Complex Analysis: Exercise 10

1. Let $g, h : \mathbb{C} \to \mathbb{C}$ be analytic functions which are both not constant. Then $f(z) = g(z)/h(z)$ defines a meromorphic function, possibly with isolated zeros and poles. Assume f has no poles on the positive real number line. Assume furthermore that there exist two numbers λ_1 and λ_2, with $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$, such that for sufficiently small positive real numbers $x > 0$, we have $f(x) < x^{-\lambda_1}$ and for sufficiently large x, we have $f(x) < x^{\lambda_2}$. Let $0 \le \phi < \pi/2$ and assume that there are no poles of f on the line given by $te^{i\phi}$, where $0 < t < \infty$.

   - Show that $\int_0^\infty f(te^{i\phi})e^{i\phi}\,dx$ exists.
   - Show that for sufficiently small ϕ, we have

   $$\lim_{\phi \to 0} \int_0^\infty f(te^{i\phi})e^{i\phi}\,dx = \int_0^\infty f(x)dx.$$

2. In theorem 39, we looked at the function $x^\lambda R(x)$, and found that it was possible to evaluate the integral

   $$\int_0^\infty x^\lambda R(x)dx$$

   by following a closed curve, consisting of four segments, then using the residue theorem. For this exercise, assume as in theorem 39 that R has no poles on the non-negative real numbers, and it has a zero of order at least two at infinity. But this time, consider the function $\log(z) \cdot R(z)$, rather than $x^\lambda R(x)$. Show that by doing so, and using the calculus of residues, it is possible to evaluate the integral

   $$\int_0^\infty R(x)dx.$$