Complex Analysis: Exercise 2

1. Assume we have two power series, $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$, both of which converge within a circle of radius $R > 0$. Assume that
$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n,$$
for all z with $|z| < R$. Show that $a_n = b_n$ for all n.

2. Let $G = \mathbb{C} \setminus \{0\}$ and let $\gamma : [0, 1] \to G$ be a continuous closed path. (Thus, in particular, $\gamma(0) = \gamma(1)$.) For each $t \in [0, 1]$, let
$$\phi(t) = \frac{\gamma(t)}{|\gamma(t)|}.$$
Therefore ϕ is a path on the unit circle in $\mathbb{C}$. Show that ϕ can only make a finite, well defined number of circuits of the circle (the *winding number*).

3. Let f be analytic in the region G and let $\gamma_1 : [a_1, b_1] \to G$ and $\gamma_2 : [a_2, b_2] \to G$ be two continuously differentiable paths in G. Assume there exists a continuously differentiable mapping $\phi : [a_1, b_1] \to [a_2, b_2]$ such that $\gamma_1(t) = \gamma_2(\phi(t))$, for all $t \in [a_1, b_1]$.[1] Do we then have
$$\int_{a_1}^{b_1} f(\gamma_1(t))\gamma_1'(t)dt = \int_{a_2}^{b_2} f(\gamma_2(t))\gamma_2'(t)dt\,?$$

4. Let G be a region, and let $f : G \to \mathbb{C}$ be analytic. Assume that for some particular $z_0 \in G$ we have both $f(z_0) = 0$ and $f'(z_0) \neq 0$.
   (a) Show that there exists an $\epsilon > 0$ such that for all z with
   $$|z - z_0| = \epsilon,$$
   we have $z \in G$ and also $f(z) \neq 0$.
   (b) Show that for sufficiently small $\epsilon > 0$ we have
   $$\int_{|z-z_0|=\epsilon} \frac{dz}{f(z)} = \frac{2\pi i}{f'(z_0)}.$$

[1] Assume also that $\phi(a_1) = a_2$ and $\phi(b_1) = b_2$.