0$, we have

$$\left| \frac{P(z)}{a_n z^n} \right| = \left| 1 + \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}} \right|$$

Applying the triangle inequality, each summand $\frac{|a_k|}{|z|^{n-k}} \to 0$ as $|z| \to \infty$, hence

$$\left| \frac{P(z)}{a_n z^n} \right| \longrightarrow 1$$

In particular, $|P(z)| \to \infty$ as $|z| \to \infty$ and the above gives the precise asymptotic equivalent.

Suppose now $f : \mathbb{C} \to \mathbb{C}$ is holomorphic with $|f(z)| \to \infty$ as $|z| \to \infty$. Then, the zeroes of f must be concentrated in a bounded region, hence there must be finitely of them because they have to be isolated.

We now consider $g(z) := \frac{1}{f(\frac{1}{z})}$. This is a well-defined, holomorphic function away from finitely many points, namely the zeroes of f and $z = 0$, in particular on a disk around

0 which does not contain zero itself. Note that as $z \to 0$, $zg(z) \to 0$ by the assumption on f; hence by Corollary II.3.11, g admits a holomorphic continuation to the same disk which now contains zero.

Therefore, there is a $n \in \mathbb{N}$ such that $\frac{g(z)}{z^n} \longrightarrow c \neq 0$ as $z \to 0$: g is not the zero function on the disk by construction and we can pick $n \in \mathbb{N}$ to be the integer where the non-zero first coefficient of the Taylor tower at 0 arises (and c is therefore that coefficient). Translating back to f, we have shown that

$$\frac{1}{z^n f\left(\frac{1}{z}\right)} \longrightarrow c$$

as $z \to 0$. Evaluating at $\frac{1}{z}$, we get that

$$\frac{f(z)}{z^n} \longrightarrow \frac{1}{c}$$

as $|z| \to \infty$. But now, $f : \mathbb{C} \to \mathbb{C}$ is homolorphic hence coincides with the limit of its Taylor tower at 0 at any point and we can use the same decomposition as in the first part of the exercise:

$$\left|\frac{f(z)}{z^n}\right| = \left|\sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}\right| + \left|\frac{1}{c} + \sum_{k=n+1}^{\infty} a_k z^{k-n}\right|$$

But now, the remainder $\left|\frac{1}{c} + \sum_{k=n+1}^{\infty} a_k z^{k-n}\right|$ is a holomorphic function $\mathbb{C} \to \mathbb{C}$ which is bounded, hence constant by Liouville's. This implies $a_k = 0$ for $k \geqslant n+1$ hence f is a polynomial as wanted.