

**Exercise 1**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  a function. To check whether  $m_f : \lambda \mapsto \lambda f$  is continuous at say  $\lambda_0 \in \mathbb{R}$ , we have to study the following supremum

$$\sup_{t \in \mathbb{R}} |\lambda f(t) - \lambda_0 g(t)|$$

If  $\lambda \neq \lambda_0$ , then this supremum is also given by

$$|\lambda - \lambda_0| \cdot \sup_{t \in \mathbb{R}} |f(t)|$$

Hence there are two cases: either  $\|f\|_\infty := \sup_{t \in \mathbb{R}} |f(t)|$  is finite i.e.  $f$  is bounded, and then  $m_f$  is clearly continuous since it is even  $\|f\|_\infty$ -Lipschitz or  $\|f\|_\infty$  is infinite, i.e.  $f$  is unbounded and then the above supremum is always  $+\infty$  unless  $\lambda = \lambda_0$  where it is zero and therefore  $m_f$  cannot be continuous at  $\lambda_0$ , for instance because  $m_f^{-1}(B(\lambda_0, 1)) = \{\lambda_0\}$  is not open.

**Exercise 2**

1. Let us write  $\widehat{d}$  for the function introduced in Question 1. We have three conditions to check on  $\widehat{d}$  to show that it is a pseudo-metric.

- (i) We first note that  $\widehat{d} \geq 0$  because this hold for  $d$ , and is preserved by inf and sums. To prove  $\widehat{d}(\bar{x}, \bar{x}) = 0$  for some equivalence class  $\bar{x}$ , it therefore suffices to show that  $\widehat{d}(\bar{x}, \bar{x}) \leq 0$ . But note that

$$\widehat{d}(\bar{x}, \bar{x}) \leq \widetilde{d}(\bar{x}, \bar{x})$$

where we used that an infimum is always smaller than any of its terms (here on the right hand side, we chose the term with  $n = 0$ ). But we can do this again:

$$\widetilde{d}(\bar{x}, \bar{x}) \leq d(x, x) = 0$$

where  $x \in X$  is any point whose equivalence class is  $\bar{x}$ . But then, we have shown that the non-negative number  $\widehat{d}(\bar{x}, \bar{x}) \leq 0$ ; this implies that it is null, as wanted.

- (ii) The formula for  $\widehat{d}$  is clearly symmetric because  $\widetilde{d}$  is and because we can relabel terms of the inf as we want.
- (iii) Let  $\bar{x}, \bar{y}, \bar{z}$  be three equivalence classes. Then

$$\widetilde{d}(\bar{x}, \bar{x}_1) + \dots + \widetilde{d}(\bar{x}_n, \bar{y}) + \widetilde{d}(\bar{y}, \bar{y}_1) + \dots + \widetilde{d}(\bar{y}_k, \bar{z}) \geq \widehat{d}(\bar{x}, \bar{z})$$

for any choice of integers  $n, k$ , and equivalence classes  $\bar{x}_i$  and  $\bar{y}_i$ . This is because the left hand side appears as one of the terms whose infimum defines the right hand side, namely the term with  $n + k + 1$  terms  $C_i$ , the first  $n$  equal to  $\bar{x}_i$ 's,  $C_{n+1} = Y$  and the last  $k$  equal to  $\bar{y}_i$ 's. Because this equality holds for any  $\bar{x}_1, \dots, \bar{x}_n$  and any choice of  $n$ , we deduce by definition of the infimum:

$$\widehat{d}(\bar{x}, \bar{y}) + \widetilde{d}(\bar{y}, \bar{y}_1) + \dots + \widetilde{d}(\bar{y}_k, \bar{z}) \geq \widehat{d}(\bar{x}, \bar{z})$$

We can do the same thing again but with the  $\bar{y}_i$  and we deduce

$$\widehat{d}(\bar{x}, \bar{y}) + \widehat{d}(\bar{y}, \bar{z}) \geq \widehat{d}(\bar{x}, \bar{z})$$

which is what we wanted. Note that at no point we used that either  $d$  or  $\widetilde{d}$  satisfied the triangle inequality.

**2.** Consider  $X = \mathbb{R}$  with the equivalence relation generated by  $x \simeq y$  if and only if  $x = y$  or  $x, y \in ]0, 1[$ . Let  $c$  denote the equivalence class of the interval  $]0, 1[$ , then  $\widetilde{d}(\bar{0}, c) = 0$  and  $\widetilde{d}(\bar{1}, c) = 0$ ; here  $\bar{0}$  and  $\bar{1}$  are singleton sets containing only 0 and 1.

But in consequence,  $\widetilde{d}(\bar{0}, \bar{1}) = 1$ , hence

$$\widetilde{d}(\bar{0}, \bar{1}) > \widetilde{d}(\bar{0}, c) + \widetilde{d}(\bar{1}, c)$$

which means  $\widetilde{d}$  does not satisfy the triangle inequality.

### Exercise 3

**1.** We have three conditions to verify:

- (i) The sets  $\emptyset$  and  $X$  are open, i.e. upwards closed. This is vacuously true for the former and obvious for the latter.
- (ii) If  $U, V$  are opens, let  $u \in U \cap V$  and  $x \in X$  with  $u \leq x$ . Then,  $x \in U$  because  $U$  is open and  $x \in V$  for the same reason. But then  $x \in U \cap V$ . Hence,  $U \cap V$  is open as wanted.
- (iii) If  $U_i$  is a set of opens, and  $u \in \cup_i U_i$  with  $u \leq x$ , then there is some  $i$  such that  $u \in U_i$  hence  $x \in U_i$  because  $U_i$  is open. But then in particular,  $x \in \cup_i U_i$  so this union is an open set in the Alexandroff topology.

**2.** Let  $f : (X, \leq) \rightarrow (Y, \leq)$ . We first suppose  $f$  monotone and we fix  $V \in Y$  an open in the Alexandroff topology. We have to show that

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$$

is open in  $X$ . But if  $x \leq y$ , then  $f(x) \leq f(y)$  by monotony. Since  $V$  is open, if  $f(x) \in V$  then  $f(y) \in V$  as well hence if  $x \in f^{-1}(V)$ , then  $y \in f^{-1}(V)$  which is precisely that  $f^{-1}(V)$  is open.

Now reciprocally, if  $f$  is continuous and  $x \leq y$  in  $X$  then, let  $V := \{z \in V \mid f(x) \leq z\}$ . This is clearly open in  $Y$  hence  $f^{-1}(V)$  is open in  $X$ . But  $x \in f^{-1}(V)$  hence  $y \in f^{-1}(V)$  which unravels to  $f(x) \leq f(y)$  as wanted.

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#### Exercise 4

1. Note that  $i(\emptyset) \subseteq \emptyset$  hence  $i(\emptyset) = \emptyset$ ; since we also assume that  $i(X) = X$ , we get that  $\emptyset, X \in \text{Fix}_i$ . Suppose now  $U, V \in \text{Fix}_i$ , then  $i(U \cap V) = i(U) \cap i(V) = U \cap V$  hence  $U \cap V \in \text{Fix}_i$ . Finally, if  $U_j$  is a collection of sets fixed by  $i$ , then of course, we have:

$$i\left(\bigcup_j U_j\right) \subseteq \bigcup_j U_j$$

but by writing  $U_j = (\cup_j U_j) \cap U_j$ , we deduce

$$i(U_j) = i\left(\bigcup_j U_j\right) \cap i(U_j)$$

But then, this necessarily implies that  $i(U_j) \subseteq i(\cup_j U_j)$  whence, using  $U_j = i(U_j)$ , we finally deduce the other inclusion

$$\bigcup_j U_j \subseteq i\left(\bigcup_j U_j\right)$$

and therefore the equality.

2. For every topology,  $X$  is open in  $X$  hence, for any  $x \in X$ , the assertion  $x \in X \subset X$  holds, which implies that  $\text{Int}_{\mathcal{T}}(X) = X$ .

Let  $U \subset X$ , then every  $x \in \text{Int}_{\mathcal{T}}(U)$  is such that  $x \in U$  because it belongs to at least one open  $V \subset U$ . Hence  $\text{Int}_{\mathcal{T}}(U) \subset U$ . Moreover, if  $x \in \text{Int}_{\mathcal{T}}(U)$ , then there exists  $V \subset U$  such that  $x \in V$ . But note that actually  $V \subset \text{Int}_{\mathcal{T}}(U)$  since every  $y \in V$  is witnessed by  $V$  itself as an element in  $\text{Int}_{\mathcal{T}}(U)$ . In particular, this shows that  $x \in \text{Int}_{\mathcal{T}}(\text{Int}_{\mathcal{T}}(U))$  thus the equality

$$\text{Int}_{\mathcal{T}}(\text{Int}_{\mathcal{T}}(U)) = \text{Int}_{\mathcal{T}}(U)$$

Finally, given  $U, V$ , then  $\text{Int}_{\mathcal{T}}(U \cap V) \subset \text{Int}_{\mathcal{T}}(U)$  because any  $x \in W \subset U \cap V$  with  $W$  open is in particular  $x \in W \subset U$ ; by symmetry  $\text{Int}_{\mathcal{T}}(U \cap V) \subseteq \text{Int}_{\mathcal{T}}(V)$  hence

$$\text{Int}_{\mathcal{T}}(U \cap V) \subseteq \text{Int}_{\mathcal{T}}(U) \cap \text{Int}_{\mathcal{T}}(V)$$

But now, pick  $W_U, W_V$  opens such that  $x \in W_U \subset U$  and  $x \in W_V \subset V$ . Then,  $x \in W_U \cap W_V$  which is open and in  $U \cap V$ . Hence the above inclusion is actually an equality.

3. We begin by checking that if  $\mathcal{T}$  is a topology, then  $U$  is open if and only if it is fixed by  $\text{Int}_{\mathcal{T}}$ . Note that

$$\text{Int}_{\mathcal{T}}(U) = \bigcup_{\substack{x \in V \subset U \\ V \text{ open}}} V$$

hence in particular  $\text{Int}_{\mathcal{T}}(U)$  is open so all of the fixed points are open. But reciprocally, if  $U$  is open, then every  $x \in U$  is witnessed in  $\text{Int}_{\mathcal{T}}(U)$  by the open  $U$  itself, hence  $\text{Int}_{\mathcal{T}}(U) = U$  since we have checked previously the previous inclusion.

Now, we have to check that given  $i$  which satisfies the Kuratowski axioms, then  $i$  coincides with  $\text{Int}_{\text{Fix}_i}$ , i.e. for every  $U \subset X$ ,

$$i(U) = \{x \in U \mid \exists V \in \text{Fix}_i \text{ with } x \in V \subseteq U\}$$

But if  $x$  is in the right hand side, then there is  $V \ni x$  with  $V = i(V)$  and  $V \subset U$ ; hence  $V = U \cap V$  and we get  $i(V) = i(U) \cap i(V)$  from which it follows that  $i(V) \subset i(U)$ . In consequence,  $x \in i(U)$  so the right hand set is included in the left one.

Now if  $x \in i(U)$  then  $i(i(U)) = i(U)$  hence  $i(U) \in \text{Fix}_i$  and  $i(U) \subseteq U$  hence  $x$  is in the right hand side. Therefore we have shown the wanted equality and we are done.