Universität Bielefeld Summer 2026

Geometry and Topology — Solutions to Sheet 8

Exercise 1

We treat the topologies in reverse order: first, consider Homeo($\mathbb{R}^n$) endowed with the **uniform topology**, which is induced by $||f||_\infty := \sup_{x\in\mathbb{R}^n} |f(x)|$. We prove that composition does *not* make it into a topological group.

To see this, note that the family of functions $f_n(x) := x + \frac{1}{n}$ converges uniformly to $\mathrm{id}_{\mathbb{R}^n}$. But if $g : \mathbb{R}^n \to \mathbb{R}^n$ is the homeomorphism which sends $(x_1, ..., x_n)$ to $(x_1^3, ..., x_n^3)$, we have for a given $n \in \mathbb{N}$:

$$||g \circ f_n - g||_\infty = +\infty$$

since, already in each coordinate, we have expressions of the form $3x_1^2\frac{1}{n} + 3x_1\frac{1}{n^2} + \frac{1}{n^3}$, which diverge in x_1. Therefore, $g \circ f_n$ does not converge to $g = g \circ \mathrm{id}_{\mathbb{R}^n}$ uniformly hence composition is not continuous for this topology.

Next, we consider instead the **locally-uniform topology**, which we have shown in a previous sheet to be metrizable by the Fréchet distance. This time, we claim that Homeo($\mathbb{R}^n$) *does* form a topological group for this topology. Let us fix f_n, f, g homeomorphisms of $\mathbb{R}^n$ such that f_n converges locally uniformly to f. We first prove that $f_n \circ g$ and $g \circ f_n$ converge locally uniformly to respectively $f \circ g$ and $g \circ f$; note that composition is not commutative so we really have to prove both.

Fix $x \in \mathbb{R}^n$ and $\varepsilon > 0$, then there exists a open neighbourhood $x \in U$ such that $||f_n - f||_{\infty,U} < \varepsilon$. Any such open neighbourhood contains an open ball $B(x, \delta)$ centred around x and $K := \overline{B(x, \frac{\delta}{2})}$ is therefore a compact neighborhood of x contained in U. The compactness of K guarantees that for all $x, y \in K$

$$|g(x) - g(y)| \leqslant \left(\sup_{z\in K} |g(z)|\right) \cdot |x - y|$$

It follows that for $y \in K$, we have:

$$|g(f_n(y)) - g(f(y))| < \left(\sup_{z\in K} |g(z)|\right) \cdot |f_n(y) - f(y)| < \left(\sup_{z\in K} |g(z)|\right) \cdot \varepsilon$$

Note that $(\sup_{z\in K} |g(z)|)$ is actually a fixed constant (making the compact K or the open U smaller does not increase this value) hence this shows that $g \circ f_n \to g \circ f$ locally uniformly around x.

For the other convergence, we note that there is an open V around $g(x)$ such that $||f_n - f||_{\infty,V} < \varepsilon$. Since g is a homemorphism, $V = g^{-1}(g(V))$ and letting $U = g(V)$, we have found an open which contains x and such that $||f_n \circ g - f \circ g||_{\infty,U} < \varepsilon$ whence $f_n \circ g$ converges locally uniformly to U.

Finally, to conclude, it remains to prove that f_n^{-1} converges locally uniformly to the inverse f^{-1}. But we can reduce to proving that $f \circ f_n^{-1}$ converges locally uniformly to id; indeed, if we have proven this, then we can post-compose by f^{-1} and deduce from the above result that $f^{-1} \circ f \circ f_n^{-1} = f_n^{-1}$ converges to f^{-1}.

But now, fixing $x \in \mathbb{R}^n$ and writing $y = f_n(f_n^{-1}(y))$, there is a neighbourhood V of $f_n^{-1}(x)$ such that for every y such that $f_n^{-1}(y) \in V$, we have

$$|f \circ f_n^{-1}(y) - y| = |f \circ f_n^{-1}(y) - f_n(f_n^{-1}(y))| < \varepsilon$$

But the set of such y's is none other than $f_n^{-1}(V)$ which is an open subset containing x. Hence the above inequality witnesses that $f \circ f_n^{-1}$ converges locally uniformly to id, which concludes for this topology.

Finally, we deal with topology of the **pointwise convergence**, which is not metrizable; but thankfully, we will prove $\mathrm{Homeo}(\mathbb{R}^n)$ is not a topological group unless $n = 1$ hence we will be able to deal with sequences.

We first treat the caveat $n = 1$: in this case, we claim that any homeomorphism $f_n : \mathbb{R} \to \mathbb{R}$ which converges pointwise to some homeomorphism f actually converges locally uniformly to f. This comes from the observation that any homeomorphism of $\mathbb{R}$ is actually monotonous by the intermediate value theorem. The result is now a theorem of Dini (namely, what is sometimes known as Dini-Pólya or Dini's second theorem): a sequence of monotonous functions on a compact interval which converge pointwise actually converges locally uniformly.

Let us include a short proof for the convenience of the reader: fix $a < b \in \mathbb{R}$. Without loss of generality we may assume that f and f_n are all increasing. For $\varepsilon > 0$, there is an integer k such that $k > \frac{f(b)-f(a)}{\varepsilon}$ and a subdivision $a_0 = a < ... < a_i < ... < a_k = b$ of $[a, b]$ such that for $f(a_{i+1}) - f(a_i) < \varepsilon$. Fix n such that $|f_n(a_i) - f(a_i)| < \varepsilon$, then if $x \in [a_i, a_{i+1}]$, we have

$$f_n(x) - f(x) \leqslant f_n(a_{i+1}) - f(a_i) < f_n(a_{i+1}) - f(a_{i+1}) + \varepsilon$$

and

$$f_n(x) - f(x) \geqslant f_n(a_i) - f(a_{i+1}) \geqslant f_n(a_i) - f(a_i) - \varepsilon$$

which together imply $|f_n(x) - f(x)| < 2\varepsilon$. This holds on all of $[a, b]$ which proves that f_n converges uniformly on every compact to f, as wanted.

Let us now turn towards a proof that for $n \geqslant 2$, $\mathrm{Homeo}(\mathbb{R}^n)$ is not a topological group for the pointwise convergence. We will construct two sequences of homeomorphism $f_n, g_n : \mathbb{R}^2 \to \mathbb{R}^2$ which converge respectively to homeomorphisms f, g but such that $g_n \circ f_n$ does not converge to $g \circ f$. This is sufficient because one can always extends such a homeomorphism to a homeomorphism $\widehat{f_n} : \mathbb{R}^n \to \mathbb{R}^n$ by sending $(x_1, ..., x_n)$ to $(f_n(x_1, x_2), x_3, ..., x_n)$ and similarly to get $\widehat{f}$, $\widehat{g_n}$ and $\widehat{g}$. Then since the convergence is pointwise, it is also coordinate-wise, hence the sequences $\widehat{f_n}$ and $\widehat{g_n}$ will share the above properties hence we will be done.

We pick $f_n(x, y) = (x + \frac{1}{n}, y)$ which converges pointwise to the identity. We now claim there is a homeomorphism which sends $(\frac{1}{n}, 0)$ to $(\frac{1}{n}, 1)$ and which is the identity outside of

$$U_n := \left\{ (x, y) \text{ such that } \exists t \in [0, 1], \sqrt{(x - \frac{1}{n})^2 + (y - t)^2} < \frac{1}{2^n} \right\}$$

Equivalently, U_n is the set of those points which are at a distance less than $\frac{1}{2^n}$ of $\frac{1}{n} \times [0, 1]$. This has nothing special with $\frac{1}{n}$ and up to shifting, we actually prove the claim for 0 because it makes the explicit formulae become easier to write. We also lt $\varepsilon = \frac{1}{2^n}$. Then, for $x \in]-\varepsilon; \varepsilon$, we let

$$\delta(x) := \sqrt{1 - \frac{x^2}{\varepsilon^2}}$$

Then, the above homeomorphism is given by the formula

$$h(x,y) := \begin{cases} (x, (2^n+1)y + \delta(x)) & \text{if } y \in]-\delta(x), 0] \\ (x, (1 - \frac{\delta(x)}{1+\varepsilon\delta(x)})y + \delta(x)) & \text{if } y \in [0, 1+\delta(x)[\end{cases}$$

and the identity otherwise. We leave the unruffled reader to check these give a well-defined, continuous, bijective function and he is poised to also find that h is a homeomorphism by investigating the inverse.

In any case, we pick g_n to be such an homeomorphism (shifted by $\frac{1}{n}$, so that it fits the initial demands). Then, g_n converges pointwise to id because every point is eventually outside of U_n (in fact, the reader is welcomed to convince himself that U_n is a sequence of shrinking oval shapes which converges towards $0 \times [0,1]$ but never includes it). But we can compute

$$(g_n \circ f_n)((0,0)) = g_n(\frac{1}{n}, 0) = (\frac{1}{n}, 1)$$

hence $g_n \circ f_n$ certainly does not converge to the identity.

Exercise 2

1. We first note that $\phi_{P,Q}$ is well-defined: for a $\mathbb{R}$-linear map $\phi : P \to Q$, $\mathrm{id}_P + \phi : P \to \mathbb{R}^n$ is necessarily injective since $P \cap Q = \{0\}$ hence its image is of the same dimension as P. Moreover, if $p + \phi(p) \in Q$ for $p \in P$, then necessarily $p = 0$ whence $\mathrm{Im}(\mathrm{id}_P + \phi) \cap Q = \{0\}$.

Write $U_{P,Q}$ for the target of $\phi_{P,Q}$ then if $V \in U_{P,Q}$, we have that $\varphi_V : V \hookrightarrow \mathbb{R}^n \twoheadrightarrow P$ is an $\mathbb{R}$-linear isomorphism. Here, the first map is the inclusion of V into all of $\mathbb{R}^n$ and the second the projection on the direct summand; in particular, it is $\mathbb{R}$-linear between vector spaces of the same dimension and since $V \cap Q = \{0\}$, it also must be injective.

We let $\Psi_{P,Q} : U_{P,Q} \to \mathrm{Hom}_{\mathbb{R}}(P,Q)$ be given by

$$\Psi_{P,Q}(V) := (\ P \xrightarrow[\varphi_V^{-1}]{\cong} V \hookrightarrow \mathbb{R}^n \twoheadrightarrow Q\ )$$

where the second map is the inclusion and the third map the projection on the direct summand Q. We check that $\Psi_{P,Q} \circ \phi_{P,Q} = \mathrm{id}$ and $\phi_{P,Q} \circ \Psi_{P,Q} = \mathrm{id}$. For the former, note that the map $\varphi_{\mathrm{Im}(\mathrm{id}_P + \phi)}$ is the isomorphism $\mathrm{id} - \phi$ hence $\mathrm{Im}(\mathrm{id}_P + \phi) \twoheadrightarrow Q$ recovers precisely ϕ. For the latter, note that $\mathrm{Im}(\mathrm{id}_P + \varphi_V) \subset V$ by construction and both vector spaces have the same dimension.

In particular, $\phi_{P,Q}$ and $\Psi_{P,Q}$ are inverse to one another. To conclude, we have check that they both are continuous and that $U_{P,Q}$ is open in $\mathrm{Gr}_k(\mathbb{R}^n)$. For this last point, we refer to the proof of I.3.30 which describes the opens $\mathrm{Gr}_k(\mathbb{R}^n)$ so that it suffices to make the following observation: fix a basis $e_{k+1}, ..., e_n$ of Q, then we note that $V \in U_{P,Q}$ if and only if there exists a basis of V $e_1, ..., e_n$ which completes the one of Q into a basis of $\mathbb{R}^n$.

To prove that $\phi_{P,Q}$ and $\Psi_{P,Q}$ are continuous, we lift them to maps between the Stiefel manifolds. First note that $\phi_{P,Q}$ is actually the composite

$$\mathrm{Hom}_{\mathbb{R}}(P, Q) \longrightarrow V_k(\mathbb{R}^n) \longrightarrow \mathrm{Gr}_k(\mathbb{R}^n)$$

where the second map is the canonical projection (in particular continuous) and the first map is given by picking a basis $e_1, ..., e_k$ of P and considering $(e_1 + \phi(e_1), ..., e_k + \phi(e_k))$ which spans a vector space of dimension k by the same arguments as previously. But this map is clearly continuous as both addition and evaluation at a point (which, in the matrix perspective is simply multiplication) are continuous using that $M_k(\mathbb{R}^n)$ is a topological ring.

For $\Psi_{P,Q}$, fix $e_{k+1}, ..., e_n$ a basis of Q and consider the open subspace V_Q of $V_k(\mathbb{R}^n)$ spanned by the $e_1, ..., e_k$ which complete the previous one into a basis of $\mathbb{R}^n$. The image of V_Q under the quotient map is precisely $U_{P,Q}$ $\pi : V_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n)$ is open, the restriction $V_Q \to \pi(V_Q) = U_{P,Q}$ is again a quotient map.

In consequence, it suffices to check the composite $W_Q \to \mathrm{Hom}_{\mathbb{R}}(P, Q)$ between subspaces of $\mathbb{R}^n$ is continuous. But now, we can simply inspect the above formula, and note that every operation is continuous: the only potential sticking point is the inversion ϕ_V^1 but we have already seen that $\mathrm{GL}_k(\mathbb{R})$ is a topological group so in particular, the inversion is continuous.

2. Fix $P \oplus Q = \mathbb{R}^n \simeq P' \oplus Q'$ two decompositions of $\mathbb{R}^n$, then we have to check that

$$\mathrm{Hom}_{\mathbb{R}}(P, Q) \cap \phi^{-1}_{P,Q}(U_{P',Q'}) \xrightarrow{\phi_{P,Q}} U_{P,Q} \cap U_{P',Q'} \xrightarrow{\phi^{-1}_{P',Q'}} \mathrm{Hom}_{\mathbb{R}}(P', Q') \cap \phi^{-1}_{P',Q'}(U_{P,Q})$$

is smooth. We can simply go through the previous formula point by point: it holds that $\mathrm{GL}_n(\mathbb{R})$ is actually a *Lie group*, i.e. that the manifold structure is such that the multiplication $\mu : \mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ and the inverse $i : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ are smooth maps. Indeed, this can be checked pointwise and for μ, it comes down to observing that the formula for multiplying matrix is linear in each coefficient hence smooth; for i, it follows again from Cramer's rule: determinants are polynomial in each variable hence smooth and so both the cofactor matrix and dividing by the determinant are smooth operations.

In consequence, we see that every of the previous operations was smooth; this includes the projection onto a subspace which are clearly smooth as well.

Exercise 3

The commutativity of the diagram forces the composite $\mathcal{M} \to \gamma_1 \to \mathbb{RP}^1$ to be $(x, t) \mapsto \mathbb{R}(\cos(\frac{x}{2}), \sin(\frac{x}{2}))$. Given that the first component has to be a point in that line, we are led to write

$$\begin{aligned} \psi : \mathcal{M} &\longrightarrow \gamma_1 \\ (x, t) &\longmapsto \Big(t(\cos(\frac{x}{2}), \sin(\frac{x}{2})), \mathbb{R}(\cos(\frac{x}{2}), \sin(\frac{x}{2}))\Big) \end{aligned}$$

This is clearly well-defined and continuous as we can write the formula as a map $[0, 2\pi] \to \gamma_1$ which clearly descends to the quotient. Note that this map is bijective: given $V \in \mathbb{RP}^1$, there is precisely one $x \in [0, 2\pi]$ such that $V = \mathbb{R}(\cos(\frac{x}{2}), \sin(\frac{x}{2}))$, except if $x = \{0, 2\pi\}$ but these two angles are identified in $\mathcal{M}$. If we write $\mathcal{L} : V \longmapsto x$, we see that this is a continuous map (as it is locally a complex logarithm) and therefore the map

$$(v, V) \longmapsto \left(|v|, \mathcal{L}\left(\frac{v}{|v|}\right)\right)$$

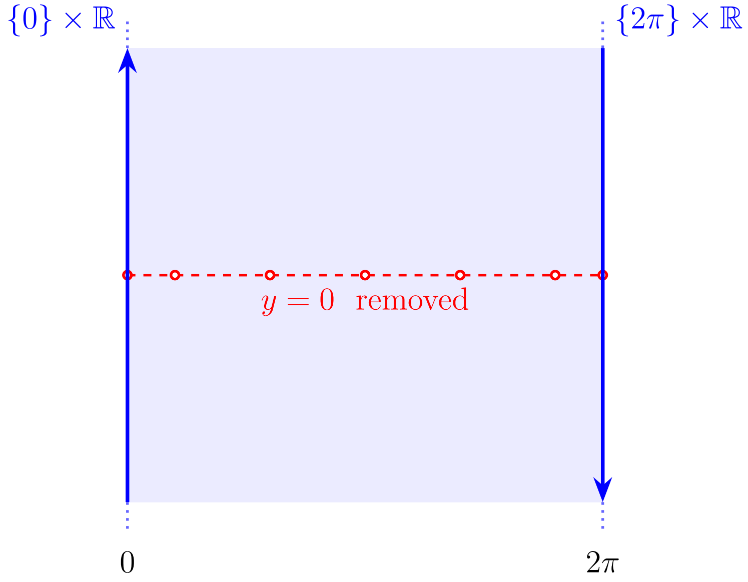
is the wanted inverse. As it is continuous, we are done.

By construction, the map we built takes $\mathcal{M}_{\leqslant 1}$ to $D(\gamma_1)$, and surjectively so, hence restricts to the wanted homeomorphism. Finally, to compute $\gamma_1 \setminus \mathrm{Im}(n)$, we can instead compute $\mathcal{M} \setminus \mathrm{Im}(\psi \circ n)$. Remark that

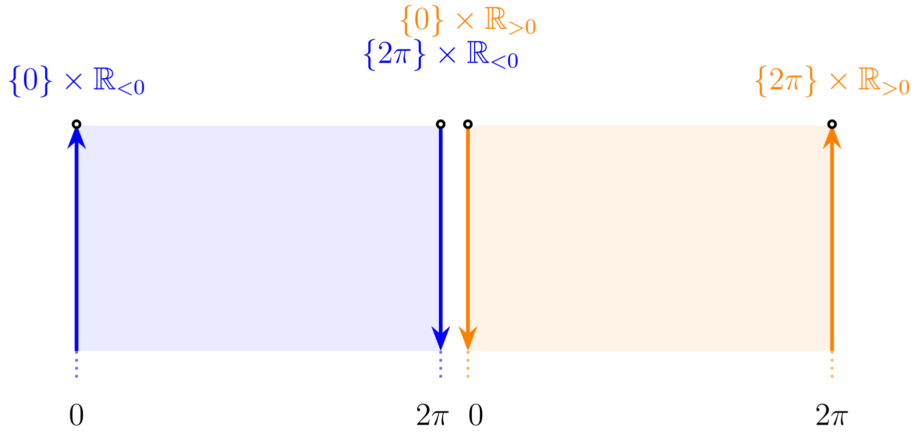
$$([0, 2\pi] \times \mathbb{R}) \setminus ([0, 2\pi] \times \{0\}) \longrightarrow \mathcal{M} \setminus \mathrm{Im}(\psi \circ n)$$

is a quotient map; differently stated, we can reverse the order of operations and first remove the subspace before taking the quotient. This is because the preimage of $\mathrm{Im}(\psi \circ n)$ is precisely $([0, 2\pi] \times \{0\})$ and it is closed in $[0, 2\pi] \times \mathbb{R}$ so the opens of the complement are simply opens of the whole space which do not meet either of the two spaces.

Courtesy of a large language model, the remainder of the proof now becomes visual. We can picture the space in question as:



and this is homeomorphic to the following, where we indicated in orange the top half that has been rotated and moved to the right of the bottom half:



The equivalence relation precisely glues along the edges that are pointing the same way (and does not involve edges that are not pointing the same way or points outside of the edges). First gluing the inner edges and then the outer edges, we see the cylinder $S^1 \times \mathbb{R}_{>0}$ appear.

Exercice 4

1. We prove that $\phi : \gamma_n \to \mathbb{RP}^{n+1}$ sending $(v, \mathbb{R}(x_1, ..., x_n))$ to $[x_1 : ... : x_n : \langle x, v\rangle]$ is a homeomorphism onto its image. Again, since the projection $V_1(\mathbb{R}^n) \to \mathbb{RP}^n$ is open, we may define this map before passing to quotients via

$$\{(v, x) \in \mathbb{R}^n \times V_1(\mathbb{R}^n) \mid v \in \mathbb{R}x\} \longrightarrow V_1(\mathbb{R}^{n+1})$$

and then the projection to $\mathbb{RP}^{n+1}$. The latter is continuous by definition and the former is sending $(v, (x_1, ..., x_n))$ to $(x_1 : ... : x_n : \langle x, v\rangle)$ which is clearly continuous. It follows that ϕ is a well-defined continuous map. Its image is readily check to be those $[x_1 : ... : x_n : x_{n+1}]$ where $[x_1 : ... : x_n] \neq 0$ in $\mathbb{RP}^n$ since the x in the above formula must span a line (and not $\{0\}$).

In fact, this image V is precisely the image under the quotient map of the subspace U of $V_1(\mathbb{RP}^{n+1})$ spanned by those $x := (x_1, ..., x_n, x_{n+1})$ where $(x_1, ..., x_n)$ is non-zero. Since the projection $V_1(\mathbb{RP}^{n+1}) \to \mathbb{RP}^n$ is open, $U \to V$ is again a quotient map. It follows that we can define and check the continuity of the purported inverse of ϕ as a map $U \longrightarrow \gamma_n$. Let then be $\psi : U \to \gamma_n$ which sends

$$(x_1, ..., x_n, x_{n+1}) \longmapsto \left(\frac{x_{n+1}}{\langle x, x\rangle}(x_1, ..., x_n), \mathbb{R}(x_1, ..., x_n)\right)$$

It is straightforward that this is a continuous map which is the inverse of ϕ, hence we are done.

2. We build now a continuous, surjective map $D(\gamma_n) \to \mathbb{RP}^{n+1}$ which sends $S(\gamma_n)$ to $[0 : ... : 0 : 1]$ and is injective elsewhere. Since $D(\gamma_n)$ is compact and $\mathbb{RP}^{n+1}$ Hausdorff, this implies that the given map is a homeomorphism as wanted.

We take the map to be induced by the following map on the Stiefel manifolds:

$$(v,(x_1,...,x_n)) \longmapsto ((1-|v|)x_1,...,(1-|v|)x_n,\langle x,v\rangle)$$

This is continuous; when $|v|=1$, it is sent to $[0:...:0:1]\in\mathbb{RP}^n$ as wanted. Moreover, this map is surjective because $|v|\mapsto\frac{|v|}{1-|v|}$ is surjective from $[0,1[$ onto $[0,+\infty[$. This concludes.