Universität Bielefeld Summer 2026
Geometry and Topology — Solutions to Sheet 4

Exercise 1

1. Suppose first that X is Hausdorff. Then, if $(x, y) \in X \times X$ is not in the diagonal, i.e. if $x \neq y$, we can find U, V opens with $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. But in particular, the last equality implies that $(U \times V) \cap \Delta_X = \emptyset$, which means that $U \times V$ is an open neighborhood of (x, y) in $X \times X$ endowed with the product topology, hence Δ_X is closed.

Now suppose Δ_X is closed. If $x \neq y$ then $(x, y) \in X \times X \setminus \Delta_X$ which is open. We can therefore pick a basic open neighborhood of (x, y) of the form $U \times V$ with U, V open. But note $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

2. The product topology on $X \times X$ is such that $f, g : Y \to X$ define a continuous function $h := (f, g) : Y \to X \times X$. Then $\{y \in Y \mid f(y) = g(y)\} = h^{-1}(\Delta_X)$. Since X is Hausdorff, Δ_X is closed and since h is continuous, so is $h^{-1}(\Delta_X)$. This concludes.

Exercise 2

1. Since G is a topological group, $f : G \times G \to G$ sending $(x, y) \mapsto xy^{-1}$ is continuous and it suffices to check that f sends $\overline{H} \times \overline{H}$ to $\overline{H}$. Since H is a group, we have

$$H \times H \subset f^{-1}(H) \subset f^{-1}(\overline{H})$$

But f is continuous, hence $f^{-1}(\overline{H})$ is closed, hence contains the closure of $H \times H$. To conclude, it suffices to see that

$$\overline{H} \times \overline{H} \subset \overline{H \times H}$$

(and therefore, the two are equal). For any point (x, y) in the open $(G \times G) \setminus \overline{H \times H}$, there is an open neighbourhood of the form $U \times V$ contained in the previous open, where U, V are opens of G. In particular, $x \in U \subset G \setminus \overline{H}$ and $y \in V \subset G \setminus \overline{H}$, whence $(x, y) \in (G \times G) \setminus (\overline{H} \times \overline{H})$. Passing to complements, we get the wanted inclusion.

2. Suppose that G/H is Hausdorff. Then, $\pi : G \to G/H$ is continuous and by a previous exercise, $\{e\} \subset G/H$ is closed. It follows that $\pi^{-1}(\{e\}) = H$ is closed in G.

Now suppose that $H \subset G$ is closed. We want to prove that $\Delta_{G/H}$ is closed; note that $(g, g') \in G \times G$ is such that $\pi(g, g') \in \Delta_{G/H}$ if and only if there exists $h \in H$ such that $g = g'h'$, i.e. $g(g')^{-1} \in H$.

We note that that $\{g, g' \in G \times G \mid g(g')^{-1} \in H\}$ is closed, as the preimage of the closed H by the continuous $(x, y) \mapsto xy^{-1}$. Note that closed sets in G/H are precisely those maps whose inverse image along π is closed in G, by definition of the quotient topology on G/H. But by the previous remark

$$\{g, g' \in G \times G \mid g(g')^{-1} \in H\} = (\pi \times \pi)^{-1}(\Delta_{G/H})$$

This proves that $\Delta_{G/H}$ is closed in the quotient topology on $(G \times G)/(H \times H)$; but note here that this is not *a priori* the same space as $G/H \times G/H$. There is a canonical, continuous map $\Phi : (G \times G)/(H \times H) \to G/H \times G/H$ which is bijective; if it is a homeomorphism, we are done by applying exercise 1 and get that G/H is Hausdorff from the closure of $\Delta_{G/H}$.

The fact that Φ is a homeomorphism follows from a result in the lecture notes, namely Lemma 3.32 and the example that follows. Indeed, in the case of topological groups, the projection $\pi : G \to G/H$ is always open, hence we get

$$(G \times G)/(H \times G) \simeq G/H \times G$$

and then we can apply this again to the other factor to get

$$((G \times G)/(H \times G))\,/(G/H \times H) \simeq G/H \times G/H$$

Finally, the left hand side is as a group, equivalent to $(G \times G)/(H \times H)$ and quotienting by $H \times G$ followed by $G/H \times H$ induces the same topology as quotienting once by $H \times H$ because both topologies are coinduced from the same map, i.e. the projection from $G \times G$. Here we are using that if $\mathcal{T}$ is a topology, then $g_! f_! \mathcal{T} = (f \circ g)_! \mathcal{T}$.

Exercise 3

We know that e is surjective by standard facts about writing numbers in base b. We first prove that e is continuous: suppose $x = e(a_0, ..., a_n, ...)$, then the following open set

$$\prod_{i=1}^{N} \{a_i\} \times \prod_{i \geqslant N+1} \{0, ..., b-1\}$$

is contained in the preimage of any interval centred around x, by making N large enough. This is because

$$\left| \sum_{i \geqslant N+1} \frac{f_k}{(b+1)^k} \right| < b \cdot \frac{\frac{1}{b^N}}{1 - \frac{1}{b}}$$

and the right hand side goes to 0 as $n \to +\infty$.

To show that e is a quotient map, it now suffices to prove that a map $g : [0,1] \to X$ is continuous, where X is any topological space, if and only if $g \circ e$ is continuous. One direction is evident and for the converse, it suffices to prove that for a subset $U \subset [0,1]$, if $e^{-1}(U)$ is open then U is open in $[0,1]$.

Suppose then $e^{-1}(U)$ is open and let us pick $x = e(a_i) \in U$ then the explicit description of opens in the product lets us find an open $V \subset e^{-1}(U)$ of the following form

$$V := \prod_{i=1}^{N} \{a_i\} \times \prod_{i \geqslant N+1} \{0, ..., b-1\}$$

by writing x in base b and truncating. Here we have to be a little more careful in some edge cases so let us modify V in the following cases: if $a_i = 0$ for large n and a_{n_0} is non-zero, then replace $\{a_{n_0}\}$ by $\{a_{n_0}, a_{n_0} - 1\}$ and dually if $a_i = b$ for n large enough and $a_{n_0} < b$, then replace $\{a_{n_0}\}$ by $\{a_{n_0}, a_{n_0} + 1\}$. Finally, this leaves the two cases where $a_i = 0$ for all i or $a_i = b$ for all i which corresponds to the case $x = 0$ and $x = 1$ respectively, where we do not change anything. We have survived the case distinction and can return to the proof.

Now we prove that

$$e(V) = \left\{ y \in [0,1] \mid y = \sum_{i \geqslant 1} \frac{a_i}{(b+1)^i} \text{ for } (a_i) \in V \right\}$$

is an interval whose interior contains x. The claim about the interior is precisely why we modified the V a little in the edge cases; here, we note that interiors are taken *in* $[0,1]$ so in the case where $x = 0$ or $x = 1$, they are only half-open intervals

To see that $e(V)$ is an interval, we prove that it is connected; note that this is *not* the case of V. Still, any decomposition $V = V_1 \sqcup V_2$ by non-empty opens V_1, V_2 has to be such that after a certain rank N, both V_1 and V_2 contain $\prod_{i \geqslant m+1} \{0, ..., b-1\}$ and therefore, at rank m, a decomposition $\{0, ..., b-1\} = V_1^m \sqcup V_2^m$ where V_i^m is the set of m^{th}-coordinates in V_i (i.e. the projection onto the m^{th} factor).

Without loss of generality, using that everything is finite, we may assume that the V_i only differ at m and $V_1^m = \{0, ..., i\}$, $V_2^m = \{i+1, ..., b\}$ with $i < b$. But note that

$$e(a_0, ..., i, b, ...) = e(a_0, ..., i+1, 0, ...)$$

where the i and the $i+1$ are in position m. In particular, any such decomposition is no longer disjoint after applying e. To conclude, note that any disjoint $e(V) = W_1 \sqcup W_2$ will induce a disjoint decomposition of $V = (e^{-1}(W_1) \cap V) \sqcup (e^{-1}(W_2) \cap V)$ which we just ruled out. This implies that $e(V)$ is connected which concludes.

Exercice 4

Call of the map of the exercise ϕ. It is clear that ϕ is continuous, and note that $\phi(x, y) = \phi(x', y')$ implies $x = x'$ and either $y = y'$ or $y = 1$ and $y' = 0$ or the converse, the latter condition we can summarize by $\{y, y'\} = \{0, 1\}$. In particular, ϕ becomes injective when passing to the quotient.

Of course, f is also surjective onto $Y :=]0,1] \times [0,1] \sqcup \{(0,1)\}$ so the real question is whether it is a homeomorphism. We claim this is not true: to see this, let us consider the following open set in $X/\sim$, which we identify *as a set* with $[0,1]^2$:

$$U = \{(x, y) \in [0,1]^2 \mid y > 1 - x\} \cup \{(0,1)\}$$

This pulls back under ϕ to the subset $\{(x, y) \in]0,1] \times [-1, 0] \mid y - 1 > 1 - x\} \sqcup [0,1] \times \{1\}$, which is open in X as both components are. Hence U is also open in the quotient $X/\sim$.

But U is not open in the usual topology on $[0,1]^2$: the point $(0,1)$ has no open neighbourhood since U contains no ball centred $(0,1)$, as any such ball has points of the form $(\varepsilon, 1-2\varepsilon)$ and those are not in U for ε sufficiently small.