Universität Bielefeld Summer 2026
Geometry and Topology — Solutions to Sheet 11

Exercise 1

We split this exercise in two distinct parts, namely we will show the following two facts:

- Manifolds are locally compact, i.e. for every $x \in M$, a manifold, there exists an open $U \ni x$ and a compact K such that $U \subset K$.
- Any locally compact, Hausdorff space is compactly generated.

For the first point, note that every open $U \subset \mathbb{R}^n$ is locally compact, as for every $x \in U$, there exists $r > 0$ such that $D(x, r) \subset U$ but then

$$x \in D\left(x, \frac{r}{2}\right) \subset \overline{D\left(x, \frac{r}{2}\right)}$$

is precisely what we want. But now, as the name suggests, being locally compact is a local property: it suffices to check that every point $x \in M$ has a neighbourhood which is a locally compact space. But every manifold is locally homeomorphic to an open of $\mathbb{R}^n$ hence the previous remark concludes.

For the second point, fix M locally compact Hausdorff. The Hausdorff hypothesis is here to guarantee that if X is closed in M, then $X \cap K$ is again closed for any compact K, since in a Hausdorff space, compact implies closed.

For the converse direction, let us fix $x \in M \setminus X$ and pick a compact neighbourhood $x \in U \subset K \subset M$. Then, $X \cap K$ is closed by assumption and therefore

$$V := U \cap M \setminus (X \cap K)$$

is an open neighbourhood of x; but note also that $V \cap X =$, since any $x \in V$ is in U hence in K and therefore, if it was also in X, would be in $X \cap K$ which is prohibited. It follows that $V \subset M \setminus X$ hence the latter is open, whence X is closed in M as wanted.

Exercise 2

For a Lie group G, let us denote $\mu_g : x \in G \mapsto g \cdot x \in G$ the left multiplication by g. This is a smooth linear map with inverse $\mu_{g^{-1}}$ hence

$$D_e\mu_g : T_eG \longrightarrow T_gG$$

is again a smooth diffeomorphism with inverse $D_g\mu_{g^{-1}}$. It follows that we can write a map $\Phi : G \times T_eG \to TG$ as follows:

$$(g, v) \longmapsto (D_e\mu_g(v))$$

Note that this is indeed a continuous map which commutes with the projection down to G, i.e. a map of bundle. Moreover, this map has an inverse given by

$$v \longmapsto (\pi(v), D_{\pi(v)}\mu_{\pi(v)^{-1}}(v)$$

where $\pi : TG \to G$ is the canonical projection. Since both maps in question are diffeomorphic, this is an equivalence of smooth bundles over G which concludes.

Exercise 3

Let us start with a vector bundle $\pi : E \longrightarrow B$ and produce a $\mathrm{GL}_n(\mathbb{R})$-structure with typical fiber $\mathbb{R}^n$ on it. Note that since $\mathrm{GL}_n(\mathbb{R})$ acts faithfully on $\mathbb{R}^n$, we never have to worry about the compatibility on triple trivializations and it suffices to provide the structure on a pair of trivialization. Moreover, we already have identifications with $\mathbb{R}^n$ of each fiber by fixing a basis.

Let indeed (ϕ, U) and (ψ, V) be two such trivializations, then the composite

$$(U \cap V) \times \mathbb{R}^n \xrightarrow{\psi^{-1}} \pi^{-1}(U \cap V) \xrightarrow{\phi} (U \cap V) \times \mathbb{R}^n$$

is a map over $U \cap V$, i.e of the form $(x, v) \longmapsto (x, \sigma_{U,V}(x, v))$ and a homeomorphism such that for every $x \in U \cap V$, $v \longmapsto \sigma_{U,V}(x, v)$ is a linear isomorphism $\mathbb{R}^n \to \mathbb{R}^n$. But all such are given by multiplication by some matrix $A^x_{U,V} \in \mathrm{GL}_n(\mathbb{R})$. In particular, this produces the map

$$x \in U \cap V \longmapsto A^x_{U,V} \in \mathrm{GL}_n(\mathbb{R})$$

and proves it is continuous since the identification $\mathrm{Iso}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^n) \simeq \mathrm{GL}_n(\mathbb{R})$ is a homeomorphism.

Dually, let us now start with a fibre bundle $\pi : E \longrightarrow B$ with typical fiber $\mathbb{R}^n$ and which we suppose equipped with maps $x \longmapsto A^x_{U\cap V} \in \mathrm{GL}_n(\mathbb{R})$, i.e. a $\mathrm{GL}_n(\mathbb{R})$-structure. Then, we can define a vector space structure on each $\pi^{-1}(x)$ via the isomorphism with $\mathbb{R}^n$: this can even be done in family, i.e. on a local trivialization hence the addition and scalar multiplications thus defined are continuous.

Moreover, we find again that the composite

$$(U \cap V) \times \mathbb{R}^n \xrightarrow{\psi^{-1}} \pi^{-1}(U \cap V) \xrightarrow{\phi} (U \cap V) \times \mathbb{R}^n$$

is of the form $(x, v) \longmapsto (x, A^x_{U\cap V} \cdot v)$. In particular, this is linear and therefore, the vector space structure does not depend on the choice of trivialization. We also get at the same time that the transition maps are linear hence the above is indeed a vector bundle structure on π. We are done

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Exercice 4

Let us first remark that the map $p : (E \times \mathbb{R})/\{\pm 1\} \to B$ is well-defined because the projection $\pi \circ \mathrm{pr}_1 : E \times \mathbb{R} \to B$ does factor through the quotient. We also learn that it is continuous at the same time. Moreover, since the fiber at some $b \in B$ of $\pi \circ \mathrm{pr}_1$ is

$$(\pi \circ \mathrm{pr}_1)^{-1}(b) = \{(e,t) \in E \times \mathbb{R} \mid \pi(e) = b\} \simeq \pi^{-1}(b) \times \mathbb{R}$$

we learn that $p^{-1}(b) \simeq \mathbb{R}$ thanks to a choice of identification $\pi^{-1}(b) \simeq \{\pm 1\}$. Of course, since π was a bundle, we can actually make this identification continuously in a local neighbourhood, i.e. there is an open $U \ni b$ such that

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi", "\simeq"'] \arrow[dr] & & U \times \{\pm 1\} \arrow[dl] \\ & U & \end{tikzcd}$$

commutes, and we can use this to produce the vector bundle structure on p. Using the previous exercise, we try instead to produce a $\mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times$-structure.

To do this, we note that given two trivializations U, V as above, the composite

$$(U \cap V) \times \mathbb{R} \xrightarrow{\psi^{-1}} \pi^{-1}(U \cap V) \xrightarrow{\phi} (U \cap V) \times \mathbb{R}$$

has two possible forms. We have two decompositions $\pi^{-1}(U) = U_+ \sqcup U_-$ and $\pi^{-1}(V) = V_+ \sqcup V_-$ and the disjointness forces either $U_+ \cap V_- = \emptyset$ or $U_+ \cap V_+ = \emptyset$ (and dually for U_-). In the first case, the above transition map is simply given by the identity $[e,t] \mapsto [e,t]$. In the other case however, we pick up a sign $[e,t] \mapsto [e,-t]$ since we first have to translate the negative part of V_- into the positive U_+.

In consequence, we find not only a $\mathbb{R}^\times$-structure but actually, a $O(1) = \{\pm 1\}$-structure since the two possible invertible elements of $\mathbb{R}^\times$ that do show up are either 1 and -1.