Universität Bielefeld Summer 2026
Geometry and Topology — Solutions to Sheet 10

Exercise 1

1. We already know that those charts make TM into a topological manifold when M is C^1. We only have to check that they upgrade to C^{l-1}-charts when M is actually C^l, for $l \geqslant 2$. Fix $(U, \phi), (V, \psi)$ two compatible charts of M, then we have to check that

$$\phi^{-1}(U \cap \psi(U \cap V)) \times \mathbb{R}^m \longrightarrow \psi^{-1}(\phi^{-1}(U \cap V) \cap V) \times \mathbb{R}^m$$

sending (x, v) to the couple $(\psi^{-1} \circ \phi(x), D_x(\psi^{-1} \circ \phi)(v))$ is a C^{l-1}-map. But by assumption, the first coordinate is in fact C^l and $(x, v) \mapsto D_x(\psi^{-1} \circ \phi)(v)$ is linear in v and C^{l-1} in the x-variable (essentially by definition of what a C^l-map $\mathbb{R}^n \to \mathbb{R}^n$ is). We are done.

2. Fixing two local C^{l-1}-charts as above, we identify Df with the composite:

$$\phi^{-1}(U \cap \psi(U \cap V)) \times \mathbb{R}^m \longrightarrow \psi^{-1}(\phi^{-1}(U \cap V) \cap V) \times \mathbb{R}^m$$

sending (x, v) to $(\psi^{-1} \circ f \circ \phi(x), D_x(\psi^{-1} \circ f \circ \phi)(v))$. The first coordinate is C^l by assumption on f and the second is C^{l-1} again by the same considerations as in the previous question.

Exercise 2

Let $\pi : E \to B$ be a fiber bundle over a connected base. If $x \in B$, $\pi^{-1}(\{x\}) \cong F$, then picking a local trivialization $U \ni x$ of π yields a diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\simeq"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

In particular, for any $y \in U$, $\pi^{-1}(\{y\}) \cong F$ hence the set $\{y \in B \mid \pi^{-1}(\{y\}) \cong F\}$ is open. In consequence, the following

$$B = \bigsqcup_{F \subset E} \{y \in B \mid \pi^{-1}(\{y\}) \cong F\}$$

is a disjoint union of opens, where the union is over the set of isomorphism classes of subspaces of E. Since B is connected, it follows that all but one of those sets is empty, which is precisely the wanted conclusion.

Exercise 3

1. Let $U \subset E$ be open. It suffices to check that $\pi(U)$ contains a small open neighbourhood around every $\pi(x)$ for $x \in U$. But we can always find $V \ni \pi(x)$ open on which we have a trivialization

$$\begin{tikzcd} \pi^{-1}(V) \arrow[rr, "\simeq", "\phi"'] \arrow[dr, "\pi"'] & & V \times F \arrow[dl, "\mathrm{pr}_1"] \\ & V & \end{tikzcd}$$

It follows that every $y \in V$ is in the image of π, i.e. $x \in V \subset \pi(U)$. Hence, π is an open map.

2. We have already seen a version of this statement during the lecture, namely that $\mathrm{Im}(\sigma)$ is closed, but let us reprove it nonetheless. We use the fact that if F is Hausdorff, then the diagonal

$$\Delta : F \longrightarrow F \times F$$

is a closed map. Fix a local trivialization as above and consider the map

$$\Phi : \pi^{-1}(V) \longrightarrow F \times F$$

which sends x to the tuple $(\mathrm{pr}_2\phi(x), \mathrm{pr}_2\phi\sigma\pi(x))$, i.e. which takes the F-components of $\phi(x)$ and $\phi\sigma\pi(x)$. If $x = \sigma(y)$ for some y, then $\Phi(x) \in \mathrm{Im}(\Delta_F)$ since $\sigma\pi\sigma(y) = \sigma(y)$ hence $\sigma\pi(x) = x$. But reciprocally, if $\Phi(x) \in \mathrm{Im}(\Delta_F)$ then since Φ is a bundle map which is in particular injective, we get $x = \sigma\pi(x)$ which implies that x is in the image of σ, i.e. we proved

$$x \in \mathrm{Im}(\sigma) \cap \pi^{-1}(V) \iff x \in \Phi^{-1}(\Delta_F)$$

which means the sets $\mathrm{Im}(\sigma) \cap \pi^{-1}(V) = \Phi^{-1}(\Delta_F)$ are homeomorphic via Φ. Since Φ is continuous and Δ_F is closed, so is $\mathrm{Im}(\sigma) \cap \pi^{-1}(V)$. This is seen to imply that $\mathrm{Im}(\sigma)$ is closed.

To prove the more general claim of the exercise, note that σ induces a homeomorphism onto its image, whose inverse is given by π restricted to $\mathrm{Im}(\sigma)$. Hence, every closed set $A \subset B$ is sent to a closed set $\sigma(A) \subset \mathrm{Im}(\sigma)$ but since the latter is closed, this implies that $\sigma(A) \subset E$ is closed again, which is what we wanted.

Exercice 4

We apply a Theorem of the lecture: $p : z \mapsto z^n$ is a smooth map $\mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$ which are both manifolds of the same dimension (namely 2); p is also a polynomial so we have seen in the lecture that it is proper. Moreover, its differential is nowhere vanishing hence every point is a regular value. It follows from the Lecture that p is locally trivial around every point.

Since $\mathbb{C} - \{0\}$ is connected, we learn that the fiber of p is constant hence by the standard computation of n^{th}-roots of unity, a set of cardinal n. This concludes.